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NULL SETS AND COMBINATORIAL COVERING PROPERTIES

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Abstract. A subset of the Cantor cube is null-additive if its algebraic sum with any null set is null. We construct a set of cardinality continuum such that: all continuous images of the set into the Cantor cube are null-additive, it contains a homeomorphic copy of a set that is not null-additive, and it has the property γ, a strong combinatorial covering property. We also construct a nontrivial subset of the Cantor cube with the property γ that is not null additive. Set-theoretic assumptions used in our constructions are far milder than used earlier by Galvin–Miller and Bartoszyński–Recław, to obtain sets with analogous properties. We also consider products of Sierpiński sets in the context of combinatorial covering properties.

§1. Introduction. Let $\mathbb{N}$ be the set of natural numbers and $\mathrm{P}(\mathbb{N})$ be the power set of $\mathbb{N}$. We identify each set in $\mathrm{P}(\mathbb{N})$ with its characteristic function, an element of the Cantor cube $\{0,1\}^{\mathbb{N}}$; in that way we introduce topology in $\mathrm{P}(\mathbb{N})$. The Cantor space $\mathrm{P}(\mathbb{N})$ with the symmetric difference operation $\oplus$ is a topological group; this operation coincides with the addition modulo 2 in $\{0,1\}^{\mathbb{N}}$. A set $X \subseteq \mathrm{P}(\mathbb{N})$ is *null-additive* in $\mathrm{P}(\mathbb{N})$ if for any null set $Y \subseteq \mathrm{P}(\mathbb{N})$ the set $X \oplus Y := \{ x \oplus y : x \in X, y \in Y \}$ is null. In an analogous way, define null-additive subsets of the real line with the addition $+$ as a group operation. As we see in the forthcoming Theorem 2.2, it is relatively consistent with ZFC that null-additive subsets of $\mathrm{P}(\mathbb{N})$ are not preserved by homeomorphisms into $\mathrm{P}(\mathbb{N})$. Subsets of the real line whose all continuous images into the real line are null-additive were considered by Galvin and Miller [3]; to this end they used combinatorial covering properties.

By *space* we mean a Tychonoff topological space. A *cover* of a space is a family of proper subsets of the space whose union is the entire space. An *open* cover of a space is a cover whose members are open subsets of the space. A cover of a space is an *ω-cover* if each finite subset of the space is contained in a set from the cover and it is a *γ-cover* if it is infinite and each point of the space belongs to all but finitely many sets from the cover. A space has the property γ if every open ω-cover of the space contains a γ-cover. This property was introduced by Gerlits and Nagy in the context of local properties of functions spaces [4]. They proved that a space X has the property γ if and only if the space $\mathrm{C}_{\mathrm{p}}(X)$ of all continuous real-valued functions defined on X with the pointwise convergence topology is *Fréchet–Urysohn*, i.e., each point in the closure of a subset of $\mathrm{C}_{\mathrm{p}}(X)$ is a limit of a sequence from the set [4, Theorem 2]. Galvin and Miller observed that for a subset of the real line X with the property γ and a meager subset of the real line Y, the set

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$X + Y := \{ x + y : x \in X, y \in Y \}$ is meager. They pointed out that they were unable to prove an analogous statement for null sets, and thus they introduced a formally stronger property than γ [3, p. 152]. For a natural number n, an open cover of a space is an *n-cover* if each n-elements subset of the space is contained in a member of the cover. A space X has the property *strongly* γ if there is an increasing sequence $f \in \mathbb{N}^{\mathbb{N}}$ such that for each sequence $\mathcal{U}_1, \mathcal{U}_2, \ldots$ where $\mathcal{U}_n$ is an $f(n)$-cover of X, there are sets $U_1 \in \mathcal{U}_1$, $U_2 \in \mathcal{U}_2, \ldots$ such that the family $\{U_n : n \in \mathbb{N}\}$ is a γ-cover of X.

Galvin and Miller proved that any subset of the real line with the property strongly γ is null-additive. The property strongly γ is preserved by continuous mappings, and thus any continuous image of a set with the property strongly γ, into the real line, is null-additive [3, Theorem 7]. Under Martin Axiom, Galvin and Miller constructed a subset of $\mathrm{P}(\mathbb{N})$ of cardinality continuum with the property strong-γ [3, Theorem 8]. In 1996, under some set-theoretic assumption, Bartoszyński and Recław [2] constructed a subset of $\mathrm{P}(\mathbb{N})$ of cardinality continuum with the property γ that is not null-additive (in particular, they separated the properties γ and strongly γ).

Assuming only an equality between some cardinal characteristics of the continuum, we construct a subset of $\mathrm{P}(\mathbb{N})$ of cardinality continuum with the property γ whose all continuous images into $\mathrm{P}(\mathbb{N})$ are null-additive and it contains a homeomorphic copy of a set that is not null-additive. We also weaken a set-theoretic assumption in the result of Bartoszyński and Recław. In both the cases we use combinatorial methods of construction of subsets of $\mathrm{P}(\mathbb{N})$ with the property γ invented by Tsaban ([7, Theorem 3.6], [8, Theorem 6]) and developed by Włudecka and the first named author [13]. We also use a combinatorial covering characterization of null-additive subsets of $\mathrm{P}(\mathbb{N})$ given by Zindulka [17].

§2. Null-additive sets with the property γ. Let $[\mathbb{N}]^\infty$ be the family of infinite subsets of $\mathbb{N}$. Each set in $[\mathbb{N}]^\infty$ we identify with an increasing function from $\mathbb{N}^{\mathbb{N}}$. Depending on the interpretation, points of $[\mathbb{N}]^\infty$ are referred to as sets or functions. For natural numbers n, m with $n < m$, define $[n, m) := \{ i \in \mathbb{N} : n \leq i < m \}$. For sets a and b, we write $a \subseteq^* b$, if the set $a \setminus b$ is finite. A *pseudointersection* of a family of infinite sets is an infinite set a with $a \subseteq^* b$ for all sets b in the family. A family of infinite sets is *centered* if the finite intersections of its elements are infinite. Let $\mathfrak{p}$ be the minimal cardinality of a subfamily of $[\mathbb{N}]^\infty$ that is centered and has no pseudointersection. Let Fin be the family of finite subsets of $\mathbb{N}$. The following notion plays a crucial role in our constructions.

Definition 2.1. (Szewczak and Włudecka [13]) A set $X \subseteq [\mathbb{N}]^\infty$ with $|X| \geq \mathfrak{p}$ is a *$\mathfrak{p}$-generalized tower* if for each function $a \in [\mathbb{N}]^\infty$, there are sets $b \in [\mathbb{N}]^\infty$ and $S \subseteq X$ with $|S| < \mathfrak{p}$ such that

$$x \cap \bigcup_{n \in b} [a(n), a(n+1)) \in \mathrm{Fin}$$

for all sets $x \in X \setminus S$.

For functions $f, g \in [\mathbb{N}]^\infty$, we write $f \leq^* g$, if the set $\{ n : f(n) > g(n) \}$ is finite. A subset of $[\mathbb{N}]^\infty$ is *unbounded* if for any function $g \in [\mathbb{N}]^\infty$, there is a function f in the set with $f \not\leq^* g$. Let $\mathfrak{b}$ be the minimal cardinality of an unbounded subset of $[\mathbb{N}]^\infty$.

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The existence of a $\mathfrak{p}$-generalized tower in $[\mathbb{N}]^\infty$ is independent of ZFC, i.e., it is equivalent to the equality $\mathfrak{p} = \mathfrak{b}$ [7, Lemma 3.3]. Let $\mathrm{non}(\mathcal{N}_{\mathrm{add}})$ be the minimal cardinality of a subset of $\mathrm{P}(\mathbb{N})$ that is not null-additive.

THEOREM 2.2. *Assume that* $\mathfrak{p} = \mathrm{non}(\mathcal{N}_{\mathrm{add}}) = \mathfrak{c}$. *There is a set* $X \subseteq [\mathbb{N}]^\infty$ *such that*

(1) *The set* X *is a* $\mathfrak{p}$*-generalized tower.*
(2) *The set* $X \cup \mathrm{Fin}$ *has the property* γ.
(3) *All continuous images of the set* $X \cup \mathrm{Fin}$ *into* $\mathrm{P}(\mathbb{N})$ *are null-additive.*
(4) *The set* X *is homeomorphic to a subset of* $\mathrm{P}(\mathbb{N})$ *that is not null-additive.*

We need the following notions and auxiliary results.

For a class $\mathcal{A}$ of covers of spaces and a space X, let $\mathcal{A}(X)$ be the family of all covers of X from the class $\mathcal{A}$. Let $\mathcal{A}_1, \mathcal{A}_2, \ldots$ and $\mathcal{B}$ be classes of covers of spaces. A space X satisfies

$$\mathsf{S}_1(\{\mathcal{A}_n\}_{n\in\mathbb{N}}, \mathcal{B})$$

if for each sequence $\mathcal{U}_1, \mathcal{U}_2, \ldots$ with $\mathcal{U}_n \in \mathcal{A}_n(X)$ for each n, there are sets $U_1 \in \mathcal{U}_1, U_2 \in \mathcal{U}_2, \ldots$ such that $\{ U_n : n \in \mathbb{N} \} \in \mathcal{B}(X)$. Let O_n be the class of all open n-covers of spaces for each n and Γ be the class of all open γ-covers of spaces. Using the above notion, the strongly γ property is the property $\mathsf{S}_1(\{\mathrm{O}_n\}_{n\in\mathbb{N}}), \Gamma)$. Let d be a metric in $\mathrm{P}(\mathbb{N})$ that coincides with the standard metric in $\{0,1\}^{\mathbb{N}}$, that is, for points $x, y \in \mathrm{P}(\mathbb{N})$, let

$$d(x,y) := \begin{cases} 2^{-\min((x\cup y)\setminus(x\cap y))}, & \text{if } x \neq y, \\ 0, & \text{if } x = y. \end{cases}$$

Let ε be a positive real number. For a point $x \in \mathrm{P}(\mathbb{N})$, let $B(x,\varepsilon) := \{ y \in \mathrm{P}(\mathbb{N}) : d(x,y) < \varepsilon \}$ and for a set $F \subseteq \mathrm{P}(\mathbb{N})$, let $B(F,\varepsilon) := \bigcup\{ B(x,\varepsilon) : x \in F \}$. Fix a natural number n. An open cover of a subspace of $\mathrm{P}(\mathbb{N})$ is a *uniform n-cover* if there is a positive real number ε such that for each n-elements subset F of the space, the set $B(F,\varepsilon)$ is contained in a member of the cover. Let $\mathrm{O}_n^{\mathsf{unif}}$ be the family of all uniform n-covers of subspaces of $\mathrm{P}(\mathbb{N})$. In 1995, Shelah characterized null-additive subsets of $\mathrm{P}(\mathbb{N})$ [10]. Using the Shelah result, Zindulka characterized null-additive subsets of $\mathrm{P}(\mathbb{N})$, in terms of combinatorial covering properties.

THEOREM 2.3 (Zindulka [17]). *A subset of* $\mathrm{P}(\mathbb{N})$ *is null-additive if and only if it satisfies* $\mathsf{S}_1(\{\mathrm{O}_n^{\mathsf{unif}}\}_{n\in\mathbb{N}}, \Gamma)$.

For a set X and a natural number n, let $\mathrm{P}_n(X)$ be the family of all n-elements subsets of X. By a similar argument as in the proof of a result of Tsaban [14, Theorem 2.1], we have the following result.

LEMMA 2.4. *For a set* $X \subseteq \mathrm{P}(\mathbb{N})$ *the following assertions are equivalent*:

(1) *The set* X *satisfies* $\mathsf{S}_1(\{\mathrm{O}_n^{\mathsf{unif}}\}_{n\in\mathbb{N}}, \Gamma)$.
(2) *There is a function* $f \in [\mathbb{N}]^\infty$ *such that the set* X *satisfies* $\mathsf{S}_1(\{\mathrm{O}_{f(n)}^{\mathsf{unif}}\}_{n\in\mathbb{N}}, \Gamma)$.
(3) *For each function* $f \in [\mathbb{N}]^\infty$, *the set* X *satisfies* $\mathsf{S}_1(\{\mathrm{O}_{f(n)}^{\mathsf{unif}}\}_{n\in\mathbb{N}}, \Gamma)$.

PROPOSITION 2.5. *Let* $X \subseteq \mathrm{P}(\mathbb{N})$ *be a set and assume that there is a function* $f \in [\mathbb{N}]^\infty$ *such that for each sequence* $\mathcal{U}_1, \mathcal{U}_2, \ldots$ *with* $\mathcal{U}_n \in \mathrm{O}_{f(n)}(X)$ *for each* n, *there are sets* $U_1 \in \mathcal{U}_1, U_2 \in \mathcal{U}_2, \ldots$ *and a set* $X' \subseteq X$ *with* $|X'| < \mathrm{non}(\mathcal{N}_{\mathrm{add}})$ *such that*

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$\{ U_n : n \in \mathbb{N} \} \in \Gamma(X \setminus X')$. *Then all continuous images of the set* X *into* $\mathrm{P}(\mathbb{N})$ *are null-additive.*

PROOF. Let $\mathcal{U}_1, \mathcal{U}_2, \dots$ be a sequence such that $\mathcal{U}_n \in \mathrm{O}^{\mathsf{unif}}_{2f(n)}(X)$ for each n. Fix a natural number n. We may assume that there is a positive real number ε_n such that

$$\mathcal{U}_n = \{ B(A, \varepsilon_n) : A \in \mathrm{P}_{2f(n)}(X) \}.$$

We have $\mathcal{U}'_n := \{ B(F, \varepsilon_n) : F \in \mathrm{P}_{f(n)}(X) \} \in \mathrm{O}^{\mathsf{unif}}_{f(n)}(X)$. By the assumption, there are sets $F_1 \in \mathrm{P}_{f(1)}(X), F_2 \in \mathrm{P}_{f(2)}(X), \dots$ and a set $X' \subseteq X$ with $|X'| < \mathrm{non}(\mathcal{N}_{\mathrm{add}})$ such that

$$\{ B(F_n, \varepsilon_n) : n \in \mathbb{N} \} \in \Gamma(X \setminus X').$$

Since $|X'| < \mathrm{non}(\mathcal{N}_{\mathrm{add}})$, there are sets $F'_1 \in \mathrm{P}_{f(1)}(X'), F'_2 \in \mathrm{P}_{f(2)}(X'), \dots$ such that

$$\{ B(F'_n, \varepsilon_n) : n \in \mathbb{N} \} \in \Gamma(X').$$

For each natural number n, there is a set $A_n \in \mathrm{P}_{2f(n)}(X)$ such that $F_n \cup F'_n \subseteq A_n$ and

$$B(F_n, \varepsilon_n) \cup B(F'_n, \varepsilon_n) \subseteq B(A_n, \varepsilon_n) \in \mathcal{U}_n.$$

We have

$$\{ B(A_n, \varepsilon_n) : n \in \mathbb{N} \} \in \Gamma(X).$$

By Lemma 2.4 and Theorem 2.3, the set X is null-additive. ⊣

By Theorem 2.3, we have the following result.

PROPOSITION 2.6. *Null-additivity in* $\mathrm{P}(\mathbb{N})$ *is preserved by uniformly continuous functions.*

LEMMA 2.7. *Let* X *be a space and* n, k *be natural numbers. If* $\mathcal{U} \in \mathrm{O}_{n+k}(X)$, *then each* n*-elements subset of* X *is contained in at least* k *pairwise different sets from* $\mathcal{U}$.

For the remaining part of this section, let $f \in [\mathbb{N}]^\infty$ be a sequence such that $f(n+1) > 2^{f(n)+2n} + n$ for all natural numbers n. For a set $d \in [\mathbb{N}]^\infty$, let $[d]^\infty$ be the family of all infinite subsets of d.

LEMMA 2.8. *Let* $\mathcal{U}_1, \mathcal{U}_2, \dots$ *be a sequence of families of open sets in* $\mathrm{P}(\mathbb{N})$ *such that* $\mathcal{U}_n \in \mathrm{O}_{f(n)}(\mathrm{Fin})$ *for all natural numbers* n *and* $d \in [\mathbb{N}]^\infty$. *There are an element* $x \in [d]^\infty$ *and pairwise different sets* $U_1 \in \mathcal{U}_1, U_2 \in \mathcal{U}_2, \dots$ *such that for each natural number* n *and each set* $y \in \mathrm{P}(\mathbb{N})$:

If $y \cap [f(n), x(n+1)) \subseteq \{x(1), x(1)+1, \dots, x(n), x(n)+1\}$, *then* $y \in U_{n+1}$.

In particular,

$$\{ U_n : n \in \mathbb{N} \} \in \Gamma(\{ y \in \mathrm{P}(\mathbb{N}) : y \subseteq^* x \cup (x+1) \}).$$

PROOF. Let $x(1) := a(1)$ and $U_1 \in \mathcal{U}_1$. Fix a natural number n and assume that natural numbers $x(1), \dots, x(n)$ and sets $U_1 \in \mathcal{U}_1, \dots, U_n \in \mathcal{U}_n$ have already been defined. We have

$$|\mathrm{P}([1, f(n)) \cup \{x(1), x(1)+1, \dots, x(n), x(n)+1\})| \leq 2^{f(n)+2n}$$

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and $\mathcal{U}_{n+1} \in \mathrm{O}_{f(n+1)}(\mathrm{Fin})$. By Lemma 2.7 there is a set $U_{n+1} \in \mathcal{U}_{n+1} \setminus \{U_1, \dots, U_n\}$ such that

$$\mathrm{P}([1, f(n)) \cup \{x(1), x(1)+1, \dots, x(n), x(n)+1\}) \subseteq U_{n+1}.$$

For each set

$$s \in \mathrm{P}([1, f(n)) \cup \{x(1), x(1)+1, \dots, x(n), x(n)+1\}),$$

there is a natural number $m_s \in a$ with $m_s > \max\{f(n), x(n)+1\}$ such that for each set $y \in \mathrm{P}(\mathbb{N})$:

$$\text{If } y \cap [1, m_s) = s, \text{ then } y \in U_{n+1}.$$

Define

$$x(n+1) := \max\{\, m_s : s \in \mathrm{P}([1, f(n)) \cup \{x(1), x(1)+1, \dots, x(n), x(n)+1\})\,\}.$$

Fix a natural number n. Let $y \in \mathrm{P}(\mathbb{N})$ be a set such that

$$y \cap [f(n), x(n+1)) \subseteq \{x(1), x(1)+1, \dots, x(n), x(n)+1\}.$$

Then the set $s := y \cap [1, x(n+1))$ belongs to $\mathrm{P}([1, f(n)) \cup \{x(1), x(1)+1, \dots, x(n), x(n)+1\})$, and

$$y \cap [1, m_s) = y \cap [1, x(n+1)) = s.$$

Thus, $y \in U_{n+1}$. ⊣

Lemma 2.9. (Folklore [15, Lemma 2.13]) *A set $X \subseteq [\mathbb{N}]^\infty$ is unbounded if and only if for each function $a \in [\mathbb{N}]^\infty$, there are a set $b \in [\mathbb{N}]^\infty$ and an element $x \in X$ such that*

$$x \cap \bigcup_{n \in b} [a(n), a(n+1)) = \emptyset,$$

for all $x \in X$.

For elements $b \in [\mathbb{N}]^\infty$ and $c \in \{0,1\}^{\mathbb{N}}$, let $2b$, $b+1$, and $b+c$ be elements in $[\mathbb{N}]^\infty$ such that $(2b)(n) := 2b(n)$, $(b+1)(n) := b(n)+1$, and $(b+c)(n) := b(n) + c(n)$ for all natural numbers n.

Proof of Theorem 2.2. Construction of a set X: Let $\{\, (\mathcal{U}_1^{(\alpha)}, \mathcal{U}_2^{(\alpha)}, \dots) : \alpha < \mathfrak{c} \,\}$ be an enumeration of all sequences of families of open sets in $\mathrm{P}(\mathbb{N})$ such that $\mathcal{U}_n^{(\alpha)} \in \mathrm{O}_{f(n)}(\mathrm{Fin})$ for all ordinal numbers $\alpha < \mathfrak{c}$ and natural numbers n. Let $[\mathbb{N}]^\infty = \{\, d_\alpha : \alpha < \mathfrak{c} \,\}$.

Apply Lemma 2.8 to the sequence $\mathcal{U}_1^{(0)}, \mathcal{U}_2^{(0)}, \dots$ and to a set $d \in [2\mathbb{N}]^\infty$ with $d_0 \leq^* d$. Then there are sets $U_1^{(0)} \in \mathcal{U}_1^{(0)}$, $U_2^{(0)} \in \mathcal{U}_2^{(0)}, \dots$ and an element $x_0 \in [2\mathbb{N}]^\infty$ such that

$$\{\, U_n^{(0)} : n \in \mathbb{N} \,\} \in \Gamma(\{\, y \in \mathrm{P}(\mathbb{N}) : y \subseteq^* x_0 \cup (x_0+1) \,\}).$$

Fix an ordinal number $\alpha < \mathfrak{c}$. Assume that elements $x_\beta \in [2\mathbb{N}]^\infty$ with $d_\beta \leq^* x_\beta$ and sets $U_1^{(\beta)} \in \mathcal{U}_1^{(\beta)}$, $U_2^{(\beta)} \in \mathcal{U}_2^{(\beta)}, \dots$ with

$$\{\, U_n^{(\beta)} : n \in \mathbb{N} \,\} \in \Gamma(\{\, y \in \mathrm{P}(\mathbb{N}) : y \subseteq^* x_\beta \cup (x_\beta+1) \,\}$$

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have already been defined for all ordinal numbers $\beta < \alpha$ such that for $\beta, \beta' < \alpha$ if $\beta < \beta'$, then $x_\beta{}^* \supseteq x_{\beta'}$. Let $d \in [\mathbb{N}]^\infty$ be a pseudointersection of the family $\{ x_\beta : \beta < \alpha \}$ with $d_\alpha \leq^* d$. Apply Lemma 2.8 to the sequence $\mathcal{U}_1^{(\alpha)}, \mathcal{U}_2^{(\alpha)}, \ldots$ and to the element d. Then there are an element $x_\alpha \in [d]^\infty$ and sets $U_1^{(\alpha)} \in \mathcal{U}_1^{(\alpha)}$, $U_2^{(\alpha)} \in \mathcal{U}_2^{(\alpha)}, \ldots$ such that

$$\{ U_n^\alpha : n \in \mathbb{N} \} \in \Gamma(\{ y \in \mathrm{P}(\mathbb{N}) : y \subseteq^* x_\alpha \cup (x_\alpha + 1) \}). \tag{2.9.1}$$

Let $\{0,1\}^{\mathbb{N}} = \{ c_\alpha : \alpha < \mathfrak{c} \}$. Define

$$X := \{ x_\alpha + c_\alpha : \alpha < \mathfrak{c} \}.$$

By the construction, for all ordinal numbers α, β with $\alpha \leq \beta < \mathfrak{c}$, we have

$$x_\beta + c_\beta \subseteq x_\beta \cup (x_\beta + 1) \subseteq^* x_\alpha \cup (x_\alpha + 1). \tag{2.9.2}$$

(1) Let $a \in [\mathbb{N}]^\infty$. Since the set $\{ x_\alpha : \alpha < \mathfrak{c} \}$ is unbounded, the set $\{ x_\alpha \cup (x_\alpha + 1) : \alpha < \mathfrak{c} \}$ is unbounded, too. By Lemma 2.9, there are an ordinal number $\alpha < \mathfrak{c}$ and a set $b \in [\mathbb{N}]^\infty$ such that

$$(x_\alpha \cup (x_\alpha + 1)) \cap \bigcup_{n \in b} [a(n), a(n+1)) = \emptyset.$$

By (2.9.2), we have

$$(x_\beta + c_\beta) \cap \bigcup_{n \in b} [a(n), a(n+1)) \subseteq^* (x_\alpha \cup (x_\alpha + 1)) \cap \bigcup_{n \in b} [a(n), a(n+1)) = \emptyset$$

for all ordinal numbers β with $\alpha \leq \beta < \mathfrak{c}$. Thus, the set X is a $\mathfrak{p}$-generalized tower.

(2) By (1), the set X is a $\mathfrak{p}$-generalized tower, and thus the set $X \cup \mathrm{Fin}$ has the property γ [13, Theorem 4.1(1)].

(3) The set $X \cup \mathrm{Fin}$ satisfies the property from Proposition 2.5: Let $\mathcal{U}_1, \mathcal{U}_2, \ldots$ be a sequence of open families in $\mathrm{P}(\mathbb{N})$ such that $\mathcal{U}_n \in \mathrm{O}_{f(n)}(X \cup \mathrm{Fin})$ for all natural numbers n. There is an ordinal number $\alpha < \mathfrak{c}$ such that this sequence is equal to the sequence $\mathcal{U}_1^{(\alpha)}, \mathcal{U}_2^{(\alpha)}, \ldots$. By (2.9.1) and (2.9.2), we have

$$\{ U_n^{(\alpha)} : n \in \mathbb{N} \} \in \Gamma(\{ x_\beta + c_\beta : \alpha \leq \beta < \mathfrak{c} \}).$$

(4) In our proof we use Rothberger's trick [6, Theorem 9.4]. The map $\Phi\colon \mathrm{P}(\mathbb{N}) \to \{0,1\}^{\mathbb{N}}$ unifying each subset of $\mathbb{N}$ with its characteristic function is a homeomorphism and an isometry with respect to the standard metric in $\{0,1\}^{\mathbb{N}}$ and the metric d on $\mathrm{P}(\mathbb{N})$ defined in Section 2. The set

$$G := \left\{ ((\Phi(x))(1),\ x(1)_{\bmod 2},\ (\Phi(x))(2),\ x(2)_{\bmod 2}, \ldots) : x \in X \right\}$$

is a homeomorphic copy of the set X. Since the projection $\pi\colon G \to \{0,1\}^{2\mathbb{N}}$ is onto and uniformly continuous, the map

$$\Phi^{-1} \circ \pi \circ \Phi\colon \Phi^{-1}[G] \to \mathrm{P}(\mathbb{N})$$

is onto and uniformly continuous, too. By Proposition 2.6, the set $\Phi^{-1}[G]$ is not null additive. The set $\Phi^{-1}[G]$ is homeomorphic to G. ⊣

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§3. A γ-set that is not null-additive. Let $\mathrm{non}(\mathcal{N})$ be the minimal cardinality of a subset of $\mathrm{P}(\mathbb{N})$ that is not null. A set $\{ x_\alpha : \alpha < \mathfrak{p} \} \subseteq [\mathbb{N}]^\infty$ is a $\mathfrak{p}$-unbounded tower if it is unbounded and for all ordinal numbers $\alpha, \beta < \mathfrak{p}$ with $\alpha < \beta$, we have $x_\alpha{}^* \supseteq x_\beta$. A $\mathfrak{p}$-unbounded tower exists if and only if $\mathfrak{p} = \mathfrak{b}$ [7, Lemma 3.3].

THEOREM 3.1. *Assume that* $\mathfrak{p} = \mathfrak{b} = \mathrm{non}(\mathcal{N})$. *There is a* $\mathfrak{p}$*-generalized tower* $X \subseteq [\mathbb{N}]^\infty$ *that is not null-additive. In particular, the set* $X \cup \mathrm{Fin}$ *is a nontrivial set with the property* γ *that is not null-additive.*

PROOF. Let $f \in [\mathbb{N}]^\infty$ be a function such that $f(n) := \sum_{i=0}^{n} 2^i$ for all natural numbers n. Define

$$Y_n := \{ x \in \mathrm{P}(\mathbb{N}) : x \cap [f(n), f(n+1)) = \emptyset \}$$

for all natural numbers n. Each set Y_n is clopen and has measure less or equal than $2^{-(n+1)}$, and thus the set $Y := \bigcap_m \bigcup_{n>m} Y_n$ is null. Let $T = \{ t_\alpha : \alpha < \mathfrak{p} \}$ be a $\mathfrak{p}$-unbounded tower in $[\mathbb{N}]^\infty$ and $Z = \{ z_\alpha : \alpha < \mathfrak{p} \}$ be a nonnull set in $\mathrm{P}(\mathbb{N})$.

Define

$$x_\alpha := z_\alpha \cap \bigcup_{n\in\mathbb{N}} [f(t_\alpha(2n)), f(t_\alpha(2n)+1) \cup \{ f(t_\alpha(2n+1)) : n \in \mathbb{N} \}$$

for all ordinal numbers $\alpha < \mathfrak{p}$ and $X := \{ x_\alpha : \alpha < \mathfrak{p} \}$.

The set X is a $\mathfrak{p}$-generalized tower: For each ordinal number α with $\alpha < \mathfrak{p}$, we have

$$\{ f(t_\alpha(2n+1)) : n \in \mathbb{N} \} \subseteq x_\alpha,$$

and thus $X \subseteq [\mathbb{N}]^\infty$. Let $a \in [\mathbb{N}]^\infty$. There is a set $c \in [\mathbb{N}]^\infty$ such that

$$|[f(c(k)), f(c(k+1))) \cap a| \geq 2$$

for all natural numbers k. By Lemma 2.9, there are a set $d \in [\mathbb{N}]^\infty$ and an ordinal number $\alpha < \mathfrak{p}$ such that

$$t_\alpha \cap \bigcup_{k\in d} [c(k), c(k+1)) = \emptyset. \tag{3.1.1}$$

Then there is a set $b \in [\mathbb{N}]^\infty$ such that

$$\bigcup_{n\in b} [a(n), a(n+1)) \subseteq \bigcup_{k\in d} [f(c(k)), f(c(k+1))).$$

Fix an ordinal number β with $\alpha \leq \beta < \mathfrak{p}$. We have

$$\begin{aligned} x_\beta \cap \bigcup_{n\in b} [a(n), a(n+1)) &\subseteq \bigcup_{n\in t_\beta} [f(n), f(n+1)) \cap \bigcup_{n\in b} [a(n), a(n+1)) \\ &\subseteq^* \bigcup_{n\in t_\alpha} [f(n), f(n+1)) \cap \bigcup_{k\in d} [f(c(k)), f(c(k+1))). \end{aligned}$$

By (3.1.1), the latter intersection is empty.

We have $Z \subseteq X \oplus Y$: Fix an ordinal number $\alpha < \mathfrak{p}$. Since

$$x_\alpha \cap \bigcup_{n\in\mathbb{N}} [f(t_\alpha(2n)), f(t_\alpha(2n)+1) = z_\alpha \cap \bigcup_{n\in\mathbb{N}} [f(t_\alpha(2n)), f(t_\alpha(2n)+1),$$

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we have

$$(x_\alpha \oplus z_\alpha) \cap \bigcup_{n\in\mathbb{N}} [f(t_\alpha(2n)), f(t_\alpha(2n)+1) = \emptyset,$$

and thus $x_\alpha \oplus z_\alpha \in Y$. Then there is an element $y_\alpha \in Y$ such that $x_\alpha \oplus z_\alpha = y_\alpha$. Thus, $z_\alpha = x_\alpha \oplus y_\alpha$.

Since X is a $\mathfrak{p}$-generalized tower, the set $X \cup \mathrm{Fin}$ has the property γ [13, Theorem 4.1(1)]. Since $Z \subseteq (X \cup \mathrm{Fin}) \oplus Y$, the set $X \cup \mathrm{Fin}$ is not null-additive. ⊣

REMARK 3.2. The assumption of Theorem 3.1 is valid assuming the Continuum Hypothesis or Martin Axiom and in the following model. Let $\mathbb{P}_{\aleph_1}$ be an $\aleph_1$-iteration with finite support of a measure algebra and G be a generic filter in $\mathbb{P}_{\aleph_1}$. Let M be a model of ZFC and $\mathfrak{c} = \aleph_2$. In the model $M[G]$, the assumption $\mathfrak{p} = \mathfrak{b} = \mathrm{non}(\mathcal{N})$ from Theorem 3.1 is true: Let $\mathrm{non}(\mathcal{M})$ be the minimal cardinality of a subset of $\mathrm{P}(\mathbb{N})$ that is not meager. Adding $\aleph_1$ Cohen reals, we have $\mathrm{non}(\mathcal{M}) = \aleph_1$ and adding $\aleph_1$ Solovay reals, we have $\mathrm{non}(\mathcal{N}) = \aleph_1$. Thus, $\mathfrak{p} = \aleph_1$. Since $\mathrm{non}(\mathcal{M}) = \aleph_1$, we have $\mathfrak{b} = \aleph_1$.

THEOREM 3.3. *Assume that* $\mathfrak{p} = \mathfrak{b} = \mathrm{non}(\mathcal{N})$. *There is a nontrivial subset of the real line with the property* γ *that is not null-additive.*

PROOF. We use notions from the paper of the second named author [16, Corollary 11]. Let $X \subseteq [\mathbb{N}]^\infty$ be a set from Theorem 3.1. Let $p\colon \mathrm{P}(\mathbb{N}) \to A$ be a homeomorphism. Since the map $f \circ g^{-1} \circ p$ is continuous, the image Y of the set $X \cup \mathrm{Fin}$ under the map $f \circ g^{-1} \circ p$ is a subset of the real line with the property γ. Suppose that the set Y is null-additive. Then the set $g \circ f^{-1}[Y]$ is null-additive [16, Theorem 12]. Since the map p^{-1} is uniformly continuous and the set $X \cup \mathrm{Fin}$ is not null-additive, the set $p[X] = (g \circ f^{-1}[Y])$ is not null-additive, too, a contradiction. ⊣

THEOREM 3.4. *It is consistent with ZFC that there is a* $\mathfrak{p}$*-unbounded tower in* $[\mathbb{N}]^\infty$ *and* $\mathrm{non}(\mathcal{N}_{\mathrm{add}}) < \mathfrak{p}$.

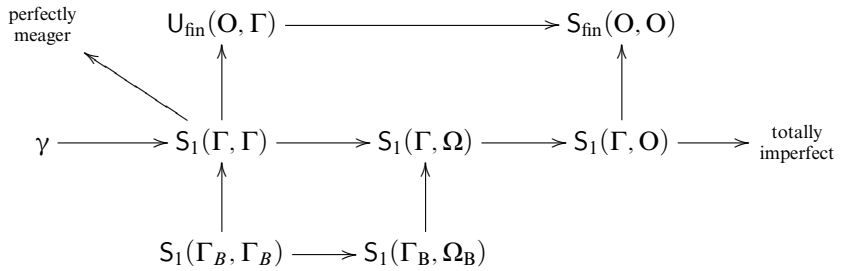
PROOF. The dual Borel conjecture is the statement that for any set $X \subseteq \mathrm{P}(\mathbb{N})$ with cardinality $\aleph_1$, there is a null set $N \subseteq \mathrm{P}(\mathbb{N})$ such that $X \oplus N = \mathrm{P}(\mathbb{N})$. By the result of Judah and Shelah [1, Theorem 8.5.23] there is a model for ZFC satisfying Martin Axiom for σ-centered sets and the dual Borel conjecture. In that model, we have $\mathfrak{p} = \aleph_2 = \mathfrak{b}$, and thus a $\mathfrak{p}$-unbounded tower exists. On the other hand, since the dual Borel conjecture holds, we have $\mathrm{non}(\mathcal{N}_{\mathrm{add}}) = \aleph_1$. ⊣

§4. Nonproductivity of Sierpiński-type sets. Let O be the class of open covers of spaces. A space X satisfies Menger's property $\mathsf{S}_{\mathrm{fin}}(\mathrm{O},\mathrm{O})$ if for each sequence $\mathcal{U}_1, \mathcal{U}_2, \ldots \in \mathrm{O}(X)$, there are finite sets $\mathcal{F}_1 \subseteq \mathcal{U}_1$, $\mathcal{F}_2 \subseteq \mathcal{U}_2, \ldots$ such that $\bigcup_n \mathcal{F}_n \in \mathrm{O}(X)$. A space X satisfies Hurewicz's property $\mathsf{U}_{\mathrm{fin}}(\mathrm{O},\Gamma)$ if for each sequence $\mathcal{U}_1, \mathcal{U}_2, \ldots \in \mathrm{O}(X)$, there are finite sets $\mathcal{F}_1 \subseteq \mathcal{U}_1$, $\mathcal{F}_2 \subseteq \mathcal{U}_2, \ldots$ such that $\{\bigcup \mathcal{F}_n : n \in \mathbb{N}\} \in \Gamma(X)$. The property $\mathsf{U}_{\mathrm{fin}}(\mathrm{O},\Gamma)$ implies $\mathsf{S}_{\mathrm{fin}}(\mathrm{O},\mathrm{O})$ and it generalizes σ-compactness. An uncountable subset of $\mathrm{P}(\mathbb{N})$ is a *Sierpiński* set if its intersection with any null set is at most countable; its existence is independent of ZFC. Any Sierpiński set satisfies $\mathsf{U}_{\mathrm{fin}}(\mathrm{O},\Gamma)$ but it is not σ-compact. Assuming that the Continuum Hypothesis holds, there are two Sierpiński sets whose product space does not even satisfy $\mathsf{S}_{\mathrm{fin}}(\mathrm{O},\mathrm{O})$ [5, p. 250].

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A category theoretic counterpart to a Sierpiński set is a *Luzin set*, i.e., an uncountable subset of $\mathrm{P}(\mathbb{N})$ whose intersection with any meager set is at most countable. Each Luzin set satisfies $\mathsf{S}_{\mathrm{fin}}(\mathrm{O},\mathrm{O})$ but no $\mathsf{U}_{\mathrm{fin}}(\mathrm{O},\Gamma)$. A *set of reals* is a space homeomorphic to a subset of the real line. Let **P** be a property of spaces. A space is *productively* **P** if its product space with any space satisfying **P** satisfies **P**. Assuming $\aleph_1 = \mathrm{cf}(\mathfrak{d})$, in the class of sets of reals, no Luzin set is productively $\mathsf{S}_{\mathrm{fin}}(\mathrm{O},\mathrm{O})$ [11, Corollary 2.11]. There are open problems ([11, Problem 7.5], [12, Problem 5.5]), whether in the class of sets of reals (or in the class of general topological spaces) for any Sierpiński set X, there is a space Y satisfying $\mathsf{U}_{\mathrm{fin}}(\mathrm{O},\Gamma)$ such that the product space $X \times Y$ does not satisfy $\mathsf{U}_{\mathrm{fin}}(\mathrm{O},\Gamma)$, and what if we assume the Continuum Hypothesis? We consider an analogous problem with respect to combinatorial covering properties, stronger than $\mathsf{U}_{\mathrm{fin}}(\mathrm{O},\Gamma)$.

Let $\mathcal{A}$, $\mathcal{B}$ be classes of covers of spaces. A space X satisfies $\mathsf{S}_1(\mathcal{A},\mathcal{B})$ if for each sequence $\mathcal{U}_1, \mathcal{U}_2, \ldots \in \mathcal{A}(X)$, there are sets $U_1 \in \mathcal{U}_1$, $U_2 \in \mathcal{U}_2, \ldots$ such that $\{ U_n : n \in \mathbb{N} \} \in \mathcal{B}(X)$. Let Ω be the class of all open ω-covers of spaces. A *Borel* cover of a space is a cover whose members are Borel subsets of the space. Let Γ_{B} and Ω_{B} be classes of all countable Borel γ-covers and countable Borel ω-covers of spaces, respectively. The property γ, considered in the previous sections, is one of the classic properties in the *selection principles* theory, it is equivalent to the property $\mathsf{S}_1(\Omega,\Gamma)$ [4, Theorem 2]. A set of reals is *totally imperfect* if it does not contain an uncountable compact set and it is *perfectly meager* if its intersection with a perfect set is meager in this perfect set. The following diagram presents relations between considered properties [5, 9].



Let $\mathrm{cov}(\mathcal{N})$ be the minimal cardinality of a family of null subsets of $\mathrm{P}(\mathbb{N})$ whose union is $\mathrm{P}(\mathbb{N})$ and $\mathrm{cof}(\mathcal{N})$ be the minimal cardinality of a family of null subsets of $\mathrm{P}(\mathbb{N})$ such that any null subset of $\mathrm{P}(\mathbb{N})$ is contained in a set from the family. For an uncountable ordinal number κ, a set $X \subseteq \mathrm{P}(\mathbb{N})$ is a *κ-Sierpiński set* if $|X| \geq \kappa$ and for any null set $Y \subseteq \mathrm{P}(\mathbb{N})$, we have $|X \cap Y| < \kappa$. Every $\mathfrak{b}$-Sierpiński set satisfies $\mathsf{S}_1(\Gamma_B,\Gamma_B)$ ([5, Theorem 2.9], [15, Theorem 2.4]).

THEOREM 4.1.

(1) *Assume that* $\mathrm{cov}(\mathcal{N}) = \mathrm{cof}(\mathcal{N}) = \mathfrak{b}$. *For every* $\mathfrak{b}$*-Sierpiński set* S, *there is a* $\mathfrak{b}$*-Sierpiński set* S' *such that the product space* $S \times S'$ *does not satisfy* $\mathsf{S}_1(\Gamma,\Gamma)$.
(2) *Assume that* $\mathrm{cov}(\mathcal{N}) = \mathrm{cof}(\mathcal{N}) = \mathfrak{d} = \mathfrak{c}$ *and the cardinal number* $\mathfrak{c}$ *is regular. For every* $\mathfrak{c}$*-Sierpiński set* S, *there is a* $\mathfrak{c}$*-Sierpiński set* S' *such that the product space* $S \times S'$ *does not satisfy* $\mathsf{S}_1(\Gamma,\mathrm{O})$.

In order to prove Theorem 4.1, we need the following lemma.

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LEMMA 4.2. *Assume that* $\mathrm{cov}(\mathcal{N}) = \mathrm{cof}(\mathcal{N})$ *and the cardinal number* $\mathrm{cov}(\mathcal{N})$ *is regular. For every* $\mathrm{cov}(\mathcal{N})$*-Sierpiński set* S *and every set* Y *of cardinality at most* $\mathrm{cov}(\mathcal{N})$*, there is a* $\mathrm{cov}(\mathcal{N})$*-Sierpiński set* S' *such that* $Y \subseteq S \oplus S'$.

PROOF. Let $\{ N_\alpha : \alpha < \mathrm{cov}(\mathcal{N}) \}$ be a cofinal family of null sets in $\mathrm{P}(\mathbb{N})$ and $Y = \{ y_\alpha : \alpha < \mathrm{cov}(\mathcal{N}) \} \subseteq \mathrm{P}(\mathbb{N})$. Fix an ordinal number $\alpha < \mathrm{cov}(\mathcal{N})$ and assume that elements $x_\beta \in \mathrm{P}(\mathbb{N})$ have already been defined for all ordinal numbers β with $\beta < \alpha$. The set $y_\alpha \oplus S$ is a $\mathrm{cov}(\mathcal{N})$-Sierpiński. Since the cardinal number $\mathrm{cov}(\mathcal{N})$ is regular, the union $\bigcup_{\beta<\alpha} N_\beta \cup \{ x_\beta : \beta < \alpha \}$ cannot cover any $\mathrm{cov}(\mathcal{N})$-Sierpiński set. Then there is an element

$$x_\alpha \in (y_\alpha \oplus S) \setminus \bigcup_{\beta<\alpha} N_\beta \cup \{ x_\beta : \beta < \alpha \}.$$

By the construction the set $S' := \{ x_\alpha : \alpha < \mathrm{cov}(\mathcal{N}) \}$ is a $\mathrm{cov}(\mathcal{N})$-Sierpiński set and for each ordinal number $\alpha < \mathrm{cov}(\mathcal{N})$, there is an element $s_\alpha \in S$ such that $y_\alpha = x_\alpha \oplus s_\alpha$. Thus, $Y \subseteq S \oplus S'$. ⊣

Let $\mathrm{non}(\mathcal{M})$ be the minimal cardinality of a nonmeager subset of $\mathrm{P}(\mathbb{N})$.

PROOF OF THEOREM 4.1.

(1) By the Cichoń diagram, we have $\mathrm{cov}(\mathcal{N}) \leq \mathrm{non}(\mathcal{M}) \leq \mathrm{cof}(\mathcal{N})$, and thus $\mathrm{non}(\mathcal{M}) = \mathfrak{b}$. Let $Y \subseteq \mathrm{P}(\mathbb{N})$ be a nonmeager set of cardinality $\mathfrak{b}$. By Lemma 4.2, there is a $\mathfrak{b}$-Sierpiński set S' such that $Y \subseteq S \oplus S'$. Every set with the property $\mathsf{S}_1(\Gamma, \Gamma)$ is meager and the property $\mathsf{S}_1(\Gamma, \Gamma)$ is preserved by continuous functions. Since $S \oplus S'$ is a nonmeager continuous image of the product space $S \times S'$, the product $S \times S'$ does not satisfy $\mathsf{S}_1(\Gamma, \Gamma)$.

(2) Every set satisfying $\mathsf{S}_1(\Gamma, \mathrm{O})$ is totally imperfect. Let $Y = [\mathbb{N}]^\infty$. Proceed analogously as in (1). ⊣

REMARK 4.3. The assumptions of Theorem 4.1 are valid assuming the Continuum Hypothesis or Martin Axiom. In a model obtained by $\aleph_2$-iteration of Sacks forcing with countable supports, we have $\mathrm{cov}(\mathcal{N}) = \mathrm{non}(\mathcal{M}) = \mathrm{cof}(\mathcal{N}) = \mathfrak{b} = \aleph_1$. In a model obtained by $\aleph_2$-iteration of amoeba forcing with finite supports, we have $\mathrm{cov}(\mathcal{N}) = \mathrm{cof}(\mathcal{N}) = \mathfrak{d} = \mathfrak{c}$.

It is not known whether, in the class of totally imperfect sets, the properties $\mathsf{S}_1(\Gamma, \mathrm{O})$ and $\mathsf{S}_{\mathrm{fin}}(\mathrm{O}, \mathrm{O})$ are different. Theorem 4.1 can be useful to solve this problem.

REMARK 4.4. Assume that the Continuum Hypothesis holds. If there is a Sierpiński set whose product space with any Sierpiński set satisfies $\mathsf{S}_{\mathrm{fin}}(\mathrm{O}, \mathrm{O})$, then, in the class of totally imperfect sets, the properties $\mathsf{S}_{\mathrm{fin}}(\mathrm{O}, \mathrm{O})$ and $\mathsf{S}_1(\Gamma, \mathrm{O})$ are different.

For a finite set $F \subseteq [\mathbb{N}]^\infty$ and a set $X \subseteq [\mathbb{N}]^\infty$ let $\max[F] := \{ \max_{f \in F} f(n) : n \in \mathbb{N} \}$, an element of $[\mathbb{N}]^\infty$, and $\mathrm{maxfin}[X] := \{ \max[F] : F \text{ is a finite subset of } X \}$. For elements $f, g \in [\mathbb{N}]^\infty$ we write $f \leq^\infty g$ if the set $\{ n : f(n) \leq g(n) \}$ is infinite.

Using similar ideas as in the proof that any Sierpiński set satisfies $\mathsf{S}_1(\Gamma, \Gamma)$ [5, Theorems 2.9 and 2.10], we obtain the following results.

PROPOSITION 4.5. *Every* $\mathfrak{d}$*-Sierpiński set satisfies* $\mathsf{S}_1(\Gamma_{\mathrm{B}}, \Omega_{\mathrm{B}})$.

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PROOF. Let S be a $\mathfrak{d}$-Sierpiński set. There is a positive real number p such that the outer measure of S is equal to p. Let B be a Borel set containing S with $\mu(B) = p$. Let $\mathcal{U}_1, \mathcal{U}_2, \ldots \in \Gamma_{\mathrm{Bor}}(S)$ be a sequence of families of Borel subsets of the set B. Let $\mathcal{U}_n = \{ U^n_m : m \in \mathbb{N} \}$ for all natural numbers n, and we may assume that each such a family is increasing (if not consider the family $\{ \bigcap_{i \geq j} U^n_i : j \in \mathbb{N} \}$, an increasing countable Borel cover of S). Fix a natural number k. There is a function $f_k \in \mathbb{N}^{\mathbb{N}}$ such that $\mu(U^n_{f_k(n)}) \geq (1 - \frac{1}{2^{n+k}})p$. For a set $A_k := \bigcap_n U^n_{f_k(n)}$, we have $\mu(A_k) \geq (1 - \frac{1}{2^k})p$. The set $A = \bigcup_k A_k$ has measure p, and thus $|S \setminus A| < \mathfrak{d}$. For each element $x \in S \setminus A$, there is a function $f_x \in \mathbb{N}^{\mathbb{N}}$ such that $x \in \bigcap_n U^n_{f_x(n)}$. There is a function $g \in \mathbb{N}^{\mathbb{N}}$ such that

$$\mathrm{maxfin}[\{ f_k : k \in \mathbb{N} \} \cup \{ f_x : x \in S \setminus A \}] \leq^{\infty} g.$$

We have $\{ U^n_{g(n)} : n \in \mathbb{N} \} \in \Omega(S)$: Let F be a finite subset of S. There is a set $a \in \mathrm{Fin}$ such that $F \cap A \subseteq \bigcup_{k \in a} A_k$. Let

$$f := \mathrm{maxfin}[\{ f_k : k \in a \} \cup \{ f_x : x \in S \setminus A \}].$$

Then $F \subseteq U^n_{f(n)}$ for all but finitely many natural numbers n. Since $f \leq^{\infty} g$, and families $\mathcal{U}_n$ are increasing, we have $F \subseteq U^n_{g(n)}$ for infinitely many natural numbers n. ⊣

The following corollary is a straightforward consequence of Theorem 4.1.

COROLLARY 4.6.

(1) *Assume that* $\mathrm{cov}(\mathcal{N}) = \mathrm{cof}(\mathcal{N}) = \mathfrak{b}$. *No* $\mathrm{cov}(\mathcal{N})$*-Sierpiński set is productively* $\mathsf{S}_1(\Gamma, \Gamma)$.

(2) *Assume that* $\mathrm{cov}(\mathcal{N}) = \mathrm{cof}(\mathcal{N}) = \mathfrak{d} = \mathfrak{c}$ *and the cardinal number* $\mathfrak{c}$ *is regular. No* $\mathfrak{c}$*-Sierpiński set* S *is productively* $\mathsf{S}_1(\Gamma, \Omega)$ *or productively* $\mathsf{S}_1(\Gamma, \mathrm{O})$.

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REFERENCES

[1] T. BARTOSZYŃSKI and H. JUDAH, ***Set Theory: On the Structure of the Real Line***, A. K. Peters, Wellesley, 1995.

[2] T. BARTOSZYŃSKI and I. RECŁAW, *Not every* γ*-set is strongly meager*. ***Contemporary Mathematics***, vol. 192 (1996), pp. 25–29.

[3] F. GALVIN and A. MILLER, γ*-sets and other singular sets of real numbers*. ***Topology and Its Applications***, vol. 17 (1984), pp. 145–155.

[4] J. GERLITS and Z. NAGY, *Some properties of* $C(X)$*, I*. ***Topology and Its Applications***, vol. 14 (1982), pp. 151–161.

[5] W. JUST, A. MILLER, M. SCHEEPERS, and P. SZEPTYCKI, *The combinatorics of open covers II*. ***Topology and Its Applications***, vol. 73 (1996), pp. 241–266.

[6] A. MILLER, *Special subsets of the real line*, ***Handbook of Set-Theoretic Topology*** (K. Kunen and J.E. Vaughan, editors), North-Holland, Amsterdam, 1984, pp. 201–233.

[7] T. ORENSHTEIN and B. TSABAN, *Linear* σ*-additivity and some applications*. ***Transactions of the American Mathematical Society***, vol. 363 (2011), pp. 3621–3637.

[8] A. OSIPOV, P. SZEWCZAK, and B. TSABAN, *Strongly sequentially separable function spaces, via selection principles*. ***Topology and Its Applications***, vol. 270 (2020), Article no. 106942.

https://doi.org/10.1017/jsl.2021.51 Published online by Cambridge University Press

[9] M. Scheepers, *Combinatorics of open covers I*: *Ramsey theory*. ***Topology and Its Applications***, vol. 69 (1996), pp. 31–62.

[10] S. Shelah, *Every null-additive set is meager-additive*. ***Israel Journal of Mathematics***, vol. 89 (1995), pp. 357–376.

[11] P. Szewczak and B. Tsaban, *Products of Menger spaces*: *A combinatorial approach*. ***Annals of Pure and Applied Logic***, vol. 168 (2017), pp. 1–18.

[12] ———, *Products of general Menger spaces*. ***Topology and Its Applications***, vol. 255 (2019), pp. 41–55.

[13] P. Szewczak and M. Włudecka, *Unbounded towers and products*. ***Annals of Pure and Applied Logic***, vol. 172 (2021), Article no. 102900.

[14] B. Tsaban, *Strong γ-sets and other singular spaces*. ***Topology and Its Applications***, vol. 153 (2005), pp. 620–639.

[15] ———, *Menger's and Hurewicz's problems*: *Solutions from "The Book" and refinements*. ***Contemporary Mathematics***, vol. 533 (2011), pp. 211–226.

[16] T. Weiss, *Addendum to "On meager additive and null additive sets in the Cantor space 2^ω and in $\mathbb{R}$"*. ***Bulletin of the Polish Academy of Sciences. Mathematics***, vol. 57 (2009), pp. 91–99.

[17] O. Zindulka, *Strong measure zero and the like via selection principles*. ***Proceedings of the Conference Frontiers of Selection Principles, Poland***, 2017, Plenary Lecture. Slides available at http://selectionprinciples.com/Talks/Zindulka.pdf.

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