Maximal runs of strictly composite integers

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As we know, an integer $n \geqslant 2$ is *prime* if its only positive factors are 1 and n itself; otherwise it is *composite*. Yet amongst the composite numbers, there are many which are merely some power of a prime number, such as $4 = 2^2$ or $81 = 3^4$. We say that an integer $n \geqslant 2$ is *strictly composite* if it is neither a prime number nor a power of a prime number. Equivalently, an integer is strictly composite if it has at least two distinct prime factors. The first six strictly composite numbers are

$$6 = 2 \times 3, \quad 10 = 2 \times 5, \quad 12 = 2^2 \times 3, \quad 14 = 2 \times 7, \quad 15 = 3 \times 5, \quad 18 = 2 \times 3^2.$$

Now the fact that there are gaps 'as large as we like' between successive prime numbers is well known. Equivalently, we can say that there exist arbitrarily long sequences of consecutive integers, all of which are composite. For example, one has

(a) 4

(b) 8, 9, 10

(c) 24, 25, 26, 27, 28

and the remarkable

(d) 114, 115, 116, 117, 118, 119, 120, 121, 122, 123, 124, 125, 126.

In this paper, the word *run* means a sequence of consecutive integers; a (*strictly*) *composite run* is a run in which each term is (strictly) composite; and such a run is *maximal* if it cannot be extended at either end to give a longer such run. Each of (a)–(d) above is a maximal composite run, but not a strictly composite run, due to the presence of a prime power in each.

The goal of this paper is to explore *maximal strictly composite runs*. We begin, however, by recalling some basic facts about composite runs.

Since the difference between consecutive primes (apart from the primes 2 and 3) is always even, every maximal composite run, being bracketed by consecutive primes on both ends, must contain an *odd* number of terms (equivalently, every such run must start and finish with an even number). See (a)–(d) above as examples. However, it remains an unsolved problem as to whether a maximal composite run of length $2n + 1$ exists for each n. Furthermore, we do not know how to find a composite run of given length and with the minimum initial term, except experimentally for small values of n.

The standard proof that there are arbitrarily long composite runs is both simple and beautiful; but it is also very well known. However, since it is relevant to what follows, it is worth another brief look.

Theorem 1: For any $n \in \mathbb{N}$, there exist n consecutive natural numbers, all of which are composite.

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Proof: Letting $m = n + 1$, consider the run of n integers

$$m! + 2,\ m! + 3,\ \ldots,\ m! + m.$$

The first of these numbers is divisible by 2, the second by 3, …, and the last by m, and hence they are all composite.

We remark that the run of n smaller integers

$$m! - m,\ m! - (m - 1),\ \ldots,\ m! - 2$$

will do equally well to establish the result. Since these two runs are separated by just the three numbers $m! - 1$, $m!$, $m! + 1$, this leads to the interesting question of the possible primality of $m! - 1$ and $m! + 1$, yet another unsolved problem in number theory. If $m! - 1$ and $m! + 1$ are both composite, as they frequently are, then, since $m!$ is certainly composite, one can now join up the previous two runs to generate a splendidly long composite run (this forms the basis for (d) above). Incidentally, based on numerical evidence, we propose the following:

Conjecture: If $m! - 1$ and $m! + 1$ are both prime, then $m = 3$.

Inspecting sequences A002981 and A002982 from [1] confirms that this is true for $m \leqslant 150{,}208$. These sequences also confirm that $m! - 1$ and $m! + 1$ are *both* composite, for each m such that $4 \leqslant m \leqslant 147{,}854$ apart from just 41 exceptions.

While Theorem 1 provides us with composite runs of any given length, the integers involved soon become enormous. Yet the runs that it generates are often not maximal; in particular, for all even values of n, the resulting even-length run is never maximal.

The little-known counterpart of Theorem 1 for strictly composite runs is the following beautiful result:

Theorem 2: For any $n \in \mathbb{N}$, there exist n consecutive natural numbers, all of which are strictly composite.

[We have been unable to find any mention of this result in textbooks or in the literature on number theory. It was set as an International Mathematics Olympiad question in 1989, and is attributed to the late Bernt Lindström of the Stockholm Institute of Technology. It really deserves to be better known — so authors of new editions of books on number theory, please take note.]

Proof: Letting $m = n + 1$, consider the run of n integers

$$(2m)! + 2,\ (2m)! + 3,\ \ldots,\ (2m)! + m.$$

As in the proof of Theorem 1, none of these numbers is a prime. Can any of them — that is, $(2m)! + k$ for any $2 \leqslant k \leqslant m$ — be a prime power?

Since the set $\{1,\ 2,\ 3,\ \ldots,\ 2m\}$ contains both the numbers k and $2k$, clearly k^2 is a factor of $(2m)!$ giving $(2m)!/k = ik$, for some integer i. Thus

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$$t = (2m)! + k = k\left[\frac{(2m)!}{k+1}\right] = k(ik+1).$$

But k and $ik + 1$ have no common factors, so t cannot be a prime power.

This result provides us with a simple constructive method to generate strictly composite runs of any length, but as in the case of Theorem 1 above, the numbers involved quickly become enormous. Moreover, the runs are generally not maximal; even for the case $n = 2$, the run 722, 723 forms only a small part of the enclosing maximal run 720, … , 726. Our goal is to find strictly composite runs which are both maximal and whose elements are of a more modest size.

We briefly digress to note that Theorem 2 has a nice application in an unexpected direction. For each $n \in \mathbb{N}$, let $n^* = \text{LCM}(1,2,3, \ldots, n)$. Thus, n^* is the smallest integer into which each of 1,2,3, … , n divides evenly. The first several values of n^* are given by:

n	1	2	3	4	5, 6	7	8	9, 10	11, 12	13, 14, 15
n^*	1	2	6	12	60	420	840	2520	27720	360360

We note that $n^* > (n-1)^*$ if, and only if, n is a prime or a prime power, since for other values, all of the prime factors of n, and their multiplicities, already occur amongst the numbers less than n. Theorem 2 thus implies:

Theorem 3: The function n^* remains constant for arbitrarily long stretches — that is, for each $c \in \mathbb{N}$, there is a number n for which

$$n^* = (n+1)^* = (n+2)^* = \ldots = (n+c)^*.$$

Returning now to our quest for maximal strictly composite runs, we note that zero-length runs of this type occur only when two adjacent integers are primes and/or prime powers. There is only one case in which two primes are adjacent, namely 2 and 3. Similarly, there is also only one case in which two (prime) powers are adjacent, namely 8 (2^3) and 9 (3^2), as given by:

Catalan's Conjecture: For integers x, a, y, b, all greater than 1, the only solution to $x^a - y^b = 1$ is $x = 3, a = 2, y = 2, b = 3$, or $3^2 - 2^3 = 1$.

[This is actually now a theorem — it was conjectured by Catalan in 1844 and finally proved by Mihăilescu in 2002.]

We now consider the case where a prime and a prime power are adjacent. Clearly, the prime in question must be odd, so the prime power must be even, and hence a power of 2. Thus, we investigate cases where $2^n \pm 1$ is prime. There is only a single case where both these neighbours of 2^n are prime:

Theorem 4: If $2^n - 1$ and $2^n + 1$ are both prime, then $n = 2$.

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Proof: One of the three consecutive numbers $2^n - 1$, 2^n, $2^n + 1$ must be divisible by 3, and it cannot be 2^n. Now any prime which is divisible by 3 must equal 3. If $2^n - 1 = 3$, then $n = 2$ (and $2^n + 1 = 5$). If $2^n + 1 = 3$, then $n = 1$, but then $2^n - 1 = 1$, which is not prime.

If $2^n - 1$ is prime, then n itself must be prime, since

$$2^{ab} - 1 = \left(2^a - 1\right)\left(1 + 2^a + 2^{2a} + \ldots + 2^{(b-1)a}\right).$$

Primes of the form $M_p = 2^p - 1$ are *Mersenne primes*, and are very rare; currently only 48 are known, from $2^2 - 1$ up to the massive $2^{57,885,161} - 1$. Incidentally, the 14 largest Mersenne primes currently known have all been discovered by a global collaborative effort involving hundreds of thousands of individual computers [2].

If $2^n + 1$ is prime, then a well-known result shows that n must be zero or a power of two. Primes of the form $F_k = 2^{2^k} + 1$ are *Fermat primes*, and again are very rare, with only four known instances:

$$F_1 = 2^2 + 1 = 5, \qquad F_2 = 2^4 + 1 = 17, \qquad F_3 = 2^8 + 1 = 257,$$

$$F_4 = 2^{16} + 1 = 65{,}537.$$

Euler discovered that $F_5 = 2^{32} + 1 = 4{,}294{,}967{,}297$ is composite, with 641 as a factor, and F_6 to F_{32} have likewise all been shown to be composite.

Recapping, we have assembled the following facts:

- the only adjacent primes are 2 and 3
- the only adjacent prime powers are 2^3 and 3^2
- a prime and a prime power can be adjacent only at $2^n \pm 1$ and 2^n
- $2^n - 1$ is prime only in the rare case of it being a Mersenne prime
- $2^n + 1$ is prime only in the rare case of it being a Fermat prime.

So apart from the very few exceptions listed above, $2^n - 1$ and $2^n + 1$ will always be strictly composite. This motivates us to focus our search for strictly composite numbers on blocks of integers directly before and after powers of two.

We see that 3 $\square$ 4 $\square$ 5 and 7 $\square$ 8 $\square$ 9 are the only two cases where a power of two is adjacent to two primes, and a prime and a prime power, respectively. We may consider these blocks as trivial runs of length zero.

Then we have 13 $\boxed{14\ \ 15}$ 16 17 $= F_2$, the first genuine maximal strictly composite run, with length 2, followed by

$$M_5 = 31 \ \square\ 32\ \boxed{33\ \ 34\ \ 35\ \ 36}\ 37,$$

the first run of length 4, and 61 $\boxed{62\ \ 63}$ 64 $\boxed{65\ \ 66}$ 67, two runs of length 2 on either side of 64. Next, we come to

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$$M_7 = 127 \boxed{\ } \underline{128} \boxed{129\ \ 130} 131$$

and

$$251 \boxed{252\ \ 253\ \ 254\ \ 255} \underline{256} \boxed{\ } 257 = F_3$$

(further runs of length 2 and 4), then

$$509 \boxed{510\ \ 511} \underline{512} \boxed{513\ \ 514\ \ 515\ \ 516\ \ 517\ \ 518\ \ 519\ \ 520} 521,$$

the first run of exact length 8, followed by

$$1021 \boxed{1022\ \ 1023} \underline{1024} \boxed{1025\ \ 1026\ \ 1027\ \ 1028\ \ 1029\ \ 1030} 1031,$$

the first run of exact length 6.

Note that all these runs have even length. In fact, this is true in general:

Theorem 5: Every maximal strictly composite run which is bordered by a power of two has even length.

Proof: One border is 2^n. The other border must be either $2^{n \pm 1}$, p or p^k, where p is an odd prime and $k \geqslant 2$. Only in the first case is the length odd; but this case is impossible, since by Bertrand's Theorem, there is always a prime between 2^n and $2^{n \pm 1}$.

The 'power of two' condition is necessary here, as the examples $5 \boxed{6} 7$ and $19 \boxed{20\ \ 21\ \ 22} 23$ clearly demonstrate.

We used a computer to explore up to 2^{62}, searching for maximal strictly composite runs directly before and after each power of two, and present the results below; for compactness, we list only the first occurrence of each run of a given length.

Length	Start	Finish
2	14	$15 = 2^4 - 1$
4	$33 = 2^5 + 1$	36
8	$513 = 2^9 + 1$	520
6	$1025 = 2^{10} + 1$	1030
16	$8193 = 2^{13} + 1$	8208
26	$16385 = 2^{14} + 1$	16410
14	65522	$65535 = 2^{16} - 1$
28	$131073 = 2^{17} + 1$	131100
20	$524289 = 2^{19} + 1$	524308
42	$16777217 = 2^{24} + 1$	16777258
38	33554394	$33554431 = 2^{25} - 1$
34	$33554433 = 2^{25} + 1$	33554466
56	268435400	$268435455 = 2^{28} - 1$
10	$536870913 = 2^{29} + 1$	536870922
40	17179869144	$17179869183 = 2^{34} - 1$
24	$17179869185 = 2^{34} + 1$	17179869208

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30	34359738338			34359738367	$= 2^{35} - 1$
52	34359738369	$= 2^{35} + 1$		34359738420	
44	274877906900			274877906943	$= 2^{38} - 1$
22	549755813889	$= 2^{39} + 1$		549755813910	
86	1099511627690			1099511627775	$= 2^{40} - 1$
54	35184372088778			35184372088831	$= 2^{45} - 1$
58	35184372088833	$= 2^{45} + 1$		35184372088890	
114	140737488355214			140737488355327	$= 2^{47} - 1$
80	562949953421232			562949953421311	$= 2^{49} - 1$
68	562949953421313	$= 2^{49} + 1$		562949953421380	
128	2251799813685120			2251799813685247	$= 2^{51} - 1$
46	4503599627370450			4503599627370495	$= 2^{52} - 1$
110	9007199254740882			9007199254740991	$= 2^{53} - 1$
32	18014398509481952			18014398509481983	$= 2^{54} - 1$
158	18014398509481985	$= 2^{54} + 1$		18014398509482142	
12	144115188075855860			144115188075855871	$= 2^{57} - 1$
130	576460752303423489	$= 2^{59} + 1$		576460752303423618	
92	1152921504606846884			1152921504606846975	$= 2^{60} - 1$
134	4611686018427387905	$= 2^{62} + 1$		4611686018427388038	

We conjecture that runs of this form exist for each even length, although we note that no example for length 18 has yet been found; however, we have explored only 118 blocks so far, adjacent to the powers 2^4 to 2^{62}.

Examining the above data, again with the help of a computer, we found that the other border of each run is always a prime, never a prime power. It is an open question whether or not this holds true in general. However, we do note that primes tend to be far more abundant than prime powers. For example, up to 2^{10}, there are about 7 primes for each prime power; up to 2^{20}, the ratio is 339:1, and up to 2^{30}, it rises to 14,289:1 (once again, the computer proved to be such a helpful tool in this exploration). Thus, the observed result may simply be due to this very large imbalance between the relative densities of primes and prime powers.

Finally, we relaxed the requirement of searching next to a power of two and performed an unrestricted search for maximal strictly composite runs. Because of the significantly greater computational effort involved, we have so far searched only up to 2^{20}. We present the results below, listing only those runs whose length exceeds those of all previous runs.

Length	Start		Finish	
1	6		6	
2	14		15	$= 2^4 - 1$
3	20		22	
4	33	$= 2^5 + 1$	36	
5	54		58	
7	90		96	

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9	140		148
11	200		210
13	294		306
19	888		906
21	1130		1150
29	1332	$= 11^3 + 1$	1360
31	5592		5622
33	8468		8500
35	9552		9586
43	15684		15726
51	19610		19660
71	31398		31468
85	155922		156006
95	360654		360748
111	370262		370372
113	492114		492226

We note that, apart from the three exceptions indicated above, each run is bordered on both sides by primes, not prime powers (possibly due to the abundance of the former relative to the latter), and that as a consequence, almost all of these maximal runs have odd length.

Unsurprisingly, this unrestricted search discovers runs which are longer (yet not hugely so) than the restricted search, but the computational effort is much greater due to the vastly increased search space.

References

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2. *Great Internet Mersenne Prime Search* accessed February 2015 at http://mersenne.org

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