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On the Laplacian Eigenvalues of $G_{n,p}$

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We investigate the Laplacian eigenvalues of sparse random graphs $G_{n,p}$. We show that in the case that the expected degree $\bar{d} = (n-1)p$ is bounded, the spectral gap of the normalized Laplacian $\mathscr{L}(G_{n,p})$ is $o(1)$. Nonetheless, w.h.p. $G = G_{n,p}$ has a large subgraph core(G) such that the spectral gap of $\mathscr{L}(\mathrm{core}(G))$ is as large as $1 - O(\bar{d}^{-1/2})$. We derive similar results regarding the spectrum of the combinatorial Laplacian $L(G_{n,p})$. The present paper complements the work of Chung, Lu and Vu [8] on the Laplacian spectra of random graphs with given expected degree sequences. Applied to $G_{n,p}$, their results imply that in the 'dense' case $\bar{d} \geqslant \ln^2 n$ the spectral gap of $\mathscr{L}(G_{n,p})$ is $1 - O(\bar{d}^{-1/2})$ w.h.p.

1. Introduction and results

Numerous heuristics for NP-hard graph problems are based on *spectral techniques*: given a graph $G = (V, E)$, the heuristic sets up a matrix representing the graph and uses the eigenvalues and eigenvectors of this matrix to construct a solution to the combinatorial problem. Spectral techniques are popular in applications, because there are rather efficient methods to compute eigenvalues and eigenvectors (see [17]). In particular, spectral techniques have been applied successfully to NP-hard graph-partitioning problems such as MAX CUT, MINIMUM BISECTION, MAXIMUM CLIQUE, or GRAPH COLOURING (*e.g.*, [1, 23]).

Though in many cases there are worst-case examples known that show that certain spectral heuristics perform badly on general instances (*e.g.*, [18]), spectral methods seem to perform well on many 'practical' inputs. Therefore, in order to provide a better understanding of spectral methods, quite a few papers deal with rigorous analyses of spectral heuristics on suitable classes of *random* graphs. For example, Alon and Kahale [2] suggested a spectral heuristic for GRAPH COLOURING, Alon, Krivelevich and Sudakov [3] dealt with a spectral method for MAXIMUM CLIQUE, and Boppana [6] and Coja-Oghlan [10] studied spectral heuristics for MINIMUM BISECTION.

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Of course, crucial ingredients to such rigorous analyses of spectral methods are results on properties of the spectra of random graphs. To be more specific, we need to spell out precisely how graphs are represented by matrices. Thus, let $G = (V, E)$ be a graph. The *adjacency matrix* $A(G) = (a_{vw})_{v,w\in V}$ represents G in the most obvious way: the entries are $a_{vw} = 1$ if $\{v, w\} \in E$, and $a_{vw} = 0$ otherwise. Furthermore, let $d_G(v)$ denote the degree of $v \in V$ in G. Then the *combinatorial Laplacian* of G is

$$L(G) = (b_{vw})_{v,w\in V}, \quad \text{where } b_{vw} = \begin{cases} d_G(v) & \text{if } v = w, \\ -1 & \text{if } \{v, w\} \in E, \\ 0 & \text{otherwise.} \end{cases}$$

Finally, the *normalized Laplacian* of G is

$$\mathscr{L}(G) = (c_{vw})_{v,w\in V}, \text{ where } c_{vw} = \begin{cases} 1 & \text{if } v = w \text{ and } d_G(v) > 0, \\ -1/\sqrt{d_G(v)d_G(w)} & \text{if } \{v, w\} \in E, \\ 0 & \text{otherwise.} \end{cases} \tag{1.1}$$

If M is a symmetric $n \times n$ matrix, then we denote its (real) eigenvalues by $\lambda_1(M) \leqslant \cdots \leqslant \lambda_n(M)$. Observe that all three matrices $A(G)$, $L(G)$, and $\mathscr{L}(G)$ are symmetric; in fact, $L(G)$ and $\mathscr{L}(G)$ are positive semidefinite. Moreover, the multiplicity of the eigenvalue 0 in both $L(G)$ and $\mathscr{L}(G)$ equals the number of connected components of G. Thus, $\lambda_1(L(G)) = \lambda_1(\mathscr{L}(G)) = 0$. The smallest positive eigenvalue is called the *spectral gap* of $L(G)$ or $\mathscr{L}(G)$. In addition, if $G = (V, E)$ is a graph on n vertices, then $\lambda_n(A(G)) - \max\{-\lambda_2(A(G)), \lambda_{n-1}(A(G))\}$ is called the spectral gap of $A(G)$. (For general background on the Laplacian matrices and the relations of their eigenvalues to other graph parameters we refer to Chung [7].)

The spectral heuristics suggested in [2, 3, 10] use the eigenvalues and eigenvectors of the input graph's adjacency matrix. Moreover, the graph-partitioning heuristics [6, 23] are based on the combinatorial Laplacian, as is the lower bound on the bisection width due to Donath and Hoffman [12]. Furthermore, Dasgupta, Hopcroft and McSherry [11] suggested a heuristic for a general graph-partitioning problem that employs the normalized Laplacian. The crucial parameter of the spectrum concerned in [2, 3, 6, 10, 11, 12, 23] is essentially the spectral gap of the corresponding matrices.

The goal of the present paper is to analyse the spectral gap of the Laplacian matrices L and $\mathscr{L}$ on random graphs. More precisely, we shall consider the well-known $G_{n,p}$ model of a random graph: the vertex set of $G_{n,p}$ is $V = \{1, \ldots, n\}$, and each of the $\binom{n}{2}$ possible edges is present with probability $0 \leqslant p \leqslant 1$ independently of all others. The $G_{n,p}$ model has received considerable attention: see [5, 20] for comprehensive treatments. We say that $G_{n,p}$ enjoys some property $\mathscr{P}$ *with high probability* ('w.h.p.') if the probability that $\mathscr{P}$ holds in $G_{n,p}$ tends to 1 as $n \to \infty$.

1.1. Results

Chung, Lu and Vu [8] investigated the Laplacian spectrum of random graphs with given expected degree sequences. Applied to $G_{n,p}$, their results imply the following: if the expected average degree $\bar{d} = (n-1)p$ is at least $\ln^2 n$, then w.h.p.

$$1 - (4 + o(1))\bar{d}^{\,-1/2} \leqslant \lambda_2(\mathscr{L}(G_{n,p})) \leqslant \lambda_n(\mathscr{L}(G_{n,p})) \leqslant 1 + (4 + o(1))\bar{d}^{\,-1/2}; \tag{1.2}$$

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thus, the spectral gap is at least $1-(4+o(1))\bar{d}^{-1/2}$.

Investigating the spectral gap of $\mathscr{L}(G_{n,p})$ and $L(G_{n,p})$ for sparse random graphs with expected degree $\bar{d} < \ln^2 n$, we obtain the following results.

- We observe that (1.2) does not hold if $\bar{d} = O(1)$; in this case the spectral gap of $\mathscr{L}(G_{n,p})$ is $o(1)$ w.h.p.
- Nonetheless, we show that, for $\bar{d} < \ln^2 n$, the random graph $G = G_{n,p}$ has a large subgraph core(G) on which a similar statement as (1.2) remains valid w.h.p.
- In addition, we prove that the combinatorial Laplacian $L(G_{n,p})$ behaves similarly: w.h.p. there is a large subgraph H of $G_{n,p}$ such that $L(H)$ has a large spectral gap.

The following proposition shows that (1.2) does not hold in the sparse case where $\bar{d} = O(1)$ as $n \to \infty$.

Proposition 1.1. *Suppose that $\bar{d} = (n-1)p$ equals a constant $c > 0$. Let $\mathscr{L} = \mathscr{L}(G_{n,p})$. Then w.h.p. $\lambda_n(\mathscr{L}) = 2$. Moreover, if $\lambda_i(\mathscr{L})$ is the smallest positive eigenvalue, then $\lambda_i(\mathscr{L}) = o(1)$ w.h.p.*

Loosely speaking, the reason why (1.2) is not true if $\bar{d}$ is constant is that in this case the random graph $G_{n,p}$ is somewhat irregular, whereas $G_{n,p}$ is 'almost' regular if $\bar{d} \gg \ln n$. Indeed, the degree of any vertex v of $G = G_{n,p}$ is binomially distributed with mean $\bar{d}$, so that in the case $\bar{d} \gg \ln n$ w.h.p. *all* vertex degrees of G are $(1+o(1))\bar{d}$ (by Chernoff bounds). By contrast, if $\bar{d} = O(1)$, then the binomial distribution can be approximated well by the Poisson distribution with mean $\bar{d}$. Hence, the degree distribution of $G_{n,p}$ has a fairly heavy tail. In fact, if $\bar{d} = O(1)$, then for any constant d there are $\Omega(n)$ vertices of degree d, and the maximum degree of $G_{n,p}$ is $\Omega(\ln n/\ln\ln n) \gg \bar{d}$ w.h.p.

More precisely, it turns out that the upper tail of the degree distribution, *i.e.*, vertices of degree significantly larger than $\bar{d}$, do not yield eigenvalues of $\mathscr{L}(G)$ outside the range $1 \pm O(\bar{d}^{-1/2})$. The reason is that in $\mathscr{L}(G)$ the entries corresponding to edges $\{v,w\}$ of $G = G_{n,p}$ are scaled down by the geometric mean $(d_G(v)d_G(w))^{1/2}$ of the vertex degrees (*cf.* (1.1)). On the other hand, vertices of degree much smaller than $\bar{d}$ induce both positive eigenvalues close to 0 and the eigenvalue 2. Therefore, we consider the following process that, given $G = G_{n,p}$, removes the vertices of small degree.

CR1. Initially, let H be the subgraph of G obtained by removing all vertices from G that have degree smaller than $\bar{d}/2$.

CR2. While there is a vertex $v \in H$ that has $\geqslant 100$ neighbours in $G - H$, remove v from H.

We call the outcome H of this process the *core* of G, denoted by core(G). Observe that by construction

$$\forall v \in \mathrm{core}(G) : d_{\mathrm{core}(G)}(v) \geqslant \frac{\bar{d}}{2} - 100. \tag{1.3}$$

While $\mathscr{L}(G_{n,p})$ does not satisfy (1.2) w.h.p., the main result of this paper is that an estimate similar to (1.2) does hold for core$(G_{n,p})$ w.h.p. Moreover, the process CR1–CR2 removes $\leqslant n\exp(-\Omega(\bar{d}))$ vertices w.h.p.

Theorem 1.2. *Suppose that $c_0 \leqslant \bar{d} = (n-1)p \leqslant 0.99n$ for a sufficiently large constant $c_0 > 0$. Let $\mathscr{L} = \mathscr{L}(\mathrm{core}(G))$. There is a constant $c_1 > 0$ such that w.h.p. the following holds.*

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(1) *Let* $v = \#V(\mathrm{core}(G))$. *Then* $v \geqslant n(1-\exp(-\bar{d}/c_1))$.
(2) *We have* $0 = \lambda_1(\mathscr{L}) < 1 - c_1\bar{d}^{-1/2} \leqslant \lambda_2(\mathscr{L}) \leqslant \lambda_v(\mathscr{L}) \leqslant 1 + c_1\bar{d}^{-1/2}$.

(Note that the first statement of Theorem 1.2 implies that $\mathrm{core}(G_{n,p}) = G_{n,p}$ w.h.p. if $\bar{d} > c_1 \ln n$. Thus, replacing the assumption $\bar{d} \geqslant \ln^2 n$ by $\bar{d} \geqslant c_1 \ln n$, we can '*en passant*' extend the result (1.2) of Chung, Lu and Vu a little.)

In comparison to $\mathscr{L}(G_{n,p})$, the combinatorial Laplacian $L(G_{n,p})$ turns out to be extremely susceptible to fluctuations of the vertex degrees. In fact, since the vertex degrees of $G = G_{n,p}$ are the diagonal entries of $L(G)$, it is not sufficient to remove vertices of very small degree as in the construction of $\mathrm{core}(G)$ (see the Remark in Section 5, p. 939, for more detailed comments). Nevertheless, the following theorem shows that if $H = (V(H), E(H))$ is a subgraph of G such that the degrees of the vertices of H deviate from their expectation $\tilde{d} = (\#V(H)-1)p$ by at most $O(\tilde{d}^{1/2})$, then the spectral gap of $L(H)$ is $\tilde{d} - O(\tilde{d}^{1/2})$.

Theorem 1.3. *For each* $\varepsilon > 0$ *there are numbers* $c_0, c^* > 0$ *such that, in the case* $c_0 \leqslant \bar{d} = (n-1)p \leqslant 0.99n$, *the random graph* $G = G_{n,p}$ *enjoys the following two properties w.h.p.*

(1) *There is an induced subgraph* H *of* G *such that, letting* $\tilde{d} = (\#V(H)-1)p$, *we have*

$$|d_H(v) - \tilde{d}| \leqslant c^*\tilde{d}^{1/2} \quad \text{for all } v \in V(H) \tag{1.4}$$

and $\#V(H) \geqslant (1-\varepsilon)n$.

(2) *Let* H *be an induced subgraph of* G *such that* $\#V(H) \geqslant \frac{n}{2}$, *and let* $\tilde{d} = (\#V(H)-1)p$. *If* H *satisfies* (1.4), *then with respect to the eigenvalues of* $L = L(H)$ *we have*

$$0 = \lambda_1(L) < \tilde{d} - 2c^*\tilde{d}^{1/2} \leqslant \lambda_2(L) \leqslant \lambda_n(L) \leqslant \tilde{d} + 2c^*\tilde{d}^{1/2}. \tag{1.5}$$

1.2. Related work

It is instructive to compare Theorems 1.2 and 1.3 with results on the spectrum of the adjacency matrix of $G_{n,p}$. With respect to the spectral gap of $A = A(G_{n,p})$, Füredi and Komlós [16] proved that

$$\max\{-\lambda_1(A), \lambda_{n-1}(A)\} \leqslant (2+o(1))(np(1-p))^{1/2},\ \lambda_n(A) \sim np \quad \text{if } np(1-p) \gg \ln^6 n \tag{1.6}$$

w.h.p. (In fact there seems to be a gap in the proof, which has been fixed by Vu [25] and independently in [9, Chapter 3].) Further, Feige and Ofek [13] showed that $\max\{-\lambda_1(A), \lambda_{n-1}(A)\} \leqslant O(np)^{1/2}$ and $\lambda_n(A) = \Theta(np)$ w.h.p. when $np \geqslant \ln n$. This result corresponds to the result (1.2) of Chung, Lu and Vu on $\mathscr{L}(G_{n,p})$.

However, in the sparse case $\bar{d} = (n-1)p = O(1)$, neither $\max\{-\lambda_1(A), \lambda_{n-1}(A)\} \leqslant O(\bar{d})^{1/2}$ nor $\lambda_n(A) = \Theta(\bar{d})$ holds w.h.p. For if $\bar{d} = O(1)$, then the upper tail of the degree distribution induces an unbounded number of both positive and negative eigenvalues as large as $\Omega(\ln n/\ln\ln n)^{1/2}$ in absolute value. Nonetheless, following an idea of Alon and Kahale [2], Feige and Ofek [13] showed that there is a constant $C > 0$ such that the graph G' obtained by removing all vertices of degree, say, $> 2\bar{d}$ from G w.h.p. satisfies

$$\max\{-\lambda_1(A(G')), \lambda_{\#V(G')-1}(A(G'))\} \leqslant C(np)^{1/2}, \quad (1-o(1))\bar{d} \leqslant \lambda_{\#V(G')}(A(G')) \leqslant 2\bar{d}. \tag{1.7}$$

Loosely speaking, we can compare (1.7) and Theorem 1.2 as follows. While vertices of degree $d \gg \bar{d}$ induce eigenvalues of $A(G_{n,p})$ of order $\pm\Theta(d^{1/2})$ (as one can see by considering the

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spectrum of a star induced by such a high-degree vertex and its neighbours), high-degree vertices do not affect the spectral gap of $\mathscr{L}(G_{n,p})$. This is due to the normalization of the entries c_{vw} of $\mathscr{L}$ (*cf.* (1.1)). However, because of this normalization, vertices of degree significantly smaller than $\bar{d}$ induce eigenvalues of $\mathscr{L}$ far from 1, while vertices of small degree do not affect the spectrum of $A(G_{n,p})$ in a similar way.

In addition to the spectral gap, the distribution of the largest eigenvalue of $A = A(G_{n,p})$ has also been investigated. Füredi and Komlós showed that $\lambda_n(A)$ has asymptotically a normal distribution with mean $(n-1)p + (1-p)$ and variance $2p(1-p)$, provided that $0 < p < 1$ remains constant as $n \to \infty$ (*cf.* Janson [19] for a recent improvement of this result). Moreover, Krivelevich and Sudakov proved that $\lambda_n(A) \sim \max\{\sqrt{\Delta(G)}, np\}$ holds w.h.p. even if $0 < p = p(n) < 1$ is any function of n; here $\Delta(G)$ signifies the maximal degree of G.

Regarding the global distribution of the eigenvalues, Chung, Lu and Vu [8] proved an analogue of Wigner's famous semicircle law [26] for $\mathscr{L}(G_{n,p})$ in the case $np \gg 1$. Moreover, a semicircle law for $A(G_{n,p})$, $np(1-p) \gg \ln^6 n$, follows from the considerations of Füredi and Komlós [16]. However, Bauer and Golinelli [4] observed that in the sparse case $np = O(1)$ the spectrum of $A(G_{n,p})$ is more complex: in the density function of the spectrum there occur 'peaks' corresponding to both the spectra of tree components and the spectra of trees attached to the giant component (in the case $np > 1$). In addition, Khorunzhiy, Kirsch and Müller [21] studied the spectrum of $L(G_{n,p})$ for extremely sparse graphs $p < 1/n$ (that consist of components of size $O(\ln n)$ w.h.p.). Both [4] and [21] are motivated by the relevance of the spectra of $A(G_{n,p})$ and $L(G_{n,p})$ in physics.

In addition to the $G_{n,p}$ model, the eigenvalues of matrices representing various further types of random graphs have been investigated. For instance, Friedman [14] determined the eigenvalue gap of random regular graphs. Note that for regular graphs G the eigenvalues of $L(G)$ and $\mathscr{L}(G)$ can be obtained from the eigenvalues of $A(G)$ directly. Moreover, Chung, Lu and Vu [8] studied a rather general model of random graphs with given expected degree sequences. They investigate the relations between the expected degree sequence of G and the eigenvalues of $A(G)$ and $\mathscr{L}(G)$ to obtain both results on the global distribution of the eigenvalues (like the semicircle law) and results on the spectral gap. With respect to the $G_{n,p}$ model, their results imply the aforementioned bound (1.2) on the spectral gap of $\mathscr{L}(G_{n,p})$ with $np \geqslant \ln^2 n$.

Dasgupta, Hopcroft and McSherry [11] considered a model of random graphs with planted partitions that generalizes the model studied in [8]. They showed that the normalized Laplacian can be used to recover the planted partition w.h.p. under certain assumptions on the parameters of the model; in particular, they need to assume that the minimum expected degree is $\gg \ln^6 n$ (where n is the number of vertices). However, Proposition 1.1 indicates that this approach cannot be generalized directly to sparse random graphs with expected degree $O(1)$ even if all expected degrees coincide, because the lower tail of the degree distribution jumbles up the spectrum of the normalized Laplacian.

1.3. Techniques and outline

At first glance, investigating the spectrum of $\mathscr{L}(G_{n,p})$ seems rather difficult, because the entries of the matrix are mutually dependent. By contrast, the entries of the adjacency matrix $A(G_{n,p})$ are mutually independent (up to the trivial dependence due to symmetry). Therefore, to prove Theorem 1.2 we will not deal with $\mathscr{L}(G_{n,p})$ directly, but rather relate its spectrum to that of $A(G_{n,p})$. Regarding the spectrum of $A(G_{n,p})$, we can build upon the following theorem.

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Theorem 1.4. *There are constants $c_1, c_2 > 0$ such that, for all $0 < p = p(n) < 1$ and all $d \geqslant \max\{c_1, 1.01 \cdot np\}$, the following holds. Let $A = (a_{vw})_{v,w=1,\ldots,n} = A(G_{n,p})$. Moreover, set $V' = \{v \in V : d_G(v) \leqslant d\}$ and let $A' = (a_{vw})_{v,w\in V'}$. Then with probability $\geqslant 1 - O(n^{-1})$ the following three statements hold.*

(1) $\#V' \geqslant n(1 - \exp(-d/c_2))$.
(2) *For all unit vectors $\xi \perp \vec{1}$, we have $\|A'\xi\| \leqslant c_2 d^{1/2}$, where $\vec{1}$ denotes the vector with all entries $= 1$.*
(3) *If $W \subset V'$ has cardinality $\geqslant (1 - d^{-10})n$, then the minor $A'' = (a_{ij})_{i,j\in W}$ satisfies $\|A''\vec{1} - np\vec{1}\| \leqslant c_2(nd)^{1/2}$.*

Theorem 1.4 is a slight modification of the main result of Feige and Ofek [13] (the theorem is stated and proved explicitly in [9, Chapter 3]). The proof of Theorem 1.4 relies on a technique invented by Kahn and Szemerédi [15], which differs significantly from the 'trace method' used, *e.g.*, by Chung, Lu and Vu [8] and Füredi and Komlós [16].

To prove Theorem 1.2, we proceed as follows. Observe that Theorem 1.4 does *not* apply to $A(\mathrm{core}(G))$, because if $\bar{d} = (n-1)p = O(1)$, then $\mathrm{core}(G)$ contains vertices of degree $\gg \bar{d}$ w.h.p. Therefore, we decompose $\mathrm{core}(G)$ into two parts S and Z, where S (essentially) consists of all vertices of $\mathrm{core}(G)$ whose degree deviates from $\bar{d}$ by at most $\varepsilon\bar{d}$, and Z consists of the remaining vertices. Then, we can study the spectrum of the submatrix of $\mathscr{L}(\mathrm{core}(G))$ induced on S by combining Theorem 1.4 with an analysis of the normalization of $\mathscr{L}(\mathrm{core}(G))$ (*cf.* (1.1)). The latter analysis requires a thorough study of the degree distribution of $\mathrm{core}(G)$. Furthermore, we shall prove that the subgraph of $\mathrm{core}(G)$ induced on Z as well as the bipartite subgraph of $\mathrm{core}(G)$ consisting of the S-Z-edges are extremely sparse, so that we can analyse the spectrum of the corresponding submatrices of $\mathscr{L}(\mathrm{core}(G))$ directly. The proof of Theorem 1.3 relies on a similar approach, but is a little simpler.

The paper is organized as follows. In Section 2, we introduce some notation and state a few general lemmas on $G_{n,p}$ that will be used repeatedly in the remaining sections. Section 3 deals with the rather simple proof of Proposition 1.1. Then, in Section 4, we prove Theorem 1.2. Furthermore, Section 5 is devoted to the proof of Theorem 1.3. Finally, in Section 6 we carry out the proofs of some technical lemmas.

2. Preliminaries

2.1. Notation

If G is a graph, then we let $V(G)$ denote the vertex set and $E(G)$ the edge set of G. Frequently we just write G instead of $V(G)$ or $E(G)$. If $S \subset V(G)$, then $G[S]$ signifies the subgraph of G induced on S. Further, if $A, B \subset V(G)$, then $e(A,B) = e_G(A,B) = \#\{e \in E(G) : e \cap A, e \cap B \neq \emptyset\}$ is the number of A-B-edges in G. Moreover, we let $e(A) = e_G(A) = e(A,A)$. For $v \in V(G)$, we let $N(v) = N_G(v) = \{w \in V : \{v,w\} \in E(G)\}$ be the neighbourhood and $d_G(v) = \#N_G(v)$ the degree of v in G. If $X \subset V$, then we let $N(X) = N_G(X) = \bigcup_{v\in X} N_G(v)$.

Throughout, we let $V = \{1, \ldots, n\}$ be a set of n labelled vertices. *We frequently assume implicitly in the computations that n is sufficiently large.* We will mainly be concerned with random graphs $G = G_{n,p}$; we then let $\bar{d} = (n-1)p$ signify the expected average degree.

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If $v \geqslant 1$ is an integer, then $\vec{E}_v$ signifies the $v \times v$ matrix whose diagonal entries are 1 and whose remaining entries are 0. Furthermore, $\vec{1}_v$ denotes a vector of size v with all entries equal to 1. We omit the index v if it is clear from the context. If T is a set and $S \subset T$, then $\vec{1}_S$ denotes the vector $(a_v)_{v \in T}$ such that $a_v = 1$ if $v \in S$ and $a_v = 0$ otherwise. If A is a symmetric $v \times v$ matrix, then $\lambda_1(A) \leqslant \cdots \leqslant \lambda_v(A)$ denote the eigenvalues of A. Moreover, $\|A\| = \max_{\|\xi\|=1} \|A\xi\|$ signifies the norm of A. By $\mathrm{diag}(a_1, \ldots, a_n)$ we denote the $n \times n$ diagonal matrix with diagonal entries $a_1, \ldots, a_n$. Finally, if $M = (m_{vw})_{v,w \in T}$ is a matrix and $S \subset T$, then we let $M_S = (m'_{vw})_{v,w \in T}$ denote the matrix with entries $m'_{vw} = m_{vw}$ if both v and w are in S, and $m'_{vw} = 0$ otherwise.

We shall use the elementary estimate

$$1 + \frac{1}{2}x - \frac{1}{8}x^2 \leqslant \sqrt{1+x} \leqslant 1 + \frac{x}{2} \quad (-1 < x). \tag{2.1}$$

Furthermore, recall that for any graph G

$$\|A(G)\| \leqslant \max_{v \in V(G)} d_G(v). \tag{2.2}$$

2.2. Chernoff bounds

Let ϕ denote the function

$$\phi(x) = (1+x)\ln(1+x) - x. \tag{2.3}$$

Using elementary calculus, it is easily verified that $\phi(x) \geqslant \phi(-x)$ for $0 \leqslant x < 1$, and that

$$\phi(x) \geqslant \frac{x^2}{2(1+x/3)} \quad (x \geqslant 0), \quad \textit{cf.}\ [20, \text{p. } 27]. \tag{2.4}$$

In terms of the function ϕ, we have the following bounds on the tails of a binomially distributed random variable X with mean μ; the proofs of both inequalities are given in [20, Section 2.1]:

$$\mathrm{P}(X \geqslant \mu + t) \leqslant \exp\left(-\mu\phi\left(\frac{t}{\mu}\right)\right) \leqslant \exp\left(-\frac{t^2}{2(\mu + t/3)}\right) \quad (0 < t), \tag{2.5}$$

$$\mathrm{P}(X \leqslant \mu - t) \leqslant \exp\left(-\mu\phi\left(\frac{-t}{\mu}\right)\right) \leqslant \exp\left(-\frac{t^2}{2\mu}\right) \quad (0 < t < \mu). \tag{2.6}$$

2.3. Preliminaries on $G_{n,p}$

To prove Theorems 1.2 and 1.3, we need a few general facts on the degree distribution of $G_{n,p}$. Since the vertex degrees in the random graph $G_{n,p}$ are binomially distributed with mean $\bar{d} = (n-1)p$, we can use the Chernoff bounds (2.5) and (2.6) to bound the probability that the degree of $v \in V$ deviates from $\bar{d}$ by a certain amount. This yields the following estimate, whose somewhat technical proof is deferred to Section 6.2.

Lemma 2.1. *Let $t \geqslant 0$. Let $G = G_{n,p}$, and let $S_t = \{v \in V : 2^t \leqslant |d_G(v) - \bar{d}| \cdot \bar{d}^{-1/2} < 2^{t+1}\}$. Moreover, set $\phi_t = \bar{d} \cdot \phi(2^t \bar{d}^{-1/2})$. Then $\mathrm{P}\big[\#S_t \leqslant 4\exp(-\phi_t/100)n\big] \geqslant 1 - n^{-1/100}$.*

Let $G = G_{n,p}$. We will apply Lemma 2.1 in the following way: given some $t \geqslant 0$, we consider the sets $S_j = \{v \in V : 2^{t+j} \leqslant |d_G(v) - \bar{d}| \cdot \bar{d}^{-1/2} < 2^{t+j+1}\}$ for $j \geqslant 0$. Since $0 \leqslant d_G(v) \leqslant n$ for all v, there are only $O(\ln n)$ values $j \geqslant 0$ such that $S_j \neq \emptyset$. Let $\phi_j = \bar{d} \cdot \phi(2^{t+j}\bar{d}^{-1/2})$. Then

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by Lemma 2.1, for all j we have $\mathrm{P}\big[\#S_j > 4\exp(-\phi_j/100)n\big] \leqslant n^{-1/100}$, whence by the union bound w.h.p. $\#S_j \leqslant 4\exp(-\phi_j/100)n$ for *all* j.

Corollary 2.2. *Suppose that $\bar{d} = (n-1)p \geqslant c_0$ for a sufficiently large constant $c_0 > 0$. Let $G = G_{n,p}$, and let $S = \{v \in V : |d_G(v) - \bar{d}| \geqslant \bar{d}/1000\}$. There is a constant $\delta > 0$ such that $\#S \leqslant n\exp(-\delta\bar{d})$ w.h.p.*

Proof. Let $\varepsilon = 1/1000$. Let $t \geqslant 0$ be such that

$$2^t = \varepsilon\bar{d}^{1/2}. \tag{2.7}$$

Let $S_j = \{v \in V : 2^{t+j}\bar{d}^{1/2} \leqslant |\bar{d} - d_G(v)| < 2^{t+j+1}\bar{d}^{1/2}\}$ for $j \geqslant 0$. Then $S \subset \bigcup_{j=0}^{\lceil \ln n\rceil} S_j$. Moreover, by Lemma 2.1, $\#S_j \leqslant 4n\exp(-\phi_{t+j})$ for all $j = 0, \ldots, \lceil \ln \bar{n}\rceil$ w.h.p., where $\phi_{t+j} = \bar{d}\phi(2^{t+j}\bar{d}^{-1/2})/100$. Hence, we obtain

$$\begin{aligned}
\frac{\#S}{n} &\leqslant \sum_{j=0}^{\lceil \ln \bar{n}\rceil} \frac{\#S_j}{n} \leqslant 4\sum_{j=0}^{\lceil \ln \bar{n}\rceil} \exp(-\phi_{t+j}) \overset{(2.4)}{\leqslant} 4\sum_{j=0}^{\infty} \exp\left[-\frac{2^{2(t+j)}}{200(1+\frac{1}{3}2^{t+j}\bar{d}^{-1/2})}\right] \\
&= 4\sum_{j:2^{t+j}\leqslant \frac{1}{3}\bar{d}^{1/2}} \exp\left(-\frac{1}{400}2^{2(t+j)}\right) + \sum_{j:2^{t+j}>\frac{1}{3}\bar{d}^{1/2}} \exp\left(-\frac{1}{400}2^{(t+j)}\bar{d}^{1/2}\right) \\
&\overset{(2.7)}{\leqslant} 4\sum_{j=0}^{\infty} \exp\left(-\frac{\varepsilon^2\bar{d}}{400}\cdot 2^j\right) \leqslant \exp\left(-\frac{\varepsilon^2\bar{d}}{800}\right),
\end{aligned}$$

thereby proving the corollary. □

If $G = G_{n,p}$ and $X \subset V$, then the expected degree of all vertices $v \in X$ is $\bar{d} = (n-1)p$. Therefore, we expect that $\#N_G(X) \leqslant \bar{d}\#X$. The following corollary shows that indeed a similar estimate holds for *all* sets $X \subset V$ simultaneously w.h.p.

Corollary 2.3. *Suppose that $\bar{d} = (n-1)p \geqslant c_0$ for a sufficiently large constant $c_0 > 0$. Then the random graph $G = G_{n,p}$ enjoys the following property w.h.p.:*

$$\forall X \subset V : \#N_G(X) \leqslant 2\bar{d}\#X + n\exp(-\bar{d}/15).$$

Proof. Let $X_{-1} = \{v \in X : d_G(v) \leqslant 2\bar{d}\}$. Moreover, let $X_j = \{v \in V : 2^j\bar{d}^{1/2} \leqslant d_v - \bar{d} \leqslant 2^{j+1}\bar{d}^{1/2}\}$ for $j \geqslant 0$. Then $\#N_G(X) \leqslant 2\bar{d}\#X_{-1} + \sum_{j:2^{j+1}\geqslant \bar{d}^{1/2}}(\bar{d} + 2^{j+1}\bar{d}^{1/2})\#X_j$. Further, letting $\phi_j = \bar{d}\cdot\phi(2^j\bar{d}^{-1/2})/100$ and invoking Lemma 2.1, we obtain that w.h.p.

$$\begin{aligned}
\sum_{j:2^j\geqslant \frac{1}{2}\bar{d}^{1/2}} 4(2^{j+1}\bar{d}^{1/2} + \bar{d})n\exp(-\phi_j) &\overset{(2.4)}{\leqslant} n\sum_{j:2^j\geqslant \frac{1}{2}\bar{d}^{1/2}} 3\cdot 2^{j+1}\bar{d}^{1/2}\exp\left[-\frac{2^j\bar{d}^{1/2}}{600}\right] \\
&\leqslant n\sum_{j:2^j\geqslant \frac{1}{2}\bar{d}^{1/2}} \exp\left[-\frac{2^j\bar{d}^{1/2}}{700}\right] \leqslant n\exp\left[-\frac{\bar{d}}{1500}\right],
\end{aligned}$$

as desired. □

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Finally, we shall apply the following lemma repeatedly, which shows that for *any* small set $X \subset V$ the subgraph of $G = G_{n,p}$ induced on Y is very sparse w.h.p.

Lemma 2.4. *Suppose that $\bar{d} = (n-1)p \geqslant c_0$ for a sufficiently large constant $c_0 > 0$. Then the random graph $G = G_{n,p}$ has the following property w.h.p.: if $X \subset V$ has cardinality $\#X \leqslant 2n\bar{d}^{-2}$, then $e_G(X) \leqslant 10\#X$.*

The proof of Lemma 2.4 can be found in [10, Section 2.2].

3. Proof of Proposition 1.1

Suppose that $\bar{d} = (n-1)p = c > 0$ remains constant as $n \to \infty$. Then w.h.p. $G = G_{n,p}$ is disconnected, and $\Omega(n)$ connected components are trees (see [5, Chapter 5]). Since, for a bipartite graph H, the largest eigenvalue of $\mathscr{L}(H)$ equals 2 (see [7, p. 8]), we conclude that $\lambda_n(\mathscr{L}(G)) \geqslant 2$ w.h.p.; on the other hand, $\|\mathscr{L}(G)\| \leqslant 2$ for any graph G (see [7, p. 7]), whence $\lambda_n(\mathscr{L}(G_{n,p})) = 2$ with high probability.

To bound the smallest positive eigenvalue of $\mathscr{L}(G_{n,p})$, we need the following lemma, which we shall prove in Section 6.1.

Lemma 3.1. *With high probability, one of the connected components of $G = G_{n,p}$ is a path of length $\Omega(\ln n)$.*

Thus, assume that $G = G_{n,p}$ has a connected component P that is a path of length $k = \Omega(\ln n)$. Then the spectrum of $\mathscr{L}(G)$ contains the spectrum $S = \{\lambda_i(\mathscr{L}(P)) : 1 \leqslant i \leqslant k\}$ of P. Furthermore, $S = \{1 - \cos(\pi j/(k-1)) : 0 \leqslant j < k\}$ (see [7, p. 6]). Hence, w.h.p. the smallest positive eigenvalue of $\mathscr{L}(G)$ is $\leqslant 1 - \cos(\pi/(k-1)) = O(k^{-2}) = O(\ln^{-2} n) = o(1)$, as claimed. (Of course, to obtain this estimate, it is not necessary that the path under consideration actually has length $\Omega(\ln n)$; any isolated path of length $\ell = \ell(n)$ such that $\ell(n) \to \infty$ as $n \to \infty$ would do.)

4. The spectral gap of the normalized Laplacian

Throughout, we let $G = G_{n,p}$ and $H = \mathrm{core}(G)$. Moreover, we assume that $c_0 \leqslant \bar{d} = (n-1)p \leqslant 0.99n$ for some sufficiently large constant $c_0 > 0$.

4.1. Outline of the proof

As a first step, we shall prove the following bound on the number of vertices in $H = \mathrm{core}(G)$, thereby establishing part (1) of Theorem 1.2.

Proposition 4.1. *There is a constant $c > 0$ such that $\#V(H) \geqslant n\exp(-c\bar{d})$ w.h.p.*

Proof. Let Y be the set of vertices deleted in step CR1. By Corollary 2.2, we have $y = \#Y \leqslant n\exp(-c'\bar{d})$ w.h.p., for a certain constant $c' > 0$. Let $z_1, \ldots, z_k$ be the vertices removed during step CR2. We claim that $k < y$ w.h.p.; note that if this is true, then $\#V(\mathrm{core}(G)) \geqslant n - (y+k) \geqslant n(1 - \exp(-c'\bar{d}/2))$, provided that $\bar{d} \geqslant c_0$ for a large enough constant c_0.

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Thus, assume that $k \geqslant y$. In the graph G each vertex z_i has at least 100 neighbours in the set $Y \cup \{z_1, \ldots, z_{i-1}\}$. Consequently, the set $T = Y \cup \{z_1, \ldots, z_y\}$ has the properties

$$e_G(T) \geqslant 100y = 50\#T \quad \text{and} \quad \#T = 2y \leqslant 2n\exp(-c'\bar{d}) \leqslant n\bar{d}^{-2}. \tag{4.1}$$

However, by Lemma 2.4 there is no set T satisfying (4.1) w.h.p. □

To prove part (2) of Theorem 1.2, we consider the matrix

$$D = \mathrm{diag}(\bar{d}^{\,1/2} d_H(v)^{-1/2})_{v \in V(H)};$$

note that D is well defined, because $d_H(v) > 0$ for all $v \in V(H)$ (by (1.3)). Furthermore, since $d_H(v) \geqslant \bar{d}/3$ for all $v \in H$ by (1.3), all entries of D are $\leqslant 2$, whence

$$\|D\| \leqslant 2. \tag{4.2}$$

Let $M = D \cdot A(H) \cdot D$. Then for $v, w \in V(H)$ the vwth entry of M is $\bar{d}(d_H(v)d_H(w))^{-1/2}$ if v, w are adjacent and 0 if v, w are not adjacent. Hence, $\mathscr{L}(H) = \hat{E} - \bar{d}^{-1}M$.

To study its eigenvalues, we decompose the matrix M into two parts. Let $X = \{v \in H : |d_H(v) - \bar{d}| \leqslant 0.01\bar{d}\}\backslash N_G(V\backslash H)$, and set $S = X\backslash N_H(V\backslash X)$. Then we will consider the submatrices M_S and $M_{V\backslash S}$ separately. To study the eigenvalues of M_S, we observe that the set S contains most of the vertices of H.

Lemma 4.2. *With high probability, we have* $\#S \geqslant (1 - \bar{d}^{-10})n$.

Proof. Let $X_0 = \{v \in V(H) : |d_H(v) - \bar{d}| \leqslant \varepsilon\bar{d}\}$. Then Corollary 2.2 entails that $\#V\backslash X_0 \leqslant \bar{d}^{-16}n$ w.h.p. Moreover, $\#V\backslash H \leqslant \bar{d}^{-16}n$ w.h.p. by Proposition 4.1. Hence, Corollary 2.3 implies that $\#N_G(V\backslash H) \leqslant \bar{d}^{-14}n$ w.h.p. Therefore, $\#V\backslash X \leqslant \#V\backslash X_0 + \#N_G(V\backslash H) \leqslant \bar{d}^{-13}n$ w.h.p. Thus, once more due to Corollary 2.3, we have $\#N_H(V\backslash X) \leqslant \bar{d}^{-11}n$ w.h.p. Hence, w.h.p.

$$\#V\backslash S \leqslant \#V\backslash X + \#N_H(V\backslash X) \leqslant (\bar{d}^{-13} + \bar{d}^{-11})n \leqslant \bar{d}^{-10}n,$$

as claimed. □

The next proposition, which we will prove in Section 4.2, summarizes the analysis of M_S.

Proposition 4.3. *There is a constant* $c_1 > 0$ *such that* $\|M_S\eta\| \leqslant c_1\bar{d}^{1/2}$ *for all unit vectors* $\eta \perp D^{-1}\vec{1}$ *w.h.p.*

With respect to the matrix $M_{V\backslash S}$, we shall prove the following slightly more general proposition in Section 4.3.

Proposition 4.4. *There is a constant* $C > 0$ *such that w.h.p. the following holds. If* $Z \subset V$ *has cardinality* $\#Z \leqslant n\bar{d}^{-8}$, *then* $\|M_Z\| \leqslant C\bar{d}^{1/2}$ *w.h.p.*

Proof of Theorem 1.2. The first assertion is an immediate consequence of Proposition 4.1. To prove the second assertion, let $\mathscr{L} = \mathscr{L}(H)$. Moreover, let $Z = (V(H)\backslash S) \cup N_H(V(H)\backslash S)$.

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Combining Corollary 2.3 and Lemma 4.2, we conclude that $\#Z \leqslant \bar{d}^{-8}n$ w.h.p. Therefore, Proposition 4.4 implies that w.h.p.

$$\|M_{Z\cap S}\| \leqslant \|M_Z\| \leqslant C\bar{d}^{\,1/2}, \tag{4.3}$$

for a certain constant $C > 0$. Moreover, after permuting the vertices suitably, the matrix M has the form

$$M = \begin{pmatrix} M_{Z\setminus S} & * & 0 \\ * & M_{Z\cap S} & * \\ 0 & * & M_{S\setminus Z} \end{pmatrix},$$

so that

$$M = M_Z - M_{Z\cap S} + M_S. \tag{4.4}$$

Let $\xi = D^{-1}\vec{1}$. Then combining (4.3), (4.4), and Proposition 4.3, we obtain that w.h.p.

$$\max_{\eta\perp\xi,\,\|\eta\|=1} \|M\eta\| \leqslant \|M_Z\| + \|M_{Z\cap S}\| + \max_{\eta\perp\xi,\,\|\eta\|=1} \|M_S\eta\| \leqslant (2C + c_1)\bar{d}^{\,1/2}. \tag{4.5}$$

Finally, as $\mathscr{L} = \vec{E} - \bar{d}^{-1}M$ and $\mathscr{L}\xi = 0$, (4.5) yields that

$$\lambda_2(L) \geqslant 1 + \bar{d}^{-1} \min_{\eta\perp\xi,\,\|\eta\|=1} \langle M\eta, \eta\rangle \geqslant 1 - \bar{d}^{-1} \max_{\eta\perp\xi,\,\|\eta\|=1} \|M\eta\| \geqslant 1 - (2C + c_1)\bar{d}^{\,-1/2},$$
$$\lambda_n(L) \leqslant 1 + \bar{d}^{-1} \max_{\eta\perp\xi,\,\|\eta\|=1} \langle M\eta, \eta\rangle \leqslant 1 + \bar{d}^{-1} \max_{\eta\perp\xi,\,\|\eta\|=1} \|M\eta\| \leqslant 1 + (2C + c_1)\bar{d}^{\,-1/2},$$

whence Theorem 1.2 follows. □

4.2. Proof of Proposition 4.3

In order to prove Proposition 4.3, we shall combine Theorem 1.4 (which concerns the eigenvalues of the adjacency matrix of $G_{n,p}$) with an investigation of the effect of the normalization $D_S \cdot A(H) \cdot D_S$. The starting point of the analysis of the normalization is the following estimate.

Lemma 4.5. *There is a constant $C > 0$ such that $\sum_{v\in S}\big(\bar{d}^{\,1/2} - d_H(v)^{1/2}\big)^2 \leqslant Cn$ w.h.p.*

Proof Let $t_v = d_H(v) - \bar{d}$. Then $|t_v| \leqslant 0.01\bar{d}$ for all $v \in S$. Therefore,

$$\left|\frac{\bar{d}^{1/2} - (\bar{d} + t_v)^{1/2}}{\bar{d}^{1/2}}\right| = \left|1 - \left(1 + \frac{t_v}{\bar{d}}\right)^{1/2}\right| \overset{(2.1)}{\leqslant} \frac{|t_v|}{2\bar{d}} + \frac{1}{8}\left(\frac{t_v}{\bar{d}}\right)^2 \leqslant \frac{|t_v|}{\bar{d}} \quad (v \in S). \tag{4.6}$$

Let $S_j = \{v \in S : 2^j\bar{d}^{1/2} \leqslant |t_v| < 2^{j+1}\bar{d}^{1/2}\}$ for $j \geqslant 0$, and set $S_{-1} = \{v \in S : |t_v| < \bar{d}^{1/2}\}$. Then

$$\sum_{v\in S}\left(\frac{\bar{d}^{\,1/2} - d_H(v)^{1/2}}{\bar{d}^{\,1/2}}\right)^2 \overset{(4.6)}{\leqslant} \sum_{v\in S}\frac{t_v^2}{\bar{d}^2} = \frac{1}{\bar{d}}\sum_{j\geqslant -1}\sum_{v\in S_j}\frac{t_v^2}{\bar{d}} \leqslant \frac{1}{\bar{d}}\sum_{j\geqslant -1}\#S_j 2^{2(j+1)}. \tag{4.7}$$

Furthermore, let ϕ denote the function defined in (2.3), and set $\phi_j = \bar{d}\cdot\phi(2^j\bar{d}^{-1/2})$. Then by Lemma 2.1 we have $\#S_j \leqslant 4n\exp(-\phi_j/100)$ for all $j \geqslant 0$ w.h.p. Hence, (2.4) yields

$$\#S_j \leqslant 4n\exp\left[-\frac{2^{2j}}{200(1 + 2^j\bar{d}^{-1/2}/3)}\right] \leqslant 4n\exp(-2^j/400) \quad (j \geqslant 0). \tag{4.8}$$

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Combining (4.7) and (4.8), we obtain that

$$\sum_{v\in S}\left(\frac{\bar{d}^{\,1/2}-d_H(v)^{1/2}}{\bar{d}^{\,1/2}}\right)^2 \leqslant \frac{4n}{\bar{d}}\left[1+\sum_{j\geqslant 0}\exp(-2^j/400)2^{2j+2}\right] \leqslant Cn\bar{d}^{-1} \tag{4.9}$$

w.h.p., for a certain constant $C>0$. □

Now, our goal is to bound $\|M_S\xi\|$ for unit vectors $\xi\perp\vec{1}_S$. The following lemma states that, for such vectors ξ, the image $D_S\xi$ is essentially perpendicular to $\vec{1}_S$.

Lemma 4.6. *There is a constant $c_2>0$ such that w.h.p. the following holds: for all unit vectors $\xi\perp\vec{1}_S$ we have $\langle D_S\xi,\vec{1}_S\rangle\leqslant c_2\bar{d}^{-1/2}\|\vec{1}_S\|$.*

Proof Since $\langle\xi,\vec{1}_S\rangle=0$, we have

$$\begin{aligned}|\langle D_S\xi,\vec{1}_S\rangle| &= |\langle D_S\xi,\vec{1}_S\rangle-\langle\xi,\vec{1}_S\rangle| = \left|\sum_{v\in S}\left[\left(\frac{\bar{d}}{d_H(v)}\right)^{1/2}-1\right]\xi_v\right| \\ &\leqslant \left[\sum_{v\in S}\left[\left(\frac{\bar{d}}{d_H(v)}\right)^{1/2}-1\right]^2\right]^{1/2}\cdot\|\xi\| \quad\text{(by Cauchy–Schwarz).}\end{aligned} \tag{4.10}$$

Furthermore, as $|d_H(v)-\bar{d}|\leqslant 0.01\bar{d}$ for all $v\in S$, we have

$$\left|\left(\frac{\bar{d}}{d_H(v)}\right)^{1/2}-1\right| = \left|\frac{\bar{d}^{1/2}-d_H(v)^{1/2}}{d_H(v)^{1/2}}\right| \leqslant 2\left|\frac{\bar{d}^{1/2}-d_H(v)^{1/2}}{\bar{d}^{1/2}}\right|. \tag{4.11}$$

Combining (4.10), (4.11), and Lemma 4.5, we conclude that w.h.p.

$$\langle D_S\xi,\vec{1}_S\rangle^2 \leqslant 4\sum_{v\in S}\left(\frac{\bar{d}^{\,1/2}-d_H(v)^{1/2}}{\bar{d}^{\,1/2}}\right)^2 \leqslant 4Cn\bar{d}^{-1}, \tag{4.12}$$

where $C>0$ is the constant from Lemma 4.5. Finally, as Lemma 4.2 implies that $\|\vec{1}_S\|^2=\#S\geqslant 0.99n$ w.h.p., the assertion follows from (4.12). □

Corollary 4.7. *There is a constant $c_3>0$ such that the following holds w.h.p.:*

$$\forall\xi\perp\vec{1}_S,\ \|\xi\|=1:\|M_S\xi\|\leqslant c_3\bar{d}^{1/2}.$$

Proof Let $\xi\perp\vec{1}_S$ be a unit vector. Then (4.2) implies that

$$\|M_S\xi\| = \|D_SA_SD_S\xi\| \leqslant \|D_S\|\cdot\|A_SD_S\xi\| \leqslant 2\|A_SD_S\xi\|. \tag{4.13}$$

Consider a vector $\eta\perp\vec{1}_S$ of norm $\leqslant 1$ and a number $-1\leqslant\alpha\leqslant 1$ such that $D_S\xi=\alpha\|\vec{1}_S\|^{-1}\vec{1}_S+\eta$. By Lemma 4.6, $|\alpha|\leqslant c_2\bar{d}^{\,-1/2}$ w.h.p. Moreover, Theorem 1.4 yields that w.h.p. $\|A_S\eta\|\leqslant C_2\bar{d}^{\,1/2}$, where C_2 signifies a suitable constant. In addition, (2.2) entails $\|A_S\vec{1}_S\|\leqslant\|A_S\|\cdot\|\vec{1}_S\|\leqslant 2\bar{d}\cdot\|\vec{1}_S\|$, because $d_H(v)\leqslant 2\bar{d}$ for all $v\in S$. Hence, we conclude that there is a constant $c_3>0$ such that

$$\|M_S\xi\| \overset{(4.13)}{\leqslant} 2\|A_SD_S\xi\| \leqslant 2\big(\|A_S\eta\|+|\alpha|\cdot\|\vec{1}_S\|^{-1}\cdot\|A_S\vec{1}_S\|\big) \leqslant 2C_2\bar{d}^{1/2}+4|\alpha|\bar{d} \leqslant c_3\bar{d}^{1/2},$$

thereby proving the corollary. □

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While Corollary 4.7 bounds $\|M_S\xi\|$ for unit vectors $\xi \perp \vec{1}_S$, our actual goal is to bound $\|M\eta\|$ for unit vectors $\eta \perp D^{-1}\vec{1}$. Therefore, the next step is to compare $\vec{1}_S$ and $D^{-1}\vec{1}$.

Lemma 4.8. *There is a constant $C > 0$ such that $\|D^{-1}\vec{1}_H - \vec{1}_S\| \leqslant C\bar{d}^{-1/2}n^{1/2}$ w.h.p.*

Proof We have

$$\|D^{-1}\vec{1}_H - \vec{1}_S\|^2 = \sum_{v\in S}\left[\left(\frac{d_H(v)}{\bar{d}}\right)^{1/2} - 1\right]^2 + \sum_{v\in H\setminus S}\frac{d_v}{\bar{d}}. \tag{4.14}$$

Furthermore, by Lemma 4.5 there is a constant $C' > 0$ such that w.h.p.

$$\sum_{v\in S}\left[\left(\frac{d_H(v)}{\bar{d}}\right)^{1/2} - 1\right]^2 = \sum_{v\in S}\left(\frac{d_H(v)^{1/2} - \bar{d}^{1/2}}{\bar{d}^{1/2}}\right)^2 \leqslant C'n\bar{d}^{-1}. \tag{4.15}$$

Moreover, let $S_j = \{v \in V : 2^j\bar{d}^{1/2} \leqslant |\bar{d} - d_G(v)| < 2^{j+1}\bar{d}^{1/2}\}$ for $j \geqslant 0$, and let $S_{-1} = \{v \in V : |\bar{d} - d_G(v)| \leqslant 0.01\bar{d}\}$. In addition, set $\phi_j = \bar{d}\cdot\phi(2^j\bar{d}^{-1/2})$, where ϕ is the function defined in (2.3). Applying Lemmas 2.1 and 4.2, we obtain that w.h.p.

$$\begin{aligned}\sum_{v\in H\setminus S}\frac{d_v}{\bar{d}} &\leqslant \sum_{j:2^j\geqslant 0.01\bar{d}^{1/2}} \#S_j 2^{j+1} + \sum_{v\in S_{-1}\setminus S}\frac{d_v}{\bar{d}}\\ &\leqslant \sum_{j:2^j\geqslant 0.01\bar{d}^{1/2}} 4n\exp(-\phi_j/100)2^{j+1} + 2\#V\setminus S\\ &\overset{(2.4)}{\leqslant} 8n\sum_{j:2^j\geqslant 0.01\bar{d}^{1/2}} 2^j\exp(-2^j/400) + 2\#V\setminus S \leqslant C'n\bar{d}^{-1},\end{aligned} \tag{4.16}$$

provided that C' is a large enough constant. Plugging (4.15) and (4.16) into (4.14), we obtain the desired bound on $\|D^{-1}\vec{1}_H - \vec{1}_S\|$. □

In addition, to prove Proposition 4.3 we need the following simple estimate.

Lemma 4.9. *Let $e = \|\vec{1}_S\|^{-1}\vec{1}_S$. Then $\|M_S e\| \leqslant 8\bar{d}$.*

Proof Since $d_H(v) \leqslant 2\bar{d}$ for all $v \in S$, (2.2) yields $\|A_S\| \leqslant 2\bar{d}$. Moreover, $\|D\| \leqslant 2$ by (4.2). Hence, $\|M_S e\| \leqslant \|D_S\|^2\cdot\|A\|\cdot\|e\| \leqslant 8\bar{d}$. □

Proof of Proposition 4.3. Let

$$\gamma = \|D^{-1}\vec{1}_H\|^{-1}D^{-1}\vec{1}_H \quad\text{and}\quad e = \|\vec{1}_S\|^{-1}\vec{1}_S.$$

Since, by Lemma 4.2, $(1-\bar{d}^{-2})n^{1/2} \leqslant \|\vec{1}_S\| = \#S^{1/2} \leqslant n^{1/2}$, Lemma 4.8 implies that

$$|\|\vec{1}_S\| - \|D^{-1}\vec{1}_H\|| \leqslant \|\vec{1}_S - D^{-1}\vec{1}_H\| \leqslant C\bar{d}^{-1/2}n^{1/2}, \quad\text{so that } \|D^{-1}\vec{1}_H\| \geqslant \frac{1}{2}n^{1/2}. \tag{4.17}$$

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Therefore, there is a constant $C > 0$ such that

$$\begin{aligned}\|\gamma - e\| &\leqslant \|D^{-1}\vec{1}_H\|^{-1} \cdot \|D^{-1}\vec{1}_H - \vec{1}_S\| + \|\vec{1}_S\| \cdot \left| \frac{\|\vec{1}_S\| - \|D^{-1}\vec{1}_H\|}{\|\vec{1}_S\| \cdot \|D^{-1}\vec{1}_H\|} \right| \\ &\overset{(4.17)}{\leqslant} 2n^{-1/2} \cdot \left[\|D^{-1}\vec{1}_H - \vec{1}_S\| + |\|\vec{1}_S\| - \|D^{-1}\vec{1}_H\|| \right] \\ &\overset{\text{Lem. 4.8, (4.17)}}{\leqslant} 4C\bar{d}^{-1/2}. \end{aligned} \tag{4.18}$$

Now, let $\eta \perp \gamma$ be a unit vector. Then there is a decomposition $\eta = \alpha e + \beta\xi$, where $\xi \perp \vec{1}_S$ is a unit vector, and $\alpha^2 + \beta^2 = 1$. Owing to (4.18),

$$|\alpha| = |\langle \eta, e\rangle| = |\langle \eta, \gamma\rangle + \langle \eta, e - \gamma\rangle| \leqslant |\langle \eta, \gamma\rangle| + \|\eta\| \cdot \|e - \gamma\| \leqslant 4C\bar{d}^{-1/2}. \tag{4.19}$$

Finally, Corollary 4.7 and Lemma 4.9 entail that w.h.p. $\|M_S\eta\| \leqslant |\alpha| \cdot \|M_S e\| + \|M_S\xi\| \leqslant 32C\bar{d}^{1/2} + c_3\bar{d}^{1/2}$, as claimed. □

4.3. Proof of Proposition 4.4

The proof of Proposition 4.4 relies on the following fact. If $Z \subset V$ is small enough (say, $\#Z \leqslant n\bar{d}^{-8}$), then the subgraph $G[Z]$ of $G = G_{n,p}$ induced on Z has a very simple structure: we can decompose $Z = Z_1 \cup \cdots \cup Z_K$ so that the vertices in Z_j have only a small number of neighbours in the sets $\bigcup_{i \geqslant j} Z_i$ of higher index.

Lemma 4.10. *With high probability, $G = G_{n,p}$ has the following property: any set $Z \subset V$ of cardinality $\#Z \leqslant n\bar{d}^{-8}$ admits a partition $Z = \bigcup_{i=1}^{K} Z_i$ such that:*

(1) $K \leqslant \ln n$,
(2) *if $1 \leqslant j \leqslant K$ and $v \in Z_j$, then $e(v, \bigcup_{i=j}^{K} Z_i) \leqslant 100$.*

In particular, the maximum degree of the subgraph $G[Z_i]$ of G induced on Z_i is $\leqslant 100$.

Proof We define the sets Z_j inductively. First, we let $Z_1 = \{v \in Z : e(v, Z) \leqslant 100\}$. Then, having defined Z_j, we set $Y_j = Z \setminus \bigcup_{i=1}^{j} Z_i$ and $Z_{j+1} = \{v \in Y_j : e(v, Y_j) \leqslant 100\}$. Note that this construction ensures that (2) holds.

In order to prove (1), we shall establish that w.h.p. $\#Z_{j+1} \geqslant \frac{4}{5}\#Y_j$ for all $j \geqslant 1$. To this end, we observe that by Lemma 2.4, $\sum_{v \in Y_j} e(v, Y_j) = 2e(Y_j) \leqslant 20 \cdot \#Y_j$ for all $j \geqslant 1$ w.h.p., because of our assumption that $\#Z \leqslant n\bar{d}^{-8}$. Hence, the estimate $100 \cdot \#Y_j \setminus Z_{j+1} \leqslant \sum_{v \in e(v,Y)} e(v, Y_j) \leqslant 20 \cdot \#Y_j$ implies that $\#Z_{j+1} \geqslant \frac{4}{5}Y_j$, as desired. □

Proof of Proposition 4.4. Decompose Z into sets $Z_1, \ldots, Z_K$ as in Lemma 4.10; as a convenience, set $Z_{>j} = \bigcup_{j<i\leqslant K} Z_i$ and $Z_{<j} = \bigcup_{1\leqslant i<j} Z_i$. Let $P = \sum_{j=1}^{K} M_{Z_j}$ and $Q = M_Z - P$. Then the submatrix P of M_Z represents the edges within the classes Z_j, while Q represents the Z_i-Z_j-edges for $i \neq j$. Since $M = P + Q$, it suffices to prove that $\|P\|, \|Q\| \leqslant C\bar{d}^{1/2}$ for a sufficiently large constant C.

With respect to Q, consider a unit vector $\xi = (\xi_v)_{v \in V}$. Then the entries of $\eta = Q\xi$ are

$$\eta_v = \sum_{w \in N_H(v) \cap Z_{>j}} \frac{\bar{d}\xi_w}{(d_H(v)d_H(w))^{1/2}} + \sum_{w \in N_H(v) \cap Z_{<j}} \frac{\bar{d}\xi_w}{(d_H(v)d_H(w))^{1/2}} \quad (v \in Z_j).$$

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Let

$$\alpha_j = \sum_{v\in Z_j}\left[\sum_{w\in N_H(v)\cap Z_{>j}} \frac{\bar{d}\xi_w}{(d_H(v)d_H(w))^{1/2}}\right]^2,$$
$$\beta_j = \sum_{v\in Z_j}\left[\sum_{w\in N_H(v)\cap Z_{<j}} \frac{\bar{d}\xi_w}{(d_H(v)d_H(w))^{1/2}}\right]^2.$$

Then

$$\|\eta\|^2 = \sum_{j=1}^{K}\sum_{v\in Z_j}\eta_v^2 \leqslant 4\sum_{j=1}^{K}\alpha_j+\beta_j. \tag{4.20}$$

Furthermore, since each vertex $v\in Z_j$ has $\leqslant 100$ neighbours in $Z_{>j}$, we get

$$\begin{aligned}\sum_{j=1}^{K}\alpha_j &\leqslant 10^4\sum_{j=1}^{K}\sum_{v\in Z_j}\sum_{w\in N_H(v)\cap Z_{>j}}\left(\frac{\bar{d}\xi_w}{(d_H(v)d_H(w))^{1/2}}\right)^2\\ &= 10^4\sum_{j=1}^{K}\sum_{w\in Z_j}\sum_{v\in N_H(w)\cap Z_{<j}}\frac{\bar{d}^2}{d_H(v)d_H(w)}\cdot\xi_w^2\\ &\leqslant 10^4\sum_{j=1}^{K}\sum_{w\in Z_j}\frac{\bar{d}^2}{\min_{v\in H} d_H(v)}\cdot\xi_w^2 \overset{(1.3)}{\leqslant} 10^5\bar{d}\cdot\|\xi\|^2 = 10^5\bar{d}.\end{aligned} \tag{4.21}$$

Moreover, invoking the Cauchy–Schwarz inequality, we can estimate β_j as follows:

$$\begin{aligned}\beta_j &\leqslant \sum_{v\in Z_j}\left[\sum_{w\in N_H(v)\cap Z_{<j}}\frac{\bar{d}^2}{d_H(v)d_H(w)}\right]\left[\sum_{w\in N_H(v)\cap Z_{<j}}\xi_w^2\right]\\ &\leqslant \sum_{v\in Z_j}\frac{\bar{d}^2}{d_H(v)}\sum_{w\in N_H(v)\cap Z_{<j}}\xi_w^2 \overset{(1.3)}{\leqslant} 10\bar{d}\sum_{v\in Z_j}\sum_{w\in N_H(v)\cap Z_{<j}}\xi_w^2.\end{aligned} \tag{4.22}$$

As every vertex $w\in Z_i$ has at most 100 neighbours in $Z_{>i}$, (4.22) entails that

$$\sum_{j=1}^{K}\beta_j \leqslant 10\bar{d}\sum_{j=1}^{K}\sum_{v\in Z_j}\sum_{w\in N(v)\cap Z_{<j}}\xi_w^2 \leqslant 1000\bar{d}\sum_{w\in V}\xi_w^2 \leqslant 1000\bar{d}. \tag{4.23}$$

Combining (4.20), (4.21), and (4.23), we conclude that $\|\eta\|^2\leqslant 10^6\bar{d}$, so that $\|Q\|\leqslant 1000\bar{d}^{1/2}$ with high probability.

Finally, to bound $\|P\|$, note that

$$\|P\| = \max_{1\leqslant j\leqslant K}\|M_{Z_j}\| \leqslant \max_{1\leqslant j\leqslant K}\|D_{Z_j}A_{Z_j}D_{Z_j}\| \overset{(4.2)}{\leqslant} 4\max_{1\leqslant j\leqslant K}\|A_{Z_j}\|.$$

Moreover, the subgraph $G[Z_j]$ has maximum degree $\leqslant 100$ (*cf.* Lemma 4.10), whence (2.2) yields that $\|A_{Z_j}\|\leqslant 100$ for all j. Thus, we conclude that $\|P\|\leqslant 400$, thereby completing the proof. □

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5. The spectral gap of the combinatorial Laplacian

Let $\varepsilon > 0$ be given, and let c^*, c_0 denote sufficiently large numbers depending only on ε (the values of c^*, c_0 will be specified implicitly later). We assume throughout that $c_0 \leqslant \bar{d} = (n-1)p \leqslant 0.99n$.

5.1. Outline of the proof

To exhibit a large subgraph H of $G = G_{n,p}$ that satisfies (1.4), we consider the following process.

H1. Let $T_0 = \{v \in V : |d_G(v) - \bar{d}| \geqslant c^* \bar{d}^{1/2}\}$ and $H = G - T_0$.

H2. Initially, set $k = 0$.

Do the following

Let $T_{k+1} = \{v \in V(H) : |e(v, T_k) - \#T_k p| \geqslant c^* \max\{(\#T_k/n)^{1/4} \bar{d}^{1/2}, 1\}\}$.

Set $H = H - T_{k+1}$ and increase k by 1.

while $\#T_k \geqslant n\bar{d}^{-2}$.

H3. Let $T = T_k$.

While there is a vertex v such that $e_G(v, T) \geqslant 100$, remove v from H and add v to T.

In step H1, the process removes all vertices v from G whose degree deviates from $\bar{d}$ significantly. Choosing c^* large enough, we can ensure that $\#T_0 \leqslant \varepsilon n/2$, say. Then, in step H2, we construct a sequence $T_1, \ldots, T_k$ of sets, where each T_i consists of those vertices of H such that $e(v, T_k)$ deviates from its expectation $\#T_k p$ considerably. In the analysis of the process we shall prove that $\#T_{i+1} \leqslant \#T_i/16$ w.h.p., so that the allowed deviation $c^*(\#T_k/n)^{1/4}\bar{d}^{1/2}$ decreases by a factor of 2 in each round. Finally, as soon as $\#T_k < n\bar{d}^{-2}$, we pass to step H3; note that w.h.p. $k \leqslant \ln \bar{d}$, because $\#T_{i+1} \leqslant \#T_i/16$ for all i w.h.p. Moreover, observe that the iterative procedure in step H3 is essentially identical to step CR2 in the construction of core(G). The following proposition, which we shall prove in Section 5.2, summarizes the analysis of the process H1–H3.

Proposition 5.1. *If $c^* = c^*(\varepsilon)$ is large enough, then the following holds w.h.p.*

(1) $\#V(H) \geqslant (1-\varepsilon)n$.

(2) *Let $\tilde{d} = (\#H - 1)p$. Then $|d_H(v) - \tilde{d}| \leqslant 5c^*\bar{d}^{1/2}$ for all $v \in V(H)$.*

□

Part (1) of Theorem 1.3 follows from Proposition 5.1 immediately. To establish part (2), we need the following lemma, whose proof we defer to Section 5.3.

Lemma 5.2. *There is a constant $c > 0$ such that $G = G_{n,p}$ has the following property w.h.p. Let H be a subgraph of G such that $\#V(H) \geqslant n/2$, let $\tilde{d} = (\#H - 1)p$, and suppose that (1.4) is satisfied. Let $X \subset V(H)$ be a set of cardinality $\#X \leqslant n\bar{d}^{-10}$. Then $\|A_X\| \leqslant c\tilde{d}^{1/2}$.*

Proof of Theorem 1.3. Part (1) of the theorem is an immediate consequence of Proposition 5.1. To prove part (2), we assume that H is an induced subgraph such that $\#V(H) \geqslant n/2$ that satisfies (1.4). Let $A = A(H)$ be the adjacency matrix, and let $D = \mathrm{diag}(d_H(v))_{v \in V(H)}$. Then $L = L(H) = D - A$. Since $L\vec{1}_H = 0$, in order to establish (1.5) we just need to prove that

$$\max_{\eta \perp \vec{1}_H, \|\eta\| = 1} |\langle L\eta, \eta\rangle - \tilde{d}| \leqslant 2c^*\tilde{d}^{1/2}. \tag{5.1}$$

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To show (5.1), note that (1.4) ensures that

$$\|D - \tilde{d}\vec{E}\| \leqslant c^*\tilde{d}^{\,1/2}. \tag{5.2}$$

In addition, we shall prove below that w.h.p.

$$\max_{\eta \perp \vec{1}_H,\, \|\eta\|=1} |\langle A\eta, \eta\rangle| \leqslant c^*\tilde{d}^{\,1/2}. \tag{5.3}$$

Combining (5.2) and (5.3), we conclude that

$$\begin{aligned}\max_{\eta \perp \vec{1}_H,\, \|\eta\|=1} |\langle L\eta, \eta\rangle - \tilde{d}| &\leqslant \max_{\eta \perp \vec{1}_H,\, \|\eta\|=1} (|\langle D\eta, \eta\rangle - \tilde{d}| + |\langle A\eta, \eta\rangle|)\\ &\leqslant \|D - \tilde{d}\vec{E}\| + \max_{\eta \perp \vec{1}_H,\, \|\eta\|=1} |\langle A\eta, \eta\rangle| \leqslant 2c^*\tilde{d}^{\,1/2},\end{aligned}$$

thereby proving (5.1).

Thus, the remaining task is to show (5.3). Let $X' = \{v \in V : d_G(v) \geqslant 2\bar{d}\}$, $X = X' \cap H$, and $Y = N_H(X) \cup X$. Then Corollary 2.2 entails that $\#X \leqslant \#X' \leqslant n\bar{d}^{-20}$ w.h.p. Hence, $\#Y \leqslant n\bar{d}^{-18}$ w.h.p. by Corollary 2.3. Therefore, Lemma 5.2 implies that w.h.p.

$$\max\{\|A_X\|, \|A_{Y\setminus X}\|\} \leqslant \|A_Y\| \leqslant C_1\tilde{d}^{\,1/2}, \tag{5.4}$$

where $C_1 > 0$ denotes a constant. Since $A = A_Y + A_{H-X} - A_{Y\setminus X}$, (5.4) yields that w.h.p.

$$\begin{aligned}\max_{\eta \perp \vec{1}_H,\, \|\eta\|=1} |\langle A\eta, \eta\rangle| &\leqslant \|A_Y\| + \|A_{Y\setminus X}\| + \max_{\eta \perp \vec{1}_H,\, \|\eta\|=1} |\langle A_{H-X}\eta, \eta\rangle|\\ &\leqslant 2C_1\tilde{d}^{\,1/2} + \max_{\eta \perp \vec{1}_H,\, \|\eta\|=1} |\langle A_{H-X}\eta, \eta\rangle|.\end{aligned} \tag{5.5}$$

Furthermore, by part (2) of Theorem 1.4 there is a constant $C_2 > 0$ such that w.h.p.

$$\max_{\zeta \perp \vec{1}_{H-X},\, \|\zeta\|=1} \|A_{H-X}\zeta\| \leqslant C_2\tilde{d}^{\,1/2}. \tag{5.6}$$

Let $e = \|\vec{1}_{H-X}\|^{-1} \cdot \vec{1}_{H-X}$. If $\eta \perp \vec{1}_H$ is a unit vector, then $|\langle \eta, e\rangle| \leqslant \|\vec{1}_{H-X}\|^{-1}\|\vec{1}_H - \vec{1}_{H-X}\| \leqslant (\#X/\#V(H)\setminus X)^{1/2} \leqslant \bar{d}^{-9}$. Hence, as (2.2) implies that $\|A\| \leqslant 2\bar{d}$ (because $d_H(v) \leqslant 2\bar{d}$ for all $v \in H$ by (1.4)), we conclude that for all unit vectors $\eta \perp \vec{1}_H$

$$\|A_{H-X}\eta\| \leqslant \|A(\eta - \langle \eta, e\rangle e)\| + \|A\| \cdot |\langle \eta, e\rangle| \leqslant \max_{\zeta \perp \vec{1}_{H-X},\, \|\zeta\|=1} \|A_{H-X}\zeta\| + 2\bar{d}^{-8}. \tag{5.7}$$

Thus, (5.3) follows from (5.5), (5.6), and (5.7). □

Remark. Let H be a subgraph of $G = G_{n,p}$ such that $v = \#V(H) \geqslant n/2$. If (1.4) does not hold, then $L = L(H)$ will in fact have eigenvalues that deviate from $\tilde{d} = (v-1)p$ by at least $(1 - o(1))c^*\tilde{d}^{\,1/2}$. For, let $\eta_v = (\eta_{vw})_{w \in V(H)}$ denote the vector with entries $\eta_{vw} = 0$ for $w \neq v$ and $\eta_{vv} = 1$. Moreover, let $\xi_v = \eta_v - \|\vec{1}_H\|^{-2}\langle \eta_v, \vec{1}_H\rangle\vec{1}_H$. Then $\|\xi_v\| = (1 - v^{-1})^{1/2} = 1 - o(1)$. Moreover, $\xi_v \perp \vec{1}_H$, and $\langle L\xi_v, \xi_v\rangle = \langle L\eta_v, \eta_v\rangle = d_H(v)$. Consequently,

$$\begin{aligned}\lambda_2(L) &= \min_{\xi \perp \vec{1}_H,\, \|\xi\|=1} \langle L\xi, \xi\rangle \leqslant \min_{v \in V(H)} \frac{\langle L\xi_v, \xi_v\rangle}{\|\xi_v\|^2} \leqslant (1 - o(1)) \min_{v \in V(H)} d_H(v),\\ \lambda_v(L) &= \max_{\xi \perp \vec{1}_H,\, \|\xi\|=1} \langle L\xi, \xi\rangle \geqslant \max_{v \in V(H)} \langle L\xi_v, \xi_v\rangle \geqslant \max_{v \in V(H)} d_H(v).\end{aligned}$$

Hence, the spectrum of L is indeed extremely susceptible to fluctuations of the vertex degrees.

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5.2. Proof of Proposition 5.1

As a first step, we shall bound the number of vertices removed from G in step H1.

Lemma 5.3. *There is a number $c = c(\varepsilon)$ such that for all $c^* \geqslant c$ we have $\#T_0 \leqslant \varepsilon n$ w.h.p.*

Proof Suppose that $c^* \geqslant c$ for a sufficiently large $c = c(\varepsilon)$. Let $t_0 = \log_2 c^*$. Letting $S_j = \{v \in V : 2^{t_0+j}\bar{d}^{1/2} \leqslant |d_G(v) - \bar{d}| < 2^{t_0+j+1}\bar{d}^{1/2}\}$ for $j \geqslant 0$, we have $T_0 \subset \bigcup_{j\geqslant 0} S_j$. Moreover, let ϕ be the function defined in (2.3), and set $\phi_j = \bar{d} \cdot \phi(2^{t_0+j}\bar{d}^{-1/2})$. Then, by Lemma 2.1, $\#S_j \leqslant 4n\exp(-\phi_j/100)$ for all $j \geqslant 0$ w.h.p. Thus, w.h.p.

$$\begin{aligned}\#T_0 &\leqslant \sum_{j\geqslant 0}\#S_j \leqslant 4n\sum_{j=0}^{\infty}\exp(-\phi_j/100) \overset{(2.4)}{\leqslant} 4n\sum_{j=0}^{\infty}\exp\left[-\frac{2^{2(t_0+j)}}{200(1+\frac{1}{3}\bar{d}^{-1/2}2^{t_0+j})}\right]\\ &\leqslant 4n\sum_{j=0}^{\infty}\exp\left[-\frac{c^*2^j}{400}\right] \leqslant 4n\sum_{j=1}^{\infty}\exp\left[-\frac{c^*j}{400}\right] = \frac{4n}{\exp(c^*/400)-1}.\end{aligned}$$

As $4/(\exp(c^*/400)-1) < \varepsilon$ if $c^* \geqslant c$ for a large enough $c = c(\varepsilon)$, the assertion follows. □

To bound the number of vertices that get removed during H2, we shall prove the following in Section 6.3.

Lemma 5.4. *Suppose that c^* is a large enough constant. Then w.h.p. the random graph $G = G_{n,p}$ has the following property. For all subsets X of cardinality $\#X \leqslant n/100$, the set*

$$Y = \{v \in V \setminus X : |e(v,X) - \#Xp| \geqslant c^*\max\{(\#X/n)^{1/4}\bar{d}^{1/2}, 1\}\}$$

has cardinality $\#Y \leqslant \#X/16$.

Proof of Proposition 5.1. Suppose that the sets produced by steps H1–H2 are $T_0, \ldots, T_t$. By Lemma 5.3, $\#T_0 < \varepsilon n/10$ w.h.p., provided that c^* is large enough. Furthermore, by Lemma 5.4 we have

$$\forall 1 \leqslant j \leqslant t : \#T_j \leqslant \#T_{j-1}/16 \tag{5.8}$$

w.h.p. Hence, in total step H2 removes at most $\sum_{j=1}^{\infty}\#T_0 \cdot 16^{-j} \leqslant \frac{\varepsilon n}{2}$ vertices w.h.p. Moreover, as $T_t < n\bar{d}^{-2}$, the same argument as in the proof of Proposition 4.1 shows that step H3 removes at most $\#T_t < n\bar{d}^{-2}$ vertices w.h.p.; that is, w.h.p. the set T constructed in step H3 satisfies

$$\#T \leqslant 2n\bar{d}^{-2}. \tag{5.9}$$

Thus, the total number of vertices removed during steps H1–H3 is at most $\frac{1}{10}\varepsilon n + \frac{1}{2}\varepsilon n + n\bar{d}^{-2} \leqslant \varepsilon n$ w.h.p., since we are assuming that $\bar{d} \geqslant c_0$ for a large enough $c_0(\varepsilon) > 0$.

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In order to prove the second assertion, let $\mathscr{T} = \bigcup_{j=0}^{t-1} T_j$. Then, by step H2, all vertices $v \in H$ satisfy

$$\begin{aligned}|e(v,\mathscr{T}) - \#\mathscr{T}p| &= \left|\sum_{j=0}^{t-1} e(v,T_j) - \#T_jp\right| \leqslant \sum_{j=0}^{t-1} |e(v,T_j) - \#T_jp| \\ &\leqslant \sum_{j=0}^{t-1} c^* \max\left\{\left(\frac{\#T_j}{n}\right)^{1/4} \bar{d}^{\,1/2}, 1\right\} \overset{(5.8)}{\leqslant} \sum_{j=0}^{t-1} c^* \max\left\{\left(\frac{\#T_0}{16^j n}\right)^{1/4} \bar{d}^{\,1/2}, 1\right\} \\ &\leqslant c^* t + c^* \bar{d}^{\,1/2} \sum_{j=0}^{\infty} 16^{-j/4} \leqslant c^*(t + 2\bar{d}^{\,1/2}). \end{aligned} \tag{5.10}$$

Moreover, (5.8) implies that $t \leqslant \ln \bar{d}$, because H2 stops as soon as $\#T_j < n\bar{d}^{-2}$. Thus, (5.10) yields that

$$|e(v,\mathscr{T}) - \#\mathscr{T}p| \leqslant c^*(\ln d + 2\bar{d}^{\,1/2}) \leqslant 3c^*\bar{d}^{\,1/2}, \quad \text{for all } v \in H \tag{5.11}$$

w.h.p. Furthermore, with respect to the set T constructed in H3 we have w.h.p.

$$|e(v,T) - \#Tp| \overset{(5.9)}{\leqslant} e(v,T) + 1 \leqslant 101 \leqslant \bar{d}^{\,1/2}, \quad \text{for all } v \in H. \tag{5.12}$$

Combining (5.11) and (5.12) and recalling that $|d_G(v) - \bar{d}| \leqslant c^*\bar{d}^{\,1/2}$ for all $v \in H$, due to H1, we finally conclude that w.h.p. for all $v \in H$ we have

$$\begin{aligned}|d_H(v) - (\#H - 1)p| &= |d_G(v) - e(v,\mathscr{T}) - e(v,T) - (\bar{d} - \#\mathscr{T}p - \#Tp)| \\ &\leqslant |d_G(v) - \bar{d}| + |e(v,\mathscr{T}) - \#\mathscr{T}p| + |e(v,T) - \#Tp| \leqslant 5c^*\bar{d},\end{aligned}$$

thereby proving the proposition. □

5.3. Proof of Lemma 5.2

We use a similar argument as in the proof of Proposition 4.4. Let $(Z_j)_{j=1,\dots,K}$ be a partition of Z such that $K \leqslant \ln n$ and $e(v, \bigcup_{i=j}^{K}) \leqslant 100$ for all $v \in Z_j$ and all $1 \leqslant j \leqslant K$; Lemma 4.10 shows that such a partition exists w.h.p. Let $Z_{>j} = \bigcup_{i=j+1}^{K} Z_i$ and $Z_{<j} = \bigcup_{1\leqslant i<j} Z_i$. Let $P = \sum_{j=1}^{K} A_{Z_j}$ be the submatrix of A_Z representing the edges within the classes Z_j. Since $e(v,Z_j) \leqslant 100$ for all $v \in Z_j$ and all $1 \leqslant j \leqslant K$, (2.2) implies that $\|P\| \leqslant 100$.

Moreover, letting $Q = A_Z - P$, we shall prove that there is a constant $c > 0$ such that $\|Q\xi\| \leqslant c\tilde{d}^{\,1/2}$ for all unit vectors $\xi = (\xi_v)_{v\in V}$. Let $\eta = Q\xi$. Then $\eta_v = \sum_{w\in N_G(v)\cap Z\setminus Z_j} \xi_w$ for all $v \in Z_j$. Set

$$\alpha_j = \sum_{v\in Z_j}\left[\sum_{w\in N_G(v)\cap Z_{>j}} \xi_w\right]^2, \qquad \beta_j = \sum_{v\in Z_j}\left[\sum_{w\in N_G(v)\cap Z_{<j}} \xi_w\right]^2.$$

Then $\|\eta\|^2 \leqslant 4\sum_{j=1}^{K} \alpha_j + \beta_j$ (*cf.* (4.20)). Since $e(v,Z_{>j}) \leqslant 100$ for all $v \in Z_j$ and $d_H(w) \leqslant 2\tilde{d}$ for all $w \in V$ by (1.4), we can bound $\sum_j \alpha_j$ as follows:

$$\sum_{j=1}^{K} \alpha_j \leqslant 10^4 \sum_{j=1}^{K} \sum_{v\in Z_j} \sum_{w\in N_G(v)\cap Z_{>j}} \xi_w^2 \leqslant 10^4 \sum_{w\in V} d_H(w)\xi_w^2 \leqslant 2\cdot 10^4 \cdot \tilde{d} \cdot \|\xi\|^2 \leqslant 10^5\tilde{d}. \tag{5.13}$$

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Furthermore, applying the Cauchy–Schwarz inequality, we obtain

$$\beta_j \leqslant \sum_{v\in Z_j} d_H(v) \sum_{w\in N_G(v)\cap Z_{<j}} \xi_w^2 \leqslant 2\tilde{d} \sum_{v\in Z_j} \sum_{w\in N_G(v)\cap Z_{<j}} \xi_w^2.$$

Thus, as $e(w, Z_{>j}) \leqslant 100$ for all $w \in Z_j$, we get

$$\sum_{j=1}^{K} \beta_j \leqslant 2\tilde{d} \sum_{j=1}^{K} \sum_{v\in Z_j} \sum_{w\in N_G(v)\cap Z_{<j}} \xi_w^2 \leqslant 200\tilde{d}\|\xi_w\|^2 \leqslant 200\tilde{d}. \tag{5.14}$$

Combining (5.13) and (5.14), we conclude that $\|\eta\|^2 \leqslant 4\sum_{j=1}^{K} \alpha_j + \beta_j \leqslant c^2\tilde{d}$ for a certain constant $c > 0$, whence $\|Q\| \leqslant c\tilde{d}^{1/2}$. Hence, $\|A_X\| \leqslant \|Q\| + \|P\| \leqslant c\tilde{d}^{1/2} + 100 \leqslant 2c\tilde{d}^{1/2}$.

6. Proofs of auxiliary lemmas

6.1. Proof of Lemma 3.1

Let $k = \delta \ln n$ for a sufficiently small number $\delta > 0$. Let X be the number of components of $G = G_{n,p}$ that are paths of length k. Moreover, for a set $S \subset V$ of size k let $X_S(G) = 1$ if S is an isolated path in G, and 0 otherwise. Then $X = \sum_S X_S$, and

$$\mathrm{P}[X_S = 1] = \frac{k!}{2} \cdot p^k \cdot (1-p)^{\binom{k}{2}-k+1+k(n-k)}. \tag{6.1}$$

Therefore,

$$\mathrm{E}(X) = \sum_S \mathrm{E}(X_S) = \Omega(\bar{d}^{k-1}\exp(-k\bar{d})n) \gg 1, \tag{6.2}$$

provided that $\delta > 0$ is small enough. Furthermore, if $S, T \subset V$ are disjoint sets of cardinality k, then the probability that in $G_{n,p}$ both S, T are isolated paths of length k can be estimated as follows:

$$\frac{\mathrm{P}[X_S = X_T = 1]}{\mathrm{P}[X_S = 1]\cdot\mathrm{P}[X_T = 1]} \overset{(6.1)}{=} \frac{\left[\frac{k!}{2}p^{k-1}(1-p)^{\binom{k}{2}-k}\right]^2 (1-p)^{2k(n-k)-k^2}}{\left[\frac{k!}{2}\cdot p^k\cdot(1-p)^{\binom{k}{2}-k+1+k(n-k)}\right]^2} = (1-p)^{-k^2}. \tag{6.3}$$

As a consequence, letting S, T range over subsets of V of size k, we get

$$\begin{aligned}
\mathrm{Var}(X) - \mathrm{E}(X) &\leqslant \sum_{S\neq T} \mathrm{P}[X_S = X_T = 1] - \mathrm{P}[X_S = 1]\cdot\mathrm{P}[X_T = 1] \\
&\overset{(6.3)}{\leqslant} \sum_{S\neq T} \mathrm{P}[X_S = 1]\cdot\mathrm{P}[X_T = 1]\cdot[(1-p)^{-k^2} - 1] \\
&\leqslant (1+o(1))\mathrm{E}(X)^2\cdot k^2 p \leqslant \mathrm{E}(X)\cdot\frac{\bar{d}^2\ln^2 n}{n} = o(\mathrm{E}(X)^2).
\end{aligned} \tag{6.4}$$

As $\mathrm{E}(X) \gg 1$ by (6.2), Chebyshev's inequality yields in combination with (6.4) that $X \geqslant \frac{1}{2}\mathrm{E}(X)$ w.h.p., so that in particular $X \geqslant 1$ w.h.p.

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6.2. Proof of Lemma 2.1

Let $G = G_{n,p}$. To prove the lemma, we decompose S_t into two subsets

$$S_t^+ = \{v \in V : 2^t \leqslant (d_G(v) - \bar{d}) \cdot \bar{d}^{-1/2} < 2^{t+1}\},$$
$$S_t^- = \{v \in V : 2^t \leqslant (\bar{d} - d_G(v)) \cdot \bar{d}^{-1/2} < 2^{t+1}\}.$$

Since the degree $d_G(v)$ of each vertex v is binomially distributed with mean $\bar{d} = (n-1)p$, the Chernoff bounds (2.5) and (2.6) entail that

$$\mathrm{E}(\#S_t^{\pm}) \leqslant n \exp(-\phi_t). \tag{6.5}$$

If $\phi_t \geqslant \ln n$, then Markov's inequality in combination with (6.5) yields

$$\mathrm{P}\left[\#S_t^{\pm} \geqslant n \exp\left[-\frac{\phi_t}{10}\right]\right] \leqslant \frac{\mathrm{E}(\#S_t^{\pm})}{n \exp[-\frac{\phi_t}{10}]} \leqslant \exp\left[-\frac{9}{10}\phi_t\right] \leqslant n^{-9/10}. \tag{6.6}$$

Now assume that $\phi_t < \ln n$. In order to bound $\#S_t$, we shall apply Chebyshev's inequality to $\#S_t^+$ and $\#S_t^-$. Thus, we need to estimate $\mathrm{Var}(\#S_t^{\pm})$. To this end, we employ the following lemma, whose proof can be found in [10, Section 2.2].

Lemma 6.1. *Let $d = \bar{d} + k$, $d' = \bar{d} + k'$. Let v, w be two distinct vertices. Then the random graph $G = G_{n,p}$ satisfies* $\mathrm{P}\big(d_G(v) = d \wedge d_G(w) = d'\big) = \mathrm{P}(d_G(v) = d) \cdot \mathrm{P}(d_G(w) = d') \cdot \big[1 + \frac{kk'}{(n-1)^2 p(1-p)}\big]$.

Corollary 6.2. *There is a constant $C > 0$ such that* $\mathrm{Var}(\#S_t^{\pm}) \leqslant \frac{C}{n} \cdot 2^{2t} \cdot \mathrm{E}(\#S_t^{\pm})^2 + \mathrm{E}(\#S_t^{\pm})$.

Proof Note that $\mathrm{E}(\#S_t^+(\#S_t^+ - 1))$ is the expected number of ordered pairs of distinct vertices in S_t^+. Therefore,

$$\begin{aligned}
\mathrm{E}(\#S_t^{+\,2}) - \mathrm{E}(\#S_t^+) = \mathrm{E}(\#S_t^+(\#S_t^+ - 1)) &= \sum_{v,w \in V,\, v \neq w} \mathrm{P}[v, w \in S_t^+] \\
&= \sum_{v \neq w} \mathrm{P}\left[\bar{d} + 2^t \bar{d}^{1/2} \leqslant d_G(v), d_G(w) < \bar{d} + 2^{t+1}\bar{d}^{1/2}\right] \\
&\overset{\text{Lem. 6.1}}{\leqslant} \sum_{v \neq w} \mathrm{P}[v \in S_t^+]\mathrm{P}[w \in S_t^+]\left(1 + \frac{2^{2t+2}\bar{d}}{\bar{d}(n-1)(1-p)}\right) \\
&\leqslant \left[\sum_v \mathrm{P}[v \in S_t^+]\right]^2 \cdot \left[1 + C \cdot \frac{2^{2t}}{n}\right] \\
&= \mathrm{E}(\#S_t^+)^2 \cdot \left[1 + C \cdot \frac{2^{2t}}{n}\right].
\end{aligned}$$

Hence, $\mathrm{Var}(\#S_t^+) = \mathrm{E}(\#S_t^{+\,2}) - \mathrm{E}(\#S_t^+)^2 \leqslant \mathrm{E}(\#S_t^+) + \mathrm{E}(\#S_t^+)^2 \cdot C \cdot \frac{2^{2t}}{n}$, as claimed. The same argument applies to $\#S_t^-$. □

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Plugging the estimate on $\mathrm{Var}(\#S_t^{\pm})$ from Corollary 6.2 into Chebyshev's inequality, we obtain

$$\begin{aligned}\mathrm{P}\big[\#S_t^{\pm} \geqslant 2n\exp(-\phi_t/10)\big] &\overset{(6.5)}{\leqslant} \mathrm{P}\big[\#S_t^{\pm} \geqslant \mathrm{E}(\#S_t^{\pm}) + n\exp(-\phi_t/10)\big] \\ &\leqslant \frac{\mathrm{Var}(\#S_t^{\pm})}{n^2\exp(-\phi_t/5)} \leqslant \frac{\mathrm{E}(\#S_t^{\pm}) + Cn^{-1}2^{2t}\cdot\mathrm{E}(\#S_t^{\pm})^2}{n^2\exp(-\phi_t/5)} \\ &\overset{(6.5)}{\leqslant} \frac{n\exp(-\phi_t) + C2^{2t}n\exp(-2\phi_t)}{n^2\exp(-\phi_t/5)} \\ &\leqslant n^{-1}\exp(4\phi_t/5) + Cn^{-1}\cdot 2^{2t}\exp(-9\phi_t/5) \\ &\overset{(2.4)}{\leqslant} n^{-1}(\exp(4\phi_t/5) + C'), \end{aligned} \tag{6.7}$$

where $C' > 0$ denotes a sufficiently large constant. Furthermore, as we are assuming that $\phi_t < \ln n$, (6.7) entails that

$$\mathrm{P}\big[\#S_t^{\pm} \geqslant 2n\exp(-\phi_t/10)\big] \leqslant n^{-1}(n^{4/5} + C') \leqslant n^{-1/10}. \tag{6.8}$$

Thus, by (6.6) and (6.8), both in the case $\phi_t \geqslant \ln n$ and in the case $\phi_t < \ln n$ we have

$$\begin{aligned}\mathrm{P}\big[\#S_t \geqslant 4n\exp(-\phi_t/10)\big] &\leqslant \mathrm{P}\big[\#S_t^{+} \geqslant 2n\exp(-\phi_t/10)\big] + \mathrm{P}\big[\#S_t^{-} \geqslant 2n\exp(-\phi_t/10)\big] \\ &\leqslant 2n^{-1/10},\end{aligned}$$

so that Lemma 2.1 follows.

6.3. Proof of Lemma 5.4

Let $1 \leqslant x \leqslant n/100$, $y = x/16$, and $\delta = c^*\max\{(x/n)^{1/4}\bar{d}^{\,1/2}, 1\}$. Let $X \subset V$ be an arbitrary set of cardinality x, and let $v \in V \setminus X$. Then $e(v, X)$ is binomially distributed with mean xp, so that the Chernoff bounds (2.5) and (2.6) entail that

$$r = \mathrm{P}\big[|e(v,X) - \#Xp| \geqslant \delta\big] \leqslant 2\exp\left[-xp\cdot\phi\left(\frac{\delta}{xp}\right)\right].$$

Moreover, the events that $|e(v, X) - \#Xp| \geqslant \delta$ are mutually independent for all $v \in V\setminus X$. Therefore, by the union bound, the probability that there is a set $X \subset V$ of cardinality x such that the set $Y = \{v \in V\setminus X : |e(v, X) - \#Xp| \geqslant \delta\}$ has cardinality $\geqslant y$ is at most

$$P_x = \binom{n}{x}\binom{n-x}{y}r^y \leqslant \binom{n}{x}^2 r^y \leqslant \exp\left[4x\ln(n/x) - xyp\cdot\phi\left(\frac{\delta}{xp}\right)\right], \tag{6.9}$$

where the last inequality sign follows from our assumption that $x \leqslant n/100$.

If $xp \geqslant \delta/100$, then (2.4) implies that

$$xyp\cdot\phi\left(\frac{\delta}{xp}\right) \geqslant \frac{\delta^2 y}{2(xp+\delta/3)} \geqslant \frac{\delta^2 y}{70xp} \geqslant \frac{c^{*2}y\bar{d}}{70(xn)^{1/2}p} \geqslant \frac{c^{*2}yn^{1/2}\bar{d}}{71x^{1/2}\bar{d}} \geqslant 8(xn)^{1/2}, \tag{6.10}$$

provided that the constant c^* is large enough. Combining (6.9) and (6.10), we obtain

$$P_x \leqslant \exp\big[4x\ln(n/x) - 8(xn)^{1/2}\big] \leqslant \exp\big[4(xn)^{1/2}\big(-(x/n)^{1/2}\ln(x/n) - 2\big)\big].$$

Further, because $-t^{1/2}\ln t \leqslant 1$ for all $0 < t < 1$, we get

$$P_x \leqslant \exp(-4(xn)^{1/2}) \leqslant \exp(-n^{1/2}). \tag{6.11}$$

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Now, assume that $xp < \delta/100$. Then $xp \cdot \phi\left(\frac{\delta}{xp}\right) = (xp+\delta)\ln\left(1+\frac{\delta}{xp}\right) - \delta \geqslant \frac{1}{2}\delta \ln\left(\frac{\delta}{xp}\right)$, so that

$$P_x \leqslant \exp\left[4x\ln\left(\frac{n}{x}\right) - \frac{\delta x}{33}\ln\left(\frac{n\delta}{2x\bar{d}}\right)\right], \quad \text{by (6.9).} \tag{6.12}$$

We consider two cases.

Case 1: $\delta = c^*$. Then $(x/n)^{1/4}\bar{d}^{\,1/2} \leqslant 1$, so that $\bar{d} \leqslant (n/x)^{1/2}$. Hence, $\ln\left(\frac{n\delta}{2x\bar{d}}\right) \geqslant \frac{1}{2}\ln\left(\frac{n}{x}\right)$, so that (6.12) implies

$$P_x \leqslant \exp\left[x\ln\left(\frac{n}{x}\right)\left[4 - \frac{c^*}{66}\right]\right] \leqslant \exp\left[-10x\ln\left(\frac{n}{x}\right)\right] \leqslant n^{-10}, \tag{6.13}$$

provided that c^* is sufficiently large.

Case 2: $\delta = c^*(x/n)^{1/4}\bar{d}^{\,1/2}$. Then

$$\ln\left(\frac{n\delta}{2x\bar{d}}\right) = \ln\left[\frac{n}{x}\cdot\frac{c^*}{2}\cdot\left(\frac{x}{n}\right)^{1/4}\bar{d}^{\,-1/2}\right] = \ln\left[\frac{c^*}{2}\cdot\left(\frac{n}{x}\right)^{3/4}\bar{d}^{\,-1/2}\right]. \tag{6.14}$$

If in addition $n/x > \bar{d}^{3/4}$, then $\ln\left[\frac{c^*}{2}\bar{d}^{\,-1/2}\left(\frac{n}{x}\right)^{3/4}\right] \geqslant \frac{1}{12}\ln(n/x)$, so that (6.14) entails that

$$\frac{\delta}{33}\ln\left(\frac{n\delta}{2x\bar{d}}\right) \geqslant \frac{c^*}{12\cdot 33}\ln(n/x) \geqslant 14\ln(n/x), \tag{6.15}$$

where we assume once more that c^* is large enough. If, on the other hand, $n/x \leqslant \bar{d}^{3/4}$, then

$$\delta = c^*\left(\frac{x}{n}\right)^{1/4}\bar{d}^{\,1/2} \geqslant c^*\bar{d}^{\,5/16}, \quad \text{while } \ln\left(\frac{n}{x}\right) \leqslant \ln\bar{d}. \tag{6.16}$$

Furthermore, since we are assuming that $100xp < \delta = c^*(x/n)^{1/4}\bar{d}^{\,1/2}$, we have

$$100\bar{d} \leqslant 100np < \frac{n}{x}\delta = c^*\left(\frac{n}{x}\right)^{3/4}\bar{d}^{\,1/2},$$

so that $(n/x)^{3/4} > \frac{100}{c^*}\bar{d}^{\,1/2}$. Hence,

$$\ln\left(\frac{n\delta}{2x\bar{d}}\right) \overset{(6.14)}{\geqslant} \ln\left[\frac{c^*}{2}\left(\frac{n}{x}\right)^{3/4}\bar{d}^{\,-1/2}\right] \geqslant \ln(50) \geqslant 1. \tag{6.17}$$

Combining (6.16) and (6.17), we obtain

$$\frac{\delta}{33}\ln\left(\frac{n\delta}{x\bar{d}}\right) \geqslant \frac{c^*}{33}\bar{d}^{\,5/16} \geqslant 14\ln\bar{d} \geqslant 14\ln(n/x). \tag{6.18}$$

Finally, plugging (6.15) and (6.18) into (6.12), we get

$$P_x \leqslant \exp(-10x\ln(n/x)) \leqslant n^{-10}. \tag{6.19}$$

Thus, (6.11), (6.13), and (6.19) yield that $P_x \leqslant n^{-10}$ for all $1 \leqslant x \leqslant n/100$. As a consequence, we obtain $\sum_{1\leqslant x\leqslant n/100} P_x = o(1)$, so that the lemma follows from the union bound.

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