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A linear/producer/consumer model of classical linear logic

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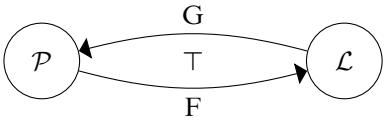
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This paper defines a new proof- and category-theoretic framework for *classical linear logic* that separates reasoning into one linear regime and two persistent regimes corresponding to ! and ?. The resulting linear/producer/consumer (LPC) logic puts the three classes of propositions on the same semantic footing, following Benton's linear/non-linear formulation of intuitionistic linear logic. Semantically, LPC corresponds to a system of three categories connected by adjunctions reflecting the LPC structure. The paper's meta-theoretic results include admissibility theorems for the cut and duality rules, and a translation of the LPC logic into category theory. The work also presents several concrete instances of the LPC model.

1. Introduction

Since its introduction by Girard in 1987, *linear logic* has been found to have a range of applications in logic, proof theory and programming languages. Its notion of 'resource consciousness' sheds light on topics as diverse as proof search (Liang and Miller 2007), memory management (Morrisett et al. 2005), alias control (Hicks et al. 2004), computational complexity (Gaboardi 2007) and security (Zdancewic and Myers 2002), amongst many others.

The power of linear logic stems from its ability to carefully manage resource usage: It makes a crucial distinction between *linear* (used exactly once) and *persistent* (unrestricted use) hypotheses, internalizing the latter via the ! connective. From a semantic point of view, the literature has converged (following Benton 1995) on an interpretation of ! as a comonad given by $! = \mathrm{F} \circ \mathrm{G}$, where $\mathrm{F} \dashv \mathrm{G}$ is a symmetric monoidal adjunction between categories $\mathcal{L}$ and $\mathcal{P}$ arranged as shown below:



Here, $\mathcal{L}$ (for 'linear') is a symmetric monoidal closed category and $\mathcal{P}$ (for 'persistent') is a cartesian category. This is, by now, a standard way of interpreting *intuitionistic* linear logic (for details, see the discussion in Melliès' 2009 paper).

If, in addition, the category $\mathcal{L}$ is $*$-autonomous, then the structure above is sufficient to interpret *classical* linear logic, where the monad ? is determined by $? = (\mathrm{F}^{op}(\mathrm{G}^{op}(-^{\perp})))^{\perp}$.

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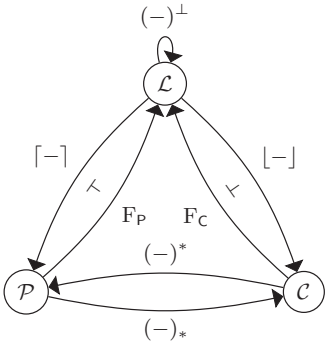


Fig. 1. Categorical model with linear, producing and consuming categories.

Whilst sound, this situation is not entirely satisfactory because it essentially commits to a particular implementation of ? in terms of $\mathcal{P}^{op}$, which, as we show, is not necessary.

With that motivation, this paper defines a proof- and category-theoretic framework for *classical linear logic* that uses *two* persistent categories: one corresponding to ! and one to ?. The resulting categorical structure is shown in Figure 1, where $\mathcal{P}$ now takes the place of the 'producing' category, in duality with $\mathcal{C}$ as the 'consuming' category. This terminology comes from the observations that

$$\begin{array}{lll} !A \vdash 1 & !A \vdash A & !A \vdash !A \otimes !A \\ \bot \vdash ?A & A \vdash ?A & ?A \parr ?A \vdash ?A. \end{array}$$

Intuitively, the top row means that $!A$ is sufficient to *produce* any number of copies of A and, dually, the bottom rows says that $?A$ can *consume* any number of copies of A.

Contributions. After an overview of the linear/non-linear (LNL) framework due to Benton (1995) in Sections 2 and 3 defines the linear/producer/consumer (LPC) presentation of classical linear logic. Similar to Benton's approach, LPC decomposes ! and ? into adjoint functors. We prove admissibility of the cut and duality rules, as well as the consistency of LPC. In addition, we highlight an interesting structural invariant of the logic that implies a surprising relationship with intuitionistic logic.

Section 4 develops the categorical model for LPC and compares it to other models from the literature. We define a categorical semantics based on this model. Section 5 presents several concrete example instances of the LPC categorical framework, and, in particular, gives an example in which $\mathcal{C}$ is not just $\mathcal{P}^{op}$.

Finally, Section 6 concludes with a discussion of related work.

2. Background

In traditional presentations of linear logic, the exponential $!A$ can be thought of as a linear proposition with persistence. This means that propositions of this form can be replicated, but only through explicit weakening and contraction:

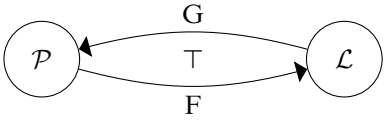
$$\frac{\Gamma \vdash B}{\Gamma, !A \vdash B} \qquad \frac{\Gamma, !A, !A \vdash B}{\Gamma, !A \vdash B}.$$

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In order to promote a linear proposition A to a persistent proposition of the form $!A$, it must be provable using only persistent hypotheses.

$$\frac{\Gamma^{!} \vdash A}{\Gamma^{!} \vdash\ !A.}$$

The logic treats linearity as the default state of the system, and persistence the exception. There are many occasions for which linearity should not always be the default, however. In the context of term calculi, it is maybe more natural to have two kinds of variables, depending on whether the variable will be used linearly or persistently (Barber 1996). This LNL or linear/persistent paradigm shifts the balance of power by placing linearity and persistence on equal footing. Benton's LNL model of intuitionistic linear logic (Benton 1995) formalizes this relationship. Categorically, the comonad ! is decomposed into a symmetric monoidal adjunction between a cartesian (non-linear) category $\mathcal{P}$ and a symmetric monoidal closed (linear) category $\mathcal{L}$:



Similarly, the LNL logic is made up of a pair of sequents – one linear and one persistent. There are also two classes of propositions:

$$A, B \ := \mathbf{I} \mid A \otimes B \mid A \multimap B \mid \mathrm{F}\, X\,;$$
$$X, Y \ := 1 \mid X \times Y \mid X \to Y \mid \mathrm{G}\, A.$$

The linear sequent may use both linear propositions A and persistent propositions X in order to prove a linear proposition; linear sequents take the form $\Gamma \vdash A$, where Γ is made up of both linear and persistent hypotheses. Non-linear sequents have the form $\Gamma^{\mathsf{P}} \Vdash X$, where Γ^{P} can contain only persistent propositions.

Based on the decomposition $!A = \mathrm{F}\,\mathrm{G}\,A$, the promotion and dereliction rules are replaced by rules that pass between the linear and persistent fragments.

$$\frac{\Gamma, X \vdash A}{\Gamma, \mathrm{F}\, X \vdash A}\ \text{F-L} \qquad \frac{\Gamma \Vdash X}{\Gamma \vdash \mathrm{F}\, X}\ \text{F-R.}$$

$$\frac{\Gamma, A \vdash A}{\Gamma, \mathrm{G}\, A \vdash A}\ \text{G-L} \qquad \frac{\Gamma^{\mathsf{P}} \vdash A}{\Gamma^{\mathsf{P}} \Vdash \mathrm{G}\, A}\ \text{G-R.}$$

There are forms of the cut rule, depending on the kind of the cut term and sequent.

$$\frac{\Gamma_1 \vdash A \qquad A, \Gamma_2 \vdash B}{\Gamma_1, \Gamma_2 \vdash B}\ \textsc{Cut}^{\mathcal{L}}_{\mathsf{L}} \qquad \frac{\Gamma_1 \Vdash X \qquad X, \Gamma_2 \vdash A}{\Gamma_1, \Gamma_2 \vdash A}\ \textsc{Cut}^{\mathcal{P}}_{\mathsf{L}}$$

$$\frac{\Gamma_1 \Vdash X \qquad X, \Gamma_2 \Vdash Y}{\Gamma_1, \Gamma_2 \Vdash Y}\ \textsc{Cut}^{\mathcal{P}}_{\mathsf{C}}.$$

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3. LPC logic

In the remainder of this paper, we seek a presentation of *classical* linear logic that distinguishes linearity and persistence following Benton's strategy. But classical linear logic has, in some sense, two different kinds of persistence. The ! operator allows weakening and contraction of hypotheses, meaning that all propositions of the form $!A$ can *produce* any number of copies of A:

$$!A \vdash 1 \qquad !A \vdash A \qquad !A \vdash !A \otimes !A.$$

The ? operator, pronounced *why not*, allows weakening and contraction of conclusions, in the sense that $?A$ can be derived from, or *consume*, any number of copies of A:

$$\bot \vdash ?A \qquad A \vdash ?A \qquad ?A \parr ?A \vdash ?A.$$

The promotion rules for ! and ? pick out not just the producer propositions, as in the case for intuitionistic linear logic, but also the consumers:

$$\frac{\Gamma^! \vdash A, \Delta^?}{\Gamma^! \vdash !A, \Delta^?} \qquad \frac{\Gamma^!, A \vdash \Delta^?}{\Gamma^!, ?A \vdash \Delta^?}.$$

The LPC logic therefore utilizes three syntactic forms for propositions, instead of two: linear propositions A, producer propositions P and consumer propositions C.

$$\begin{array}{rcl} A & ::= & 0 \mid A_1 \oplus A_2 \mid \top \mid A_1 \,\&\, A_2 \\ & \mid & 1_{\mathsf{L}} \mid A_1 \otimes A_2 \mid \bot_{\mathsf{L}} \mid A_1 \parr A_2 \\ & \mid & F_! P \mid F_? C \\ P & ::= & 1_{\mathsf{P}} \mid P_1 \otimes P_2 \mid \lceil A \rceil \\ C & ::= & \bot_{\mathsf{C}} \mid C_1 \parr C_2 \mid \lfloor A \rfloor \end{array}$$

The syntactic form of a proposition is called its *mode* – one of linear L, producing P or consuming C. The meta-variable X ranges over propositions of any mode, and the tagged meta-variable X^{m} ranges over propositions of mode m. The word *persistent* refers to propositions that are either producers or consumers.

LPC replaces the usual constructors ! and ? with two pairs of connectives: $F_!$ and $\lceil - \rceil$ for !, and $F_?$ and $\lfloor - \rfloor$ for ?. If A is a linear proposition, $\lceil A \rceil$ is a producer and $\lfloor A \rfloor$ is a consumer. On the other hand, a producer proposition P can be 'frozen' as a linear proposition $F_! P$, dropping its persistent characteristics. Similarly, for a consumer C, $F_? C$ is linear. The usual connectives $!A$ and $?A$ are encoded as linear propositions $F_! \lceil A \rceil$ and $F_? \lfloor A \rfloor$, respectively.

The inference rules of the logic are defined in Figures 2 through 5. There are two judgements: the linear sequent $\Gamma \vdash \Delta$ and the persistent sequent $\Gamma \Vdash \Delta$. In the linear sequent, the (unordered) contexts Γ and Δ may be made up of propositions of any mode; in the persistent sequent, the contexts may contain only persistent propositions. Γ^{P} indicates that Γ contains only producers, and Δ^{C} indicates that Δ contains only consumers.

Figures 2 and 3 present rules for the units and the linear operators $\oplus$, $\&$, $\otimes$ and $\parr$. It is worth noting that the multiplicative product $\otimes$ is defined only on linear and producer

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$$\frac{}{A \vdash A}\ \mathrm{Ax}^{\vdash} \qquad \frac{}{\Gamma \vdash \Delta, \top}\ \top^{\vdash}_{\mathsf{L}}\text{-R} \qquad \frac{}{\Gamma, 0 \vdash \Delta}\ 0^{\vdash}_{\mathsf{L}}\text{-L}$$

$$\frac{\Gamma, A \vdash \Delta}{\Gamma, A \,\&\, B \vdash \Delta}\ \&^{\vdash}_{\mathsf{L}}\text{-L1} \qquad \frac{\Gamma, B \vdash \Delta}{\Gamma, A \,\&\, B \vdash \Delta}\ \&^{\vdash}_{\mathsf{L}}\text{-L2} \qquad \frac{\Gamma \vdash \Delta, A}{\Gamma \vdash \Delta, A \oplus B}\ \oplus^{\vdash}_{\mathsf{L}}\text{-R1} \qquad \frac{\Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \oplus B}\ \oplus^{\vdash}_{\mathsf{L}}\text{-R2}$$

$$\frac{\Gamma \vdash \Delta, A_1 \quad \Gamma \vdash \Delta, A_2}{\Gamma \vdash \Delta, A_1 \,\&\, A_2}\ \&^{\vdash}_{\mathsf{L}}\text{-R} \qquad \frac{\Gamma, A_1 \vdash \Delta \quad \Gamma, A_2 \vdash \Delta}{\Gamma, A_1 \oplus A_2 \vdash \Delta}\ \oplus^{\vdash}_{\mathsf{L}}\text{-L}$$

$$\frac{\Gamma, {X_1}^{\mathsf{m}}, {X_2}^{\mathsf{m}} \vdash \Delta \quad \mathsf{m} \in \{\mathsf{L}, \mathsf{P}\}}{\Gamma, {X_1}^{\mathsf{m}} \otimes {X_2}^{\mathsf{m}} \vdash \Delta}\ \otimes^{\vdash}\text{-L} \qquad \frac{\Gamma_1 \vdash \Delta_1, A_1 \quad \Gamma_2 \vdash \Delta_2, A_2}{\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2, A_1 \otimes A_2}\ \otimes^{\vdash}\text{-R}$$

$$\frac{\Gamma \vdash \Delta \quad \mathsf{m} \in \{\mathsf{L}, \mathsf{P}\}}{\Gamma, 1_{\mathsf{m}} \vdash \Delta}\ 1^{\vdash}\text{-L} \qquad \frac{}{\cdot \vdash 1_{\mathsf{L}}}\ 1^{\vdash}\text{-R}$$

$$\frac{\Gamma_1, A_1 \vdash \Delta_1 \quad \Gamma_2, A_2 \vdash \Delta_2}{\Gamma_1, \Gamma_2, A_1 \parr A_2 \vdash \Delta_1, \Delta_2}\ \parr^{\vdash}\text{-L} \qquad \frac{\Gamma \vdash \Delta, {X_1}^{\mathsf{m}}, {X_2}^{\mathsf{m}} \quad \mathsf{m} \in \{\mathsf{L}, \mathsf{C}\}}{\Gamma \vdash \Delta, {X_1}^{\mathsf{m}} \parr {X_2}^{\mathsf{m}}}\ \parr^{\vdash}\text{-R}$$

$$\frac{}{\bot_{\mathsf{L}} \vdash \cdot}\ \bot^{\vdash}\text{-L} \qquad \frac{\Gamma \vdash \Delta \quad \mathsf{m} \in \{\mathsf{L}, \mathsf{C}\}}{\Gamma \vdash \Delta, \bot_{\mathsf{m}}}\ \bot^{\vdash}\text{-R}$$

Fig. 2. Inference rules for the linear sequent.

$$\frac{}{P \Vdash P}\ \mathrm{Ax}^{\Vdash}_{\mathsf{P}} \qquad \frac{}{C \Vdash C}\ \mathrm{Ax}^{\Vdash}_{\mathsf{C}}$$

$$\frac{\Gamma \Vdash \Delta}{\Gamma, 1_{\mathsf{P}} \Vdash \Delta}\ 1^{\Vdash}_{\mathsf{P}}\text{-L} \qquad \frac{}{\bot_{\mathsf{C}} \Vdash \cdot}\ \bot^{\Vdash}_{\mathsf{C}}\text{-L}$$

$$\frac{}{\cdot \Vdash 1_{\mathsf{P}}}\ 1^{\Vdash}_{\mathsf{P}}\text{-R} \qquad \frac{\Gamma \Vdash \Delta}{\Gamma \Vdash \Delta, \bot_{\mathsf{C}}}\ \bot^{\Vdash}_{\mathsf{C}}\text{-R}$$

$$\frac{\Gamma, P_1, P_2 \Vdash \Delta}{\Gamma, P_1 \otimes P_2 \Vdash \Delta}\ \otimes^{\Vdash}_{\mathsf{P}}\text{-L} \qquad \frac{\Gamma \Vdash \Delta, C_1, C_2}{\Gamma \Vdash \Delta, C_1 \parr C_2}\ \parr^{\Vdash}_{\mathsf{C}}\text{-R}$$

$$\frac{\Gamma_1 \Vdash \Delta_1, P_1 \quad \Gamma_2 \Vdash \Delta_2, P_2}{\Gamma_1, \Gamma_2 \Vdash \Delta_1, \Delta_2, P_1 \otimes P_2}\ \otimes^{\Vdash}_{\mathsf{P}}\text{-R}$$

$$\frac{\Gamma_1, C_1 \Vdash \Delta_1 \quad \Gamma_2, C_2 \Vdash \Delta_2}{\Gamma_1, \Gamma_2, C_1 \parr C_2 \Vdash \Delta_1, \Delta_2}\ \parr^{\Vdash}_{\mathsf{C}}\text{-L}$$

Fig. 3. Linear rules for the persistent sequent.

propositions, whilst the multiplicative sum $\parr$ is defined only on linear and consumer propositions.[†]

Weakening and contraction can be applied for producers on the left-hand side and consumers on the right-hand side of both the linear and persistent sequents. For producers,

[†] The persistent operators are not in general restricted to the unit and product shown here. Other operators, like $\rightarrow$ or $\vee$, could be incorporated so long as every producer operator has consumer dual.

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$$\frac{\Gamma, P \vdash \Delta}{\Gamma, F_! P \vdash \Delta}\ F_!\text{-L} \qquad \frac{\Gamma \vdash \Delta, C}{\Gamma \vdash \Delta, F_? C}\ F_?\text{-R}$$

$$\frac{\Gamma^{\mathsf{P}} \Vdash \Delta^{\mathsf{C}}, P}{\Gamma^{\mathsf{P}} \vdash \Delta^{\mathsf{C}}, F_! P}\ F_!\text{-R} \qquad \frac{\Gamma^{\mathsf{P}}, C \Vdash \Delta^{\mathsf{C}}}{\Gamma^{\mathsf{P}}, F_? C \vdash \Delta^{\mathsf{C}}}\ F_?\text{-L}$$

$$\frac{\Gamma, A \vdash \Delta}{\Gamma, \lceil A \rceil \vdash \Delta}\ \lceil - \rceil\text{-L} \qquad \frac{\Gamma \vdash \Delta, A}{\Gamma \vdash \Delta, \lfloor A \rfloor}\ \lfloor - \rfloor\text{-R}$$

$$\frac{\Gamma^{\mathsf{P}} \vdash \Delta^{\mathsf{C}}, A}{\Gamma^{\mathsf{P}} \Vdash \Delta^{\mathsf{C}}, \lceil A \rceil}\ \lceil - \rceil\text{-R} \qquad \frac{\Gamma^{\mathsf{P}}, A \vdash \Delta^{\mathsf{C}}}{\Gamma^{\mathsf{P}}, \lfloor A \rfloor \Vdash \Delta^{\mathsf{C}}}\ \lfloor - \rfloor\text{-L}$$

Fig. 4. Adjunction inference rules.

the rules are as follows:

$$\frac{\Gamma \vdash \Delta}{\Gamma, P \vdash \Delta}\ \mathrm{W}^{\vdash}\text{-L} \qquad \frac{\Gamma \Vdash \Delta}{\Gamma, P \Vdash \Delta}\ \mathrm{W}^{\Vdash}\text{-L} \qquad \frac{\Gamma, P, P \vdash \Delta}{\Gamma, P \vdash \Delta}\ \mathrm{C}^{\vdash}\text{-L} \qquad \frac{\Gamma, P, P \Vdash \Delta}{\Gamma, P \Vdash \Delta}\ \mathrm{C}^{\Vdash}\text{-L}.$$

The operators $F_!$, $F_?$, $\lceil - \rceil$ and $\lfloor - \rfloor$ are defined in Figure 4. The dereliction and promotion rules from linear logic are derived by passing through the adjunction induced by the following rules:

$$\frac{\Gamma, A \vdash \Delta}{\Gamma, !A \vdash \Delta} \quad \Rightarrow \quad \frac{\dfrac{\Gamma, A \vdash \Delta}{\Gamma, \lceil A \rceil \vdash \Delta}}{\Gamma, F_! \lceil A \rceil \vdash \Delta} \qquad \frac{\Gamma^! \vdash \Delta^?, A}{\Gamma^! \vdash \Delta^?, !A} \quad \Rightarrow \quad \frac{\dfrac{\Gamma^{\mathsf{P}} \vdash \Delta^{\mathsf{C}}, A}{\Gamma^{\mathsf{P}} \Vdash \Delta^{\mathsf{C}}, \lceil A \rceil}}{\Gamma^{\mathsf{P}} \vdash \Delta^{\mathsf{C}}, F_! \lceil A \rceil}.$$

3.1. *Displacement*

Producers and consumers play opposite roles in the sequent calculus – producers are 'persistent' on the left-hand side of a judgement, and consumers are 'persistent' on the right-hand side. When one such proposition is not subject to weakening or contraction because it is on the wrong side of the judgement, we call it *displaced*.

Definition 3.1. In a derivation of $\Gamma \vdash \Delta$ or $\Gamma \Vdash \Delta$, a producer P is displaced if it appears in Δ, and a consumer C is displaced if it appears in Γ.[‡]

Proposition 3.2 (Displacement). For any derivation $\mathcal{D}$ in LPC,

1. if $\mathcal{D}$ is a (linear) derivation of $\Gamma \vdash \Delta$, it does not contain any displaced propositions;
2. if $\mathcal{D}$ is a (persistent) derivation of $\Gamma \Vdash \Delta$, it contains exactly one displaced proposition.

[‡] In previous iterations of this paper, the displacement property did not hold for linear derivations, and we could construct derivations $\cdot \vdash P$ if P did not contain any of the adjoint operators. The revised formulation ensures that displaced propositions can only occur in persistent judgements.

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$$\frac{\Gamma_1 \vdash \Delta_1, A \quad A, \Gamma_2 \vdash \Delta_2}{\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2}\ \textsc{Cut}^{\vdash}_{\mathsf{L}}$$

$$\frac{\Gamma_1^{\mathsf{P}} \Vdash \Delta_1^{\mathsf{C}}, P \quad P, \Gamma_2 \vdash \Delta_2}{\Gamma_1^{\mathsf{P}}, \Gamma_2 \vdash \Delta_1^{\mathsf{C}}, \Delta_2}\ \textsc{Cut}^{\vdash}_{\mathsf{P}} \qquad \frac{\Gamma_1^{\mathsf{P}} \Vdash \Delta_1^{\mathsf{C}}, P \quad P, \Gamma_2 \Vdash \Delta_2}{\Gamma_1^{\mathsf{P}}, \Gamma_2 \Vdash \Delta_1^{\mathsf{C}}, \Delta_2}\ \textsc{Cut}^{\Vdash}_{\mathsf{P}}$$

$$\frac{\Gamma_1 \vdash \Delta_1, C \quad C, \Gamma_2^{\mathsf{P}} \Vdash \Delta_2^{\mathsf{C}}}{\Gamma_1, \Gamma_2^{\mathsf{P}} \vdash \Delta_1, \Delta_2^{\mathsf{C}}}\ \textsc{Cut}^{\vdash}_{\mathsf{C}} \qquad \frac{\Gamma_1 \Vdash \Delta_1, C \quad C, \Gamma_2^{\mathsf{P}} \Vdash \Delta_2^{\mathsf{C}}}{\Gamma_1, \Gamma_2^{\mathsf{P}} \Vdash \Delta_1, \Delta_2^{\mathsf{C}}}\ \textsc{Cut}^{\Vdash}_{\mathsf{C}}$$

Fig. 5. Cut inference rules.

Displacement is important for understanding the structure of valid LPC derivations, and it reveals an interesting connection to intuitionistic logic, which we will explore in Section 3.4.

3.2. *Cut rules*

In this section, we derive a collection of admissible cut rules, shown in Figure 5. The rules are structured in such a way to preserve the displacement property: Whenever a cut term is displaced in a subderivation, that derivation must satisfy the restrictions of Proposition 3.2.

To show admissibility of these rules, it is sufficient to show admissibility of an equivalent set of rules called Cut+. For linear cut terms, the Cut+ rule is identical to the corresponding Cut rule. For persistent cut terms, Cut+ uses the observation that when a persistent proposition is *not* displaced in a sequent, it can be replicated any number of times. That is, for any n, the linear derivations

$$\frac{\Gamma, (P)_n \vdash \Delta}{\Gamma, P \vdash \Delta} \qquad \text{and} \qquad \frac{\Gamma \vdash \Delta, (C)_n}{\Gamma \vdash \Delta, C}$$

are admissible in LPC, and similarly for the persistent derivation. The Cut+ rules incorporate this observation as follows:

$$\frac{\Gamma_1^{\mathsf{P}} \Vdash \Delta_1^{\mathsf{C}}, P \quad (P)_n, \Gamma_2 \vdash \Delta_2}{\Gamma_1^{\mathsf{P}}, \Gamma_2 \vdash \Delta_1^{\mathsf{C}}, \Delta_2}\ \textsc{Cut+}^{\vdash}_{\mathsf{P}} \qquad \frac{\Gamma_1^{\mathsf{P}} \Vdash \Delta_1^{\mathsf{C}}, P \quad (P)_n, \Gamma_2 \Vdash \Delta_2}{\Gamma_1^{\mathsf{P}}, \Gamma_2 \Vdash \Delta_1^{\mathsf{C}}, \Delta_2}\ \textsc{Cut+}^{\Vdash}_{\mathsf{P}}$$

$$\frac{\Gamma_1 \vdash \Delta_1, (C)_n \quad C, \Gamma_2^{\mathsf{P}} \Vdash \Delta_2^{\mathsf{C}}}{\Gamma_1, \Gamma_2^{\mathsf{P}} \vdash \Delta_1, \Delta_2^{\mathsf{C}}}\ \textsc{Cut+}^{\vdash}_{\mathsf{C}} \qquad \frac{\Gamma_1 \Vdash \Delta_1, (C)_n \quad C, \Gamma_2^{\mathsf{P}} \Vdash \Delta_2^{\mathsf{C}}}{\Gamma_1, \Gamma_2^{\mathsf{P}} \Vdash \Delta_1, \Delta_2^{\mathsf{C}}}\ \textsc{Cut+}^{\Vdash}_{\mathsf{C}}.$$

It is easy to see that the Cut and Cut+ rules are equivalent in strength.

Lemma 3.3 (Cut+ Admissibility). The Cut+ rules are admissible in LPC.

Proof. Let $\mathcal{D}_1$ and $\mathcal{D}_2$ be the hypotheses of one of the cut rules. The proof is by induction on the cut term primarily and the sum of the depths of $\mathcal{D}_1$ and $\mathcal{D}_2$ secondly.

1. Suppose $\mathcal{D}_1$ or $\mathcal{D}_2$ ends in a weakening or contraction rule on the cut term. In particular, consider the weakening case where the cut term is a producer and $\mathcal{D}_2$ is a

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linear judgement. In this case, $\mathcal{D}_1$ is a derivation of $\Gamma_1^{\mathsf{P}} \Vdash \Delta_1^{\mathsf{C}}, P$ and $\mathcal{D}_2$ is the derivation

$$\mathcal{D}_2 = \dfrac{\dfrac{\mathcal{D}_2'}{\Gamma_2, (P)_n \vdash \Delta_2}}{\Gamma_2, (P)_{n+1} \vdash \Delta_2}\ \text{W-L}.$$

By the inductive hypothesis on P, $\mathcal{D}_1$ and $\mathcal{D}_2'$, there exists a cut-free derivation of $\Gamma_1^{\mathsf{P}}, \Gamma_2 \vdash \Delta_1^{\mathsf{C}}, \Delta_2$.

2. If $\mathcal{D}_1$ or $\mathcal{D}_2$ is an axiom, the case is trivial.
3. Suppose the cut term is the principle formula in both $\mathcal{D}_1$ and $\mathcal{D}_2$ (excluding weakening and contraction rules). We consider a few of the subcases here:

($\otimes_{\mathsf{L}}$)
$$\mathcal{D}_1 = \dfrac{\dfrac{\mathcal{D}_{11}}{\Gamma_{11} \vdash \Delta_{11}, A_1} \quad \dfrac{\mathcal{D}_{12}}{\Gamma_{12} \vdash \Delta_{12}, A_2}}{\Gamma_{11}, \Gamma_{12} \vdash \Delta_{11}, \Delta_{12}, A_1 \otimes A_2}\ \otimes_{\mathsf{L}}^{\vdash}\text{-R}, \qquad \mathcal{D}_2 = \dfrac{\dfrac{\mathcal{D}_2'}{\Gamma_2, A_1, A_2 \vdash \Delta_2}}{\Gamma_2, A_1 \otimes A_2 \vdash \Delta_2}\ \otimes_{\mathsf{L}}^{\vdash}\text{-L}.$$

By the inductive hypothesis on A_2, there exists a derivation $\mathcal{E}$ of $\Gamma_{12}, \Gamma_2, A_1 \vdash \Delta_{12}, \Delta_2$. Then, the desired derivation of $\Gamma_{11}, \Gamma_{12}, \Gamma_2 \vdash \Delta_{11}, \Delta_{12}, \Delta_2$ exists by the inductive hypothesis on A_1.

($\otimes_{\mathsf{P}}$)
$$\mathcal{D}_1 = \dfrac{\dfrac{\mathcal{D}_{11}}{\Gamma_{11}^{\mathsf{P}} \Vdash \Delta_{11}^{\mathsf{C}}, P_1} \quad \dfrac{\mathcal{D}_{12}}{\Gamma_{12}^{\mathsf{P}} \Vdash \Delta_{12}^{\mathsf{C}}, P_2}}{\Gamma_{11}^{\mathsf{P}}, \Gamma_{12}^{\mathsf{P}} \Vdash \Delta_{11}^{\mathsf{C}}, \Delta_{12}^{\mathsf{C}}, P_1 \otimes P_2}\ \otimes_{\mathsf{P}}^{\Vdash}\text{-R},$$
$$\mathcal{D}_2 = \dfrac{\dfrac{\mathcal{D}_2'}{\Gamma_2, (P_1 \otimes P_2)_n, P_1, P_2 \vdash \Delta_2}}{\Gamma_2, (P_1 \otimes P_2)_{n+1} \vdash \Delta_2}\ \otimes_{\mathsf{P}}^{\vdash}\text{-L}.$$

The inductive hypothesis on $P_1 \otimes P_2$, $\mathcal{D}_1$ itself, and $\mathcal{D}_2'$ gives us a derivation $\mathcal{E}$ of

$$\Gamma_{11}^{\mathsf{P}}, \Gamma_{12}^{\mathsf{P}}, \Gamma_2, P_1, P_2 \vdash \Delta_{11}^{\mathsf{C}}, \Delta_{12}^{\mathsf{C}}, \Delta_2.$$

Multiple applications of the inductive hypothesis yields the following:

$$\dfrac{\dfrac{\mathcal{D}_{11}}{\Gamma_{11}^{\mathsf{P}} \Vdash \Delta_{11}^{\mathsf{C}}, P_1} \quad \dfrac{\dfrac{\mathcal{D}_{12}}{\Gamma_{12}^{\mathsf{P}} \Vdash \Delta_{12}^{\mathsf{C}}, P_2} \quad \dfrac{\mathcal{E}}{\Gamma_{11}^{\mathsf{P}}, \Gamma_{12}^{\mathsf{P}}, \Gamma_2, P_1, P_2 \vdash \Delta_{11}^{\mathsf{C}}, \Delta_{12}^{\mathsf{C}}, \Delta_2}\ \text{IH}(P_2)}{\Gamma_{12}^{\mathsf{P}}, \Gamma_{11}^{\mathsf{P}}, \Gamma_{12}^{\mathsf{P}}, \Gamma_2, P_1 \vdash \Delta_{12}^{\mathsf{C}}, \Delta_{11}^{\mathsf{C}}, \Delta_{12}^{\mathsf{C}}, \Delta_2}}{\Gamma_{11}^{\mathsf{P}}, \Gamma_{12}^{\mathsf{P}}, \Gamma_{11}^{\mathsf{P}}, \Gamma_{12}^{\mathsf{P}}, \Gamma_2 \vdash \Delta_{11}^{\mathsf{C}}, \Delta_{12}^{\mathsf{C}}, \Delta_{11}^{\mathsf{C}}, \Delta_{12}^{\mathsf{C}}, \Delta_2}\ \text{IH}(P_1).$$

Because the replicated contexts are made up exclusively of non-displaced propositions, it is possible to apply contraction multiple times to obtain the desired sequent.

($F_!$)
$$\mathcal{D}_1 = \dfrac{\dfrac{\mathcal{D}_1'}{\Gamma_1^{\mathsf{P}} \Vdash \Delta_1^{\mathsf{C}}, P}}{\Gamma_1^{\mathsf{P}} \vdash \Delta_1^{\mathsf{C}}, F_! P}\ F_!\text{-R} \qquad \text{and} \qquad \mathcal{D}_2 = \dfrac{\dfrac{\mathcal{D}_2'}{\Gamma_2, P \vdash \Delta}}{\Gamma_2, F_! P \vdash \Delta}\ F_!\text{-L}.$$

Because $\mathcal{D}_1'$ is a persistent derivation, we can apply the inductive hypothesis for P with $n = 1$ to obtain the desired derivation.

4. Suppose the cut term is *not* the principle formula in $\mathcal{D}_1$ or $\mathcal{D}_2$. If the cut term is a producer, then $\mathcal{D}_1$ is a persistent judgement, so it cannot be the case that the last rule

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of $\mathcal{D}_1$ is an $F_!$ rule, a $\lceil - \rceil$-L rule or a $\lfloor - \rfloor$-R rule. But it also cannot be the case that the last rule in $\mathcal{D}_1$ is a $\lceil - \rceil$-R or $\lfloor - \rfloor$-L rule, because the cut formula is the only displaced proposition in $\mathcal{D}_1$.

Suppose on the other hand that the cut term is a consumer and $\mathcal{D}_1$ is the derivation to the right. Then, $\mathcal{D}_2$ is a derivation of $\Gamma_2^{\mathsf{P}}, C \Vdash \Delta_2^{\mathsf{C}}$. By the inductive hypothesis on C, $\mathcal{D}_1'$ and $\mathcal{D}_2$, there is a derivation $\mathcal{E}$ of $\Gamma_1^{\mathsf{P}}, \Gamma_2^{\mathsf{P}} \vdash \Delta_1^{\mathsf{C}}, A, \Delta_2^{\mathsf{C}}$. Because the contexts in $\mathcal{D}_2$ were undisplaced, it is possible to apply the $\lceil - \rceil$-R rule to $\mathcal{E}$ to obtain a derivation of $\Gamma_1^{\mathsf{P}}, \Gamma_2^{\mathsf{P}} \Vdash \Delta_1^{\mathsf{C}}, \lceil A \rceil, \Delta_2^{\mathsf{C}}$.

$$\mathcal{D}_1 = \dfrac{\dfrac{\mathcal{D}_1'}{\Gamma_1^{\mathsf{P}} \vdash \Delta_1^{\mathsf{C}}, A, (C)_n}}{\Gamma_1^{\mathsf{P}} \Vdash \Delta_1^{\mathsf{C}}, \lceil A \rceil, (C)_n}\ \lceil - \rceil\text{-R}$$

□

Theorem 3.4 (Cut admissibility). The Cut rules in Figure 5 are admissible in LPC.

3.3. *Duality*

Unlike traditional presentations of classical linear logic, LPC does not contain an explicit negation operator $(-)^\perp$ or $\neg$, nor a linear implication with which to encode duality. Instead we define $(-)^\perp$ to be a meta-operation on propositions and prove that the following duality rules are admissible in LPC:

$$\dfrac{\Gamma \vdash \Delta, A}{\Gamma, A^\perp \vdash \Delta}\ (-)^\perp\text{-L}, \qquad \dfrac{\Gamma, A \vdash \Delta}{\Gamma \vdash \Delta, A^\perp}\ (-)^\perp\text{-R}.$$

In fact, there are three versions of this duality operation: $(-)^\perp$ for linear propositions, $(-)^*$ for producers and $(-)_*$ for consumers. These operators have the property that for a linear proposition A, $A^\perp$ is linear, but for a producer P, P^* is a consumer, and for a consumer C, C_* is a producer. We define these duality operations as follows:

$$\begin{array}{cccc}
\top^\perp := 0 & 0^\perp := \top & (F_!\, P)^\perp := F_?\, P^* & (F_?\, C)^\perp := F_!\, C_* \\
(A \mathbin{\&} B)^\perp := A^\perp \oplus B^\perp & (A \oplus B)^\perp := A^\perp \mathbin{\&} B^\perp & 1_{\mathsf{P}}^* := \bot_{\mathsf{C}} & \bot_{\mathsf{C}*} := 1_{\mathsf{P}} \\
1_{\mathsf{L}}^\perp := \bot_{\mathsf{L}} & \bot_{\mathsf{L}}^\perp := 1_{\mathsf{L}} & (P \otimes Q)^* := P^* \parr Q^* & (C \parr D)_* := C_* \otimes D_* \\
(A \otimes B)^\perp := A^\perp \parr B^\perp & (A \parr B)^\perp := A^\perp \otimes B^\perp & \lceil A \rceil^* := \lfloor A^\perp \rfloor & \lfloor A \rfloor_* := \lceil A^\perp \rceil
\end{array}$$

The duality rules are given in Figure 6. Notice that the rules for $(-)^{*\vdash}$-R and $(-)_*{}^{\vdash}$-L are missing, which would have been

$$\dfrac{\Gamma, C \vdash \Delta}{\Gamma \vdash \Delta, C_*}\ , \qquad \dfrac{\Gamma \vdash \Delta, P}{\Gamma, P^* \vdash \Delta}\ .$$

By the displacement theorem, there are no linear derivations with displaced hypotheses, so these rules are vacuous.

Lemma 3.5. The following axioms hold in LPC:

$$\dfrac{}{A, A^\perp \vdash \cdot} \qquad \dfrac{}{P, P^* \Vdash \cdot} \qquad \dfrac{}{C, C_* \Vdash \cdot}$$

The proof is by mutual induction on the structures of A, P and C. Similarly, we can prove $\cdot \vdash A, A^\perp$, $\cdot \Vdash P, P^*$ and $\cdot \Vdash C, C_*$.

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$$\frac{\Gamma \vdash \Delta, A}{\Gamma, A^{\perp} \vdash \Delta}\ (-)^{\perp}\text{-L} \qquad \frac{\Gamma \vdash \Delta, C}{\Gamma, C_* \vdash \Delta}\ (-)_*{}^{\vdash}\text{-L} \qquad \frac{\Gamma \Vdash \Delta, P}{\Gamma, P^* \Vdash \Delta}\ (-)^{*\Vdash}\text{-L} \qquad \frac{\Gamma \Vdash \Delta, C}{\Gamma, C_* \Vdash \Delta}\ (-)_*{}^{\Vdash}\text{-L}$$

$$\frac{\Gamma, A \vdash \Delta}{\Gamma \vdash \Delta, A^{\perp}}\ (-)^{\perp}\text{-R} \qquad \frac{\Gamma, P \vdash \Delta}{\Gamma \vdash \Delta, P^*}\ (-)^{*\vdash}\text{-R} \qquad \frac{\Gamma, P \Vdash \Delta}{\Gamma \Vdash \Delta, P^*}\ (-)^{*\Vdash}\text{-R} \qquad \frac{\Gamma, C \Vdash \Delta}{\Gamma \Vdash \Delta, C_*}\ (-)_*{}^{\Vdash}\text{-R}$$

Fig. 6. Duality inference rules.

Theorem 3.6. The duality inference rules given in Figure 6 are admissible in LPC.

Proof. We show how to derive the four left rules using cut; the constructions are the same for the right rules.

$$\frac{\Gamma \vdash \Delta, A \qquad \overline{A, A^{\perp} \vdash \cdot}}{\Gamma, A^{\perp} \vdash \Delta}\ \textsc{Cut}^{\vdash}_{\mathsf{L}} \qquad \frac{\Gamma \vdash \Delta, C \qquad \overline{C, C_* \Vdash \cdot}}{\Gamma, C_* \vdash \Delta}\ \textsc{Cut}^{\vdash}_{\mathsf{C}}$$

$$\frac{\Gamma \Vdash \Delta, C \qquad \overline{C, C_* \Vdash \cdot}}{\Gamma, C_* \Vdash \Delta}\ \textsc{Cut}^{\Vdash}_{\mathsf{C}} \qquad \frac{\Gamma^{\mathsf{P}} \Vdash \Delta^{\mathsf{C}}, P \qquad \overline{P, P^* \Vdash \cdot}}{\Gamma^{\mathsf{P}}, P^* \Vdash \Delta^{\mathsf{C}}}\ \textsc{Cut}^{\Vdash}_{\mathsf{P}}.$$

In the last case, notice that arbitrary derivations of $\Gamma \vdash \Delta, P$ must have Γ and Δ non-displaced, so the result is in fact equivalent to the rule in Figure 6. □

3.4. *Intuitionistic persistence*

The duality rules shed an interesting light on the notion of displacement discussed in Section 3.1. Consider any persistent derivation $\Gamma \Vdash \Delta$. The derivation contains exactly one displaced proposition, which means it either has the form $\Gamma^{\mathsf{P}}, P^* \Vdash \Delta^{\mathsf{C}}$ or $\Gamma^{\mathsf{P}} \Vdash P, \Delta^{\mathsf{C}}$. By duality, this derivation is equivalent to $\Gamma^{\mathsf{P}}, \Delta^{\mathsf{C}}{}_* \Vdash P$. Intuitively, the persistent fragment is *intuitionistic*; it is equivalent to a propositional sequent calculus over producers.

More formally, consider ILPC, consisting of only linear and producer propositions:

$$A ::= \cdots \mid F_! P \mid F_? P^*,$$
$$P ::= \cdots \mid \lceil A \rceil.$$

Linear derivations again have the form $\Gamma \vdash \Delta$, but persistent derivations are intuitionistic, with judgements $\Gamma \Vdash P$, where Γ contains only producers. This does *not* mean that ILPC degenerates into LNL. Indeed, the linear fragment is classical due to the non-standard operator $F_?(-)^*$, whose inference rules are

$$\frac{\Gamma \Vdash P}{\Gamma, F_? P^* \vdash \cdot}\ F_?(-)^*\text{-L} \qquad \frac{\Gamma, P \vdash \Delta}{\Gamma \vdash F_? P^*, \Delta}\ F_?(-)^*\text{-R}.$$

Proposition 3.7. There is an isomorphism between derivations of LPC and ILPC.

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3.5. *Consistency*

We close the section by pointing out that there is no (cut-free) proof of 0 in LPC, which implies the logic is consistent.

Theorem 3.8 (Consistency). There is no proof of 0 in LPC.

4. Categorical model

In categorical formulations of linear logic, ! corresponds to a comonad and ?, its dual, corresponds to a monad. For LPC, the categorical model decomposes the comonad and monad into two adjunctions between three categories, where each mode of proposition has its own category. The result is summed up nicely in Figure 1.

4.1. *Linear category*

The linear category should be able to interpret the inference rules for Figure 2 as well as the linear duality. The multiplicative fragment of linear logic is typically modelled in the literature as a *-autonomous category. For LPC, we use an equivalent notion that puts the tensor $\otimes$ and co-tensor $\parr$ on equal footing, by modelling the linear fragment $\mathcal{L}$ as a symmetric linearly distributive category with negation.

We start with basic definitions about symmetric monoidal structures.

Definition 4.1. A *symmetric monoidal category* is a category $\mathcal{C}$ equipped with a bifunctor $\otimes$, an object 1, and the following natural isomorphisms:

$$\begin{aligned}\alpha_{A_1,A_2,A_3} &: (A_1 \otimes A_2) \otimes A_3 \to A_1 \otimes (A_2 \otimes A_3)\\ \lambda_A &: 1 \otimes A \to A\\ \rho_A &: A \otimes 1 \to A\\ \sigma_{A,B} &: A \otimes B \to B \otimes A\end{aligned}$$

satisfying the usual coherence conditions.

Definition 4.2. Let $(\mathcal{C}, \otimes, 1, \alpha, \lambda, \rho, \sigma)$ and $(\mathcal{C}', \otimes', 1', \alpha', \lambda', \rho', \sigma')$ be symmetric monoidal categories. A functor $\mathrm{F} : \mathcal{C} \Rightarrow \mathcal{C}'$ is *symmetric monoidal* if there exists a map and natural transformation

$$m_1^{\mathrm{F}} : 1' \to \mathrm{F}\,1 \quad \text{and} \quad m_{A,B}^{\mathrm{F}} : \mathrm{F}(A) \otimes' \mathrm{F}(B) \to \mathrm{F}(A \otimes B)$$

that commute naturally with α, λ, ρ and σ. F is *symmetric comonoidal* if there exist

$$n_1^{\mathrm{F}} : \mathrm{F}\,1 \to 1' \quad \text{and} \quad n_{A,B}^{\mathrm{F}} : \mathrm{F}(A \otimes B) \to \mathrm{F}(A) \otimes' \mathrm{F}(B)$$

that commute appropriately.

Definition 4.3. Let F and G be symmetric monoidal (resp. comonoidal) functors $\mathrm{F}, \mathrm{G} : \mathcal{C} \Rightarrow \mathcal{C}'$. A natural transformation $\tau : \mathrm{F} \to \mathrm{G}$ is *(co-)monoidal* if it commutes with the (co-)monoidal components m^{F} and m^{G} (resp. n^{F} and n^{G}).

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Definition 4.4. A *symmetric monoidal* (resp. *comonoidal*) *adjunction* is an adjunction $F \dashv G$ between symmetric (co-)monoidal functors F and G, where the unit and counit of the adjunction are symmetric (co-)monoidal natural transformations.

4.1.1. *Linearly Distributive categories.* The LPC logic treats $\otimes$ and $\parr$ as distinct operators with their own inferences, and only then derives that they are De Morgan duals. Linearly distributive categories, introduced by Cockett and Seely (1997), mirror this relationship between $\otimes$ and $\parr$ by taking these two monoidal structures to be primitive components. They are related via a weak notion of distribution, which can be thought of as a linearization of the classical distributivity axiom $A \wedge (B \vee C) \equiv (A \wedge B) \vee (A \wedge C)$.

Definition 4.5. Let $\mathcal{L}$ be a category with two symmetric monoidal structures: $(\otimes, 1, \alpha^{\otimes}, \lambda^{\otimes}, \rho^{\otimes}, \sigma^{\otimes})$ and $(\parr, \bot, \alpha^{\parr}, \lambda^{\parr}, \rho^{\parr}, \sigma^{\parr})$. Let

$$\delta_{A_1,A_2,A_3} : A_1 \otimes (A_2 \parr A_3) \to (A_1 \otimes A_2) \parr A_3$$

be a natural transformation in $\mathcal{L}$, called a linear distributivity. Using the symmetries, we define variations $\delta^{\mathrm{L,L}}$, $\delta^{\mathrm{L,R}}$, $\delta^{\mathrm{R,R}}$ and $\delta^{\mathrm{R,L}}$ from δ following the paths shown below:

$$\begin{array}{ccccccc}
A_1 \otimes (A_2 \parr A_3) & \xleftarrow{\mathrm{id} \otimes \sigma^{\parr}} & A_1 \otimes (A_3 \parr A_2) & \xleftarrow{\sigma^{\otimes}} & (A_3 \parr A_2) \otimes A_1 & \xleftarrow{\sigma^{\parr} \otimes \mathrm{id}} & (A_2 \parr A_3) \otimes A_1 \\
\downarrow{\scriptstyle \delta^{\mathrm{L,L}} = \delta} & & \downarrow{\scriptstyle \delta^{\mathrm{L,R}}} & & \downarrow{\scriptstyle \delta^{\mathrm{R,R}}} & & \downarrow{\scriptstyle \delta^{\mathrm{R,L}}} \\
(A_1 \otimes A_2) \parr A_3 & \xrightarrow[\sigma^{\parr}]{} & A_3 \parr (A_1 \otimes A_2) & \xrightarrow[\mathrm{id} \parr \sigma^{\otimes}]{} & A_3 \parr (A_2 \otimes A_1) & \xrightarrow[\sigma^{\parr}]{} & (A_2 \otimes A_1) \parr A_3
\end{array}$$

Then, $\mathcal{L}$ is a *symmetric linearly distributive category* if the distributivities satisfy the following coherence conditions:

Unit and Distribution

$$\lambda^{\otimes} = (\lambda^{\otimes} \parr \mathrm{id}) \circ \delta^{\mathrm{L,L}} : 1_{\mathrm{L}} \otimes (A \parr B) \to A \parr B$$
$$\mathrm{id} \otimes \rho^{\parr} = \rho^{\parr} \circ \delta^{\mathrm{L,L}} : A \otimes (B \parr \bot_{\mathrm{L}}) \to A \otimes B$$

Associativity and Distribution

$$(\alpha^{\otimes} \parr \mathrm{id}) \circ \delta^{\mathrm{L,L}} = \delta^{\mathrm{L,L}} \circ (\mathrm{id} \otimes \delta^{\mathrm{L,L}}) \circ \alpha^{\otimes} : (A_1 \otimes A_2) \otimes (A_3 \parr A_4) \to (A_1 \otimes (A_2 \otimes A_3)) \parr A_4$$

$$\alpha^{\parr} \circ \delta^{\mathrm{R,R}} = (\mathrm{id} \parr \delta^{\mathrm{R,R}}) \circ \delta^{\mathrm{R,R}} \circ (\alpha^{\parr} \otimes \mathrm{id}) : ((A_1 \parr A_2) \parr A_3) \otimes A_4 \to A_1 \parr (A_2 \parr (A_3 \otimes A_4))$$

Distribution and Distribution

$$\alpha^{\parr} \circ (\delta^{\mathrm{R,R}} \parr \mathrm{id}) \circ \delta^{\mathrm{L,L}} = (\mathrm{id} \parr \delta^{\mathrm{L,L}}) \circ \delta^{\mathrm{R,R}} : (A_1 \parr A_2) \otimes (A_3 \parr A_4) \to A_1 \parr ((A_2 \otimes A_3) \parr A_4)$$
$$\delta^{\mathrm{L,L}} \circ (\mathrm{id} \otimes \delta^{\mathrm{R,R}}) \circ \alpha^{\otimes} = \delta^{\mathrm{R,R}} \circ (\delta^{\mathrm{L,L}} \otimes \mathrm{id}) : (A_1 \otimes (A_2 \parr A_3)) \otimes A_4 \to (A_1 \otimes A_2) \parr (A_3 \otimes A_4)$$

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Coassociativity and Distribution

$$\alpha^{\parr} \circ (\delta^{\mathrm{L,R}} \parr \mathrm{id}) \circ \delta^{\mathrm{L,L}} = (\mathrm{id} \parr \delta^{\mathrm{L,L}}) \circ \delta^{\mathrm{L,R}} \circ (\mathrm{id} \otimes \alpha^{\parr})$$
$$: A_1 \otimes ((A_2 \parr A_3) \parr A_4) \to A_2 \parr ((A_1 \otimes A_3) \parr A_4)$$

$$\delta^{\mathrm{L,L}} \circ (\mathrm{id} \otimes \delta^{\mathrm{R,L}}) \circ \alpha^{\otimes} = (\alpha^{\otimes} \parr \mathrm{id}) \circ \delta^{\mathrm{R,L}} \circ (\delta^{\mathrm{L,L}} \otimes \mathrm{id})$$
$$: (A_1 \otimes (A_2 \parr A_3)) \otimes A_4 \to (A_1 \otimes (A_2 \otimes A_4)) \parr A_3$$

Definition 4.6. A symmetric linearly distributive category $\mathcal{L}$ is said to *have negation* if there exists a map $(-)^{\perp}$ on objects of $\mathcal{L}$, and families of maps

$$\gamma^{\perp}_A : A^{\perp} \otimes A \to \bot \qquad \gamma^{1}_A : 1 \to A \parr A^{\perp}$$

commuting with distribution as follows:

$$\begin{array}{ccccc} A^{\perp} \otimes 1 & \xrightarrow{\mathrm{id} \otimes \gamma^1} & A^{\perp} \otimes (A \parr A^{\perp}) & & \\ \downarrow{\scriptstyle \rho^{\otimes}} & & \downarrow{\scriptstyle \delta} & & \\ A^{\perp} & \xleftarrow[\lambda^{\parr}]{} & \bot \parr A^{\perp} & \xleftarrow[\gamma^{\perp} \parr \mathrm{id}]{} & (A^{\perp} \otimes A) \parr A^{\perp} \end{array} \qquad \begin{array}{ccccc} A \otimes 1 & \xrightarrow{\mathrm{id} \otimes \underline{\gamma^1}} & A \otimes (A^{\perp} \parr A) & & \\ \downarrow{\scriptstyle \rho^{\otimes}} & & \downarrow{\scriptstyle \delta} & & \\ A & \xleftarrow[\lambda^{\parr}]{} & \bot \parr A & \xleftarrow[\underline{\gamma^{\perp}} \parr \mathrm{id}]{} & (A \otimes A^{\perp}) \parr A, \end{array}$$

where $\underline{\gamma^{\perp}} = \gamma^{\perp} \circ \sigma^{\otimes}$ and $\underline{\gamma^{1}} = \sigma^{\parr} \circ \gamma^{1}$ are induced from the symmetry.

Presentations of linearly distributive categories in the literature (Cockett and Seely 1997) typically start with monoidal structures that are not necessarily symmetric. In our presentation, Definitions 4.5 and 4.6 are significantly simplified, as many of the coherence conditions induced by symmetries in the original formulation can be derived in the symmetric monoidal case. The two presentations are equivalent.

Lemma 4.7 (Cockett and Seely 1997). Symmetric linearly distributive categories with negation correspond to *-autonomous categories.

4.1.2. *Additives.* To encode the additives, we need only require that the linear category has finite products, &, with unit $\top$, and finite coproducts, $\oplus$, with unit 0 (Bierman 1995; Melliès 2009). This is sufficient to ensure the usual isomorphisms of linear logic are preserved:

$$A \otimes (B_1 \oplus B_2) \cong (A \otimes B_1) \oplus (A \otimes B_2) \qquad A \otimes 0 \cong 0$$
$$!(A \,\&\, B) \cong !A \otimes !B \qquad !\top \cong 1_{\mathsf{L}}.$$

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As a demonstration, consider the claim that $A \otimes (B_1 \oplus B_2) \cong (A \otimes B_1) \oplus (A \otimes B_2)$. A diagram chase shows that the following morphisms are in fact inverses:

$$\begin{aligned}(A \otimes B_1) \oplus (A \otimes B_2) &\xrightarrow{[\mathrm{id}\otimes \iota_1, \mathrm{id}\otimes \iota_2]} A \otimes (B_1 \oplus B_2)\\ A \otimes (B_1 \oplus B_2) &\xrightarrow{\mathrm{id}\otimes [f_1, f_2]} A \otimes (A^\perp \parr ((A \otimes B_1) \oplus (A \otimes B_2)))\\ &\xrightarrow{\delta^{\mathrm{L,L}}} (A \otimes A^\perp) \parr ((A \otimes B_1) \oplus (A \otimes B_2))\\ &\xrightarrow{\lambda \circ (\gamma^\perp \parr \mathrm{id})} (A \otimes B_1) \oplus (A \otimes B_2),\end{aligned}$$

where f_i is

$$B_i \xrightarrow{(\gamma^1 \otimes \mathrm{id}) \circ \lambda} (A^\perp \parr A) \otimes B_i \xrightarrow{\delta^{\mathrm{R,R}}} A^\perp \parr (A \otimes B_i) \xrightarrow{\mathrm{id} \parr \iota_i} A^\perp \parr ((A \otimes B_1) \oplus (A \otimes B_2)).$$

4.2. *Persistent categories*

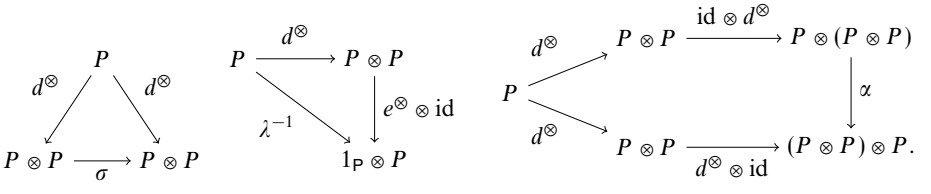
The producer and consumer categories must model weakening and contraction, but they must also be related via a categorical duality that respects the monoidal structures.

Definition 4.8. We say two symmetric monoidal categories $\mathcal{P}$ and $\mathcal{C}$ are *in duality with each other* if $\mathcal{P}$ is equivalent to $\mathcal{C}^{op}$ whilst respecting the monoidal structure. That is, there exist contravariant functors $(-)^* : \mathcal{P} \Rightarrow \mathcal{C}$ and $(-)_* : \mathcal{C} \Rightarrow \mathcal{P}$, where $(-)^*$ is monoidal and $(-)_*$ is comonoidal, and natural isomorphisms

$$\epsilon^*_{*C} : (C_*)^* \to C \quad \text{and} \quad \eta^*_{*P} : P \to (P^*)_*,$$

where ϵ^* is comonoidal and η^* is monoidal.

Definition 4.9. Let $(\mathcal{P}, \otimes, 1_{\mathsf{P}})$ be a symmetric monoidal category. A cocommutative comonoid in $\mathcal{P}$ is an object P in $\mathcal{P}$ along with two morphisms $e^\otimes : P \to 1_{\mathsf{P}}$ and $d^\otimes : P \to P \otimes P$ such that the following commuting conditions are satisfied:



4.3. *The* LPC *model*

Definition 4.10. An LPC model consists of the following:

1. A symmetric linearly distributive category $(\mathcal{L}, \otimes, \parr)$ with negation $(-)^\perp$, finite products & and finite coproducts $\oplus$.
2. Symmetric monoidal categories $(\mathcal{P}, \otimes)$ and $(\mathcal{C}, \parr)$, in duality by means of contravariant functors

$$(-)^* : \mathcal{P} \Rightarrow \mathcal{C} \qquad \text{and} \qquad (-)_* : \mathcal{C} \Rightarrow \mathcal{P}.$$

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3. Monoidal natural transformations

$$e_P^{\otimes} : P \to 1_{\mathsf{P}} \qquad \text{and} \qquad d_P^{\otimes} : P \to P \otimes P$$

in $\mathcal{P}$ and comonoidal natural transformations

$$e_C^{\parr} : \bot_{\mathsf{C}} \to C \qquad \text{and} \qquad d_C^{\parr} : C \parr C \to C$$

in $\mathcal{C}$, interchanged under duality, such that

a. for every P, $(P, d_P^{\otimes}, e_P^{\otimes})$ forms a cocommutative comonoid in $\mathcal{P}$; and

b. for every C, $(C, d_C^{\parr}, e_C^{\parr})$ forms a commutative monoid in $\mathcal{C}$.

4. Symmetric monoidal functors

$$\lceil - \rceil : \mathcal{L} \Rightarrow \mathcal{P} \qquad \text{and} \qquad F_! : \mathcal{P} \Rightarrow \mathcal{L},$$

and symmetric comonoidal functors

$$\lfloor - \rfloor : \mathcal{L} \Rightarrow \mathcal{C} \qquad \text{and} \qquad F_? : \mathcal{C} \Rightarrow \mathcal{L},$$

which respect the dualities in that

$$(F_! P)^{\perp} \simeq F_? (P^*) \qquad \text{and} \qquad \lceil A \rceil \simeq \lfloor A^{\perp} \rfloor,$$

and that form monoidal/comonoidal adjunctions

$$\lceil - \rceil \dashv F_! \qquad \text{and} \qquad F_? \dashv \lfloor - \rfloor.$$

We can make a few observations about the LPC characterization.

First, the monoidal components $m^{F_!}$ of the $F_!$ functor are necessarily isomorphisms, and so $F_!$ is both monoidal and comonoidal (and similarly for $F_?$).

Second, the condition that every object in $\mathcal{P}$ forms a cocommutative comonoid is equivalent to the condition that $\mathcal{P}$ is cartesian. The long form of the definition here highlights the fact that the comonoid structures in $\mathcal{P}$ induce the respective structures in $\mathcal{L}$ for the comonad !. Similarly, Condition 3(b) is equivalent to stating that $\mathcal{C}$ is cocartesian.

4.4. *LPC and other models of linear logic*

Since LPC is inspired by Benton's LNL paradigm, we take this time to formalize the relationship between LPC and LNL.

Definition 4.11 (Melliés 2003). An LNL model consists of (1) a symmetric monoidal closed category $\mathcal{L}$; (2) a cartesian category[§] $\mathcal{P}$ and (3) functors $\mathrm{G} : \mathcal{L} \Rightarrow \mathcal{P}$ and $\mathrm{F} : \mathcal{P} \Rightarrow \mathcal{L}$ that form a symmetric monoidal adjunction $\mathrm{F} \dashv \mathrm{G}$.

We can then conclude:

[§] The LNL model given by Benton (1995) differs from Melliés' in that it requires the cartesian category also to be closed.

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Proposition 4.12. Every LPC model is an LNL model.

In addition, a *-autonomous category in an LNL model induces a consumer category:

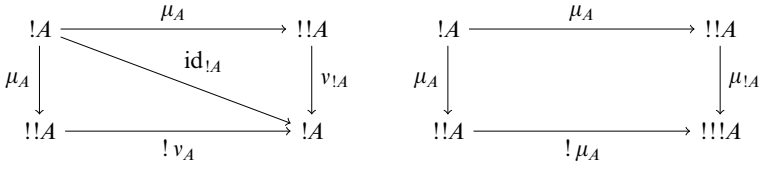
Proposition 4.13. If the category $\mathcal{L}$ in an LNL model is *-autonomous, then the categories $\mathcal{L}$, $\mathcal{P}$ and $\mathcal{P}^{op}$ form an LPC model.

Next, we prove that every LPC model contains a classical linear category as defined by Schalk (2004).

Definition 4.14. A *comonad* $(!, \mu, \nu)$ consists of a functor $!$ and natural transformations

$$\nu_A : !A \to A \qquad \text{and} \qquad \mu_A : !A \to !!A$$

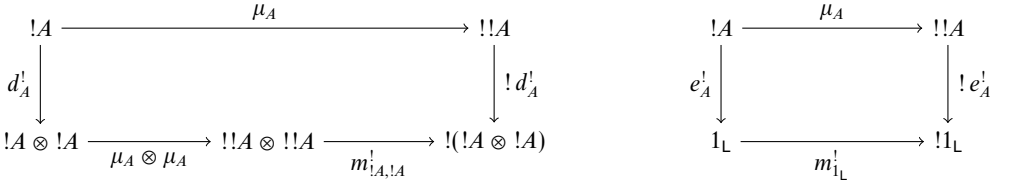
such that the following diagrams commute:



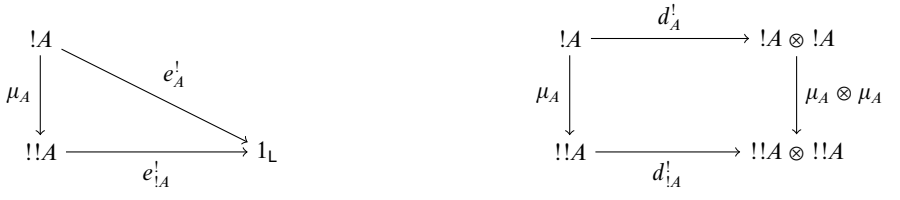
A comonad $(!, \mu, \nu)$ is *monoidal* if $!$ is a monoidal functor and μ and ν are monoidal natural transformations.

Definition 4.15. A symmetric monoidal category $\mathcal{L}$ *has a linear exponential* comonad if it has a monoidal comonad $(!, \mu, \nu)$ such that

1. there exist monoidal natural transformations $e^!_A : !A \to A$ and $d^!_A : !A \to !A \otimes !A$ such that for every object A in $\mathcal{L}$, $(!A, e^!_A, d^!_A)$ forms a cocommutative comonoid;
2. the morphisms $e^!_A$ and $d^!_A$ are coalgebra morphisms, meaning that they satisfy



3. every morphism μ_A is a morphism of comonoids, meaning that it satisfies



Definition 4.16 (Schalk 2004). A category $\mathcal{L}$ is a model for classical linear logic if and only if it

1. is *-autonomous;

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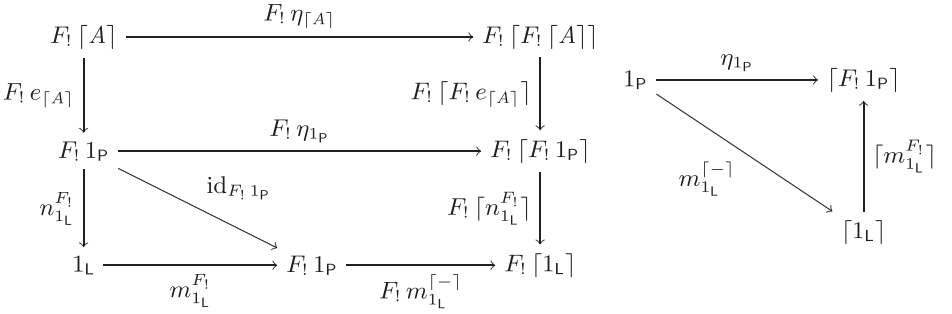


Fig. 7. $e^!_A$ is a coalgebra morphism.

2. has finite products and thus finite coproducts and
3. has a linear exponential comonad ! and thus a linear exponential monad ?.

Proposition 4.17. Every model for classical linear logic forms an *LPC* category.

The producer category can be derived from either the Kleisli or Eilenberg–Moore constructions, as demonstrated by Benton (1995), and dually for the consumer category.

Proposition 4.18. Any *LPC* category $\mathcal{L}$ is a model for classical linear logic.

Proof. Lemma 4.7 states that $\mathcal{L}$ is *-autonomous, so it suffices to show that $\mathcal{L}$ has a linear exponential comonad. The proof is similar to that of Bierman (1995), so we will simply provide a proof sketch here.

The adjunction $F_! \dashv \lceil - \rceil$ is known to form a (monoidal) comonad $F_! \lceil - \rceil$ in $\mathcal{L}$ with

$$v_A := F_! \lceil A \rceil \xrightarrow{\epsilon_A} A \qquad \text{and} \qquad \mu_A := F_! \lceil A \rceil \xrightarrow{F_! \eta_{\lceil A \rceil}} F_! \lceil F_! \lceil A \rceil \rceil,$$

where ϵ is the unit of the adjunction, and η the counit.

The components of the monoid are derived from the monoid in $\mathcal{P}$:

$$e^!_A := F_! \lceil A \rceil \xrightarrow{F_! e_{\lceil A \rceil}} F_! 1_\mathsf{P} \xrightarrow{n^{F_!}_{1_\mathsf{L}}} 1 \qquad d^!_A := F_! \lceil A \rceil \xrightarrow{F_! d_{\lceil A \rceil}} F_! (\lceil A \rceil \otimes \lceil A \rceil) \xrightarrow{n^{F_!}_{\lceil A \rceil, \lceil A \rceil}} F_! \lceil A \rceil \otimes F_! \lceil A \rceil.$$

To show that $e^!_A$ is a coalgebra morphism, we expand out the diagram in Definition 4.15 to the rectangle in Figure 7. The top rectangle commutes by the naturality of η, and the bottom left triangle commutes due to the fact that $n^{F_!}_{1_\mathsf{L}}$ and $m^{F_!}_{1_\mathsf{L}}$ are inverses. The remaining polygon can be seen to commute by reducing it to the diagram on the right. The fact that η is a monoidal natural transformation tells us that $\eta_{1_\mathsf{P}} = m^{\lceil F_! \rceil}_{1_\mathsf{P}}$, which by definition is $\lceil m^{F_!}_{1_\mathsf{L}} \rceil \circ m^{\lceil - \rceil}_{1_\mathsf{L}}$.

The proof that $d^!_A$ is a coalgebra morphism is similar.

Finally, the fact that μ_A forms a morphism of comonoids stems easily from the facts that e, d and $m^{F_!}$ are natural transformations. □

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4.5. *Interpretation of the logic*

The categorical semantics of LPC maps propositions to objects in the categories, and derivations to morphisms. For objects, the $[\![-]\!]_{\mathsf{L}}$ interpretation function maps every mode of proposition into the linear category. The interpretation of linear propositions is straightforward, and for persistent propositions, we define

$$[\![P]\!]_{\mathsf{L}} = F_! [\![P]\!]_{\mathsf{P}} \qquad [\![C]\!]_{\mathsf{L}} = F_? [\![C]\!]_{\mathsf{C}}.$$

The functions $[\![-]\!]_{\mathsf{P}}$ and $[\![-]\!]_{\mathsf{C}}$ map propositions into the producer and the consumer categories $\mathcal{P}$ and $\mathcal{C}$, respectively, but they are defined only on the persistent propositions. To map producers into the consumer category and vice versa, we define

$$[\![C]\!]_{\mathsf{P}} = ([\![C]\!]_{\mathsf{C}})_* \qquad [\![P]\!]_{\mathsf{C}} = ([\![P]\!]_{\mathsf{P}})^*.$$

Linear contexts are interpreted in the linear category. The comma is represented by the tensor $\otimes$ if the context appears on the left-hand side of a sequent, and by the cotensor $\parr$ if the context appears on the right-hand side. These interpretations of linear contexts are represented as $[\![\Gamma]\!]_{\mathsf{L}}^{\otimes}$ and $[\![\Delta]\!]_{\mathsf{L}}^{\parr}$, respectively. In the producer category, there is no cotensor and vice versa for the consumer category, so $[\![\Gamma^{\mathsf{P}}]\!]_{\mathsf{P}}$ interprets the comma as the tensor in $\mathcal{P}$, and $[\![\Gamma^{\mathsf{C}}]\!]_{\mathsf{C}}$ interprets the comma as the cotensor in $\mathcal{C}$.

In this way, a linear derivation $\mathcal{D}$ of the form $\Gamma \vdash \Delta$ will be interpreted as a morphism $[\![\mathcal{D}]\!]_{\mathsf{L}} : [\![\Gamma]\!]_{\mathsf{L}}^{\otimes} \to [\![\Delta]\!]_{\mathsf{L}}^{\parr}$. However, it is not clear in which category we should interpret a persistent sequent of the form $\Gamma \Vdash \Delta$, since Γ and Δ may contain both producer and consumer propositions. By displacement, we can choose either interpretation. Proposition 3.2 states that every such derivation $\mathcal{D}$ contains exactly one displaced proposition. This means that $\mathcal{D}$ is either of the form $\Gamma^{\mathsf{P}} \Vdash \Delta^{\mathsf{C}}, P$ or $\Gamma^{\mathsf{P}}, C \Vdash \Delta^{\mathsf{C}}$. In the category $\mathcal{P}$, this derivation will be interpreted as a morphism

$$[\![\mathcal{D}]\!]_{\mathsf{P}} : [\![\Gamma^{\mathsf{P}}]\!]_{\mathsf{P}} \otimes [\![\Delta^{\mathsf{C}}]\!]_{\mathsf{P}} \to [\![P]\!]_{\mathsf{P}} \quad \text{or} \quad [\![\mathcal{D}]\!]_{\mathsf{P}} : [\![\Gamma^{\mathsf{P}}]\!]_{\mathsf{P}} \otimes [\![\Delta^{\mathsf{C}}]\!]_{\mathsf{P}} \to [\![C]\!]_{\mathsf{P}},$$

respectively. In the same way, every derivation can be interpreted as a morphism in $\mathcal{C}$.

The interpretation is defined by mutual induction on the derivations.

1. The inference rules in Figures 2 and 3 are trivial.

2. The interpretation of weakening and contraction rules is defined using the monoid in $\mathcal{C}$ and comonoid in $\mathcal{P}$. For weakening in the linear sequent, suppose $\mathcal{D}$ is the derivation to the right. The interpretation of $\mathcal{D}$ inserts the comonoidal component $e^{\otimes}$ in $\mathcal{P}$ into the linear category:

$$\mathcal{D} = \dfrac{\dfrac{\mathcal{D}'}{\Gamma \vdash \Delta}}{\Gamma, P \vdash \Delta}\ \mathrm{W}^{\vdash}\text{-L}$$

$$[\![\mathcal{D}]\!]_{\mathsf{L}} : [\![\Gamma]\!]_{\mathsf{L}}^{\otimes} \otimes F_! [\![P]\!]_{\mathsf{P}} \xrightarrow{[\![\mathcal{D}']\!]_{\mathsf{L}} \otimes F_! e^{\otimes}} [\![\Delta]\!]_{\mathsf{L}}^{\parr} \otimes F_! 1_{\mathsf{P}} \xrightarrow{\mathrm{id} \otimes n^{F_!}} [\![\Delta]\!]_{\mathsf{L}}^{\parr} \otimes 1_{\mathsf{L}} \xrightarrow{\rho^{\otimes}} [\![\Delta]\!]_{\mathsf{L}}^{\parr}$$

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3. If the last rule in the derivation is an $F_!$-L or $F_?$-R rule, its interpretation is just its subderivation. On the other hand, if the last rule is the right $F_!$ rule, the inductive hypothesis states that there exists a morphism $[\![\mathcal{D}']\!]_{\mathsf{P}} : [\![\Gamma^{\mathsf{P}}]\!]_{\mathsf{P}} \otimes [\![\Delta^{\mathsf{C}}]\!]_{\mathsf{P}} \to [\![P]\!]_{\mathsf{P}}$. It is necessary to undo this duality transformation for interpretation in the linear category.

$$\mathcal{D} = \dfrac{\dfrac{\mathcal{D}'}{\Gamma^{\mathsf{P}} \Vdash \Delta^{\mathsf{C}}, P}}{\Gamma^{\mathsf{P}} \vdash \Delta^{\mathsf{C}}, F_! P}\, F_!\text{-R}$$

Notice that for any persistent context Γ, there is an isomorphism $\pi : [\![\Gamma]\!]^{\otimes}_{\mathsf{L}} \cong F_! [\![\Gamma]\!]_{\mathsf{P}}$ given by the monoidal components of $F_!$. Furthermore, there is an isomorphism τ between $([\![\Delta]\!]^{\parr}_{\mathsf{L}})^{\perp}$ and $F_! [\![\Delta]\!]_{\mathsf{P}}$ given by the isomorphism $(F_? C)^{\perp} \cong F_! C_*$. Using π and τ, we define the interpretation of $\mathcal{D}$:

$$\begin{aligned}
[\![\mathcal{D}]\!]_{\mathsf{L}} : [\![\Gamma^{\mathsf{P}}]\!]^{\otimes}_{\mathsf{L}} &\xrightarrow{\rho^{\otimes};(\mathrm{id}\otimes\gamma^{1})} [\![\Gamma^{\mathsf{P}}]\!]^{\otimes}_{\mathsf{L}} \otimes (([\![\Delta^{\mathsf{C}}]\!]^{\parr}_{\mathsf{L}})^{\perp} \parr [\![\Delta^{\mathsf{C}}]\!]^{\parr}_{\mathsf{L}}) \\
&\xrightarrow{\pi\otimes(\tau\parr\mathrm{id})} F_! [\![\Gamma^{\mathsf{P}}]\!]_{\mathsf{P}} \otimes (F_! [\![\Delta^{\mathsf{C}}]\!]_{\mathsf{P}} \parr [\![\Delta^{\mathsf{C}}]\!]^{\parr}_{\mathsf{L}}) \xrightarrow{\delta} (F_! [\![\Gamma^{\mathsf{P}}]\!]_{\mathsf{P}} \otimes F_! [\![\Delta^{\mathsf{C}}]\!]_{\mathsf{P}}) \parr [\![\Delta^{\mathsf{C}}]\!]^{\parr}_{\mathsf{L}} \\
&\xrightarrow{m^{F_!}\parr\mathrm{id}} F_! ([\![\Gamma^{\mathsf{P}}]\!]_{\mathsf{P}} \otimes [\![\Delta^{\mathsf{C}}]\!]_{\mathsf{P}}) \parr [\![\Delta^{\mathsf{C}}]\!]^{\parr}_{\mathsf{L}} \\
&\xrightarrow{F_! [\![\mathcal{D}']\!]_{\mathsf{P}}\parr\mathrm{id}} F_! [\![P]\!]_{\mathsf{P}} \parr [\![\Delta^{\mathsf{C}}]\!]^{\parr}_{\mathsf{L}} \xrightarrow{\sigma^{\parr}} [\![\Delta^{\mathsf{C}}]\!]^{\parr}_{\mathsf{L}} \parr [\![F_! P]\!]_{\mathsf{L}}
\end{aligned}$$

4. Suppose the last rule in $\mathcal{D}$ is the left $\lceil - \rceil$ rule. The interpretation of $\mathcal{D}$ should be a morphism from $[\![\Gamma]\!]^{\otimes}_{\mathsf{L}} \otimes F_! (\lceil [\![A]\!]_{\mathsf{L}} \rceil)$ to $[\![\Delta]\!]^{\parr}_{\mathsf{L}}$; we use the unit of the adjunction, $\epsilon : F_! \lceil A \rceil \to A$ to 'cancel out' the exponentials.

$$\mathcal{D} = \dfrac{\dfrac{\mathcal{D}'}{\Gamma, A \vdash \Delta}}{\Gamma, \lceil A \rceil \vdash \Delta}\, \lceil - \rceil\text{-L}$$

$$[\![\mathcal{D}]\!]_{\mathsf{L}} : [\![\Gamma]\!]^{\otimes}_{\mathsf{L}} \otimes F_! (\lceil [\![A]\!]_{\mathsf{L}} \rceil) \xrightarrow{\mathrm{id}\otimes\epsilon} [\![\Gamma]\!]^{\otimes}_{\mathsf{L}} \otimes [\![A]\!]_{\mathsf{L}} \xrightarrow{[\![\mathcal{D}']\!]_{\mathsf{L}}} [\![\Delta]\!]^{\parr}_{\mathsf{L}}$$

Similarly, the $\lceil - \rceil$-R rule uses the counit of the adjunction, along with the isomorphisms π and τ defined previously. If the last rule in $\mathcal{D}$ is the $\lceil - \rceil$-R rule, its interpretation is defined as follows:

$$\mathcal{D} = \dfrac{\dfrac{\mathcal{D}'}{\Gamma^{\mathsf{P}} \vdash \Delta^{\mathsf{C}}, A}}{\Gamma^{\mathsf{P}} \Vdash \Delta^{\mathsf{C}}, \lceil A \rceil}\, \lceil - \rceil\text{-R}$$

$$\begin{aligned}
[\![\mathcal{D}]\!]_{\mathsf{P}} : [\![\Gamma^{\mathsf{P}}]\!]_{\mathsf{P}} \otimes [\![\Delta^{\mathsf{C}}]\!]_{\mathsf{P}} &\xrightarrow{\eta\otimes\eta} \lceil F_! [\![\Gamma^{\mathsf{P}}]\!]_{\mathsf{P}} \rceil \otimes \lceil F_! [\![\Delta^{\mathsf{C}}]\!]_{\mathsf{P}} \rceil \xrightarrow{m^{\lceil - \rceil}} \lceil F_! [\![\Gamma^{\mathsf{P}}]\!]_{\mathsf{P}} \otimes F_! [\![\Delta^{\mathsf{C}}]\!]_{\mathsf{P}} \rceil \\
&\xrightarrow{\lceil \pi^{-1}\otimes\tau^{-1} \rceil} \lceil [\![\Gamma^{\mathsf{P}}]\!]^{\otimes}_{\mathsf{L}} \otimes ([\![\Delta^{\mathsf{C}}]\!]^{\parr}_{\mathsf{L}})^{\perp} \rceil \\
&\xrightarrow{\lceil [\![\mathcal{D}']\!]_{\mathsf{L}}\otimes\mathrm{id} \rceil} \lceil ([\![\Delta^{\mathsf{C}}]\!]^{\parr}_{\mathsf{L}} \parr [\![A]\!]_{\mathsf{L}}) \otimes ([\![\Delta^{\mathsf{C}}]\!]^{\parr}_{\mathsf{L}})^{\perp} \rceil \\
&\xrightarrow{\lceil \delta \rceil} \lceil ([\![\Delta^{\mathsf{C}}]\!]^{\parr}_{\mathsf{L}} \otimes ([\![\Delta^{\mathsf{C}}]\!]^{\parr}_{\mathsf{L}})^{\perp}) \parr [\![A]\!]_{\mathsf{L}} \rceil \\
&\xrightarrow{\gamma^{\perp}\parr\mathrm{id};\lceil\lambda^{\parr}\rceil} \lceil [\![A]\!]_{\mathsf{L}} \rceil = [\![\lceil A \rceil]\!]_{\mathsf{P}}
\end{aligned}$$

5. Examples

The results of Section 4.4 show that LPC is an adequate model for both LNL and classical linear logic, but LPC also admits new models. The following chart summarizes

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three concrete examples of LPC categories.

	$\mathcal{L}$	$\mathcal{P}$	$\mathcal{C}$
Vectors	FinVect	FinSet	FinSet^{op}
Relations	Rel	Set	Set^{op}
Bool. Alg.	FinBoolAlg	FinPoset	FinLat

5.1. *Vector spaces*

Linear algebra invokes ideas of linear logic, especially with regards to the tensor product and duality of vector spaces. Indeed, we can construct an LPC model, taking $\mathcal{L}$ to be the category of finite-dimensional vector spaces over a finite field $\mathbb{F}$, $\mathcal{P}$ be the category of finite sets and functions and $\mathcal{C}$ be the opposite category of $\mathcal{P}$.

The units 1_{L} and $\bot_{\mathsf{L}}$ of $\mathcal{L}$ may be any one-dimensional vector space; for concreteness, let them be generated by the basis $\{\mathbb{1}\}$.

The free vector space $\mathbf{Free}(X)$ of a finite set X over $\mathbb{F}$ is the vector space with vectors the formal sums $\alpha_1 x_1 + \cdots + \alpha_n x_n$. A basis for $\mathbf{Free}(X)$ is the set $\{\delta_x \mid x \in X\}$, where δ_x is the free sum x.

The dual of a vector space V (with basis B) over $\mathbb{F}$ is the set $V^\bot$ of linear maps from V to $\mathbb{F}$. For any vector $v \in V$, we can define $\overline{v} \in V^\bot$ to be the linear map acting on basis elements $x \in B$ by

$$\overline{v}[x] = \begin{cases} 1 & x = v \\ 0 & x \neq v. \end{cases}$$

Then, $\{\overline{x} \mid x \in B\}$ is a basis for $V^\bot$.

We interpret the tensor product as usual, and define $U \parr V = (U^\bot \otimes V^\bot)^\bot$. Over finite fields, every vector space is isomorphic to its dual, however, and so in this case, $\mathcal{L}$ is compact closed.

The additives & and $\oplus$ are embodied by the notions of the direct product and direct sum, which in the case of finite-dimensional vector spaces, coincide.

Lemma 5.1. The category FinVect is a symmetric linearly distributive category with negation, products and coproducts.

Proof. Since $\otimes$ and $\parr$ coincide up to isomorphism, distributivity δ is simply associativity. The coherence conditions for linear distribution thus depend on the commutativity of tensor, associativity and swap morphisms. To show the category has negation, we define $\gamma_A^\bot : A^\bot \otimes A \to \bot$ and $\gamma_A^1 : 1 \to A \parr A^\bot$ as follows, where B is a basis for A:

$$\gamma_A^\bot(\delta_u \otimes v) = \delta_u[v] \cdot \mathbb{1}, \qquad \gamma_A^1(\mathbb{1}) = \sum_{v \in B} v \otimes \overline{v}.$$

The coherence conditions for negation are easily checked. □

We will present only the adjunction between FinVect and the producer category FinSet; the other can be inferred from the opposite category. Define $\lceil - \rceil : \text{FinVect} \Rightarrow \text{FinSet}$ to

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be the forgetful functor, that takes a vector space to its underlying set of vectors. It is a monoidal functor with components $m_1^{\lceil - \rceil} : 1_{\mathsf{P}} \to \lceil 1 \rceil$ and $m_{A,B}^{\lceil - \rceil} : \lceil A \rceil \times \lceil B \rceil \to \lceil A \otimes B \rceil$ defined by $m_{1_{\mathsf{L}}}^{\lceil - \rceil}(\varnothing) = \mathbb{1}$ and $m_{A,B}^{\lceil - \rceil}(u, v) = u \otimes v$.

On objects, the functor $F_! : \textsc{FinSet} \Rightarrow \textsc{FinVect}$ takes a set X to the free vector space generated by X. For a morphism $f : X_1 \to X_2$ in FinSet, we define $F_! f : \mathbf{Free}(X_1) \to \mathbf{Free}(X_2)$ to be $F_! f(\delta_x) = \delta_{f(x)}$. Then, $F_!$ is monoidal with components $m_1^{F_!} : 1 \to F_! 1_{\mathsf{P}}$ and $m_{X_1,X_2}^{F_!} : F_! X_1 \otimes F_! X_2 \to F_!(X_1 \times X_2)$ defined as

$$m_{1_{\mathsf{L}}}^{F_!}(\mathbb{1}) = \delta_{\varnothing} \qquad m_{X_1,X_2}^{F_!}(\delta_{x_1} \otimes \delta_{x_2}) = \delta_{(x_1,x_2)}.$$

Lemma 5.2. The functors $\lceil - \rceil$ and $F_!$ form a symmetric monoidal adjunction $\lceil - \rceil \dashv F_!$.

Proof. We define the unit $\epsilon_A : F_! \lceil A \rceil \to A$ and counit $\eta_P : P \to \lceil F_! P \rceil$ as follows:

$$\epsilon_A(\delta_v) = v, \qquad \eta_P(x) = \delta_x.$$

It is easy to check that ϵ and η form an adjunction, and are both monoidal natural transformations. □

Corollary 5.3. FinVect, FinSet and $\textsc{FinSet}^{op}$ together form an *LPC* model.

Linear algebra has been considered as a model for linear logic multiple times in the literature. Ehrhard (2005) presents finiteness spaces, where the objects are spaces of vectors with finite support. In his model, the ! operator sends a space A to the space supported by finite multisets over A; it takes some effort to show that this comonad respects the finiteness conditions. Pratt (1994) proves that finite-dimensional vector spaces over a field of characteristic 2 is a Chu space and thus a model of linear logic. Valiron and Zdancewic (2014) show that the LPC model of FinVect is a sound and complete semantic model for an algebraic λ-calculus.

5.2. *Relations*

Let Rel be the category of sets and relations, and let Set be the category of sets and functions. (Notice that the sets in either category here may be infinite, unlike in the FinVect case.) It is easy to see that Rel is linearly distributive where the tensor and the cotensor are both cartesian product, and distributivity is just associativity. The unit is a singleton set, and negation on Rel is the identity operation.

Set is cartesian and its opposite category $\textsc{Set}^{op}$, cocartesian. The $F_!$ and $F_?$ functors are the forgetful functors which interpret a function as a relation. The $\lceil - \rceil$ functor takes a set to its powerset. Suppose R is a relation between A and B. The function $\lceil R \rceil : \lceil A \rceil \to \lceil B \rceil$ is defined as

$$\lceil R \rceil(X) = \{y \in B \mid \exists x \in X, \quad (x, y) \in R\}.$$

Then, $\lceil - \rceil$ has monoidal components $m_1^{\lceil - \rceil} : 1_{\mathsf{P}} \to \lceil 1_{\mathsf{L}} \rceil$ and $m_{A,B}^{\lceil - \rceil} : \lceil A \rceil \times \lceil B \rceil \to \lceil A \times B \rceil$

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defined by

$$m_{1_{\mathsf{L}}}^{\lceil - \rceil}(\varnothing) = \varnothing \qquad m_{A,B}^{\lceil - \rceil}(X_1, X_2) = X_1 \times X_2.$$

The dual notion $\lfloor - \rfloor$ is just the inverse.

Melliés (2003) discusses a non-model of linear logic based on Rel, where the exponential takes a set X to the finite subsets of X. That 'model' fails because the comonad unit $\epsilon_A : !A \to A$ is not natural. In the LPC formulation, ϵ is derived from the adjunction ensuring naturality.

5.3. *Boolean algebras*

Next, we consider an example of an LPC model where $\mathcal{P}$ and $\mathcal{C}$ are related by a non-trivial duality. The relationship is based on Birkhoff's representation theorem (Birkhoff 1937), which can be interpreted as a duality between the categories of finite partial orders and order-preserving maps ($\mathcal{P}$) on the one hand, and finite distributive lattices with bounded lattice homomorphisms ($\mathcal{C}$) on the other hand.

The linear category $\mathcal{L}$ consists of finite boolean algebras with bounded lattice homomorphisms. For the monoidal structure, the units are both the singleton lattice $\{\varnothing\}$, and the tensors $A \otimes B$ and $A \parr B$ are the boolean algebra with base set $A \times B$ and lattice structure as follows:

$$\bot = (\bot, \bot) \qquad \top = (\top, \top)$$
$$(x_1, y_1) \vee (x_2, y_2) = (x_1 \vee x_2, y_1 \vee y_2) \qquad (x_1, y_1) \wedge (x_2, y_2) = (x_1 \wedge x_2, y_1 \wedge y_2)$$
$$\neg(x, y) = (\neg x, \neg y)$$

Given a partially ordered set $(P, \leqslant)$, a subset $X \subseteq P$ is called *lower* if it is downwards closed with respect to $\leqslant$. The collection of all lower sets of P forms a lattice with $\top = P$, $\bot = \varnothing$, meet as union, and join and intersection. Let P^* refer to this lattice.

Meanwhile, given a lattice C, an element x is join-irreducible if x is neither $\bot$ nor the join of any two elements less than x. That is, $x \neq y \vee z$ for $y, z \neq x$. Let C_* be the partially ordered set with base set the join-irreducible elements of C, with the ordering $x \leqslant y$ iff $x = y \wedge x$. The operators $(-)^*$ and $(-)_*$ extend to functors forming a duality between $\mathcal{P}$ and $\mathcal{C}$ (Stanley 2011).

The monoidal structures on $\mathcal{P}$ and $\mathcal{C}$ are both given by the cartesian product. For every lattice C in $\mathcal{C}$, there exists a commutative monoid with components $e_C^{\parr} : \bot_{\mathsf{C}} \to C$ and $d_C^{\parr} : C \parr C \to C$ as follows:

$$e_C^{\parr}(\varnothing) = \bot, \qquad d_C^{\parr}(x, y) = x \wedge y.$$

The components of the monoid in $\mathcal{P}$ are easily derivable, and along with the components of $\mathcal{C}$ they are interchanged under the Birkhoff duality.

Next, we define the symmetric monoidal functors. Define $\lceil - \rceil : \mathcal{L} \Rightarrow \mathcal{P}$ and $\lfloor - \rfloor : \mathcal{L} \Rightarrow \mathcal{C}$ to be forgetful functors. For $\lceil - \rceil$ in particular, the order induced by the boolean algebra is $x \leqslant y$ iff $x = y \wedge x$.

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Define $F_!$ and $F_?$ to be the powerset algebra, which takes a structure with base set X to the boolean algebra with base set X. On morphisms, define

$$F_! f\,(X) = F_? f\,(X) = \{f\,(x) \mid x \in X\}.$$

It is easy to check that these functors respect the dualities in that $(F_!\,P)^\perp \simeq F_?\,P^*$ and $\lceil A \rceil^* \simeq \lfloor A^\perp \rfloor$. To prove $F_! \dashv \lceil - \rceil$, it suffices to show a bijection of homomorphism sets $\mathrm{Hom}(F_!\,P, A) \cong \mathrm{Hom}(P, \lceil A \rceil)$. Suppose $f : F_!\,P \to A$ in $\mathcal{L}$. Then, define $f^\sharp : P \to \lceil A \rceil$ by

$$f^\sharp\,(x) = f\,(\{z \in X \mid z \leqslant x\}).$$

This morphism is in fact order-preserving. Next, for $g : P \to \lceil A \rceil$ define $g^\flat : F_!\,P \to A$ as

$$g^\flat\,(X) = \bigvee_{x \in X} g\,(x).$$

Again, $g^\flat$ is a lattice homomorphism. It remains to check that $(f^\sharp)^\flat = f$ and $(g^\flat)^\sharp = g$.

From these definitions, the unit $\epsilon_A : F_!\lceil A \rceil \to A$ and counit $\eta_P : P \to \lceil F_!\,P \rceil$ of the adjunction are

$$\epsilon_A\,(X) = \mathrm{id}^\flat_{\lceil A \rceil}\,(X) = \bigvee_{x \in X} \mathrm{id}_{\lceil A \rceil}\,(x) = \bigvee X, \qquad \eta_P\,(x) = (\mathrm{id}_{F_!\,P})^\sharp\,(x) = \{z \mid z \leqslant x\}.$$

To show the adjunction is monoidal, it suffices to prove ϵ and η are monoidal natural transformations.

The proof of the comonoidal adjunction $\lfloor - \rfloor \dashv F_?$ is similar.

5.4. *Non-degenerate models*

Although the three examples given here all have compact closed linear categories, there is no reason to expect all interesting models to have this property. For example, using the Kleisli or Eilenberg–Moore constructions (as in Proposition 4.17), we can construct an LPC model from any linear model, including non-degenerate ones. Consider the LNL model where the linear category is the category of coherence spaces, and its Kleisli construction is the category of qualitative domains and stable functions (Girard 1986; Melliés 2003). Since the category of coherence spaces is *-autonomous, this adjunction yields an LPC model as well.

6. Discussion and related work

The LPC system builds on a significant body of work about categorical and logical formulations of linear logic. This section highlights some of the pieces most influential for LPC.

Girard (1987) first introduced linear logic to mix the constructivity of intuitionistic propositional logic with the duality of classical logic. There exist a number of categorical interpretations of Girard's sequent calculus. Benton et al.'s *linear category* (Benton et al. 1993) consists of a symmetric monoidal closed category with products and a linear exponential comonad !. The Seely category (Seely 1989) is based on the distributivity between $!A \otimes !B$ and $!(A \,\&\, B)$, but was later proved unsound by Wadler (1992). Bierman

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(1995) defined a new Seely category by adding a symmetric monoidal adjunction between the category and its co-Kleisli category.

Except for Seely's original formulation, these works deal with the intuitionistic fragment of linear logic. The multiplicative fragment (with just $\otimes$ and $\parr$) of classical linear logic is usually modelled by *-autonomous categories, introduced by Barr (1991). Schalk (2004) adapted the linear category to the classical case by requiring that the symmetric monoidal closed category be *-autonomous. The coproduct $\parr$ and coexponential ? are then induced from the duality.

Cockett and Seely (1997), seeking to study $\otimes$ and $\parr$ as independent structures unobscured by duality, introduced linearly distributive categories, which make up the linear category in the LPC model. The authors extended this motivation to the exponentials by modelling ! and ? as linear functors (Cockett and Seely 1999), meaning that ? is not derived from ! and $(-)^{\perp}$. The LPC model reflects that work by allowing ! and ? to have different adjoint decompositions. The significance of this distinction can be seen in applications of linear logic. In the process interpretation, for example, ! is the type of servers whilst ? is the type of clients. In modal logic, ! corresponds to necessity and ? to possibility, which do not always overlap in the intuitionistic formulation.

Later variations of linear logic, notably Girard's 1993 Logic of Unity (LU), distinguish persistent propositions from linear ones. The sequent $\Gamma;\Gamma' \vdash \Delta';\Delta$ describes a derivation where Γ' and Δ' are persistent and admit weakening and contraction. Ramifying LU's separation in the intuitionistic case, Benton (1995) developed the LNL logic and categorical model described in Section 4.4. In Proposition 4.12, we state that every LPC model is an LNL model.

Barber (1996) used this model as the semantics for a term calculus called DILL. A Lafont category (Lafont 1988) is a canonical instance of an LNL model where $!A$ is the free cocommutative comonoid generated by A. The exponential decomposes naturally into an adjunction between $\mathcal{L}$ and the category of cocommutative comonoids over $\mathcal{L}$. The LNL and LPC models have an advantage over Lafont categories by allowing a much greater range of interpretations for the exponentials. Lafont's construction excludes traditional models of linear logic, including coherence spaces and the category Rel.

Reed (2009) extends the ideas from LNL to an arbitrary framework of intuitionistic logics called adjoint logic. The idea is to abstract away the comonadic ! operator to other relationships that can be modelled by an adjunction, including modal logic and hierarchies of substructural logics. Pfenning and Griffith (2015) use adjoint logic as a type system for (intuitionistic) session-typed processes; we could imagine using the ILPC logic, described in Section 3.4, as a type system for classical session types, similar to the variation described by Wadler (2012).

Conclusion. The LPC formulation described in this work provides a novel way of understanding persistence in classical linear logic. We define a logic reminiscent of Benton (1995) and prove cut admissibility, duality admissibility and consistency. We then define a categorical model, prove its compatibility with other models from the literature, and translate derivations from the logic into the categories. Finally, we present three examples of concrete models for LPC.

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