Triangles around a given triangle

STAN DOLAN

1. *Introduction*

If equilateral triangles are constructed on each side of *any* given triangle, then the centroids of the equilateral triangles themselves form an equilateral triangle.

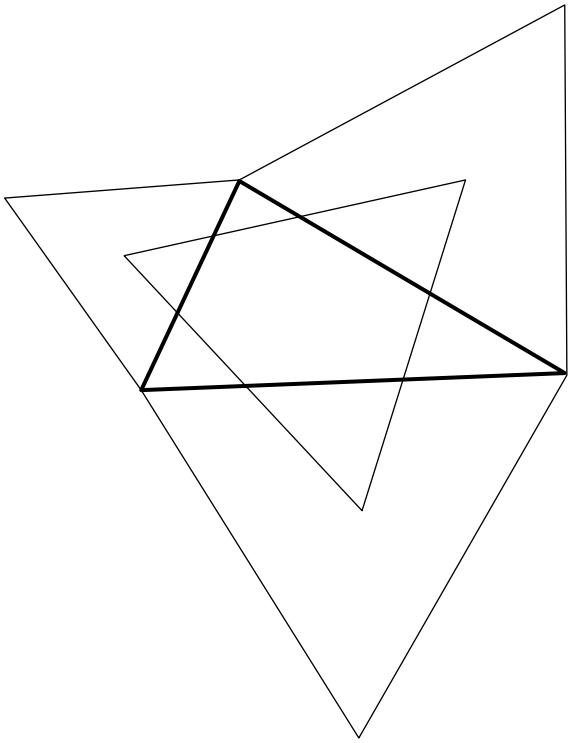


FIGURE 1

This well-known result is called Napoleon's Theorem and is just one of many attractive results concerning the drawing of triangles around a given triangle. See, for example, [1], [2], [3] and [4]. In this paper, our main purpose will be to derive a theorem of surprising generality which will enable the interested reader to generate a considerable variety of specific results.

2. *Initial examples*

The constructions illustrated below can be made around *any* given triangle *ABC* and can be investigated using geometry software programs.

https://doi.org/10.1017/mag.2015.80 Published online by Cambridge University Press

Before reading on, the interested reader might like to consider one or two of these constructions and prove that the resultant triangle $A'B'C'$ has a shape that is independent of the shape of triangle ABC.

The proofs of these results are given later as special cases of a general result.

Example 1: Let the sum of the non-zero angles α, β and γ be 90°. If three triangles are constructed on the sides of any triangle ABC, as shown, then the shape of triangle $A'B'C'$ is the same for all triangles ABC.

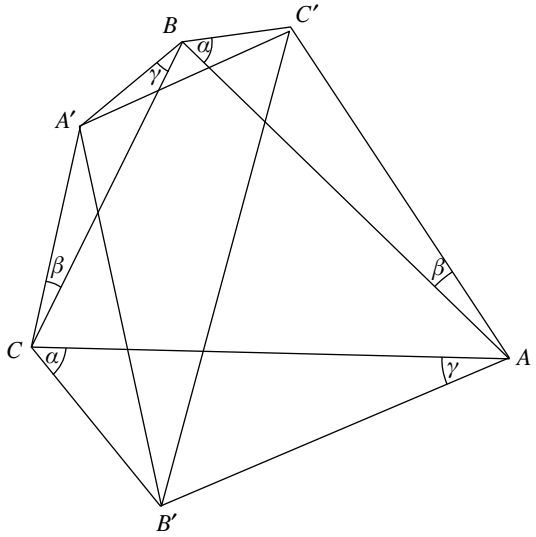


FIGURE 2

Irrespective of the shape of triangle ABC, the angles of triangle $A'B'C'$ at A', B' and C' are 2α, 2β and 2γ, respectively. The case $\alpha = \beta = \gamma = 30°$ is, of course, equivalent to Napoleon's Theorem.

Example 2: In Figure 3, the constructed triangles are mirror images of the triangles CBA', ACB' and BAC' of Example 1.

In this case, the angles of triangle $A'B'C'$ at A', B' and C' are $\beta + \gamma$, $\gamma + \alpha$ and $\alpha + \beta$, respectively, irrespective of the shape of triangle ABC.

https://doi.org/10.1017/mag.2015.80 Published online by Cambridge University Press

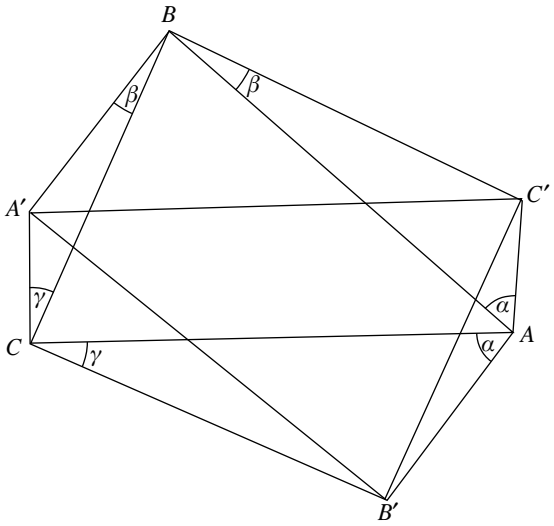


FIGURE 3

Example 3: Some or all of the triangles can be drawn inwards. In Figure 4, the equilateral triangle and the $(30°,\ 30°,\ 120°)$ triangle are drawn outwards but the $(60°,\ 30°,\ 90°)$ triangle is drawn inwards.

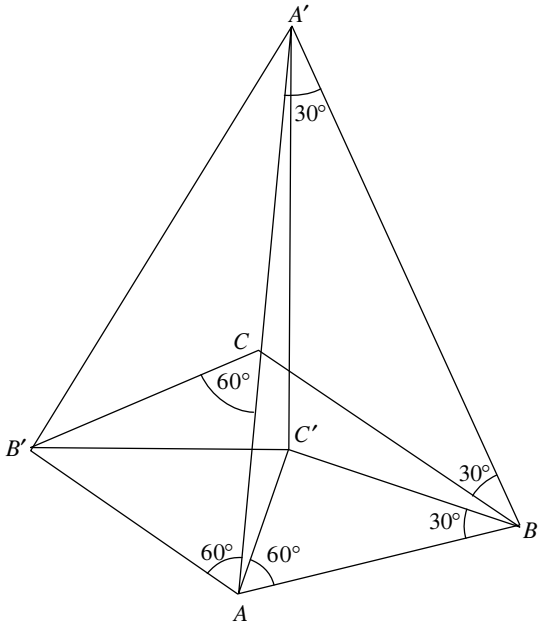


FIGURE 4

Triangle $A'B'C'$ is then always a $(30°,\ 60°,\ 90°)$ triangle.

https://doi.org/10.1017/mag.2015.80 Published online by Cambridge University Press

Example 4: This illustrates how a different arrangement around triangle *ABC* of the three triangles used in Example 3 leads to a very different result for $A'B'C'$.

In Example 3, two 30° angles were adjacent. In this example, two 30° angles are each adjacent to 60° angles.

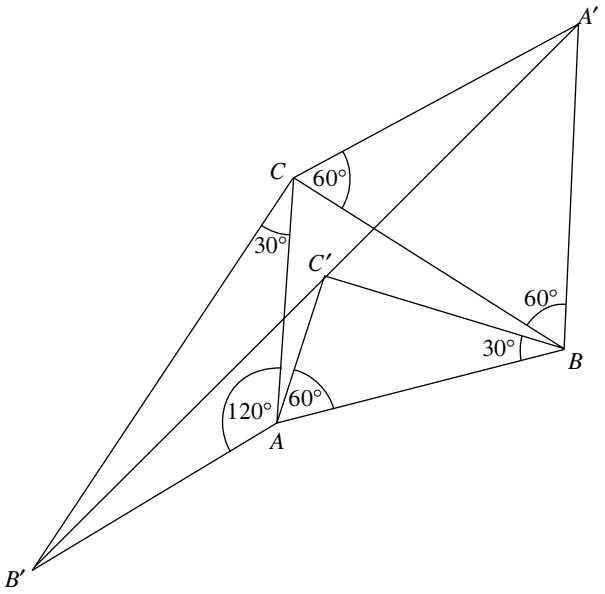


FIGURE 5

With this arrangement, C' is always the mid-point of $A'B'$.

3. *Complex numbers and triangles*

Throughout this paper we shall adopt the following conventions.

- Upper case letters will be used for points and the corresponding lower case letters will be used for the complex numbers represented by these points.
- Any ordered triple of points will be called a triangle even if all the points are identical.
- Triangle *PQR* will be said to be *directly similar* to triangle *ABC* if $A = B = C$ or if there is a combination of translation, rotation and enlargement that maps A to P, B to Q and C to R.

It should be noted that direct similarity is a reflexive and symmetric relation and that *PPP* (i.e. a single point) is directly similar to all other triangles. Furthermore, if *PQR* is directly similar to both *ABC* and *DEF*, then either *PQR* is a point or *ABC* and *DEF* are directly similar.

https://doi.org/10.1017/mag.2015.80 Published online by Cambridge University Press

Let $\mathcal{T}_1$, $\mathcal{T}_2$ and $\mathcal{T}_3$ be three given triangles such that the first two points of any $\mathcal{T}_i$ are distinct. Suppose then that ABC is a given triangle and that CBA', ACB' and BAC' are directly similar to $\mathcal{T}_1$, $\mathcal{T}_2$ and $\mathcal{T}_3$, respectively. The results of Examples 1 to 4 suggest the following general question:

Under what conditions on $\mathcal{T}_1$, $\mathcal{T}_2$ and $\mathcal{T}_3$ is the shape of triangle $A'B'C'$ independent of that of ABC and what then is this shape?

We shall answer this question in the next section. It is, in fact, a surprisingly easy question to answer using the result of [5] concerning the representations of triangles in the Argand plane. We shall use this result in the following form.

Triangle Lemma: Triangles XYZ and UVW are directly similar if, and only if,

$$u(z - y) + v(x - z) + w(y - x) = 0.$$

Proof: Triangles XYZ and UVW are directly similar if, and only if, either $u = v = w$ or there are complex numbers r and s such that

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = r\begin{pmatrix} u \\ v \\ w \end{pmatrix} + s\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\Leftrightarrow \quad \begin{vmatrix} x & u & 1 \\ y & v & 1 \\ z & w & 1 \end{vmatrix} = 0$$

$$\Leftrightarrow \quad u(z - y) + v(x - z) + w(y - x) = 0.$$

4. *The general case*

The examples of Section 2 all depended upon particular choices for the constructed triangles. There is, however, a result which is true for *all* choices of constructed triangles. An example will illustrate this general result.

In Figure 6, the constructed triangles are a $(30°, 30°, 120°)$ triangle, the straight line $AC'B$ and a $(30°, 150° - \alpha, \alpha)$ triangle, where $\sin\alpha = \frac{1}{2\sqrt{7}}$. The construction then transforms an equilateral triangle into a $(30°, 60°, 90°)$ triangle.

https://doi.org/10.1017/mag.2015.80 Published online by Cambridge University Press

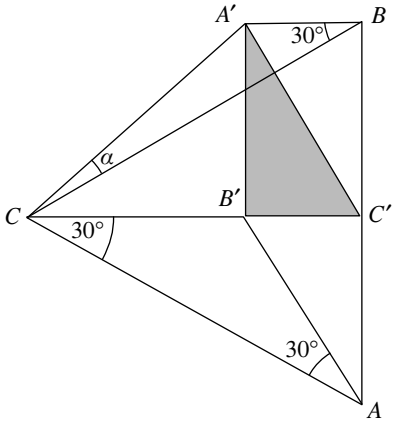


FIGURE 6

In Figure 7, ABC is a $(30°,\ 60°,\ 90°)$ triangle and the construction gives an equilateral triangle.

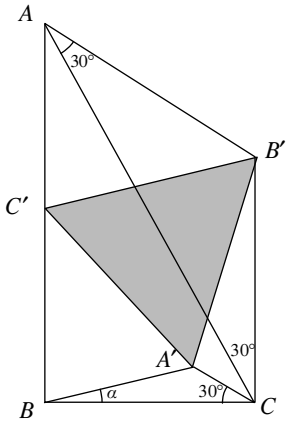


FIGURE 7

Before reading on, the interested reader might like to check the inverse relationship illustrated above for one or two other examples.

To establish the general result, some extra notation is useful. Let $\mathcal{T}_1$, $\mathcal{T}_2$ and $\mathcal{T}_3$ be three fixed triangles such that the first two points of any $\mathcal{T}_i$ are distinct. For any triangle ABC, if CBA', ACB' and BAC' are directly similar to $\mathcal{T}_1$, $\mathcal{T}_2$ and $\mathcal{T}_3$ respectively, denote $A'B'C'$ by $\mathbf{\Phi}(ABC)$.

For $\mathcal{T}_i = UVW$, define $\mathcal{T}_i^* = UVW^*$, where $w^* = u + v - w$. Then define $\mathbf{\Phi}^*$ in the same way as $\mathbf{\Phi}$ but with the $\mathcal{T}_i$ replaced by the $\mathcal{T}_i^*$.

Theorem 1: Let $\mathbf{\Phi}$ and $\mathbf{\Phi}^*$ be defined as above. For any two triangles $\mathcal{S}$ and $\mathcal{T}$, $\mathbf{\Phi}^*(\mathcal{S})$ and $\mathcal{T}$ are directly similar if, and only if, $\mathbf{\Phi}(\mathcal{T})$ and $\mathcal{S}$ are directly similar.

https://doi.org/10.1017/mag.2015.80 Published online by Cambridge University Press

Proof: Let $\mathcal{T} = ABC$ and $\mathcal{S} = XYZ$. CBA' and YZX' are both directly similar to $\mathcal{T}_1$ and are therefore directly similar to each other. By the Triangle Lemma,

$$y(a' - b) + z(c - a') + x'(b - c) = 0.$$

Similarly,

$$z(b' - c) + x(a - b') + y'(c - a) = 0$$

$$x(c' - a) + y(b - c') + z'(a - b) = 0.$$

Adding,

$$x'(c-b) + y'(a-c) + z'(b-a) = a'(z-y) + b'(x-z) + c'(y-x).$$

Therefore $X'Y'Z'$ and ABC are directly similar if, and only if, $A'B'C'$ and XYZ are directly similar.

Corollary 1: For any triangle $\mathcal{T}$, $\mathbf{\Phi}(\mathbf{\Phi}^*(\mathcal{T}))$ and $\mathbf{\Phi}^*(\mathbf{\Phi}(\mathcal{T}))$ are both directly similar to $\mathcal{T}$.

This is an immediate consequence of Theorem 1 and demonstrates the sense in which $\mathbf{\Phi}$ and $\mathbf{\Phi}^*$ can act as inverses of each other when acting upon equivalence classes of directly similar triangles that are not points.

However, it may happen that there is a triangle $\mathcal{S}$ which is not a point but such that either $\mathbf{\Phi}(\mathcal{S})$ or $\mathbf{\Phi}^*(\mathcal{S})$ is a point. This gives rise, for example, to the configuration of which Examples 1, 2, 3 and 4 are special cases.

Corollary 2: If, for some $\mathcal{S}$ which is not a point, $\mathbf{\Phi}^*(\mathcal{S})$ is a point, then $\mathbf{\Phi}(\mathcal{T})$ is directly similar to $\mathcal{S}$ for all triangles $\mathcal{T}$.

Proof: Let $\mathcal{T}$ be any triangle. Then $\mathcal{T}$ and $\mathbf{\Phi}^*(\mathcal{S})$ are directly similar and so $\mathbf{\Phi}(\mathcal{T})$ is directly similar to $\mathcal{S}$, as required.

Now suppose $\mathbf{\Phi}^*(UVW) = TTT$, where U, V and W are distinct points. Then it is immediate from the definition of $\mathbf{\Phi}^*$ that $\mathcal{T}_1 = VWT$, $\mathcal{T}_2 = WUT$ and $\mathcal{T}_3 = UVT$. This gives the following general method of constructing examples such as Examples 1, 2, 3 and 4.

Theorem 2: In a plane, let T, U, V and W be points with U, V and W distinct. Suppose that ABC is a given triangle and let triangles CBA', ACB' and BAC' be directly similar to the triangles VWT, WUT and UVT, respectively. Then $A'B'C'$ is directly similar to UVW, independent of the shape of ABC.

Proofs 'without words' can now be given for Examples 1 to 4 simply by considering the points T, U, V and W in each case.

https://doi.org/10.1017/mag.2015.80 Published online by Cambridge University Press

Example 1:

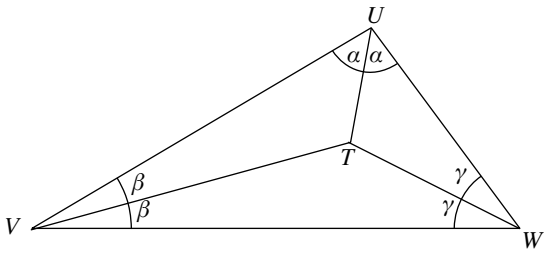


FIGURE 8

Example 2:

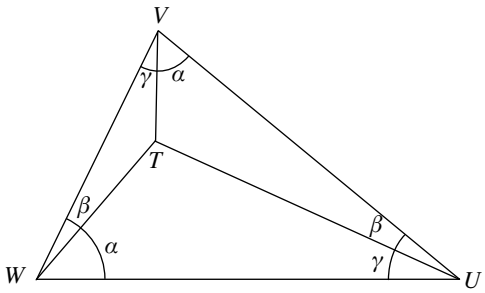


FIGURE 9

Example 3:

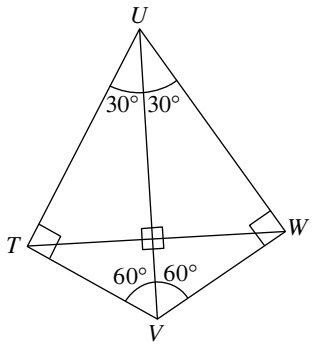


FIGURE 10

https://doi.org/10.1017/mag.2015.80 Published online by Cambridge University Press

Example 4:

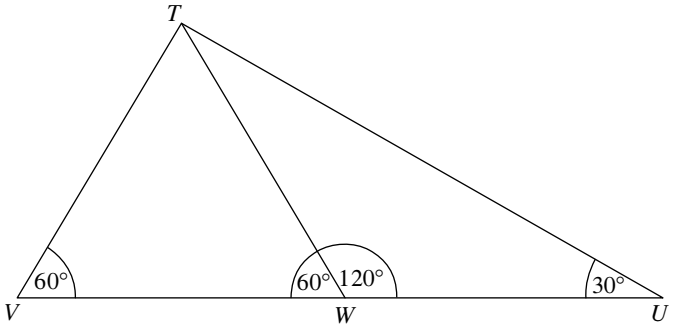


FIGURE 11

5. *Conditions on angles*

Theorem 2 shows that the shape of triangle $A'B'C'$ is independent of that of triangle ABC when triangles $\mathcal{T}_1$, $\mathcal{T}_2$ and $\mathcal{T}_3$ are based upon a point T and a triangle UVW as shown below.

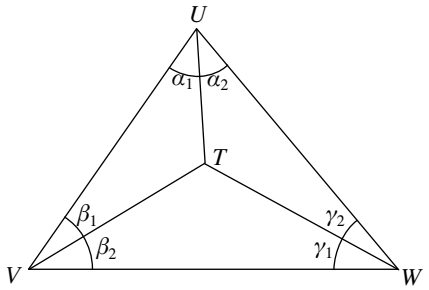


FIGURE 12

Applying the trigonometric form of Ceva's Theorem, we then have the conditions

- $\alpha_1 + \alpha_2 + \beta_1 + \beta_2 + \gamma_1 + \gamma_2 = \pm 180°$,
- $\sin\alpha_1 \sin\beta_1 \sin\gamma_1 = \sin\alpha_2 \sin\beta_2 \sin\gamma_2$.

Furthermore, when these conditions are satisfied, then the angles of $A'B'C'$ are, respectively,

- $\alpha_1 + \alpha_2$, $\beta_1 + \beta_2$ and $\gamma_1 + \gamma_2$.

One of the strengths of the complex analysis proof of Theorem 2 was that it did not matter whether the points A', B' and C' were inside or outside triangle ABC. However, care must be taken with the signs of the angles used above, using negative angles when a constructed triangle is drawn inwards.

Nevertheless, as can be seen in Example 5, these conditions on angles are very easy to apply to specific examples.

https://doi.org/10.1017/mag.2015.80 Published online by Cambridge University Press

Example 5: Let ABC be any given triangle and construct triangles CBA', ACB' and BAC' as shown below, for *any* two angles α and β such that $\alpha + \beta = 60°$. Prove that $A'B'C'$ is an equilateral triangle.

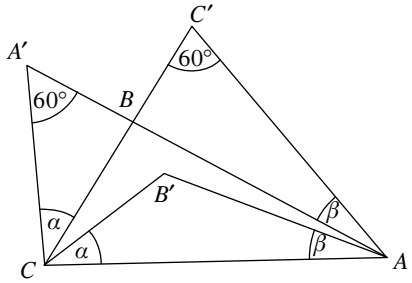


FIGURE 13

Using the same ordering as in the angle conditions stated above, the angles are

$$-\alpha,\ 120° - \beta, \beta, \alpha,\ 120° - \alpha, -\beta.$$

Now $(120° - \alpha) + (120° - \beta) = 180°$ and both conditions are clearly satisfied. The angles of $A'B'C'$ are then $120° - \alpha - \beta$, $\alpha + \beta$ and $120° - \alpha - \beta$, i.e. are all 60°, as required.

Note that the construction of Example 5 can be obtained directly from Theorem 2 by making triangle UVW equilateral, with T appropriately placed on its circumcircle on the lesser arc UW.

6. *Duality*

We have, so far, used Theorem 1 to derive an elementary method of constructing examples such as those of Section 2. Finally, let us return to Theorem 1 and prove some further general results about the transformations Φ and Φ^*.

Lemma 1: There is a triangle $\mathcal{T}$ such that $\Phi^*(\mathcal{T})$ is not a point.

Proof: For any distinct points A and C, the definition of Φ^* implies that there are points X, Y and Z such that

$$\Phi^*(AAC) = XYA, \qquad \Phi^*(ACC) = CYZ.$$

If both XYA and CYZ were points then we would have the contradiction $A = Y = C$.

Lemma 2: If there is a triangle $\mathcal{S}$ which is not a point but such that $\Phi^*(\mathcal{S})$ is a point, then there is a triangle $\mathcal{S}^*$ which is not a point but such that $\Phi(\mathcal{S}^*)$ is a point.

https://doi.org/10.1017/mag.2015.80 Published online by Cambridge University Press

Proof: From Lemma 1, we can choose a triangle $\mathscr{X}$ such that $\mathcal{S}^* = \mathbf{\Phi}^*(\mathscr{X})$ is not a point. Therefore $\mathscr{X}$ is not directly similar to $\mathcal{S}$.

By Corollary 1, $\mathbf{\Phi}(\mathbf{\Phi}^*(\mathscr{X}))$ is directly similar to $\mathscr{X}$. By Corollary 2, $\mathbf{\Phi}^*(\mathbf{\Phi}(\mathscr{X}))$ is directly similar to $\mathcal{S}$. Therefore $\mathbf{\Phi}^*(\mathbf{\Phi}(\mathscr{X}))$ is a point, as required.

Using Theorem 1, we can therefore see that there are just two cases:

Case 1. When considered as acting upon equivalence classes of directly similar triangles, $\mathbf{\Phi}$ and $\mathbf{\Phi}^*$ are inverses of each other.

Case 2. There are unique (up to direct similarity) triangles $\mathcal{S}$ and $\mathcal{S}^*$ such that

- $\mathcal{S}$ and $\mathcal{S}^*$ are not points,
- $\mathbf{\Phi}^*(\mathcal{S})$ and $\mathbf{\Phi}(\mathcal{S}^*)$ are points,
- $\mathbf{\Phi}$ maps all triangles onto triangles directly similar to $\mathcal{S}$,
- $\mathbf{\Phi}^*$ maps all triangles onto triangles directly similar to $\mathcal{S}^*$.

Perhaps the most important aspect of Case 2 not considered so far is the relationship between $\mathcal{S}$ and its dual, $\mathcal{S}^*$.

Suppose first that $\mathcal{S}$ has three distinct points.

Theorem 3: Let $\mathcal{S} = UVW$ be a triangle with three distinct points such that $\mathbf{\Phi}^*(\mathcal{S}) = TTT$. Then $\mathbf{\Phi}(\mathcal{S}^*) = TTT$ where $\mathcal{S}^*$ is the image of $\mathcal{S}$ under inversion in a unit circle centre T and reflection in a line through T.

Proof: This is the configuration of Theorem 2. First suppose that T is distinct from U, V and W.

Let the images of U, V and W under the inversion and reflection be X, Y and Z. Without loss of generality, we can take T as the origin and reflect in the real axis. In modulus-argument form, let U and V be $(|u|, \theta)$ and $(|v|, \theta')$, respectively. Then, X and Y are $\left(\frac{1}{|u|}, -\theta\right)$ and $\left(\frac{1}{|v|}, -\theta'\right)$, respectively.

An enlargement centre T of scale factor $|u||v|$ and a rotation centre T of $\theta + \theta'$ maps YXT to UVT. From Theorem 2, $\mathbf{\Phi}(\mathcal{S}^*) = X'Y'Z'$, where YXZ' is directly similar to UVT and thus to YXT. Therefore $Z' = T$ and, similarly, $X' = Y' = T$. Therefore $\mathbf{\Phi}(\mathcal{S}^*) = TTT$, as required.

Now suppose that one of U, V or W is equal to T. Without loss of generality, let $U = T$. The proof given above then extends naturally to $\mathcal{S}^* = XYZ$, where $Y = Z = T$.

The initial examples of this paper illustrate this duality. Examples 1 and 2 are based upon two dual triangles, as are Examples 3 and 4.

In the case that neither $\mathcal{S} = UVW$ nor $\mathcal{S}^* = XYZ$ have three distinct points, the relationship between these two triangles is especially simple. We

https://doi.org/10.1017/mag.2015.80 Published online by Cambridge University Press

can suppose, without loss of generality, that $\mathcal{S} = UUW$, where $\mathbf{\Phi}^*(\mathcal{S}) = UUU$. It is then a straightforward exercise to show that $\mathcal{S}^* = XZZ$, where $\mathbf{\Phi}(\mathcal{S}^*) = ZZZ$.

Acknowledgement: The author is indebted to an anonymous referee for improvements in the presentation and proofs of this paper.

References

1. E. A. Maxwell, *Geometry for advanced pupils*, Oxford University Press (1953) pp. 43-47.
2. David Wells, *Penguin dictionary of curious and interesting geometry*, Penguin (1991) p. 156.
3. Michael Fox, Napoleon triangles and adventitious angles, *Math. Gaz.*, **82** (November 1998) pp. 413-422.
4. Problem 98.I, *Math. Gaz.*, **98** (November 2014) p. 536.
5. Stan Dolan, Complex numbers and triangles, *Math. Gaz.*, **99** (July 2015) pp. 357-358.

doi:10.1017/mag.2015.80

STAN DOLAN
126A Harpenden Road, St Albans AL3 6BZ

In the Pipeline

The following articles are scheduled to appear in the March 2016 issue.

Remembering spherical trigonometry	*John Conway and Alex Ryba*
Defining trigonometrical functions via complex sequences	*Jan Gustavsson and Mikael P Sundqvist*
Should we play on your lawn or mine?	*JNS Matthews*
Packing a rectangle with m by $(m + 1)$ rectangles	*Richard Ellard and Des MacHale*
Picking genuine zeros of cubics in the Tschirnhaus method	*Dr Raghavendra G Kulharni*
Some properties of the Kiepert lines of a triangle	*Michael Fox*
On the classification of convex quadrilaterals	*Martin Josefsson*

https://doi.org/10.1017/mag.2015.80 Published online by Cambridge University Press