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Infinite ergodic theory for Kleinian groups

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Abstract. In this paper we use infinite ergodic theory to study limit sets of essentially free Kleinian groups which may have parabolic elements of arbitrary rank. By adapting a method of Adler, we construct a section map S for the geodesic flow on the associated hyperbolic manifold. We then show that this map has the Markov property and that it is conservative and ergodic with respect to the invariant measure induced by the Liouville–Patterson measure. Furthermore, we obtain that S is rationally ergodic with respect to different types of return sequences (a_n), which are governed by the exponent of convergence δ and the maximal possible rank $k_{\max}$ of the parabolic elements of the group as follows

$$a_n \asymp \begin{cases} n^{2\delta - k_{\max}} & \text{for } \delta < (k_{\max}+1)/2, \\ n/\log n & \text{for } \delta = (k_{\max}+1)/2, \\ n & \text{for } \delta > (k_{\max}+1)/2. \end{cases}$$

Subsequently, we give a discussion of an associated canonical map T, which is an analogue of the Bowen–Series map in the Fuchsian case. We show that T is pointwise dual ergodic with respect to these return sequences (a_n), which then allows us to determine the index of variation $\beta = \min\{1, 2\delta - k_{\max}\}$, and to deduce that the ergodic sums $S_n(f)/a_n$ converge strongly distributional to the Mittag–Leffler distribution of index β. We then give applications to number theory and to the statistics of cuspidal windings. Also, as a corollary we obtain a special case of Sullivan's result that the geodesic flow on a geometrically finite hyperbolic manifold is ergodic with respect to the Liouville–Patterson measure.

0. *Introduction and statement of main results*

In this paper we study the action of a Kleinian group on its limit set by using methods from non-invertible infinite ergodic theory. Recall that a Kleinian group is a discrete subgroup

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of the group of orientation-preserving isometries of hyperbolic 3-space $\mathbb{H} = \mathbb{H}^3$ (for which we mainly use the Poincaré ball model equipped with the hyperbolic metric $d = d_{\mathbb{H}}$ (see, e.g., [**9**])). Throughout, we exclusively consider essentially free Kleinian groups G, that is we assume that G admits the choice of a Poincaré polyhedron F (see [**20**]) with finitely many faces such that if two faces s and t of F intersect inside $\mathbb{H}$, then the two associated generators g_s and g_t of G commute. By Poincaré's theorem (see [**12**]), we hence have that an essentially free Kleinian group has no relations other than those which originate from cusps of rank 2. Also, note that groups in this class are, in particular, geometrically finite. Our first aim is to construct a coding map T associated with G. This construction generalizes the well-known Bowen–Series map (see [**11, 28**]) to Kleinian groups of the second kind, that is to groups G whose limit set $L(G)$ does not coincide with the whole boundary $\partial\mathbb{H}$ of hyperbolic space. In particular, T is an endomorphism of the radial limit set $L_{\mathrm{r}}(G)$, which is the intersection of $L(G)$ with the complement of the set of parabolic fixed points of G.

In order to obtain a canonical T-invariant measure ν, we then employ the well-known Patterson measure and its associated Liouville–Patterson measure (see [**25, 32**]). More precisely, by specifying a Poincaré section, we show how to obtain a measure $\tilde{\nu}$ which is invariant under the first-return map S. The map S will also be referred to as a section map. It then turns out that T is a factor of S, and we obtain our measure ν by a straight-forward disintegration procedure.

The following theorem gives the main results of this paper. In here $k_{\max}$ refers to the maximal possible rank of the parabolic fixed points of G, and $\delta = \delta(G)$ denotes the exponent of convergence of G, that is the abzissa of convergence of the Poincaré series

$$\sum_{g\in G} \exp(-sd(0, g(0))).$$

MAIN THEOREM. *The coding map $T : L_{\mathrm{r}}(G) \to L_{\mathrm{r}}(G)$ is a topologically mixing Markov map which is conservative and ergodic with respect to ν. If G has no parabolic elements, then ν is finite. If G has parabolic elements, then ν is infinite if and only if $\delta \le (k_{\max}+1)/2$. Moreover, the following holds. Here $\widehat{T}$ refers to the dual of T.*

1. *If ν is finite then the return sequence (a_n) of T is given by Birkhoff's theorem. That is $a_n = n$ for all $n \in \mathbb{N}$ and:*
   - $\dfrac{1}{n}\displaystyle\sum_{k=0}^{n-1} f \circ T^k \to \frac{1}{\nu(L_{\mathrm{r}}(G))}\int f\,d\nu$ *for all $f \in L^1(\nu)$ and ν-a.e.;*
   - $\dfrac{1}{n}\displaystyle\sum_{k=0}^{n-1} \widehat{T}^k(f) \to \frac{1}{\nu(L_{\mathrm{r}}(G))}\int f\,d\nu$ *for all $f \in L^1(\nu)$ and ν-a.e.*
2. *If ν is infinite, then T is pointwise dual ergodic with respect to ν. In this case the return sequence (a_n) is given by*

$$a_n \asymp \begin{cases} n^{2\delta-k_{\max}} & \text{for } \delta < \dfrac{k_{\max}+1}{2}, \\ \dfrac{n}{\log n} & \text{for } \delta = \dfrac{k_{\max}+1}{2}. \end{cases}$$

In particular, we hence have (where Y_β refers to the Mittag–Leffler distribution of index $\beta := 2\delta - k_{\max}$ and 's.d.' refers to strong distributional convergence):

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- $\dfrac{1}{a_n}\displaystyle\sum_{i=0}^{n-1} f\circ T^i \xrightarrow{s.d.} Y_\beta \int f\,d\nu$ *for all* $f\in L^1(\nu)$ *and* $f\geq 0$*;*
- $\dfrac{1}{a_n}\displaystyle\sum_{k=0}^{n-1} \widehat{T}^k(f) \to \int f\,d\nu$ *for all* $f\in L^1(\nu)$ *and* ν*-a.e.*

Throughout, we use the notation $a \asymp b$ to describe that the ratio of two positive real numbers a and b is uniformly bounded away from zero and infinity. Also, we write $a \ll b$ if a/b is uniformly bounded from above. The reader might like to recall that Y_β is given in terms of its generating function by

$$E(e^{zY_\beta}) = \sum_{n=0}^{\infty} \frac{(\Gamma(1+\beta))^n}{\Gamma(1+\beta n)} z^n \quad \text{for } z\in\mathbb{C}.$$

In particular, for $\delta = (k_{\max}+1)/2$ we have that $\beta = 1$, and therefore that $Y_1 = 1$. Furthermore, for $\delta = (2k_{\max}+1)/4$ the corresponding distribution of $Y_{1/2}$ is half-Gaussian. Also, recall that the strong distributional convergence as stated in the theorem means that for every probability measure m absolutely continuous with respect to ν, we have the following weak convergence, where P_β denotes the distribution of Y_β (see [**2**, p. 112]),

$$m\circ\left(\frac{1}{a_n}\sum_{i=0}^{n-1} f\circ T^i\right)^{-1} \xrightarrow{\text{weak}} P_\beta \int f\,d\nu.$$

For the proof of the theorem, we first show that the coding map T is a topologically mixing Markov map (Proposition 2). We then give some estimates for the measure ν (Lemma 1), which then allow us to determine the wandering rate for a certain set B of finite measure (Theorem 1). This gives that the induced map T_B is well defined (implying that T is conservative), as well as the criterion for the finiteness of ν. We then show that T_B has the Gibbs–Markov property with respect to the measure ν restricted to B (Theorem 2), which then implies that T_B is ergodic and hence that T is ergodic. Furthermore, the Gibbs–Markov property gives that B is a Darling–Kac set and therefore T is pointwise dual ergodic (Theorem 3). Combining this with the estimates for the wandering rate of B, we obtain our estimates for the return sequence of T (Theorem 4) as well as the convergence to the Mittag–Leffler distribution (Corollary 8).

0.1. *Further conclusions.* We remark that, using standard techniques from infinite ergodic theory (see [**2**, §3.3]), our main theorem allows us to deduce the following interesting facts for the coding map T.

- The map T is rationally ergodic with respect to ν. That is, there exists a set $A\subset L_{\mathrm{r}}(G)$ with $0<\nu(A)<\infty$ such that for all $n\in\mathbb{N}$,
$$\int_A \left(\sum_{i=0}^{n-1} 1_A\circ T^i\right)^2 d\nu \ll \left(\int_A \sum_{i=0}^{n-1} 1_A\circ T^i\,d\nu\right)^2. \tag{$*$}$$
- The map T has the following mixing property. For every A with $\nu(A)<\infty$ such that $(*)$ holds, we have for all $U, V\subset A$,
$$\lim_{n\to\infty}\frac{1}{a_n}\sum_{i=0}^{n-1}\nu(U\cap T^{-i}V) = \nu(U)\nu(V). \tag{$**$}$$

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Also, our analysis of T has some interesting consequences for the section map S.

- The map S is the natural extension of T (Proposition 3). Therefore, S is a topologically mixing Markov map.
- The fact that S is invertible immediately implies that S cannot be pointwise dual ergodic (see [**2**]). However, since T is rationally ergodic and S is the natural extension of T, we nevertheless have that S is rationally ergodic with respect to the associated canonical measure $\tilde{\nu}$. Moreover, S has the mixing property as stated in $(**)$ with respect to $\tilde{\nu}$ and with respect to the same return sequence (a_n).
- Combining the previous remark, Proposition 1 and the fact that S is conservative and ergodic, a result of [**15**] gives that the geodesic flow on $\mathbb{H}/G$ is ergodic with respect to the Liouville–Patterson measure. This is a special case of Sullivan's result for general geometrically finite hyperbolic manifolds (see [**32**]).
- The fact that S is conservative implies that for A such that $\tilde{\nu}(A) > 0$, the induced map S_A is well defined. The map S_A is an alternative section map, and therefore it gives rise to another representation of the geodesic flow by means of a special flow. In particular, for a suitably chosen set A with $\tilde{\nu}(A) < \infty$, the map S_A preserves a finite measure and the associated Markov partition has infinitely many atoms.

We end this introduction by giving the following applications and remarks.

0.2. *Applications to elementary number theory.* For this we employ the well-known relation between the regular continued fraction expansions of real numbers and codings of geodesics on the three-sheated cover $\mathcal{M}$ of the modular surface (see, e.g., [**14, 21, 26**]). For $\mathcal{M}$ we have that the invariant measure is infinite and that the relevant Mittag–Leffler distribution is of index 1. Using the fact that the hyperbolic area of $\mathcal{M}$ is equal to π, a result of [**28, 29**] implies that the return sequence is given precisely by $a_n = n/(12 \log n)$. Combining this with Corollary 8, we obtain the following statement, where λ refers to the Lebesgue measure on the unit interval, $[a_1(\xi), a_2(\xi), a_2(\xi), \dots]$ refers to the regular continued fraction expansion of $\xi \in [0, 1] \setminus \mathbb{Q}$ and $\mathcal{C}_N(\xi, n) := \operatorname{card}\{i : a_i(\xi) \geq N, 1 \leq i \leq n\}$ denotes the number of entries of the expansion of ξ which are greater than or equal to $N \in \mathbb{N}$.

For every $N \in \mathbb{N}$ and $\epsilon > 0$, we have that

$$\lim_{n\to\infty} \lambda\left(\xi \in [0,1] : \left|\frac{\log \sum_{i=1}^{n} a_i(\xi)}{\sum_{i=1}^{n} a_i(\xi)} \mathcal{C}_N(\xi, n) - \frac{1}{3}\log\left(\frac{2N}{2N-1}\right)\right| < \epsilon\right) = 1.$$

0.3. *Applications to statistics of cuspidal windings.* As an example we consider a group G for which $\delta < (k_{\max}+1)/2$. Let K be the subset of a fundamental domain F of G which corresponds to the thick part of the convex core of $\mathbb{H}/G$. The ray from the origin to any $\xi \in L_{\mathrm{r}}(G)$ is clearly covered by the G-orbit of F. For $n \in \mathbb{N}$, let $k_n(\xi)$ be the number of visits to $G(K)$ after the ray has met n copies of F. Hence, $l_n(\xi) := n - k_n(\xi)$ denotes the number of cuspidal windings after n visits. An immediate application of the second part of our main theorem gives rise to the following result.

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There exist constants $c_1, c_2 > 0$ depending on K such that for the Patterson measure μ we have, for all $\theta \in (0, \infty)$,

$$P_{2\delta-k_{\max}}((0, c_1\theta)) \leq \lim_{n\to\infty} \mu\left(\left\{\xi \in L_{\mathrm{r}}(G) : \frac{k_n(\xi)}{l_n(\xi)^{2\delta-k_{\max}}} < \theta\right\}\right) \leq P_{2\delta-k_{\max}}((0, c_2\theta)).$$

Remark. Finally, we remark that results similar to the conclusions obtained in this paper were derived for parabolic rational maps in [**3**, §9]. There it was shown that a parabolic rational map T acting on the Julia set $J(T)$ of Hausdorff dimension h is conservative and ergodic with respect to an invariant measure, which is equivalent to the h-conformal measure. Furthermore, the map T was shown to be pointwise dual ergodic with respect to a certain return sequence (b_n). As in our case, these sequence depends on two parameters, namely the Hausdorff dimension h and the maximal number of petals of the parabolic periodic points $p_{\max}$. More precisely,

$$b_n \asymp \begin{cases} n^{h(p_{\max}+1)/p_{\max}-1} & \text{for } h < 2p_{\max}/(p_{\max}+1), \\ n/\log n & \text{for } h = 2p_{\max}/(p_{\max}+1), \\ n & \text{for } h > 2p_{\max}/(p_{\max}+1). \end{cases}$$

In particular, the invariant measure is finite if and only if $h > 2p_{\max}/(p_{\max}+1)$. Therefore, a combination of our analysis with the results of [**3**] gives a further extension of Sullivan's famous dictionary translating between the theories of Kleinian groups and rational maps (see [**33**]).

1. *Canonical coding and conformal measure theory*

Codings of geodesics on hyperbolic surfaces were originally studied by Artin, Hedlund and Morse (see, e.g., [**7, 16, 17, 22, 23**]), where the following two methods were considered. The first construction relies on the fact that, for a Fuchsian group G, any oriented geodesic whose endpoints are in $L_{\mathrm{r}}(G)$ is uniquely determined by its successive visits to G-images of a fundamental polygon of G. For the second approach, a coding map acting on a subset of $L_{\mathrm{r}}(G)$ has to be constructed such that this map and the group G are orbit-equivalent on the latter subset. Note that the basic example for this is the continued fraction transformation. In [**11**], coding maps for arbitrary cofinite Fuchsian groups were introduced and employed to derive a new proof of the well-known result of Hopf which states that the geodesic flow is conservative and ergodic. For geometrically finite Fuchsian groups a discussion of these two types of codings was given in [**27**]. In there, the first approach is realized as the first return map to the following canonical Poincaré section for the geodesic flow. This section is defined by those elements of the unit tangent bundle of the universal cover whose base points are in the boundary of some fundamental polygon, and which are pointing outwards. By the main result of [**27**], this first return map is conjugated to an invertible Markov map acting on a subset of $L_{\mathrm{r}}(G) \times L_{\mathrm{r}}(G)$, and the non-invertible factor of this map coincides with the map introduced in [**11**].

Observe that the choice of a different Poincaré section gives rise to a different coding according to the first method, and to a different coding map, if there exists a non-invertible

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factor of the first return map. This approach can be found, for instance, in [**13, 18, 26**], where different subsets of the canonical section are employed to construct a coding for the flow.

Our first aim is to construct the canonical Poincaré section for the geodesic flow associated with an essentially free Kleinian group G, which is inspired by the methods in [**5, 26, 28, 29**]. Subsequently, we shall see that the associated section map is a measure theoretical analogue of the coding obtained by the successive visits to copies of a fundamental domain. Also, it will turn out that a factor of this map is a two-dimensional version of the above coding map.

1.1. *Geodesic flow and Liouville–Patterson measure.* In order to introduce the Liouville–Patterson measure, we first have to recall the notion of the Patterson measure μ (for a detailed discussion of μ we refer to [**24, 25, 31, 32**]). It is well-known that μ is a probability measure supported on the limit set $L(G)$. Furthermore, the measure μ is δ-conformal, that is for all $g \in G$ and $\xi \in L(G)$ we have the following identity, which relates the Radon–Nikodym derivative to the Poisson kernel $\mathcal{P}$:

$$\frac{d\mu \circ g}{d\mu}(\xi) = \mathcal{P}(g^{-1}(0), \xi)^{\delta}.$$

Using the representation $\{(\xi, \eta, t) : \xi, \eta \in \partial\mathbb{H}, \xi \neq \eta, t \in \mathbb{R}\}$ for the unit tangent bundle $T^1\mathbb{H}$, the Liouville–Patterson measure $\tilde{\mu}$ is a straight-forward generalization of the Liouville measure. Namely, with λ referring to the Lebesgue measure on $\mathbb{R}$,

$$d\tilde{\mu}(\xi, \eta, t) := \frac{d\mu(\xi)\, d\mu(\eta)\, d\lambda(t)}{|\xi - \eta|^{2\delta(G)}}.$$

Since G is geometrically finite, the limit set $L(G)$ splits into the radial limit set $L_{\mathrm{r}}(G)$ and the set of bounded parabolic fixed points of G, where the latter set is at most countable (see [**10**]). Since μ is non-atomic, it then follows that $\mu(L_{\mathrm{r}}(G)) = 1$ and, therefore,

$$T^1\mathbb{H} \stackrel{\tilde{\mu}}{=} \{(\xi, \eta, t) : \xi, \eta \in L_{\mathrm{r}}(G), \xi \neq \eta, t \in \mathbb{R}\}.$$

Recall that G acts on $T^1\mathbb{H}$ and that $\tilde{\mu}$ is invariant under this action as well as under the action of the geodesic flow. This gives rise to the projected Liouville–Patterson measure, which is invariant under the geodesic flow $(\phi_t)_{t\in\mathbb{R}}$ on the quotient space $T^1\mathbb{H}/G$. For ease of notation, this projected measure will also be denoted by $\tilde{\mu}$.

1.2. *Canonical section maps.* Let F be the Poincaré polyhedron as stated in the definition of 'essentially free'. The polyhedron F gives rise to a fundamental domain for the action of G on $T^1\mathbb{H}$ as follows. For $\xi, \eta \in \partial\mathbb{H}$, let $\gamma_{\xi,\eta} : \mathbb{R} \to \mathbb{H}$ refer to the directed geodesic from η to ξ normalized such that $\gamma_{\xi,\eta}(0)$ is the summit of $\gamma_{\xi,\eta}$. We now have (where $\mathrm{Cl}(\cdot)$ denotes the closure in $\mathbb{H}$),

$$T^1\mathbb{H}/G \stackrel{\tilde{\mu}}{=} \mathcal{F}/G, \quad \text{where } \mathcal{F} := \{(\xi, \eta, t) : \xi, \eta \in L_{\mathrm{r}}(G), \xi \neq \eta, \gamma_{\xi,\eta}(t) \in \mathrm{Cl}(F)\}.$$

Note that, measure theoretically, $\mathcal{F}$ coincides with $\mathcal{F}/G$. The only ambiguity occurs at the boundary of $\mathcal{F}$, which is a set of measure zero. For $(\xi, \eta, t) \in \mathcal{F}$, the entrance time $t^-_{\xi,\eta}$

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and the exit time $t^+_{\xi,\eta}$ are defined by

$$t^-_{\xi,\eta} := \inf\{s : \gamma_{\xi,\eta}(s) \in \mathrm{Cl}(F)\} \text{ and } t^+_{\xi,\eta} := \sup\{s : \gamma_{\xi,\eta}(s) \in \mathrm{Cl}(F)\}.$$

Since $\xi, \eta \in L_{\mathrm{r}}(G)$, we clearly have that $|t^-_{\xi,\eta}| < \infty$ and $|t^+_{\xi,\eta}| < \infty$. We now define the set

$$Y := \{(\xi, \eta) : \xi, \eta \in L_{\mathrm{r}}(G), \xi \neq \eta \text{ and } \exists\, t \in \mathbb{R} : \gamma_{\xi,\eta}(t) \in \mathrm{Cl}(F)\},$$

as well as the measure $\tilde{\nu}$ on Y

$$d\tilde{\nu}(\xi, \eta) := \frac{d\mu(\xi)\, d\mu(\eta)}{|\xi - \eta|^{2\delta}}.$$

In order to construct a map $S : Y \to Y$, we use the combinatorial structure of F as follows. Recall that by Poincaré's polyhedron theorem (see [**12**]), the set $\mathcal{S}$ of faces of F carries an involution $\mathcal{S} \to \mathcal{S}$, which is given by $s \mapsto s'$ such that $s'' = s$. Also, for each $s \in \mathcal{S}$ there is a unique face-pairing transformation $g_s \in G$ such that $g_s(\mathrm{Cl}(F)) \cap \mathrm{Cl}(F) = \mathrm{Cl}(s')$. The map S is then defined by

$$S(\xi, \eta) := (g_s(\xi), g_s(\eta)) \quad \text{for all } (\xi, \eta) \in Y \text{ such that } \gamma_{\xi,\eta}(t^+_{\xi,\eta}) \in s, \text{ for some } s \in \mathcal{S}.$$

The following notion of a special flow and its section map is a measure-theoretical analogue of a suspension flow (see [**6**]). For $h : Y \to \mathbb{R}_+$ measurable, the special flow $(Y_h, (\psi_t)_{t\in\mathbb{R}}, \tilde{\nu} \times \lambda)$ over S with height function h is given by

$$Y_h := \{(y, \theta) : y \in Y, 0 \leq \theta < h(y)\} \quad \text{and} \quad \psi_t(y, \theta) := (S^n y, \theta + t - h_n(y)).$$

Here, the number $n \in \mathbb{Z}$ is uniquely determined, for $(y, \theta) \in Y_h$ and $t \in \mathbb{R}$, by

$$h_n(y) \leq \theta + t < h_{n+1}(y),$$

where we have set

$$h_n(y) := \begin{cases} 0 & \text{for } n = 0, \\ \displaystyle\sum_{k=0}^{n-1} h(S^k(y)) & \text{for } n \geq 1, \\ \displaystyle -\sum_{k=n}^{-1} h(S^k(y)) & \text{for } n < 0. \end{cases}$$

In this situation the map S is called a section map for (ψ_t). Throughout, we consider exclusively the special height function h, which is given by $h(\xi, \eta) := t^+_{\xi,\eta} - t^-_{\xi,\eta}$. We now have the following result.

PROPOSITION 1. *The geodesic flow $(T^1\mathbb{H}/G, (\phi_t)_{t\in\mathbb{R}}, \tilde{\mu})$ is measure-theoretically isomorphic to the special flow $(Y_h, (\psi_t)_{t\in\mathbb{R}}, \tilde{\nu} \times \lambda)$ over the section map S with the special height function h.*

Proof. The product structure of $\tilde{\mu}$ gives the measure-theoretical isomorphism

$$(T^1\mathbb{H}/G, \tilde{\mu}) \cong (\mathcal{F}, \tilde{\nu} \times \lambda).$$

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Since $\mathcal{F} = \{(\xi, \eta, t) \mid (\xi, \eta) \in Y, t^-_{\xi,\eta} \leq t \leq t^+_{\xi,\eta}\}$, we also have the measure-theoretical isomorphism

$$\mathcal{F} \to Y_h, \quad (\xi, \eta, t) \mapsto (\xi, \eta, t - t^-_{\xi,\eta}).$$

Hence, we have that $(T^1\mathbb{H}/G, \tilde{\mu}) \cong (Y_h, \tilde{\nu} \times \lambda)$. In order to obtain that the latter isomorphism transfers the two flows into each other, we have to remove the following flow invariant set of measure zero. This complication only occurs if G has parabolic elements of rank 2. Namely, we have to remove the set of geodesics on $\mathbb{H}/G$ that intersect the projections of the faces of F of codimension 2. This gives rise to a set of measure zero, since in the presence of a rank 2 cusp we always have that the Hausdorff dimension δ of $L(G)$ exceeds 1 (recall that by [**8**], we have that $\delta > k_{\max}/2$) and, therefore, the intersection of $L(G)$ with any arbitrary circle in $\partial\mathbb{H}$ is necessarily of Patterson measure zero. It follows that the section map S is well defined on a set of full measure and that the two flows coincide on the corresponding sets of full measure. □

Combining the latter proposition and results of [**6, 15**], we obtain the following result.

COROLLARY 1. *The measure $\tilde{\nu}$ is S-invariant. Moreover, the section map S is ergodic with respect to $\tilde{\nu}$ if and only if the geodesic flow is ergodic with respect to $\tilde{\mu}$.*

1.3. *Markov coding.* In this section we derive a Markov coding for the geodesic flow. For this we show that the section map S admits a Markov partition. We introduce the following collection of subsets of $\partial\mathbb{H}$. For $s \in \mathcal{S}$, let H_s refer to the closed hyperbolic halfspace for which $F \subset H_s$ and $s \subset \partial H_s$. We then define the projections a_s of the side s to $\partial\mathbb{H}$ by

$$a_s := (\mathrm{Cl}_{\overline{\mathbb{H}}}(H_s) \cap \partial\mathbb{H})^c.$$

If there are no parabolic fixed points of rank 2, then $a_s \cap a_t = \emptyset$ for all distinct $s, t \in \mathcal{S}$. Hence, by the convexity of F, we have $\gamma_{\xi,\eta}(t^+_{\xi,\eta}) \in s$ if and only if $\xi \in a_s$. In other words, $S(\xi, \eta) = (g_s\xi, g_s\eta)$ for all $\xi \in a_s$. This immediately gives that the projection π onto the first coordinate of Y leads to a canonical factor T of S, that is we obtain the map $T : L_{\mathrm{r}}(G) \to L_{\mathrm{r}}(G)$ which is given by $T|_{a_s \cap L_{\mathrm{r}}(G)} := g_s$ and which has the property that $\pi \circ S = T \circ \pi$. Since $T(a_s) = g_s(a_s) = (a_{s'})^{\mathbf{c}}$, it follows that T is a non-invertible Markov map with respect to the partition $\{a_s \cap L_{\mathrm{r}}(G) : s \in \mathcal{S}\}$. The image measure $\nu := \tilde{\nu} \circ \pi^{-1}$ is given by disintegration as follows, for all $s \in \mathcal{S}$ and $\xi \in a_s$:

$$d\nu(\xi) = \int_{a_s^c} \frac{d\mu(y)}{|\xi - y|^{2\delta}}\, d\mu(\xi).$$

Clearly, we have that ν is T-invariant. Moreover, it will turn out that $(Y, \tilde{\nu}, S)$ is the natural extension of $(L_{\mathrm{r}}(G), \nu, T)$. This will then imply that S is conservative and ergodic if and only if T has these properties.

If there are parabolic points of rank 2, then S has no canonical factor. In this situation the idea is to construct an invertible Markov map $\tilde{S}$ which is isomorphic to S and which has a canonical factor. We now introduce the following notation. Let P be the set of parabolic points given by $P := \mathrm{Cl}_{\overline{\mathbb{H}}} F \cap L(G)$, let $k(p)$ refer to the rank of $p \in P$,

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let $\mathcal{S}_p := \{s \in \mathcal{S} \mid p \in \mathrm{Cl}(a_s)\}$ and let $P_i := \{p \in P : k(p) = i\}$ for $i = 1, 2$. Moreover, we define the set $A \subset Y$ by

$$A := Y \setminus \{(\xi, \eta) \in Y : \xi \in a_s, \eta \in a_t \text{ where } s, t \in \mathcal{S}_p \text{ for some } p \in P_2\}.$$

For $(\xi, \eta) \in Y \setminus A$ there exists $p \in P_2$ and $s, t \in \mathcal{S}_p$ such that $\xi \in a_s$ and $\eta \in a_t$. Since p is a repelling fixed point for the action of g_s on a_s, one easily verifies that there exists $n \in \mathbb{N}$ such that $S^n(\xi, \eta) \in A$. Let $\phi_A^S : Y \to \mathbb{N}$ refer to the first return time to A, which is given for $(\xi, \eta) \in Y$ by

$$\phi_A^S(\xi, \eta) := \min\{n \in \mathbb{N} : S^n(\xi, \eta) \in A\}.$$

Since $\phi_A^S(\xi, \eta) < \infty$ for all $(\xi, \eta) \in Y$, we can now define the induced map $S_A : A \to A$ by $S_A(\xi, \eta) := S^{\phi_A^S(\xi,\eta)}(\xi, \eta)$.

In order to construct our Markov map $\tilde{S}$ associated with S, we introduce the convention

$$\tau : \bigcup_{p \in P_2} \{(s, t) \in \mathcal{S}_p \times \mathcal{S}_p : a_s \cap a_t \neq \emptyset, s \neq t\} \to \bigcup_{p \in P_2} \mathcal{S}_p,$$

where τ has the properties that $\tau(s, t) = \tau(t, s) \in \{s, t\}$, and $\tau(s, t) \neq \tau(s, t')$ for (s, t) in the domain of τ (see Figure 1). With this convention, the map $\tilde{S}$ is given by

$$\tilde{S}(\xi, \eta) := \begin{cases} (g_s\xi, g_s\eta) & \text{if } \xi \in a_s \setminus \bigcup_{t \in \mathcal{S}, t \neq s} a_t \text{ for some } s \in \mathcal{S}, \\ (g_{\tau(s,t)}\xi, g_{\tau(s,t)}\eta) & \text{if } \xi \in a_s \cap a_t \text{ for distinct } s, t \in \mathcal{S}. \end{cases}$$

Note that $\tilde{S}$ is defined on $L_{\mathrm{r}}(G) \times L_{\mathrm{r}}(G)$ and invertible on $\{(\xi, \eta) \in a_s \times a_t : s, t \in \mathcal{S}, a_s \cap a_t = \emptyset\}$. Moreover, the latter set is invariant under $\tilde{S}$. In order to relate $\tilde{S}$ to S, we restrict $\tilde{S}$ to the set

$$\tilde{Y} := \bigcup_{n \in \mathbb{Z}} \tilde{S}^n A.$$

By construction we then have that $\tilde{S}(\tilde{Y}) = \tilde{Y}$ and that $\tilde{S}|_{\tilde{Y}}$ is invertible. Also, by the same argument as for S above, the first return time $\phi_A^{\tilde{S}}$ of $\tilde{S}$ to A is finite. It is then easy to see that $\phi_A^{\tilde{S}} = \phi_A^S$ and that the induced map $\tilde{S}_A$ coincides with S_A. Combining these observations, it now follows that S and $\tilde{S}$ are isomorphic, and that the invariant measure for $\tilde{S}$ is given also by $d\mu(\xi)\, d\mu(\eta)/|\xi - \eta|^{2\delta}$. We are now able to extend our coding map T and its associated measure ν to the case in which there are rank 2 cusps. Namely, we define

$$T : L_{\mathrm{r}}(G) \to L_{\mathrm{r}}(G)$$
$$\xi \mapsto \begin{cases} g_s\xi & \text{if } \xi \in a_s \setminus \bigcup_{t \in \mathcal{S}, t \neq s} a_t \text{ for some } s \in \mathcal{S}, \\ g_{\tau(s,t)}\xi & \text{if } \xi \in a_s \cap a_t \text{ for distinct } s, t \in \mathcal{S}, \end{cases}$$
$$d\nu(\xi) := \int_{\{y : (\xi, y) \in \tilde{Y}\}} \frac{d\mu(y)}{|\xi - y|^{2\delta}}\, d\mu(\xi).$$

PROPOSITION 2. *The map T is a topological mixing Markov map with respect to the partition α which is generated by $\{a_s \cap L_{\mathrm{r}}(G) : s \in \mathcal{S}\}$. Furthermore, the measure ν is invariant under T.*

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Proof. Using the fact that two faces $s, t \in \mathcal{S}$ intersect if and only if $g_s(p) = g_t(p) = p$ for some $p \in P_2$, we have the following complete description of the behaviour of an arbitrary atom b of the partition α.

Case 1. If $b = a_s$ for some $s \notin \bigcup_{p\in P_2} \mathcal{S}_p$, then $T(b) = g_s(a_s) = (a_{s'})^{\mathbf{c}} = \bigcup_{t\neq s'} a_t$.

Case 2. If $b = a_s \setminus \bigcup_{t\neq s} a_t$ for some $s \in \mathcal{S}_p$ and $p \in P_2$, then $T(b) = (a_{s'})^{\mathbf{c}} \setminus (a_u \cup a_{u'})$. Here u refers to an element of $\mathcal{S}_p$ for which $u \neq s, s'$. Note that in this situation we have that $g_s(a_u) = a_u$ and $g_s(a_{u'}) = a_{u'}$. This follows since the stabilizer G_p of p is Abelian and is generated by g_s and g_u. Furthermore, we have that $b = a_s \setminus (a_u \cup a_{u'})$, and hence $T(b) = g_s(a_s \setminus (a_u \cup a_{u'})) = (a_{s'})^{\mathbf{c}} \setminus (a_u \cup a_{u'})$.

Case 3. If $b = a_s \cap a_t$ for some distinct $s, t \in \mathcal{S}_p$ and for some $p \in P_2$, then $T(b) = (a_{\tau(s,t)'})^{\mathbf{c}} \cap a_u$. Here, u refers to an element of $\mathcal{S}_p$ for which $u \neq \tau(s, t)$ and $u \in \{s, t\}$. The fact that $T(b)$ is of this particular form follows as in the previous case.

It follows that $T(b)$ is measurable with respect to the σ-field generated by α, for each $b \in \alpha$. Also, recall that for every sequence (g_n) of pairwise disjoint elements of G, the Euclidean diameter of $g_n(F)$ tends to zero for n tending to infinity. Therefore, the diameters of the atoms of the refined partition $\bigvee_{k=0}^{n-1} T^{-k}\alpha$ (generated by $T^{-k}(\alpha)$ for $0 \leq k < n$) tend to zero for n tending to infinity. Combining these observations with the fact that $T|_b$ is injective for every $b \in \alpha$, it follows that T has the Markov property.

Finally, note that the incidence graph of T is explicitly given by the above description of the images of the atoms of α. One easily verifies that this graph is aperiodic, which then implies that T is topologically mixing. □

In order to link the properties of T and $\tilde{S}$, we show that the map S is the natural extension of T. For this we have to show that the projection π has the following two properties (see, e.g., [**2**]).

(NE1) $\pi \circ \tilde{S} = T \circ \pi$ and $\tilde{\nu} \circ \pi^{-1} = \nu$.

(NE2) $\bigvee_{n=1}^{\infty} \tilde{S}^n(\pi^{-1}\alpha)$ is the Borel σ-field of $\tilde{Y}$ up to sets of measure zero.

Also, recall the notion of a cylinder set associated with α. Namely, for $b_0, \ldots, b_n \in \alpha$ such that $T(b_i) \subset b_{i+1}$ for all i, let $[b_1 \ldots b_n]$ refer to the cylinder of length n which is given by

$$[b_1 \ldots b_n] := \{\xi \in L_{\mathrm{r}}(G) : T^{i-1}\xi \in b_i \text{ for all } i = 1, \ldots, n\}.$$

PROPOSITION 3. *The map $\tilde{S}$ is the natural extension of T.*

Proof. By the construction of T and ν, the properties in (NE1) are clearly satisfied. In order to obtain (NE2), let $[b_1 \ldots b_n]$ be some arbitrary cylinder, and let $s_i \in \mathcal{S}$ refer to the face for which $T|_{b_i} = g_{s_i}$, for each $i \in \{1, \ldots n\}$. We then have, for $m < n$,

$$\begin{aligned}\tilde{S}^m \circ \pi^{-1}[b_1 \ldots b_n] &\subset T^m([b_1 \ldots b_n]) \times g_{s_m} \cdots g_{s_1}(a_{s_1}^{\mathbf{c}}) \\ &= [b_{m+1} \ldots b_n] \times g_{s_m} \cdots g_{s_1} g_{s_1'}(a_{s_1'}) \\ &= [b_{m+1} \ldots b_n] \times (g_{s_2'} \cdots g_{s_m'})^{-1}(a_{s_1'}).\end{aligned}$$

Clearly, for suitably chosen $n, m \in \mathbb{N}$, the Euclidean diameter of the latter expression tends to zero for n, m tending to infinity. □

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Recall that the natural extension of the system $(L_{\mathrm{r}}(G), \nu, T)$ is uniquely determined up to an isomorphism. Moreover, by a standard result in infinite ergodic theory (see, e.g., [**2**]) we have the following corollary.

COROLLARY 2. *The map T is conservative and ergodic with respect to ν if and only if S has these properties with respect to $\tilde{\nu}$. Moreover, T is rationally ergodic with respect to some return sequence if and only if S is rationally ergodic with respect to the same return sequence.*

2. *Non-invertible ergodic theory*

2.1. *Wandering rates.* In this section we obtain some estimates for the wandering rates of the T-invariant measure ν. For this we first give some estimates for the measure ν on certain fundamental cells associated with the stabilizers of the parabolic points. Note that throughout this section we always assume that the elements of P are fixed points of T. This can be assumed without loss of generality, since otherwise one simply replaces T by a suitable power of T.

Let d_E refer to the Euclidean distance on the unit sphere. For $p \in P$, let G_p denote the stabilizer of p, and let Q_p be defined by

$$Q_p := \bigcap_{u \in \mathcal{S}_p} a_u^{\mathbf{c}} \cap L_{\mathrm{r}}(G).$$

LEMMA 1. *Let $h \in G_p \setminus \{\mathrm{id}\}$ such that $d_E(p, h(Q_p)) \asymp 1/n$ for some $n \in \mathbb{N}$.*

- *If $p \in P_1$, then we have*

$$\nu(h(Q_p)) \asymp \frac{1}{n^{2\delta-1}}.$$

- *If $p \in P_2$, then we have*

$$\nu(h(Q_p)) \asymp \begin{cases} \dfrac{1}{n^{2\delta-1}} & \text{if } h(Q_p) \subset a_s \cap a_t \text{ for distinct } s, t \in \mathcal{S}_p, \\ \dfrac{1}{n^{2\delta-2}} & \text{otherwise.} \end{cases}$$

Proof. Let h be given as stated in the lemma. For $k(p) = 1$ and $s \in \mathcal{S}$ such that $h(Q_p) \subset a_{s'}$, we have

$$\{(\xi, \eta) \in \tilde{Y} : \xi \in h(Q_p)\} = h(Q_p) \times \left(\left(\bigcup_{u \notin \mathcal{S}_p} a_u \cup \bigcup_{m=1}^{\infty} g_s^m(Q_p) \right) \cap L_{\mathrm{r}}(G) \right).$$

Using a well-known estimate from hyperbolic geometry (see [**30**, Lemma 2]), it follows that if $\xi \in h(Q_p)$ and $\eta \in g_s^m(Q_p)$, for some $m \in \mathbb{N}$, then $|\xi - \eta| \asymp 1/n + 1/m$. Also, by a result of [**31**, Lemma 3.1] we have that $\mu(g_s^m(Q_p)) \asymp m^{-2\delta}$.

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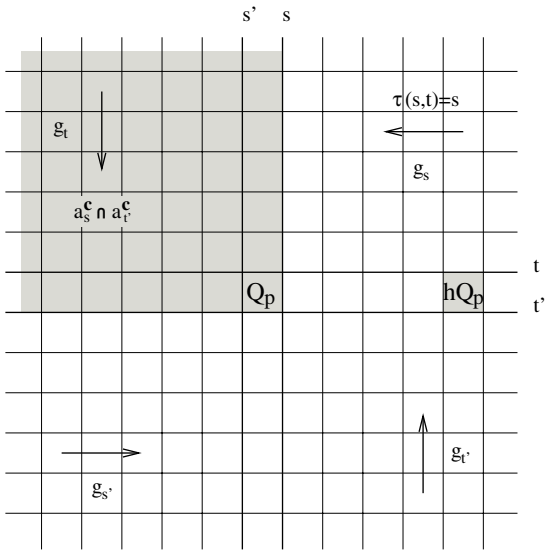


FIGURE 1. A rank 2 cusp at $p = \infty$.

Combining these observations, we obtain

$$\begin{aligned} \nu(h(Q_p)) &= \tilde{\nu}\Big(h(Q_p) \times \bigcup_{u \notin \mathcal{S}_p} a_u\Big) + \sum_{m \in \mathbb{N}} \tilde{\nu}(h(Q_p) \times g_s^m(Q_p)) \\ &\asymp \mu(h(Q_p)) + \sum_{m \in \mathbb{N}} \left(\frac{1}{n} + \frac{1}{m}\right)^{-2\delta} \mu(h(Q_p))\mu(g_s^m(Q_p)) \\ &\asymp \sum_{k=n}^{\infty} k^{-2\delta} + n^{-2\delta} \asymp n^{-2\delta+1}. \end{aligned}$$

For $k(p) = 2$, we first consider the case in which $h(Q_p) \subset a_s \cap a_t$ for distinct $s, t \in \mathcal{S}_p$. Assume that $\tau(s, t) = s$. We then have

$$\{(\xi, \eta) \in \tilde{Y} : \xi \in h(Q_p)\} = h(Q_p) \times \Big(\bigcap_{u \in \mathcal{S}_p \setminus \{s'\}} a_u^{\mathbf{c}} \cap L_{\mathrm{r}}(G)\Big).$$

Clearly, we can now employ a similar argument as above to obtain the same estimate.

For the remaining case we have that $h(Q_p) \subset \bigcap_{u \in \mathcal{S}_p \setminus \{s\}} a_u^{\mathbf{c}} \setminus Q_p$ for some $s \in \mathcal{S}_p$. Again, for ease of exposition, we assume that $\tau(s, t) = s$. In this situation we have that (see Figure 1)

$$\{(\xi, \eta) \in \tilde{Y} : \xi \in h(Q_p)\} = h(Q_p) \times (a_s^{\mathbf{c}} \cap a_{t'}^{\mathbf{c}} \cap L_{\mathrm{r}}(G)).$$

Using the above observations concerning the geometry and the Patterson measure in a

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neighbourhood of p, we now have

$$\begin{aligned}\nu(h(Q_p)) &= \tilde{\nu}(h(Q_p) \times (a_s^{\mathbf{c}} \cap a_{t'}^{\mathbf{c}})) \\ &= \sum_{\substack{g \in G_p \\ g(Q_p) \subset a_s^{\mathbf{c}} \cap a_{t'}^{\mathbf{c}}}} \tilde{\nu}(h(Q_p) \times g(Q_p)) \\ &\asymp \sum_{m \in \mathbb{N}} \left(\frac{1}{n} + \frac{1}{m}\right)^{-2\delta} \frac{m}{m^{2\delta}} \frac{1}{n^{2\delta}} \\ &\asymp n^{-2\delta+2}. \end{aligned}$$

□

Recall that the wandering rate of ν with respect to a set $A \subset L_{\mathrm{r}}(G)$ is given by

$$w_n(A) := \nu\left(\bigcup_{m=0}^{n} T^{-m}(A)\right).$$

For our purposes it is sufficient to investigate the behaviour of ν around each of the parabolic points. Therefore, we introduce the following set, for $p \in P$ and $n \in \mathbb{N}$,

$$B_p(n) := \bigcup_{m=0}^{n-1} \bigcup_{*_1} T_{\omega,p}^{-m}(Q_p),$$

where $\bigcup_{*_1}$ refers to the union over all inverse branches $T_{\omega,p}^{-m}$ of T^m fixing p. Also, let $B \subset L_{\mathrm{r}}(G)$ refer to the set which is given, for $n_0 \in \mathbb{N}$ sufficiently large, by

$$B := \bigcap_{p \in P} B_p(n_0) \cap L_{\mathrm{r}}(G).$$

Since B is bounded away from P, we have that $0 < \nu(B) < \infty$. Furthermore, by construction, one easily verifies that

$$\bigcup_{m=0}^{\infty} T^{-m}(B) = L_{\mathrm{r}}(G).$$

Note that later we shall see that the set B is a Darling–Kac set for T, which will then imply that T is pointwise dual ergodic with respect to ν. For the following theorem, we consider the wandering rate $(w_n^p(B))$ associated with $p \in P$, which is given by

$$w_n^p(B) := \nu(B_p(n_0 + n)).$$

THEOREM 1. *With the notation above we have, for all $n \in \mathbb{N}$ and $p \in P$,*

$$w_n^p(B) \asymp \begin{cases} n^{k(p)-2\delta+1} & \text{for } \delta < (k(p)+1)/2, \\ \log n & \text{for } \delta = (k(p)+1)/2, \\ 1 & \text{for } \delta > (k(p)+1)/2. \end{cases}$$

In particular, we hence have for the wandering rate $w_n(B) := \sum_{p \in P} w_n^p(B)$ of B,

$$w_n(B) \asymp \begin{cases} n^{k_{\max}-2\delta+1} & \text{for } \delta < (k_{\max}+1)/2, \\ \log n & \text{for } \delta = (k_{\max}+1)/2, \\ 1 & \text{for } \delta > (k_{\max}+1)/2. \end{cases}$$

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Proof. First note that for $p \in P_2$ we have by Lemma 1, for each $m \in \mathbb{N}$,

$$\sum_{\substack{h\in G_p,\\ d_E(p,h(Q_p))\asymp 1/m}} \nu(h(Q_p)) = \sum_{*_2} \nu(h(Q_p)) + \sum_{*_3} \nu(h(Q_p)) \asymp \sum_{*_2} \frac{1}{m^{2\delta-1}} + \frac{4}{m^{2\delta-2}} \asymp \frac{1}{m^{2\delta-2}}.$$

Here $\sum_{*_2}$ refers to the summation over all elements $h \in G_p$ for which $d_E(p, h(Q_p)) \asymp 1/m$ and $h(Q_p) \subset a_s \cap a_t$ for distinct $s, t \in \mathcal{S}_p$ (clearly, $\sum_{*_3}$ then refers to the summation over the remaining summands). Using this observation and once more Lemma 1, we now have

$$\begin{aligned}
\nu\Big(\bigcup_{m=0}^{n} T^{-m}B\Big) &= \nu(B) + \sum_{p\in P}\sum_{m=n_0}^{n_o+n}\sum_{*_1} \nu(T_{\omega,p}^{-m}(Q_p)) \\
&\asymp \sum_{p\in P}\sum_{m=n_0}^{n_0+n} \sum_{\substack{h\in G_p,\\ d_E(p,h(Q_p))\asymp 1/m}} \nu(h(Q_p)) \\
&\asymp \sum_{p\in P_1}\sum_{m=n_0}^{n_0+n} \sum_{\substack{h\in G_p,\\ d_E(p,h(Q_p))\asymp 1/m}} \frac{1}{m^{2\delta-1}} + \sum_{p\in P_2}\sum_{m=n_0}^{n_0+n} \frac{1}{m^{2\delta-2}} \\
&\asymp \sum_{p\in P}\sum_{m=n_0}^{n_0+n} \frac{m^{k(p)-1}}{m^{2\delta-1}} \\
&\asymp \begin{cases} n^{k_{\max}-2\delta+1} & \text{for } \delta < (k_{\max}+1)/2, \\ \log n & \text{for } \delta = (k_{\max}+1)/2, \\ 1 & \text{for } \delta > (k_{\max}+1)/2. \end{cases}
\end{aligned}$$

□

As an immediate consequence of the latter theorem, we obtain the following result of [**19**].

COROLLARY 3. *The measure ν is infinite if and only if $\delta \le (k_{\max}+1)/2$. Moreover, if G has parabolic elements of rank 1 as well as of rank 2, then ν gives finite mass to small neighbourhoods of the parabolic fixed points of rank 1.*

2.2. *Gibbs–Markov property and ergodicity.* In this section we give a finer analysis for the induced map T_B. We show that T_B has the Gibbs–Markov property (see [**2, 4, 28, 29**]) with respect to ν_B, where ν_B refers to the restriction of ν to B. Using standard results from ergodic theory, this then allows to deduce that T_B is ergodic and that T is pointwise dual ergodic and conservative.

Recall that by definition T_B has the Gibbs–Markov property with respect to ν_B if and only if there exists a number $\theta \in (0, 1)$ such that the following holds.

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For arbitrary cylinders ω_1 of length n and ω_2 of length m such that $[\omega_2] \subset T_B^n([\omega_1])$, we have for ν_B-almost every pair $x, y \in [\omega_2]$,

$$\left|\log \frac{d\nu_B \circ T_{B,\omega_1}^{-n}}{d\nu_B}(x) - \log \frac{d\nu_B \circ T_{B,\omega_1}^{-n}}{d\nu_B}(y)\right| \ll \theta^m.$$

Here T_{B,ω_1}^{-n} refers to the inverse branch of T_B^n which maps $T_B^n([\omega_1])$ to $[\omega_1]$.

THEOREM 2. *The map T_B has the Gibbs–Markov property with respect to the measure ν_B.*

Proof. Let ω_1, ω_2 be given as above. We then have, for $x, y \in [\omega_2]$,

$$\begin{aligned}\log \frac{((d\nu \circ T_{B,\omega_1}^{-n})/d\nu)(x)}{((d\nu \circ T_{B,\omega_1}^{-n})/d\nu)(y)} &= \log \frac{((d\nu \circ T_{B,\omega_1}^{-n})/d\mu \circ T_{B,\omega_1}^{-n})(x)}{((d\nu \circ T_{B,\omega_1}^{-n})/d\mu \circ T_{B,\omega_1}^{-n})(y)} \\ &\quad + \log \frac{((d\mu \circ T_{B,\omega_1}^{-n})/d\mu)(x)}{((d\mu \circ T_{B,\omega_1}^{-n})/d\mu)(y)} + \log \frac{(d\mu/d\nu)(x)}{(d\mu/d\nu)(y)}.\end{aligned}$$

We split the estimate of the modulus of this expression into separate parts according to the three summands in the latter expression.

For the third summand, let $C_0 := \max\{|z-w|^{-1} : z \in a_t \cap B,\ w \in (a_t)^{\mathbf{c}}, t \in \mathcal{S}\}$. For $s \in \mathcal{S}$ such that $[\omega_1] \subset a_s$, we then have

$$\begin{aligned}&\left|\log \frac{d\nu}{d\mu}(x) - \log \frac{d\nu}{d\mu}(y)\right| \\ &\quad = \left|\log \int_{\{\eta:(\xi,\eta)\in\tilde{Y}\}} \frac{1}{|x-\eta|^{2\delta}}\, d\mu(\eta) - \log \frac{d\nu}{d\mu}(y)\right| \\ &\quad \le \left|\log \int_{\{\eta:(\xi,\eta)\in\tilde{Y}\}} \frac{1}{|y-\eta|^{2\delta}} \left(\frac{|x-\eta|+|x-y|}{|x-\eta|}\right)^{2\delta} d\mu(\eta) - \log \frac{d\nu}{d\mu}(y)\right| \\ &\quad \le \left|\log \left((1+C_0|x-y|)^{2\delta} \int_{\{\eta:(\xi,\eta)\in\tilde{Y}\}} \frac{1}{|y-\eta|^{2\delta}}\, d\mu(\eta)\right) - \log \frac{d\nu}{d\mu}(y)\right| \\ &\quad \ll |x-y|.\end{aligned}$$

For the first summand, note that for $z \in [\omega_1]$ we have $(d\nu/d\mu)(z) = (d\nu \circ T_{B,\omega_1}^{-n}/d\mu \circ T_{B,\omega_1}^{-n})(T_B^n z)$. Therefore, for a suitably chosen $\tilde{\omega}_1$ of length n, the modulus of the first summand is bounded from above by $|T_{B,\tilde{\omega}_1}^{-n}(x) - T_{B,\tilde{\omega}_1}^{-n}(y)|$, which is clearly less than $|x-y|$.

For the second summand, let g_{ω_1} be the element of G which corresponds to T_{B,ω_1}^{-n}. Now observe that by the triangle inequality we have $|g_{\omega_1}^{-1}(0)-x| \le |g_{\omega_1}^{-1}(0)-y|+|x-y|$. Moreover, by construction of B, there exists a constant $C_1 > 0$ such that $|g_{\omega_1}^{-1}(0)-z|^{-1} < C_1$, for all $z \in [\omega_2]$. Therefore, it follows that

$$\frac{|g_{\omega_1}^{-1}(0)-y|}{|g_{\omega_1}^{-1}(0)-x|} < 1 + \frac{|x-y|}{|g_{\omega_1}^{-1}(0)-x|} < 1 + C_1|x-y|.$$

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Combining this observation with the δ-conformality of μ, we now have

$$\begin{aligned}\left|\log\frac{((d\mu\circ g_{\omega_1})/d\mu)(x)}{((d\mu\circ g_{\omega_1})/d\mu)(y)}\right| &= \delta\left|\log\frac{\mathcal{P}(g_{\omega_1}^{-1}(0),x)}{\mathcal{P}(g_{\omega_1}^{-1}(0),y)}\right|\\ &= \delta\left|\log\frac{|g_{\omega_1}^{-1}(0)-y|}{|g_{\omega_1}^{-1}(0)-x|}\right|\\ &< \delta\,|\log(1+C_1|x-y|)|\\ &\ll |x-y|.\end{aligned}$$

Finally, by the uniform expansiveness of T_B, there exists a constant $\theta\in(0,1)$ depending on the choice of B such that $|x-y|<\theta^m$. This completes the proof. □

The following two statements can be deduced immediately from the proof of the previous proposition. They are not essential for the purposes of this paper, nevertheless they might be of interest elsewhere.

COROLLARY 4. *The map T_B has the Gibbs–Markov property with respect to the Patterson measure restricted to B.*

COROLLARY 5. *The logarithm of the density $d\nu/d\mu$ is Lipschitz continuous on B with respect to d_E.*

The Gibbs–Markov property of T_B with respect to ν_B allows to employ the following standard chain of arguments from ergodic theory. Namely, it is well-known that for the following three statements we have (i) ⇒ (ii) ⇒ (iii) ⇒ (iv) (see [**2, 3, 34**]).

(i) T_B has the Gibbs–Markov property with respect to the invariant measure ν_B.

(ii) For the dual operator $\widehat{T}_B$ there exists $\rho\in(0,1)$ such that for all $f\in L^1(B)$ and $n\in\mathbb{N}$, we have
$$\left\|\widehat{T}_B^n f-\int_B f\,d\nu\right\|_{\mathrm{L}}\ll\rho^n\|f\|_{\mathrm{L}}.$$
Here $\|\cdot\|_{\mathrm{L}}$ refers to the Lipschitz norm (see, e.g., [**3**, p. 541]).

(iii) T_B is continued fraction mixing (see, e.g., [**3**, p. 500]).

(iv) The set B is a Darling–Kac set for T. This means that there exists a sequence $(a_n)_{n\in\mathbb{N}}$ such that
$$\frac{1}{a_n}\sum_{i=0}^{n-1}\widehat{T}^i 1_B(x)\to\nu(B)\quad\text{uniformly for }\nu\text{-a.e. } x\in B.$$
The sequence (a_n) is usually referred to as the return sequence of T.

Using [**2**, Proposition 3.7.5], we can now complete this chain of arguments and obtain the following result.

THEOREM 3. *The map T is pointwise dual ergodic with respect to ν. That is, with $(a_n)_{n\in\mathbb{N}}$ referring to the return sequence of T, we have*

$$\frac{1}{a_n}\sum_{i=0}^{n-1}\widehat{T}^i f\to\int f\,d\nu\quad\textit{for all } f\in L^1(\nu).$$

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In particular, we also have the following immediate consequences (see [**2**]).

COROLLARY 6. *The map T is rationally ergodic with respect to ν. That is, there exists a set $A \subset L_{\mathrm{r}}(G)$ with $0 < \nu(A) < \infty$ such that for all $n \in \mathbb{N}$,*

$$\int_A \left(\sum_{i=0}^{n-1} 1_A \circ T^i \right)^2 d\nu \ll \left(\int_A \sum_{i=0}^{n-1} 1_A \circ T^i \, d\nu \right)^2. \tag{$*$}$$

COROLLARY 7. *The map T has the following mixing property. For A with $\nu(A) < \infty$ such that $(*)$ holds, we have for all $U, V \subset A$,*

$$\lim_{n\to\infty} \frac{1}{a_n} \sum_{i=0}^{n-1} \nu(U \cap T^{-i} V) = \nu(U)\nu(V).$$

The results of Theorem 1 concerning the wandering rate $w_n(B)$ of B now allow us to determine the return sequence (a_n) of T explicitly. For this, recall that a sequence (s_n) is called a regularly varying sequence with index of variation κ if $\lim_{m\to\infty} s_{nm}/s_m = n^\kappa$, for all $n \in \mathbb{N}$. Using Theorem 1, one easily verifies that $(w_n(B))$ is a regularly varying sequence with index of variation α given by

$$\alpha = \max\{0, k_{\max} - 2\delta + 1\}.$$

Since T is pointwise dual ergodic and B is a Darling–Kac set, we may now apply a result of [**2**, Proposition 3.8.7], which gives

$$a_n \, w_n(B) \sim \frac{n}{\Gamma(2-\alpha)\Gamma(1+\alpha)}.$$

Therefore, combining this result and our estimates for $w_n(B)$ in Theorem 1, we now obtain the following theorem.

THEOREM 4. *For the return sequence (a_n) of T we have, for each $n \in \mathbb{N}$,*

$$a_n \asymp \begin{cases} n^{2\delta - k_{\max}} & \text{for } \delta < (k_{\max}+1)/2, \\ n/\log n & \text{for } \delta = (k_{\max}+1)/2, \\ n & \text{for } \delta > (k_{\max}+1)/2. \end{cases}$$

Clearly, the sequence a_n is also a regularly varying sequence. A straight-forward calculation shows that its index of variation β is given by

$$\beta = \min\{1, 2\delta - k_{\max}\}.$$

Applying a result of [**1**, Theorem 1], we can now conclude that for $\delta \le (k_{\max}+1)/2$, the ergodic sums $S_n(f)/a_n$ have the following remarkable statistical behaviour. Note that for $\delta > (k_{\max}+1)/2$, the measure ν is finite and, hence, the behaviour of these sums is given by Birkhoff's theorem as stated in the first part of our main theorem.

COROLLARY 8. *If $\delta \le (k_{\max}+1)/2$ then the ergodic sums $S_n(f)/a_n$ converge strongly distributional to the Mittag–Leffler distribution Y_β of index β, for each $f \in L^1_+(\nu)$. That is, we have*

$$\frac{1}{a_n} \sum_{i=0}^{n-1} f \circ T^i \xrightarrow{s.d.} Y_\beta \int_{L_{\mathrm{r}}(G)} f \, d\nu.$$

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Note that the upper bound of the domain where these distributional laws hold is precisely $(k_{\max}+1)/2$. By a result of [**19**], this value coincides with the value at which the system exhibits a thermodynamical phase transition.

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