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The simplicial extension theorem

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Familiarity with the category of simplicial sets and the Milnor realization functor, as in, e.g. (5), is assumed. We say $f\colon |Y| \to |Z|$ is *realized* if $f = |g|$ for some g.

Simplicial Extension Theorem. Let $X \subset Y$ and Z be simplicial sets, with Z Kan, and suppose given $f\colon |Y| \to |Z|$ so that $f\,||X|$ is realized. Then there exists a homotopy $f \simeq f' \operatorname{rel} |X|$ so that f' is realized.

The proof given below uses a minimal amount of simplicial technique, and avoids the usual (see (2)) appeal to ordinary simplicial approximations coupled with delicate notions of subdivision. Subdivision in the semisimplicial context is a rather unnatural notion and has given considerable trouble in the past, see (1), (3), (4), (7). In fact, we deduce a relative simplicial approximation theorem:

Corollary.† *Let $L \subset K$ and M be finite simplicial complexes and suppose given $f\colon |K| \to |M|$ so that $f\,||L|$ is simplicial. Then there exists an integer r, an rth derived K_r of K, and a homotopy $f \simeq f' \operatorname{rel} L$ so that $f'\colon K_r \to M$ is simplicial.*

Proof of Theorem. By an induction over skeletons, it is enough to prove the theorem in the case Y is an n-simplex Δ_n and X is its boundary $\dot{\Delta}_n$. Further, we can suppose Z is connected. Choose a base point $* \subset Z$. The case $n = 1$ is easily dealt with so suppose $n > 1$.

We find $g\colon |\Delta_n \times \Delta_1| \to |Z|$ extending f on $|\Delta_n| = |\Delta_n \times 1|$ and so that the restriction to $|\Delta_n \times 0 \cup \dot{\Delta} \times \Delta_1|$ is realized. The restriction may then be extended to a realized map g' on $|\Delta_n \times \Delta_1|$, since Z is Kan. Combining g with g' gives the required homotopy. It remains to construct g.

Let $\Lambda_n^k \subset \Delta_n$ be the kth horn of Δ_n, and define $A = \Lambda_n^0 \times 0 \cup 0 \times \Delta_1 \cup \dot{\Delta}_n \times 1$. Then $A \subset \Delta_n \times \Delta_1$. It follows from connectivity of Z that there is a map $h_0\colon A \to Z$ so that $h_0(\Lambda_n^0 \times 0) \subset *$ and $|h_0|$ agrees with f on $|\dot{\Delta}_n \times 1|$. Since Z is Kan, h_0 may be extended to $h_1\colon \dot{\Delta}_n \times \Delta_1 \to Z$. Now comes the 'minimal' technique. Replace $* \subset Z$ by a minimal fibration $p\colon E \to Z$. Then, in particular, E and $|E|$ are contractible and $|p|\colon |E| \to |Z|$ is locally trivial and therefore a Serre fibration. Lift $h_1\,|\,\Lambda_n^1 \times 0 \to Z$ to $l_1\colon \Lambda_n^1 \times 0 \to E$, then lift $|h_1|$ on $\dot{\Delta}_n \times \Delta_1$ and f on $|\Delta_n \times 1|$ to get $l_2\colon |\dot{\Delta}_n \times \Delta_1 \cup \Delta_n \times 1| \to |E|$ extending $|l_1|$. Now $l_2|\colon |\delta_1 \Delta_n \times 0| \to |F|$, where F is the fibre of p, and, by induction, there is a homotopy H of $l_2\,|\bmod |(\delta_1 \Delta_n)^{\cdot}|$ to some realized map. Combine H and l_2 to find a map $l_4\colon |\dot{\Delta}_n \times \Delta_1 \cup \Delta_n \times 1| \to |E|$ so that in particular $l_4\,|\dot{\Delta}_n \times 0$ is realized and $l_4\,|\Delta_n \times 1|$

† For the original proof, see (8). The definition of K_r used here will be slightly different from that of (8), but the theorem of (8) is easily deduced from the corollary here.

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covers f. Use the fact that E is contractible to extend l_4 to

$$l_5: |\Delta_n \times 0 \cup \dot{\Delta}_n \times \Delta_1 \cup \Delta_n \times 1| \to |E|$$

so l_5 is realized on $\Delta_n \times 0$. Then use the fact that $|E|$ is contractible to extend finally to $l_6: |\Delta_n \times \Delta_1| \to |E|$. The required g' is then a slight modification of $|p|\, l_6$.

Proof of Corollary. For our purposes, a derived subdivision K_1 of K is defined as follows. Let ${}^1A, {}^2A, \ldots, {}^vA$ be some of the simplexes of $K - L$ listed in order of increasing dimension so that if ${}^iA < B$ then $B =$ some ${}^jA$. Choose $a_i \in {}^i\mathring{A}\, i = 1, \ldots, v$ and replace each ${}^iA$ in turn by the cone $a_i({}^i\dot{A})_1$.

Now suppose given an elementary collapse of M to $J \subset M$ across B^n from A^{n-1}. Choose $b \in \mathring{B}$ and $a \in \mathring{A}$. Then we have a derived subdivision M_1 of M, so that $J \subset M_1$, and a simplicial retraction $s: M_1 \to J$ defined by $s(a) = s(b) =$ (vertex of B opposite A).

Suppose given a simplicial map $t: K \to M$, then by choosing a suitable interior point for each of those simplexes which map to A or B we have a simplicial map $t_1: K_1 \to M_1$. Further $|st_1| \simeq |t| \operatorname{rel} t^{-1}J$.

Now recall the process of making a simplicial set X Kan (p. 68 of (5)). For each k, n and $u: \Lambda_n^k \to X$ attach a copy of Δ_n. Iterate the construction indefinitely and the result is a Kan complex, $\Lambda(x)$.

The proof can now be completed as follows. Totally order the vertices of M. Then via f, the vertices of L receive a partial ordering. Choose a total ordering of the vertices of K, compatible with the given partial ordering. Let $\underline{L}, \underline{K}, \underline{M}$ be the associated simplicial sets. Then $|\underline{L}| = |L|$ etc. and the theorem can be applied to give a realized map $f'': |\underline{K}| \to |\Lambda(\underline{M})|$. By the technique indicated above, find a further homotopy

$$f'' \simeq f' \operatorname{rel} |\underline{L}|$$

so that f' is as required. Finally find the required homotopy in $|\underline{M}|$ by using a strong deformation retraction of $|\Lambda(\underline{M})|$ onto $|\underline{M}|$.

Remark. The well known equivalence (via. the singular and realization functors) of the homotopy categories of simplicial sets and *cw* complexes may be quickly deduced from the extension theorem together with Whitehead's theorem (isomorphism on π_* implies homotopy equivalence) in the *cw* category.

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