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Three-wave interaction and Manley–Rowe relations in quantum hydrodynamics

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The theory for nonlinear three-wave interaction in magnetized plasmas is reconsidered using quantum hydrodynamics. The general coupling coefficients are calculated for the generalized Bohm de Broglie term. It is found that the Manley–Rowe relations are fulfilled only if the form of the particle dispersive term coincides with the standard expression. The implications of our results are discussed.

1. Introduction

During the last decade much work has been devoted to quantum plasmas (see e.g. Manfredi 2005; Glenzer and Redmer 2009; Shukla and Eliasson 2010, 2011; Haas 2011 and references therein). Laboratory applications include quantum wells (Manfredi and Hervieux 2007), spintronics (Wolf et al. 2001), and plasmonics (Atwater 2007). Quantum plasma effects can also be of interest in experiments with solid density targets (Glenzer and Redmer 2009) as well as in astrophysics (Kouveliotou et al. 1998; Palmer et al. 2005; Harding and Lai 2006).

Nonlinear wave–wave interaction in plasmas has been studied since the 1960s (see e.g. Kadomtsev 1965; Sjolund and Stenflo 1967; Sagdeev and Galeev 1969; Tsytovich 1970). Of special interest here are the three-wave interaction processes that have a wide range of applications, including, for example, stimulated Brillouin scattering in the ionosphere (Dysthe et al. 1977; Stenflo 2004) and various processes in laser-plasma experiments (Lashmore-Davies 1981; Kruer 1988; Mironov et al. 1990). From a theoretical point of view, the Manley–Rowe relations (Manley and Rowe 1956) are of much interest when three-wave interaction processes are studied (Larsson and Stenflo 1973; Stenflo and Larsson 1977; Weiland and Wilhelmsson 1977; Brodin and Stenflo 1988b). These relations put important constraints on the dynamics; for example, for a background plasma in thermodynamic equilibrium, the pump wave may only decay into waves with lower frequencies.

In the present work, three-wave interaction in a homogenous magnetized plasma is studied using the simplest form of quantum hydrodynamic equations, but with a slight generalization of the Bohm de Broglie term such that it depends on a free parameter. The exchange of wave energies in the three waves is calculated, and the conditions under which the Manley–Rowe relations are fulfilled are found. The results are compared with previous works (Larsson and Stenflo 1973; Stenflo and Larsson 1977; Shahid et al. 2013), and our findings are used to draw general conclusions regarding the mathematical structure of quantum hydrodynamics.

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2. Quantum hydrodynamics and the Manley–Rowe relations

The most simple quantum hydrodynamic equations (Manfredi 2005; Lundin et al. 2007; Haas 2011) read

$$\frac{\partial n}{\partial t} + \nabla \cdot (n\mathbf{v}) = 0, \tag{2.1}$$

$$\left(\frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla\right)\mathbf{v} = \frac{q}{m}(\mathbf{E} + \mathbf{v}\times\mathbf{B}) - \frac{\nabla P}{nm} + \frac{\hbar^2}{2m^2}\nabla\left(\frac{1}{\sqrt{n}}\nabla^2\sqrt{n}\right), \tag{2.2}$$

where n is the number density, $\mathbf{v}$ is the fluid velocity, q and m are the particle charge and mass, $\mathbf{E}$ and $\mathbf{B}$ are the electric and magnetic fields respectively, P is the pressure and $h = 2\pi\hbar$ is Planck's constant. The last term in (2.2) is the Bohm de Broglie force, which normally can be neglected for ions due to mass dependence. Equations (2.1) and (2.2) for each species are complemented by the standard Maxwell equations and an equation of state for the pressure. An often used simple relation is

$$\left(\frac{P}{P_0}\right) = \left(\frac{n}{n_0}\right)^{\gamma}, \tag{2.3}$$

which includes isothermal ($\gamma = 1$), classical adiabatic ($\gamma = 3$), or Fermi pressures ($\gamma = 5/3$) as special cases. Here P_0 and n_0 are the unperturbed pressure and number density respectively. Typically, when the Bohm de Broglie force is significant, the thermodynamic temperature T is smaller than the Fermi temperature $T_F = (\hbar^2/2m_e)(3\pi^2)^{2/3}n^{2/3}/k_B$, which makes the Fermi pressure the favored equation of state in quantum plasmas. While the expression for the Fermi pressure $P_F = (\hbar^2/5m_e)(3\pi^2)^{2/3}n^{5/3}$ is well established, there is still a degree of uncertainty regarding the most accurate factor in the equation of state for a degenerate plasma. The reason is that for a weak collisional system (as is typically appropriate for a plasma), the system is not in local thermodynamic equilibrium during compression by electromagnetic forces, in which case there is no firm bases for any type of pressure model. Comparisons with kinetic theories based on the Wigner function (Manfredi 2005) can then favor values of $\gamma \neq 5/3$ even for $T \ll T_F$. We will not be concerned with the best value of γ in the rest of the paper, and simply note that for a degenerate plasma we have $1 \lesssim \gamma \lesssim 3$.

Equations (2.1) and (2.2) can be derived from the Schrödinger equation using a Madelung ansatz for the wave function (Manfredi 2005; Manfredi & Haas 2001), where the wave function amplitude become the square root of the number density and the gradient of the phase is closely related to the fluid velocity. While the Bohm de Broglie force comes out straightforwardly from the single particle Schrödinger equation, the derivation of (2.2) depends on the possibility to interchange the ordering between averaging over particles and taking spatial derivatives (see e.g. (4.30) in Manfredi 2005). While such an interchange sometimes can be justified, this step becomes questionable when the Bohm de Broglie force is large, in which case (2.2) lacks a firm base.

Another means to derive quantum hydrodynamic equations is to take moments of the Wigner function (Gardner 1994; Manfredi 2005; Haas et al. 2010a,b). Such a procedure can to some extent lend support to (2.2), but depending on technical details it may also generate evolution equations that deviate considerably from the ones presented here. In particular, the quantum effect may occur first in the heat flux equation, not already in the momentum equation (Haas et al. 2010a,b). A general problem when using moment expansions is that typically truncation of the series

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depends on physical insights rather than mathematical rigor. In the limit of small collisions the truncation must necessarily involve rather crude approximations, since the effects of wave–particle interaction (which is dropped in the fluid limit) is not small in general. In such a scenario when no rigorous justification from first principles can be made, the credence of the fluid equations can be determined on two grounds. First, that there is a reasonable agreement with kinetic theory in most situations. Second, that the mathematical structure of the fluid equations is sound. The first criterion is discussed, for example, in Manfredi (2005), where a good agreement of (2.1) and (2.2) with kinetic theory is found for some model problems. The second criterion is usually deemed to be fulfilled if proper conservation laws for momentum, energy, and angular momentum are obeyed. Here we would like to extend these requirements on the mathematical structure, and also demand that the basic equations fulfill the Manley–Rowe relations (Manley and Rowe 1956) when nonlinear three-wave interaction (Larsson and Stenflo 1973; Stenflo and Larsson 1977; Weiland and Wilhelmsson 1977; Brodin and Stenflo 1988b) is studied.

Let us consider three waves with frequencies and wave numbers $(\omega_{(i)}, \mathbf{k}_{(i)})$ $i = 1, 2, 3$ that propagate in a homogenous magnetized plasma. We let the frequencies and wave numbers be related through

$$\omega_{(3)} = \omega_{(1)} + \omega_{(2),} \tag{2.4}$$

$$\mathbf{k}_{(3)} = \mathbf{k}_{(1)} + \mathbf{k}_{(2),} \tag{2.5}$$

which correspond to energy and momentum conservation, respectively, in case we make a quantum mechanical interpretation. The consistency of a quantum mechanical interpretation depends on the Manley–Rowe relations, however. According to the Manley–Rowe relations, the change in energy (denoted by $W_{(i)}$) of each wave must be in direct proportion to its frequency such that we can imagine wave interaction taking place one quanta at a time. Thus, in terms of the wave energies the Manley–Rowe relations can be written as

$$\frac{1}{\omega_{(3)}}\frac{dW_{(3)}}{dt} = -\frac{1}{\omega_{(1)}}\frac{dW_{(1)}}{dt} = -\frac{1}{\omega_{(2)}}\frac{dW_{(2)}}{dt}. \tag{2.6}$$

All the common classical plasma models lead to coupling coefficients for three-wave interactions that are consistent with the Manley–Rowe relations, including the Vlasov equation and the multifluid equations of type (2.1) and (2.2), *but without the Bohm de Broglie term* (see e.g. Larsson and Stenflo 1973; Stenflo and Larsson 1977; Stenflo 1994). Furthermore, requiring that (2.6) is fulfilled, can be used as a means for separating useful plasma models from less physical ones. For a concrete example, see for example, Brodin and Stenflo (1989), where a class of pressure tensor models was investigated, and only the sub-class consistent with (2.6) was deemed appropriate. In the next section we will demonstrate that the fluid equations, including the Bohm de Broglie term, in general lead to coupling coefficients that fulfill the Manley–Rowe relations. It should be stressed that this depends on the detailed mathematical structure of the quantum force. To emphasize this point, we will consider a slightly generalized Bohm de Broglie term given by

$$\frac{\hbar^2}{2m}\nabla\left(\frac{1}{n^\xi}\nabla^2 n^\xi\right). \tag{2.7}$$

As we will see, the Manley–Rowe relations will be fulfilled if and only if $\xi = 1/2$, in which case (2.7) agrees with the standard form displayed in (2.2). This supports

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the idea that fulfillment of the Manley–Rowe relations is a highly useful criterion in separating physical models from non-physical ones.

The existence of Manley–Rowe relations is closely related to the underlying Hamiltonian structure, e.g. the existence of a Lagrangian density. We will not pursue this topic here; however, we refer to Larsson (1996), Dodin et al. 2008, and Dodin (2013) for a more thorough discussion on such issues.

3. The coupled three-wave equations

3.1. *Preliminaries*

In general, we consider our variables as given by the sum of an unperturbed background and a small perturbation, e.g. $n = n_0 + \delta n$, where index 0 denotes the unperturbed value. The background plasma is time-independent and homogenous with zero drift velocities, and the unperturbed magnetic field is $\mathbf{B}_0 = B_0\hat{\mathbf{z}}$. The perturbations consist of contributions from all waves, $\delta n = \sum_{j=1}^{3} n_{(j)}(t)\exp[i(\mathbf{k}_{(j)}\cdot\mathbf{r} - \omega_{(j)}t)] + \text{c.c.}$, where c.c. denotes the complex conjugate, and the time-dependence of the amplitudes is assumed to be slow compared to the wave frequency. First, limiting ourselves to only time-dependent amplitudes simplifies some of the technical aspects of the derivation. A generalization to a weak spatial dependence of amplitudes is easily included by the substitution $\partial/\partial t \to \partial/\partial t + \mathbf{v}_{g(j)}\cdot\nabla$. Here the index (j) on the group velocity $\mathbf{v}_{g(j)}$ is $j = 1, 2, 3$ depending on which wave amplitude the derivative is acting. Next, we make an amplitude expansion keeping only up to second-order terms. Writing linear quantities on the left-hand side, and nonlinear on the right-hand side, the momentum equation reads as

$$\begin{aligned}
&\frac{\partial \delta\mathbf{v}}{\partial t} - \frac{q}{m}(\delta\mathbf{E} + \delta\mathbf{v}\times\mathbf{B}_0) + \frac{\gamma P_0}{n_0^2 m}\nabla\delta n - \frac{\xi\hbar^2}{2m^2 n_0}\nabla\nabla^2\delta n \\
&= -(\delta\mathbf{v}\cdot\nabla)\delta\mathbf{v} + \frac{q}{m}\delta\mathbf{v}\times\delta\mathbf{B} - \frac{\gamma(\gamma-2)P_0}{n_0^3 m}\delta n\nabla\delta n \\
&\quad - \frac{\xi\hbar^2}{2m^2 n_0^2}\left(\delta n\nabla\nabla^2\delta n + \nabla^2\delta n\nabla\delta n + 2(1-\xi)\left(\nabla\delta n\cdot\nabla\right)\nabla\delta n\right). \qquad (3.1)
\end{aligned}$$

Before we start the nonlinear analysis, it is convenient to introduce expressions for wave energy densities. These are

$$\begin{aligned}
W_{(i)} &= \frac{\epsilon_0}{2}\mathbf{E}_{(i)}\cdot\mathbf{E}^*_{(i)} + \frac{1}{2\mu_0}\mathbf{B}_{(i)}\cdot\mathbf{B}^*_{(i)} \\
&\quad + \sum_s\left[\frac{m_s n_{0s}}{2}\mathbf{v}_{(i)s}\cdot\mathbf{v}^*_{(i)s} + \left(\frac{\gamma_s P_{0s}}{2n_{0s}^2} + \frac{\xi\hbar^2 k_{(i)}^2}{4m_s n_{0s}}\right)n_{(i)s}n^*_{(i)s}\right], \qquad (3.2)
\end{aligned}$$

where the star denotes complex conjugate. Expressions for the wave energy densities can be deduced by demanding that $W_{(i)}$ is conserved to all orders in the slow time derivative (i.e. acting on wave amplitudes) when the nonlinearities are neglected. From the dispersion relation, where the wave frequency becomes real in the absence of dissipative mechanisms, one can of course deduce that different sub-parts of the wave energy are conserved separately in the absence of nonlinear interactions. However, the wave energy (3.2) is the unique expression that can be shown to be conserved without using the linear dispersion relation.

Formally all species are treated equivalently in (3.1) and (3.2). The fact that electrons may be described quantum mechanically and ion classically can be accounted for in

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the final result by choosing γ_s differently for electrons and ions, and dropping the Bohm de Broglie term altogether for ions.

3.2. *The Manley–Rowe relations*

Including only the linearized terms of the left-hand side in (3.1), as well as in the continuity equation and Maxwell's equations, the wave energy of each wave is conserved. Including the quadratically nonlinear terms of the right-hand side, the rate of change of each wave energy becomes proportional to terms that are cubic in amplitude. Only the resonant cubic terms that survive averaging over several wave periods are kept. Thus, the energy change of wave 3 directly associated with the electric field can be written as follows:

$$\frac{\epsilon_0}{2}\mathbf{E}^*_{(3)}\cdot\frac{\partial\mathbf{E}_{(3)}}{\partial t}+\text{c.c}$$
$$=\frac{\epsilon_0 c^2}{2}\mathbf{E}^*_{(3)}\cdot\left[\nabla\times\mathbf{B}_{(3)}-\mu_0\sum_s q_s(n_{0s}\mathbf{v}_{(3)s}+n_{(1)s}\mathbf{v}_{(2)s}+n_{(2)s}\mathbf{v}_{(1)s})\right]+\text{c.c.} \quad (3.3)$$

in accordance with (2.4) and (2.5). As the terms that are quadratic in the wave fields will cancel when all source terms are considered, only the cubic terms on the right-hand side are of interest here. Treating the other energy terms in the same manner, we thus find that $\mathrm{d}W_3/\mathrm{d}t$ becomes proportional to a large number of cubic terms. Simplifying this expression using linear approximations (e.g. $n_{(j)s}=n_{0s}\mathbf{k}_{(j)}\cdot\mathbf{v}_{(j)s}/\omega_{(j)}$ etc) in the cubic terms, we obtain after some lengthy algebra,

$$\frac{\mathrm{d}W_{(3)}}{\mathrm{d}t}=\omega_{(3)}\sum_s\left[-\frac{im_s}{2}\left(n_{(1)s}\mathbf{v}_{(2)s}\cdot\mathbf{v}^*_{(3)s}+n_{(2)s}\mathbf{v}_{(1)s}\cdot\mathbf{v}^*_{(3)s}+n^*_{(3)s}\mathbf{v}_{(1)s}\cdot\mathbf{v}_{(2)s}\right)\right.$$
$$-\frac{i\gamma_s(\gamma_s-2)P_{0s}}{n^3_{0s}}n_{(1)s}n_{(2)s}n^*_{(3)s}$$
$$+\frac{i\xi\hbar^2}{8m_s n^2_{0s}}\left[k^2_{(1)}+k^2_{(2)}+k^2_{(3)}-(2\xi-1)\mathbf{k}_{(1)}\cdot\mathbf{k}_{(2)}\right]n_{(1)s}n_{(2)s}n^*_{(3)s}$$
$$\left.-\frac{m_s\omega_{cs}}{2\omega_{(3)}}n_{0s}\left(\frac{k_{(2)z}}{\omega_{(2)}}-\frac{k_{(1)z}}{\omega_{(1)}}\right)\mathbf{v}^*_{(3)s}\cdot(\mathbf{v}_{(1)s}\times\mathbf{v}_{(2)s})\right]+\text{c.c.} \quad (3.4)$$

Equation (3.4) is our main result, together with similar expressions for $dW_{(1,2)}/dt$ that can be obtained directly from (3.4) using the symmetry between $\omega_{(1)}$, $\omega_{(2)}$, and $-\omega_{(3)}$ as well as between $\mathbf{k}_{(1)}$, $\mathbf{k}_{(2)}$, and $-\mathbf{k}_{(3)}$. When $\hbar\to 0$, (3.4) agrees with Stenflo and Larsson (1977). Furthermore, the corresponding expression for $W_{(1,2)}$ confirms that the Manley–Rowe relations (2.6) are fulfilled when $\hbar\to 0$. At a first glance the last term of (3.4) seems to be in conflict with (2.6) (i.e. the symmetry between waves 1, 2, and 3 is not explicit), but simple manipulations using (2.4) and (2.5) quickly confirms that the term is in full agreement with (2.6). The quantum term on the other hand has two very different contributions. The first term (proportional to $k^2_{(1)}+k^2_{(2)}+k^2_{(3)}$) is obviously in agreement with (2.6). However, the second term proportional to $(2\xi-1)$ must vanish for the Manley–Rowe relations to hold, i.e. we must have $\xi=1/2$. Thus, we can confirm that fulfillment of (2.6) can be used as a criterion for disregarding non-physical models. From now on we limit ourselves to the standard Bohm de Broglie term with $\xi=1/2$, in which case

$$\frac{1}{\omega_{(3)}}\frac{dW_{(3)}}{dt}=-\frac{1}{\omega_{(1,2)}}\frac{dW_{(1,2)}}{dt}=V+\text{c.c.} \quad (3.5)$$

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with

$$V = \sum_s \Bigg[-\frac{im_s}{2}\left(n_{(1)s}\mathbf{v}_{(2)s}\cdot\mathbf{v}^*_{(3)s} + n_{(2)s}\mathbf{v}_{(1)s}\cdot\mathbf{v}^*_{(3)s} + n^*_{(3)s}\mathbf{v}_{(1)s}\cdot\mathbf{v}_{(2)s}\right)$$
$$-\frac{i\gamma_s(\gamma_s-2)P_{0s}}{n_{0s}^3}n_{(1)s}n_{(2)s}n^*_{(3)s} + \frac{i\hbar^2}{16m_s n_{0s}^2}\left[k^2_{(1)}+k^2_{(2)}+k^2_{(3)}\right]n_{(1)s}n_{(2)s}n^*_{(3)s}$$
$$-\frac{m_s\omega_{cs}}{2\omega_{(3)}}n_{0s}\left(\frac{k_{(2)z}}{\omega_{(2)}}-\frac{k_{(1)z}}{\omega_{(1)}}\right)\left[\mathbf{v}^*_{(3)s}\cdot(\mathbf{v}_{(1)s}\times\mathbf{v}_{(2)s})\right]. \tag{3.6}$$

Property (3.5) has important consequences. First, it means that a quantum interpretation of the three-wave interaction process is possible, as noted above. This has the further implication that parametric decay occurs from higher to lower frequencies unless the wave energy density is negative, which can only occur if the background plasma has a free energy source. Three-wave interaction in homogenous plasmas using quantum hydrodynamic equations has been considered previously by Shahid et al. (2013), specifically focusing on the parametric decay of Langmuir waves in magnetized plasmas. However, their calculations did not produce the Manley–Rowe symmetric formulas, and thus our above result is an improvement in this respect. Furthermore, (3.6) covers all types of waves (Alfvén waves, whistler waves, extra-ordinary, etc.) and thus represents an extensive generalization of previous work.

3.3. *Three-wave equations*

In the previous section we showed that the Manley–Rowe relations are fulfilled for the physical case of $\xi = 1/2$. However, in order to do practical calculations of wave interactions (e.g. to find growth rates for parametric instabilities), we first need to rewrite the equations in terms of wave amplitudes rather than wave energy densities. For this purpose, we note that the wave energy densities can be written as $W = \varepsilon_0 E^*_j \partial(\omega D_{ij})/\partial\omega E_i$ for each wave, where we denote Cartesian components x, y, z with indices $1, 2, 3$ respectively in order to use the summation convention (a closely related and often used expression for the wave energy is equivalent to $W = \varepsilon_0(1/\omega)E^*_j\partial(\omega^2\varepsilon_{ij})/\partial\omega E_i$, where ε_{ij} is the dielectric tensor). The electric field eigenvectors fulfill $D_{ij}E_j = 0$ with

$$D_{ij} = \left(1-\frac{k^2c^2}{\omega^2}\right)\delta_{ij} + \frac{k_i k_j c^2}{\omega^2} + \sum_s \chi_{ij}, \tag{3.7}$$

and the susceptibility tensor χ_{ij} for each species is given by

$$\chi = -\frac{\omega_p^2}{\Omega_s^4}$$
$$\times\begin{bmatrix} \omega^2-(k^2-k_x^2)V_s^2 & k_xk_yV_s^2+\frac{i\omega_{cs}}{\omega}(\omega^2-k_z^2V_s^2) & k_xk_zV_s^2+\frac{i\omega_{cs}}{\omega}k_yk_zV_s^2 \\ k_xk_yV_s^2-\frac{i\omega_{cs}}{\omega}(\omega^2-k_z^2V_s^2) & \omega^2-(k^2-k_y^2)V_s^2 & k_yk_zV_s^2-\frac{i\omega_{cs}}{\omega}k_xk_zV_s^2 \\ k_xk_zV_s^2-\frac{i\omega_{cs}}{\omega}k_yk_zV_s^2 & k_yk_zV_s^2+\frac{i\omega_{cs}}{\omega}k_xk_zV_s^2 & \omega^2-\omega_{cs}^2-V_s^2(k^2-k_z^2)\end{bmatrix}, \tag{3.8}$$

where we have defined

$$\Omega_s^4 = \omega^2(\omega^2-V_s^2k^2)-\omega_{cs}^2(\omega^2-V_s^2k_z^2), \tag{3.9}$$

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$$V_s^2 = \frac{\gamma P_{0s}}{m_s n_{0s}} + \frac{\hbar^2 k^2}{4m_s^2}. \tag{3.10}$$

The linear susceptibility in a fluid theory including the Bohm de Broglie force, has been computed in Lundin et al. (2007). Here we have generalized this expression to arbitrary Cartesian coordinate axes, since we cannot choose a coordinate axis along the perpendicular wavenumber for more than one of the interacting waves in general. Finally, the dispersion relation for each wave is determined by

$$D(\omega, \mathbf{k}) \equiv \det D_{ij} = 0. \tag{3.11}$$

Now we want to express all quantities appearing in (3.5) and (3.6) in terms of a single variable representing the wave amplitude of each wave. Somewhat arbitrarily, we can pick the z-component of the electric field, but the procedure outlined below works for any component of the electric field. First, using $D_{ij}E_j = 0$ we can express E_x and E_y in terms of E_z for each wave. Together with $v_i = -i\omega\epsilon_0\chi_{ij}E_j/qn_0$, this gives all velocity components in terms of E_z, and the density perturbation is obtained in terms of $\delta n = n_0 k_i v_i/\omega$. The remaining quantity needed is the wave energy density, which with the help of $W = \varepsilon_0 E_j^* \partial(\omega D_{ij})/\partial\omega) E_i$ is written as

$$W_{(3)} = \frac{\varepsilon_0\omega_3}{(D_{xx}D_{yy} - D_{xy}D_{yx})} \frac{\partial D(\omega_3, \mathbf{k}_3)}{\partial\omega_3} E_{(3)z} E^*_{(3)z} \tag{3.12}$$

for wave 3. As a consequence, all variables appearing in (3.5) and (3.6) can be expressed in terms of the z-component of electric field amplitudes, in which case (3.5) and (3.6) can be rewritten as

$$\frac{dE_{(1,2)z}}{dt} = \alpha_{(1,2)} E_{(3)z} E^*_{(2,1)z}, \tag{3.13}$$

$$\frac{dE_{(3)z}}{dt} = \alpha_{(3)} E_{(1)z} E_{(2)z}, \tag{3.14}$$

where we now allow for spatially dependent amplitudes such that

$$\frac{dE_{(j)z}}{dt} = \left(\frac{\partial}{\partial t} + \mathbf{v}_{g(j)} \cdot \nabla\right) E_{(j)z}. \tag{3.15}$$

It is straightforward to find the general expressions for the coupling coefficients $\alpha_{(1,2,3)}$ from formulas (3.5), (3.6), (3.12), and the procedure outlined above. However, in order to obtain comparatively simple and illustrative formulas, we consider a special case where the plasma is unmagnetized, $B_0 = 0$, and waves 1 and 3 are Langmuir waves. Furthermore, we let the plasma be degenerate, i.e. for a 3D Fermi gas $P_0 = n_0 m v_F^2/5$ and $\gamma = 5/3$, where we have used the thermodynamic equilibrium pressure (see discussion in Sec. 2). In this case the general dispersion relation (3.11) reduces to

$$\omega_{(1,3)}^2 = \omega_p^2 + \frac{1}{3}k_{(1,3)}^2 v_F^2 + \frac{\hbar^2 k_{(1,3)}^4}{4m^2} \tag{3.16}$$

when the corrections due to the ion motion is neglected. Wave 2 is a low-frequency ion-acoustic wave fulfilling the approximate dispersion relation

$$\omega_{(2)}^2 = \frac{\omega_{pi}^2}{1 + \omega_{pe}^2/(\frac{1}{3}k_{(2)}^2 v_F^2 + \hbar^2 k_{(2)}^4/4m_e^2)}, \tag{3.17}$$

where we have set the ion temperature to zero, and let $m_e/m_i \to 0$, but avoided the approximation of quasi-neutrality in order to allow for short wavelengths. Making the

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corresponding approximations in (3.6) and (3.12), we obtain the coupled equations:

$$\frac{\partial \phi_{(3)}}{\partial t} = -\frac{iq_e}{2m_e\omega_{(3)}k_{(3)}^2\omega_{pe}^2}\tilde{V}\phi_{(1)}\phi_{(2)}, \tag{3.18}$$

$$\frac{\partial \phi_{(2)}}{\partial t} = \frac{iq_e\omega_{(2)}^3}{2m_e k_{(2)}^2\omega_{pi}^2\omega_{pe}^4}\tilde{V}\phi_{(1)}^*\phi_{(3)}, \tag{3.19}$$

$$\frac{\partial \phi_{(1)}}{\partial t} = \frac{iq_e}{2m_e\omega_{(1)}k_{(1)}^2\omega_{pe}^2}\tilde{V}\phi_{(3)}\phi_{(2)}^*, \tag{3.20}$$

where

$$\tilde{V} = \left(1 - \frac{\omega_{pi}^2}{\omega_{(2)}^2}\right)\left(\omega_{(1)}\omega_{(3)}k_{(2)}^2\mathbf{k}_{(1)}\cdot\mathbf{k}_{(3)} - \left[\frac{v_F^2}{9} + \frac{\hbar^2}{8m_e^2}(k_{(1)}^2 + k_{(2)}^2 + k_{(3)}^2)\right]k_{(1)}^2k_{(2)}^2k_{(3)}^2\right). \tag{3.21}$$

As usual these equations can be used to compute growth rates and threshold values for parametric instabilities. However, to get a nonzero threshold value we need to add a finite width of the pump wave or a damping mechanism due to e.g. collisions (see e.g. Weiland & Wilhelmsson (1977); Brodin & Stenflo 1988a).

4. Concluding discussion

In this paper, we have focused on the Manley–Rowe relations in quantum hydrodynamics. Our starting point has been that basic equations that are physically sound should produce coupling coefficients for three-wave interaction that obeys these relations. As discussed, for example, in Larsson (1996); Dodin et al. (2008); and Dodin (2013), fulfillment of the Manley–Rowe relations comes from an underlying Hamiltonian structure. For classical plasmas, it is illustrated rather clearly in Stenflo (1994) that the Manley–Rowe relations are satisfied for arbitrary wave propagation in hot magnetized plasmas with a uniform background. Moreover, the Manley–Rowe relations are more general than expected, i.e. they are sometimes applicable outside their expected range of validity (see Kaufman and Stenflo 1975). The generalized Manley–Rowe relations are valid for non-uniform plasmas (Kaufman and Stenflo 1979; Lindgren et al. 1981; Aliev and Brodin 1990) and, somewhat surprisingly, also for turbulent plasmas (see the rather instructive paper by Vladimirov and Stenflo 1997).

Nevertheless, the derivation of standard quantum hydrodynamic equations contains steps that can be questioned when both pressure and particle dispersive effects are large. Hence, it is not obvious that such equations preserve a physically sound structure, i.e. obey the Manley–Rowe relations. As demonstrated by (3.5)–(3.6), however, these relations are indeed fulfilled when the standard Bohm de Broglie term is used to describe particle dispersive effects. This is not the case, however, if the Bohm de Broglie term is replaced by a slightly generalized expression, which demonstrates that fulfillment of the Manley-Rowe relations is a useful criterion in separating acceptable models from unphysical ones. Besides these theoretical aspects, we note that our results extend previous works on three-wave interaction based on classical fluid equations (Stenflo and Larsson 1977) to cover quantum hydrodynamics. The quantum contribution to the coupling coefficient is important for short wavelengths (of the order of the thermal de Broglie wavelength), and for a quantum parameter $H = \hbar\omega_p/k_BT$, which is not much smaller than unity. For a thorough discussion

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of systems that fit this description, see Manfredi (2005); Shukla and Eliasson (2010, 2011); and Haas (2011).

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