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UNIVERSAL THEORIES AND COMPACTLY EXPANDABLE MODELS

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Abstract. Our aim is to solve a quite old question on the difference between expandability and compact expandability. Toward this, we further investigate the logic of countable cofinality.

§1. Introduction. In this article we solve an open problem on expandability of models by an application of some new results we obtain on the logic $L(Q^{\mathrm{cf}}_{\aleph_0})$, first-order logic with the additional quantifier $Q^{\mathrm{cf}}_{\aleph_0}$ of cofinality $\aleph_0$. The syntax of $L(Q^{\mathrm{cf}}_{\aleph_0})$ allows the construction of formulas of the form $Q^{\mathrm{cf}}_{\aleph_0} xy\varphi(x, y, \overline{z})$. The meaning of this formula in a structure M is given by the rule: $M \models Q^{\mathrm{cf}}_{\aleph_0} xy\varphi(x, y, \overline{a})$ if and only if the relation $\{(b, c) : M \models \varphi(b, c, \overline{a})\}$ is a linear ordering of the set $\{c : M \models \exists x\varphi(x, c, \overline{a})\}$ and has cofinality $\aleph_0$.

The second author has introduced $L(Q^{\mathrm{cf}}_{\aleph_0})$ in [4] and has proved that is a fully compact logic, i.e., a set Σ of $L(Q^{\mathrm{cf}}_{\aleph_0})$-sentences (of any cardinality) has a model if every finite subset of Σ has a model. In the same article it is proved that $L(Q^{\mathrm{cf}}_{\aleph_0})$ satisfies the Löwenheim–Skolem theorem down to $\aleph_1$, in the following particularly strong form combining downward an upward Löwenheim–Skolem theorems: if Σ is a set of $L(Q^{\mathrm{cf}}_{\aleph_0})$-sentences having an infinite model, then Σ has a model in every cardinal $\kappa \geq \aleph_1, |\Sigma|$. See Theorems 2.5 and 2.6 of [4] applied to $n = 0$, $\mu = 1$ and $\lambda_0 = \aleph_0$. Compactness of $L(Q^{\mathrm{cf}}_{\aleph_0})$ will be used here to show that for any cardinal $\kappa = 2^{<\kappa} > \aleph_0$, $L(Q^{\mathrm{cf}}_{\aleph_0})$ has a κ-universal theory. The Löwenheim–Skolem theorem won't be used until the very end, in the proof of Proposition 3.6.

In Section 2 we define some classes of $L(Q^{\mathrm{cf}}_{\aleph_0})$-theories, in particular the class $\mathbf{T}^{\mathbf{ec}}_{<\kappa}$ of theories with vocabulary of cardinality $<\kappa$ which are in some sense an analog of existentially closed models and its extension $\mathbf{T}^{\mathbf{ab}}_{<\kappa}$, the class of theories with vocabulary of cardinality $<\kappa$ which are amalgamation bases. We prove that if $\kappa = 2^{<\kappa} > \aleph_0$, then any amalgamation base $T_* \in \mathbf{T}^{\mathbf{ab}}_{<\kappa}$ can be extended to some existentially closed $T \in \mathbf{T}^{\mathbf{ec}}_{<\kappa^+}$ which is universal over T_*, meaning that every consistent extension of T_* of cardinality $\leq\kappa$ can be embedded over T_* in T. The proof only uses compactness and some basic facts of the logic $L(Q^{\mathrm{cf}}_{\aleph_0})$, such as finitary character and possibility of renaming, and they can be easily generalized to other similar compact logics (see Remark 2.8).

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The notions of expandability and compact expandability were introduced by the first author in [1] and further discussed in [2]. They are notions of largeness for models related to first-order theories, similar to resplendency but without the use of parameters. In these articles the existence of a compactly expandable model which is not expandable is left as an open problem. In Section 3 we solve the problem using the tools developed in Section 2.

To simplify notation, we will only consider relational languages, but the results can be easily adapted to languages with constants and function symbols. A vocabulary τ is a set of symbols, predicates in our case. If T is a theory (a first-order theory or a $L(Q^{\rm cf}_{\aleph_0})$-theory) $\tau(T)$ will be the vocabulary of T. $L(Q^{\rm cf}_{\aleph_0})(\tau)$ is the set of $L(Q^{\rm cf}_{\aleph_0})$-sentences of vocabulary τ. Along the whole article κ, μ are infinite cardinal numbers. Along all this article, consistent means finitely satisfiable. Since we are dealing with a compact logic, this is the same thing as being satisfiable.

§2. Universal theories in $L(Q^{\rm cf}_{\aleph_0})$. In this section we work with sentences and theories of the compact logic $L(Q^{\rm cf}_{\aleph_0})$ of countable cofinality.

DEFINITION 2.1. Let τ_1, τ_2 be vocabularies.

1. An *embedding* of τ_1 in τ_2 is a one-to-one mapping $f : \tau_1 \to \tau_2$ that preserves arities. It is *over* $\tau_0 \subseteq \tau_1$ if every symbol of τ_0 remains fixed by f. It is an *isomorphism* if it is moreover surjective. If $\overline{R} = \langle R_1, \ldots, R_n \rangle$ and $\overline{S} = \langle S_1, \ldots, S_m \rangle$ are tuples of predicates, we write $\overline{R} \approx \overline{S}$ if they have the same length $n = m$ and the mapping defined by $R_i \mapsto S_i$ is an isomorphism of vocabularies, i.e., it is one-to-one and preserves arities.
2. Any embedding $f : \tau_1 \to \tau_2$ induces a *renaming*, a mapping from $L(Q^{\rm cf}_{\aleph_0})(\tau_1)$ into $L(Q^{\rm cf}_{\aleph_0})(\tau_2)$ for which we will use the same notation f. If $\sigma \in L(Q^{\rm cf}_{\aleph_0})(\tau_1)$, $f(\sigma)$ is the sentence obtained by substitution of every symbol $R \in \tau_1$ of σ by the corresponding symbol $f(R) \in \tau_2$.
3. Assume $\tau_1 \subseteq \tau_2$. The notation $\psi(\overline{R}, \overline{S})$ will be used for $L(Q^{\rm cf}_{\aleph_0})(\tau_2)$-sentences with the understanding that $\overline{R} \subseteq \tau_1$ and $\overline{S} \subseteq \tau_2 \setminus \tau_1$ are tuples of predicates without repetitions and they include all predicates appearing in the sentence.

DEFINITION 2.2. 1. Let $\mathbf{T}_{<\kappa}$ be the class of consistent $L(Q^{\rm cf}_{\aleph_0})$-theories T such that $\tau(T)$ has cardinality $<\kappa$.

2. Let $\mathbf{T}^{\rm c}_{<\kappa}$ be the class of all theories $T \in \mathbf{T}_{<\kappa}$ which are complete in $L(Q^{\rm cf}_{\aleph_0})(\tau(T))$, so in particular are closed under conjunction.
3. Let $\mathbf{T}^{\rm ab}_{<\kappa}$ be the class of all $T_0 \in \mathbf{T}^{\rm c}_{<\kappa}$ which are amalgamation bases in the following sense: if $\tau_0 = \tau(T_0)$ and τ_1, τ_2 are vocabularies with $\tau_0 = \tau_1 \cap \tau_2$ and $T_l \in \mathbf{T}_{<\kappa}$ for $l = 1, 2$ are theories with $\tau(T_l) = \tau_l$ and $T_0 = T_1 \cap T_2$, then $T_1 \cup T_2$ is consistent in $L(Q^{\rm cf}_{\aleph_0})$.
4. Finally, let $\mathbf{T}^{\rm ec}_{<\kappa}$ be the class of all theories $T \in \mathbf{T}^{\rm c}_{<\kappa}$ which are existentially closed in the following sense: for any vocabulary $\tau_1 \supseteq \tau(T)$, for any $L(Q^{\rm cf}_{\aleph_0})$-sentence $\psi(\overline{R}, \overline{S})$, where $\overline{R} \subseteq \tau(T)$ and $\overline{S} \subseteq \tau_1 \setminus \tau(T)$ are tuples of predicates without repetitions, if $T \cup \{\psi(\overline{R}, \overline{S})\}$ is consistent, then $\psi(\overline{R}, \overline{S}') \in T$ for some tuple of predicates $\overline{S}' \subseteq \tau(T)$ such that $\overline{S} \approx \overline{S}'$.

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Existentially closed theories are a parallel of existentially closed models. We do not mean, of course, that they are theories of existentially closed models.

LEMMA 2.3. 1. $\mathbf{T}_{<\kappa} \supseteq \mathbf{T}^{\mathbf{c}}_{<\kappa} \supseteq \mathbf{T}^{\mathbf{ab}}_{<\kappa} \supseteq \mathbf{T}^{\mathbf{ec}}_{<\kappa}$.

2. *Any* $T \in \mathbf{T}_{<\kappa}$ *can be extended to a member of* $\mathbf{T}^{\mathbf{c}}_{<\kappa}$ *with the same vocabulary.*

3. *Assume* $\tau_i \cap \tau_j = \tau$ *for* $i < j < \mu$, $T_i \in \mathbf{T}_{<\kappa}$, $T \in \mathbf{T}^{\mathbf{ab}}_{<\kappa}$, $\tau(T_i) = \tau_i$, $\tau(T) = \tau$, *and* $T \subseteq T_i$. *Then* $\bigcup_{i<\mu} T_i$ *is consistent.*

4. *If* $\kappa > \aleph_0$ *and* $T \in \mathbf{T}_{<\kappa}$, *then it can be extended to a member of* $\mathbf{T}^{\mathbf{ec}}_{<\kappa}$.

5. *For* $l = \mathbf{c}, \mathbf{ab}, \mathbf{ec}$: *if* f *is an isomorphism from* τ_1 *onto* τ_2, *and* $T \in \mathbf{T}^{l}_{<\kappa}$, *then* $f(T) \in \mathbf{T}^{l}_{<\kappa}$.

PROOF. 1. We check that $\mathbf{T}^{\mathbf{ec}}_{<\kappa} \subseteq \mathbf{T}^{\mathbf{ab}}_{<\kappa}$. Let $T_0 \in \mathbf{T}^{\mathbf{ec}}_{<\kappa}$ be of vocabulary τ_0, let τ_1, τ_2 be vocabularies such that $\tau_0 = \tau_1 \cap \tau_2$ and let $T_l \in \mathbf{T}^{\mathbf{c}}_{<\kappa}$ for $l = 1, 2$ be corresponding theories with $\tau(T_l) = \tau_l$ and $T_0 = T_1 \cap T_2$. Toward a contradiction, assume $\psi(\overline{R}, \overline{S}) \in T_1$, $\overline{R} \subseteq \tau_0$, $\overline{S} \subseteq \tau_1 \setminus \tau_0$ have no repetitions and $\{\psi(\overline{R}, \overline{S})\} \cup T_2$ is inconsistent. Since $\{\psi(\overline{R}, \overline{S})\} \cup T_0$ is consistent and $T_0 \in \mathbf{T}^{\mathbf{ec}}_{<\kappa}$, for some tuple of predicates $\overline{S}' \approx \overline{S}$ we have $\psi(\overline{R}, \overline{S}') \in T_0$ and hence $\{\psi(\overline{R}, \overline{S}')\} \cup T_2$ is consistent. Since the predicates of $\overline{S}$ do not belong to τ_2, we may rename again the formula, showing the consistency of $\{\psi(\overline{R}, \overline{S})\} \cup T_2$, which is against our assumption.

2. By compactness of the logic $L(Q^{\mathrm{cf}}_{\aleph_0})$.

3. The amalgamation of finitely many theories with common intersection in $\mathbf{T}^{\mathbf{ab}}_{<\kappa}$ can be proved by induction. The general case follows by compactness.

4. Fix a countable vocabulary τ_∞ disjoint of $\tau(T)$ and containing for each natural number $n \geq 1$ infinitely many n-ary predicates. Let $T_0 = T$ and $\tau_0 = \tau(T)$. We claim that for some vocabulary $\tau_1 \supseteq \tau_0$ of cardinality $|\tau_0| + \aleph_0$ and disjoint of τ_∞, there is some complete $L(Q^{\mathrm{cf}}_{\aleph_0})(\tau_1)$-theory $T_1 \supseteq T_0$ such that for every $L(Q^{\mathrm{cf}}_{\aleph_0})$-sentence $\psi(\overline{R}, \overline{S})$, if $T_1 \cup \{\psi(\overline{R}, \overline{S})\}$ is consistent and $\overline{R} \subseteq \tau_0$ and $\overline{S} \subseteq \tau_\infty$, then there is a tuple of predicates $\overline{S}' \subseteq \tau_1$ such that $\overline{S}' \approx \overline{S}$ and $\psi(\overline{R}, \overline{S}') \in T_1$. For this purpose, let $\mu = |\tau_0| + \aleph_0$ and let us enumerate $(\sigma_i : i < \mu)$ all $L(Q^{\mathrm{cf}}_{\aleph_0})(\tau_0 \cup \tau_\infty)$-sentences. We inductively define a continuous ascending chain $(\Sigma_i : i < \mu)$ of sets of $L(Q^{\mathrm{cf}}_{\aleph_0})$-sentences. Start with $\Sigma_0 = T_0$. If $\sigma_i = \psi(\overline{R}, \overline{S})$ (with $\overline{R} \subseteq \tau_0$ and $\overline{S} \subseteq \tau_\infty$) is consistent with Σ_i, take a new tuple $\overline{S}' \approx \overline{S}$ of predicates and put $\Sigma_{i+1} = \Sigma_i \cup \{\psi(\overline{R}, \overline{S}')\}$. Otherwise, $\Sigma_{i+1} = \Sigma_i$. In the limit case take the union. Then let τ_1 be the vocabulary of $\bigcup_{i<\mu} \Sigma_i$ and let T_1 be a complete $L(Q^{\mathrm{cf}}_{\aleph_0})(\tau_1)$-theory extending $\bigcup_{i<\mu} \Sigma_i$.

Iterating, we define T_{i+1} for $i < \omega$ as a complete extension of T_i in a vocabulary $\tau_{i+1} \supseteq \tau_i = \tau(T_i)$ of cardinality $|\tau_{i+1}| = |\tau_i| + \aleph_0$ which is disjoint with τ_∞. We require that for every $L(Q^{\mathrm{cf}}_{\aleph_0})(\tau_i \cup \tau_\infty)$-sentence $\psi(\overline{R}, \overline{S})$, if $T_{i+1} \cup \{\psi(\overline{R}, \overline{S})\}$ is consistent and $\overline{R} \subseteq \tau_i$ and $\overline{S} \subseteq \tau_\infty$, then there is a tuple of predicates $\overline{S}' \subseteq \tau_{i+1}$ such that $\overline{S}' \approx \overline{S}$ and $\psi(\overline{R}, \overline{S}') \in T_{i+1}$. Then $T' = \bigcup_{i<\omega} T_i$ has the required properties: $T' \in \mathbf{T}^{\mathbf{ec}}_{<k}$ extends T. In fact, its vocabulary $\tau' = \bigcup_{i<\omega} \tau_i$ verifies $|\tau'| = |\tau(T)| + \aleph_0 < \kappa$. Moreover, if $T' \cup \{\psi(\overline{R}, \overline{S})\}$ is consistent, $\overline{R} \subseteq \tau'$ and $\overline{S} \cap \tau' = \emptyset$, we may assume that $\overline{S} \subseteq \tau_\infty$ and we may fix $i < \omega$ such that $\overline{R} \subseteq \tau_i$. Hence $\psi(\overline{R}, \overline{S}') \in T_{i+1} \subseteq T'$ for some $\overline{S}' \approx \overline{S}$.

5. Obvious. ⊣

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LEMMA 2.4. 1. *Assume* $l = \mathbf{c}, \mathbf{ab}$ *or* $\mathbf{ec}$ *and* $T_i \in \mathbf{T}^l_{<\kappa}$ *for every* $i < \delta$. *If* $T_i \subseteq T_j$ *for all* $i < j < \delta$ *and* $\bigcup_{i<\delta} T_i$ *has vocabulary* τ_δ *with* $|\tau_\delta| < \kappa$ (*in particular, if* $\mathrm{cf}(\delta) < \mathrm{cf}(\kappa)$), *then* $\bigcup_{i<\delta} T_i \in \mathbf{T}^l_{<\kappa}$.

2. *Assume* $l = \mathbf{c}, \mathbf{ab}$ *or* $\mathbf{ec}$, $T_0 \subseteq T_2 \in \mathbf{T}^l_{<\kappa}$, $\mu < \kappa$ *and* $|T_0| \leq \mu$. *Then* $T_0 \subseteq T_1 \subseteq T_2$ *for some* $T_1 \in \mathbf{T}^l_{<\mu^+}$.

3. *If* $\aleph_0 < \kappa_1 < \kappa_2$ *and* $|\tau(T)| < \kappa_1$, *then for* $l = \mathbf{c}, \mathbf{ab}$ *or* $\mathbf{ec}$: $T \in \mathbf{T}^l_{<\kappa_1}$ *if and only if* $T \in \mathbf{T}^l_{<\kappa_2}$.

4. *Let* $|\tau(T)| < \kappa$. *Then for* $l = \mathbf{c}, \mathbf{ab}$ *or* $\mathbf{ec}$: $T \in \mathbf{T}^l_{<\kappa}$ *if and only if for some club* $E \subseteq [\tau(T)]^{\leq \aleph_0}$, *for every* $\tau \in E$, $T \cap L(Q^{\mathrm{cf}}_{\aleph_0})(\tau) \in \mathbf{T}^l_{<\aleph_1}$.

PROOF. 1. Clear, using compactness of $L(Q^{\mathrm{cf}}_{\aleph_0})$.

2. This is obvious for complete theories (case $l = \mathbf{c}$). Let us consider existentially closed theories (case $l = \mathbf{ec}$). Fix, as above, a countable vocabulary τ_∞ disjoint of $\tau(T_2)$ and containing for each $n \geq 1$ infinitely many n-ary predicates. Let $(\sigma_i : i < \mu)$ be an enumeration of all $L(Q^{\mathrm{cf}}_{\aleph_0})(\tau(T_0) \cup \tau_\infty)$-sentences $\sigma_i = \psi(\overline{R}, \overline{S})$ consistent with T_0 with $\overline{R} \subseteq \tau(T_0)$ and $\overline{S} \subseteq \tau_\infty$. We inductively define a corresponding sequence $(\sigma'_i : i < \mu)$ of sentences $\sigma'_i \in T_2$. If $\sigma_i = \psi(\overline{R}, \overline{S})$ is consistent with T_2, we use the fact that $T_2 \in \mathbf{T}^{\mathbf{ec}}_{<\kappa}$ to find some tuple $\overline{S}' \subseteq \tau(T_2)$ such that $\overline{S}' \approx \overline{S}$ and $\psi(\overline{R}, \overline{S}') \in T_2$, and we put $\sigma'_i = \psi(\overline{R}, \overline{S}')$. If σ_i is not consistent with T_2 we choose as σ'_i some sentence in T_2 inconsistent with σ_i. Let τ_0 be the vocabulary of $\{\sigma'_i : i < \mu\}$ and let $\Sigma_0 = T_2 \restriction \tau_0$. Notice that $T_0 \subseteq \Sigma_0$. Notice that every $L(Q^{\mathrm{cf}}_{\aleph_0})$-sentence $\psi(\overline{R}, \overline{S})$ with $\overline{R} \subseteq \tau(T_0)$ and $\overline{S} \subseteq \tau_\infty$ which is consistent with Σ_0 is also consistent with T_2, and therefore there is some $\overline{S}' \subseteq \tau_0$ such that $\overline{S}' \approx \overline{S}$ and $\psi(\overline{R}, \overline{S}') \in \Sigma_0$. Now we iterate this construction ω times, obtaining an ascending chain of vocabularies $(\tau_i : i < \omega)$ with $\tau(T_0) \subseteq \tau_0$, $\tau_i \subseteq \tau(T_2)$ and $|\tau_i| = \mu$ together with corresponding theories $\Sigma_i = T_2 \restriction \tau_i$ such that for every $L(Q^{\mathrm{cf}}_{\aleph_0})$-sentence $\psi(\overline{R}, \overline{S})$ consistent with Σ_{i+1} and with $\overline{R} \subseteq \tau_i$ and $\overline{S} \subseteq \tau_\infty$, there is some $\overline{S}' \subseteq \tau_{i+1}$ such that $\overline{S}' \approx \overline{S}$ and $\psi(\overline{R}, \overline{S}') \in \Sigma_{i+1}$. Then $T_1 = \bigcup_{i<\omega} \Sigma_i$ satisfies the requirements.

The case of an amalgamation base ($l = \mathbf{ab}$) is similar. We extend T_0 to $T_1 \in \mathbf{T}^{\mathbf{c}}_{<\mu^+}$ such that $T_1 \subseteq T_2$ and every sentence $\psi(\overline{R}, \overline{S})$ consistent with T_1 with $\overline{R} \subseteq \tau(T_1)$ and $\overline{S} \subseteq \tau_\infty$ is consistent with T_2, and then one easily checks that $T_1 \in \mathbf{T}^{\mathbf{ab}}_{<\mu^+}$.

3 is clear and 4 follows from 1 and 2. ⊣

LEMMA 2.5. *Let* $T_0, T_1, T_2 \in \mathbf{T}^{\mathbf{ec}}_{<\kappa}$ *and assume* $T_0 \subseteq T_1$ *and the embedding* $f : \tau(T_0) \to \tau(T_2)$ *maps* T_0 *into* T_2. *Then there is some* $T_3 \in \mathbf{T}^{\mathbf{ec}}_{<\kappa}$ *such that* $T_2 \subseteq T_3$ *and there is some embedding* $g : \tau(T_1) \to \tau(T_3)$ *extending* f *and mapping* T_1 *into* T_3.

PROOF. Extend f to an embedding g with domain $\tau(T_1)$ and such that $g(\tau(T_1)) \cap \tau(T_2) = f(\tau(T_0))$, define T'_0 and T'_1 as the images of T_0 and T_1 by g respectively, and apply items 5 and 1 of Lemma 2.3 to T_0 to show that $T'_0 \in \mathbf{T}^{\mathbf{ab}}_{<\kappa}$. Now, obviously, $T'_0 \subseteq T'_1 \in \mathbf{T}_{<\kappa}$, $T'_0 \subseteq T_2 \in \mathbf{T}_{<\kappa}$ and hence, by definition of $T'_0 \in \mathbf{T}^{\mathbf{ab}}_{<\kappa}$, $T'_1 \cup T_2$ is consistent. By item 4 of Lemma 2.3, it can be extended to some $T_3 \in \mathbf{T}^{\mathbf{ec}}_{<\kappa}$. Clearly, g maps T_1 into T_3. ⊣

DEFINITION 2.6. Assume $T_* \in \mathbf{T}_{<\kappa}$, $T \in \mathbf{T}^{\mathbf{ec}}_{<\kappa^+}$, and $T_* \subseteq T$. We call the theory T *κ-universal over* T_* if for every $T' \in \mathbf{T}^{\mathbf{ec}}_{<\kappa^+}$ such that $T_* \subseteq T'$, there is some embedding $f : \tau(T') \to \tau(T)$ over $\tau_* = \tau(T_*)$ mapping T' into T.

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THEOREM 2.7. *Assume $T_* \in \mathbf{T}^{\mathbf{ab}}_{<\kappa}$, i.e., it is an amalgamation base. If $\kappa = 2^{<\kappa} > \aleph_0$, then there exists some κ-universal theory $T \in \mathbf{T}^{\mathbf{ec}}_{<\kappa^+}$ over T_*.*

PROOF.

CASE 1. κ is regular.

Let τ_∞ be some vocabulary such that $\tau_* \subseteq \tau_\infty$, $|\tau_\infty| = \kappa$ and for every $n \geq 1$, τ_∞ has κ many n-ary predicates. Let $(S_i : i < \kappa)$ be a enumeration of all theories in $\mathbf{T}^{\mathbf{ec}}_{<\kappa}$ whose vocabulary is a subset of τ_∞. Notice that every theory in $\mathbf{T}^{\mathbf{ec}}_{<\kappa}$ is a renaming of some S_i.

We are going to construct a continuous ascending chain of $L(Q^{\mathrm{cf}}_{\aleph_0})$-theories $(T_i : i < \kappa)$ extending T_* such that for every $i < \kappa$:

1. $T_i \in \mathbf{T}^{\mathbf{ec}}_{<\kappa^+}$.
2. If $j, l < i$, $S_j \subseteq S_l$ and f is an embedding of : $\tau(S_j)$ into $\tau(T_i)$ mapping S_j into T_i, then f can be extended to an embedding $g : \tau(S_l) \to \tau(T_{i+1})$ mapping S_l into T_{i+1}.

We start by choosing some arbitrary initial theory $T_0 \in \mathbf{T}^{\mathbf{ec}}_{<\kappa}$ with $T_* \subseteq T_0$ and $\tau_0 = \tau(T_0) \subseteq \tau_\infty$. By Lemma 2.4, we may take unions at limit stages. Now we show how to obtain T_{i+1}. Let us consider a particular case of $j, l < i$, such that $S_j \subseteq S_l$ and $f : \tau(S_j) \to \tau(T_i)$, an embedding mapping S_j into T_i. By Lemma 2.5, there is some $T_{j,l,f} \in \mathbf{T}^{\mathbf{ec}}_{<\kappa^+}$ extending T_i and some embedding $g : \tau(S_l) \to \tau(T_{j,l,f})$ that maps S_l into $T_{j,l,f}$ and extends f. The number of the possible triples (j, l, f) is $\leq \kappa$ and hence, by iteration and taking unions at limits, we obtain $T_{i+1} \in \mathbf{T}^{\mathbf{ec}}_{<\kappa^+}$ as desired.

Now let $T = \bigcup_{i<\kappa} T_i$. By Lemma 2.4, $T \in \mathbf{T}^{\mathbf{ec}}_{<\kappa^+}$.

CLAIM 1. *If $S, S' \in \mathbf{T}^{\mathbf{ec}}_{<\kappa}$, $S \subseteq S'$ and $f : \tau(S) \to \tau(T)$ is an embedding mapping S into T, then there is an embedding $f' : \tau(S') \to \tau(T)$ extending f that maps S' into T.*

Proof of Claim 1. Since for some $l < \kappa$ there is some isomorphism between $\tau(S_l)$ and $\tau(S')$ mapping S_l onto S', we may assume that $S' = S_l$ and $S = S_j$ for some $j, l < \kappa$. Now choose $i < \kappa$ such that $i > j, l$ and the range of f is contained in $\tau(T_i)$ and notice that f maps S_j into T_i. By construction of T_{i+1}, there is some extension $f' : \tau(S_l) \to \tau(T_{i+1})$ of f mapping S_l into T_{i+1} and hence into T. This proves the claim.

We show now that T is κ-universal over T_*. Let $T' \in \mathbf{T}^{\mathbf{ec}}_{<\kappa^+}$ be such that $T_* \subseteq T'$ and decompose it (using Lemma 2.4) as $T' = \bigcup_{i<\kappa} T'_i$ where $(T'_i : i < \kappa)$ is a continuous ascending chain of theories $T'_i \in \mathbf{T}^{\mathbf{ec}}_{<\kappa}$. Since $T_* \in \mathbf{T}^{\mathbf{ab}}_{<\kappa}$, we may assume that $T_0 \subseteq T'$, and hence, without loss of generality, $T'_0 = T_0$. We claim that there is a continuous ascending chain $(f_i : i < \kappa)$ of embeddings $f_i : \tau(T'_i) \to \tau(T)$ over τ_* mapping T'_i into T. Notice that in this case $f = \bigcup_{i<\kappa} f_i$ will be an embedding of $\tau(T')$ into $\tau(T)$ over τ_* mapping T' into T. We start taking f_0 as the identity in $\tau(T'_0)$ and we take unions at limit stages. If $f_i : \tau(T'_i) \to \tau(T)$ has been obtained, then by Claim 1 we can extend f_i to f_{i+1} mapping T'_{i+1} into T.

CASE 2. κ is singular.

By König's Theorem on cofinality, κ is a strong limit. Let $\theta = \mathrm{cf}(\kappa)$ and choose $(\kappa_i : i < \theta)$, an increasing sequence of cardinal numbers which is cofinal in κ and

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such that $|\tau_*| < \kappa_0^+$ and $2^{\kappa_i} \leq \kappa_{i+1}$ for all $i < \theta$. For each $i < \theta$ let $\tau_{i,\infty}$ be some vocabulary such that $\tau_* \subseteq \tau_{i,\infty}$, $|\tau_{i,\infty}| = \kappa_i$ and for every $n \geq 1$, $\tau_{i,\infty}$ has κ_i many n-ary predicates. Let $(S_{i,j} : j < 2^{\kappa_i})$ be a enumeration of all theories in $\mathbf{T}^{\mathbf{ec}}_{<\kappa_i^+}$ whose vocabulary is a subset of $\tau_{i,\infty}$. Every theory in $\mathbf{T}^{\mathbf{ec}}_{<\kappa_i^+}$ is a renaming of some $S_{i,j}$.

We inductively construct a continuous ascending chain of $L(Q^{\mathrm{cf}}_{\aleph_0})$-theories $(T_i : i < \theta)$ such that for every $i < \theta$:

1. $T_i \in \mathbf{T}^{\mathbf{ec}}_{<\kappa_i^+}$.
2. If $j, l < 2^{\kappa_i}$, $S_{i,j} \subseteq S_{i,l}$, and $f : \tau(S_{i,j}) \to \tau(T_i)$ is an embedding mapping $S_{i,j}$ into T_i, then f can be extended to an embedding $g : \tau(S_{i,l}) \to \tau(T_{i+1})$ mapping $S_{i,l}$ into T_{i+1}.

We choose $T_0 \in \mathbf{T}^{\mathbf{ec}}_{<\kappa_0^+}$ arbitrary with $T_* \subseteq T_0$ and $\tau_0 = \tau(T_0) \subseteq \tau_{0,\infty}$ and we take unions at limit stages. This is possible by Lemma 2.4, since for any limit ordinal $\delta < \theta$, $|\bigcup_{i<\delta} \tau(T_i)| \leq \kappa_\delta$. In order to obtain T_{i+1}, we consider a particular case of $j, l < 2^{\kappa_i}$ such that $S_{i,j} \subseteq S_{i,l}$ and there is some embedding $f : \tau(S_{i,j}) \to \tau(T_i)$ mapping $S_{i,j}$ into T_i. By Lemma 2.5, there is some $T_{j,l,f} \in \mathbf{T}^{\mathbf{ec}}_{<\kappa}$ extending T_i and some embedding $g : \tau(S_{i,l}) \to \tau(T_{j,l,f})$ that extends f and maps $S_{i,l}$ into $T_{j,l,f}$. By Lemma 2.4 we may find such $T_{j,l,f}$ with the additional property that $|\tau(T_{j,l,f})| < \kappa_i^+$. Note that the number of possible embeddings f from $\tau(S_{i,j})$ into $\tau(T_i)$ is $\leq \kappa_i^{\kappa_i} \leq \kappa_{i+1}$ and therefore the number of triples (j, l, f) is $\leq \kappa_{i+1}$. We may assume that $\tau(T_{j,l,f}) \cap \tau(T_{j',l',f'}) = \tau(T_i)$ whenever $(j, l, f) \neq (j', l', f')$. By item 3 of Lemma 2.3, we see that the union of all $T_{j,l,f}$ is consistent. Since it has cardinality $\leq \kappa_{i+1}$, by item 4 of Lemma 2.3 this union can be extended to $T_{i+1} \in \mathbf{T}^{\mathbf{ec}}_{<\kappa_{i+1}^+}$ as required.

Now let $T = \bigcup_{i<\theta} T_i$. It is clear that $T \in \mathbf{T}^{\mathbf{ec}}_{<\kappa^+}$.

Claim 2. *If $S, S' \in \mathbf{T}^{\mathbf{ec}}_{<\kappa_i}$, $S \subseteq S'$ and $f : \tau(S) \to \tau(T_i)$ is an embedding mapping S into T_i, then there is an embedding $f' : \tau(S') \to \tau(T_{i+1})$ extending f that maps S' into T_{i+1}.*

Proof of Claim 2. Like in the proof of Claim 1, we may assume that $S' = S_{i,l}$ and $S = S_{i,j}$ for some $j, l < 2^{\kappa_i}$.

We finally check that also in this case T is κ-universal over T_*. Let $T' \in \mathbf{T}^{\mathbf{ec}}_{<\kappa^+}$ and decompose it (using Lemma 2.4) as $T' = \bigcup_{i<\theta} T'_i$ where $(T'_i : i < \theta)$ is a continuous ascending chain of theories $T'_i \in \mathbf{T}^{\mathbf{ec}}_{<\kappa_i^+}$. As before, we may assume that $T_0 \subseteq T'$ and hence that $T'_0 = T_0$. We claim that there is a continuous ascending chain $(f_i : i < \theta)$ of embeddings $f_i : \tau(T'_i) \to \tau(T_{i+1})$ over τ_0 mapping T'_i into T_{i+1}. Notice that in this case $f = \bigcup_{i<\theta} f_i$ will be an embedding of $\tau(T')$ into $\tau(T)$ over τ_* mapping T' into T. We start by taking f_0 as the identity in τ_0 and we take unions at limit stages. If $f_i : \tau(T'_i) \to \tau(T_{i+1})$ has been obtained, we use Claim 2 (applied to $i + 1$) to extend f_i to f_{i+1} mapping T'_{i+1} into T_{i+2}. ⊣

The proof of Theorem 2.7 and the preceeding lemmas use only a few properties of the logic $L(Q^{\mathrm{cf}}_{\aleph_0})$. The reader can easily check that, besides compactness and the possibility of building negations and conjunctions of sentences, we only need to rename symbols and to give small upper bounds to the number of sentences in given vocabularies. We are only interested here in applications of the logic $L(Q^{\mathrm{cf}}_{\aleph_0})$, but it may be convenient to state the main results of this section in the more general

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setting of abstract model theory. This is done in the next observation, whose proof is left to the reader.

REMARK 2.8. Let $\mathcal{L}$ be a compact logic, in the sense of abstract model theory. Assume $\mathcal{L}$ admits renaming, is closed under boolean operators, in every sentence only finitely many symbols occur, and in every finite vocabulary there are only countably many (or $\leq\theta$) sentences. Define the classes $\mathbf{T}^{l}_{<\kappa}$ and the notion of κ-universal theory in an analogous way for $\mathcal{L}$. If $T_* \in \mathbf{T}^{\mathbf{ab}}_{<\kappa}$ and $\kappa = 2^{<\kappa} > \aleph_0$ (or $>\theta$), then there exists some κ-universal theory $T \in \mathbf{T}^{\mathbf{ec}}_{<\kappa^+}$ over T_*. And similarly the other claims of this section.

§3. Compact expandability. The following definition and the facts stated subsequently are given for models of countable vocabularies. This is only for simplification purposes, everything can be formulated with full generality with a few modifications.

DEFINITION 3.1. Let M be a model of countable vocabulary τ and of cardinality κ.

1. M is *expandable* if for every vocabulary $\tau' \supseteq \tau$ of cardinality $\leq\kappa$, if Σ is a first-order set of sentences of vocabulary τ' consistent with the first-order theory $\mathrm{Th}(M)$ of M, then there is some expansion M' of M to τ' such that $M' \models \Sigma$.
2. Call a set of first-order sentences Σ of vocabulary $\tau' \supseteq \tau$ *finitely satisfiable in* M if for every finite subset $\Sigma_0 \subseteq \Sigma$ there is an expansion of M that satisfies Σ_0.
3. M is *compactly expandable* if for every vocabulary $\tau' \supseteq \tau$ of cardinality $\leq\kappa$, if Σ is a first-order set of sentences of vocabulary τ' finitely satisfiable in M, then there is some expansion M' of M to τ' such that $M' \models \Sigma$.

The motivation for studying compactly expandable models came originally from the interest in restricted forms of the compactness theorem for logics with standard part. These logics were considered by M. Morley in [3] as generalizations of ω-logic. In ω-logic some notions concerning the natural numbers remain fixed, in M-logic the structure M replaces the structure of natural numbers. If M is a model of cardinality κ, for trivial reasons the M-logic does not satisfy the compactness theorem for sets of sentences of cardinality larger than κ. One can prove that the model M is compactly expandable if and only if M-logic is κ-compact, that is, if it satisfies the compactness theorem for sets of sentences of cardinality at most κ. For more on this see [1] and [2].

For a proof of the following list of facts, see [1].

FACT 3.2.
1. *Saturated and special models are expandable.*
2. *Expandable models are compactly expandable.*
3. *Compactly expandable models are ω-saturated and universal.*
4. *The countable compactly expandable models are the countable saturated models.*
5. *If T is superstable and does not have the finite cover property, then every compactly expandable model of T of cardinality $\geq 2^{\aleph_0}$ is saturated.*
6. *If T is $\aleph_0$-stable and does not have the finite cover property, then every compactly expandable model of T is saturated.*
7. *If T is superstable and has the finite cover property, T has compactly expandable models which are not saturated.*

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8. *Every unsuperstable theory having a saturated model of cardinality* $\kappa > \aleph_0$, *has a compactly expandable model of cardinality* κ *which is not saturated.*

The methods used in [1] to obtain compactly expandable models which are not saturated nor special are based on ultrapowers and chains of ultrapowers. They always provide expandable models. In [2] several examples of theories are discussed where every compactly expandable model is expandable. The question of whether in some theory there exists a compactly expandable model which is not expandable was asked in [1] in general and also particularly for the theory of linear dense orders without endpoints. We will give now an affirmative answer.

We are going to apply the results of the previous section to the particular case of the vocabulary $\tau_< = \{<\}$ and to some complete $L(Q^{\rm cf}_{\aleph_0})$-theory $T_<$ in this vocabulary. We only require from $T_<$ to extend the first-order theory DLO of the dense linear order without endpoints and to contain the $L(Q^{\rm cf}_{\aleph_0})$-sentences expressing that $<$ has cofinality $\aleph_0$ while $>$ has cofinality larger than $\aleph_0$ and for every point a, both $(\{b : b < a\}, <)$ and $(\{b : b > a\}, >)$ have cofinality larger than $\aleph_0$. More precisely:

Definition 3.3. Let $\tau_< = \{<\}$ and let $T_<$ be the $L(Q^{\rm cf}_{\aleph_0})(\tau_<)$-theory with the following axioms:

1. $\forall xyz(x < y \wedge y < z \to x < z) \wedge \forall x\,(\neg x < x) \wedge \forall xy(x < y \vee y < x \vee x = y)$.
2. $\forall xy(x < y \to \exists z(x < z \wedge z < y)) \wedge \forall x \exists yz(y < x \wedge x < z)$.
3. $Q^{\rm cf}_{\aleph_0} xy\,(x < y) \wedge \neg Q^{\rm cf}_{\aleph_0} yx\,(x < y)$.
4. $\forall x(\neg Q^{\rm cf}_{\aleph_0} zy(x < y \wedge y < z) \wedge \neg Q^{\rm cf}_{\aleph_0} yz(y < z \wedge z < x))$.

Remark 3.4. $T_<$ is consistent and complete.

Proof. Let $(A, <_A)$ be an $\aleph_1$-saturated dense linear ordering without endpoints and consider the lexicographic order of the product $(\omega, <) \times (A, <_A)$. All the axioms hold in this ordering. For completeness, use the quantifier elimination of the first-order theory of linear dense orders without endpoints. ⊣

Remark 3.5. There is a countable $T_* \supseteq T_<$ with vocabulary $\tau_* = \tau(T_*) \supseteq \tau_<$ such that $T_* \in \mathbf{T}^{\rm ec}_{<\aleph_1}$; in particular, T_* is an amalgamation base.

Proof. $T_< \in \mathbf{T}_{<\aleph_1}$, and by items 4 and 1 of Lemma 2.3, it can be extended to some $T_* \in \mathbf{T}^{\rm ec}_{<\aleph_1} \subseteq \mathbf{T}^{\rm ab}_{<\aleph_1}$. ⊣

Proposition 3.6. *Let* $\kappa = 2^{<\kappa} > \aleph_0$ (*for instance* $\kappa = \beth_\omega$) *and let* $T \in \mathbf{T}^{\rm ec}_{<\kappa^+}$ *be* κ*-universal over the theory* T_* *from Remark* 3.5 (*and hence extend* $T_* \supseteq T_<$). *Then*:

1. T *has a model of cardinality* κ.
2. *If* M *is a model of* T *of cardinality* κ *and* $M_< = M \restriction \{<\}$, *then*
   (a) $M_<$ *is compactly expandable, and*
   (b) $M_<$ *is not expandable.*

Proof. 1. Because $L(Q^{\rm cf}_{\aleph_0})$ satisfies the Löwenheim–Skolem theorem down to any uncountable cardinal.

2 (a). Let T' be a first-order theory in a vocabulary τ' containing the symbol $<$ and such that $|\tau'| \leq \kappa$, and assume that T' is finitely satisfiable in $M_<$. We can assume that $\tau_* \cap \tau(T') = \{<\}$ and then $T_* \cup T'$ is finitely satisfiable in $M_<$ and can be extended to some $T'' \in \mathbf{T}^{\rm ec}_{<\kappa^+}$. Since T is κ-universal over T_*, there is some embed-

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ding $f : \tau(T'') \to \tau(T)$ over $\tau_< = \{<\}$ (and even over τ_*) mapping T'' into T. Since $M \models T, T''$ holds in an expansion of M and therefore in an expansion of $M_<$.

2 (b). Notice that the first-order theory of $M_<$ is DLO, the theory of dense linear orders without endpoints. There are models of DLO, hence elementarily equivalent to $M_<$, where every open interval is isomorphic to the whole model, e.g., the ordering of the real numbers. Borrowing terminology from permutation group theory, they are sometimes called doubly transitive linear orders. This property can be expressed adding a new predicate to the language, that is, there is a finite vocabulary τ' containing $<$ and some first-order theory T' of vocabulary τ' which is consistent with DLO and in every model of T' every open interval is order isomorphic to the whole model. Since $M_<$ has cofinality $\aleph_0$ and all open intervals have cofinality $>\aleph_0$, it is not doubly transitive and, hence, no expansion of $M_<$ satisfies T'. ⊣

COROLLARY 3.7. *There are compactly expandable linear dense orderings without endpoints which are not expandable.*

PROOF. By Proposition 3.6. ⊣

REMARK 3.8. In the proof of Proposition 3.6 we can use a DLO isomorphic to its inverse. There are different choices for the theory $T_<$ of Definition 3.3. We can specify the cofinality of the reverse order $>$ to be $\aleph_0$. In fact, we can even not require density of the order.

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