BOUNDED-WEAK TOPOLOGIES AND COMPLETENESS IN VECTOR SPACES

By G. T. ROBERTS

Received 17 July 1952

1. Suppose that X is any vector space on which it is possible to recognize a class of sets of such a nature that it is natural to call them 'bounded'. (Precise conditions for such a class of sets are given in § 2.) Let L be any vector space of linear functionals on X which map each 'bounded' set into a bounded set. We say that a filter $\mathscr{F}$ is *boundedly-weakly convergent* if it is convergent in the weak topology of the linear system $[X, L]$ and contains a 'bounded' set. If M is a vector space of boundedly-weakly continuous linear functionals on X which includes L, we say that a subset S of M is *limited* if $\langle \mathscr{F}, f \rangle$ converges uniformly for $f \in S$ whenever $\mathscr{F}$ is a boundedly-weakly convergent filter. We will show that the set of all polar sets of limited sets forms a base of neighbourhoods of the origin for the strongest of a class of topologies on X that induces the same topology on each 'bounded' set as that induced by the weak topology of the linear system $[X, L]$. M can be topologized by taking the polar sets of all 'bounded' sets as a base of neighbourhoods of the origin. The principal result of this paper is that, providing the weak closure of any 'bounded' set is 'bounded', the limited sets are just the totally-bounded sets of M in this topology. This result is a generalization of (3) Theorem 1, (4) Théorème 5, and (6) Satz 14. We deduce from this and a result due to Grothendieck ((5), Proposition 2, Corollaire 3) that under certain conditions the completion of M is the union of the weak closures in the space of all linear functionals on X of the totally-bounded subsets of M.

2. Throughout this paper X will denote an abstract real or complex vector space of any dimension. X may be topologized; but no topology is implied in the symbol X. X^* will denote the set of all scalar-valued linear functionals on X. L and M will be vector subspaces of X^*, which will always be assumed to be total sets of functionals on X. Thus X and L (or M) form a linear system in the sense of Mackey(7), which we will denote by $[X, L]$ (or $[X, M]$). We will identify X with its canonical image in L^* or M^* without further comment. If S is a subset of X (or L), the subset S^0 of L (or X) consisting of those elements f such that $\sup |\langle f, S \rangle| \leqslant 1$ will be called the *polar of S in L* (or X). The polar in L (or X) of the polar in X (or L) of S will be called the *polar closure of S with respect to* $[X, L]$. If this is equal to S, S is said to be *polar closed*.

Lemma 1. *If S, S_1 and S_2 are subsets of X and $\{S_\alpha : \alpha \in A\}$ is any family of such sets,*

(i) *S^0 is polar closed,*

(ii) *if $S_1 \subseteq S_2$, $S_2^0 \subseteq S_1^0$,*

(iii) $(\bigcup \{S_\alpha : \alpha \in A\})^0 = \bigcap \{S_\alpha^0 : \alpha \in A\}$,

where the polarity is with respect to the members of the same linear system throughout.

https://doi.org/10.1017/S0305004100028267 Published online by Cambridge University Press

These results follow easily from the definitions.

We say that a subset S of X is *absolutely convex*, if for any pair of scalars λ and μ such that $|\lambda|+|\mu| = 1$ and any pair of elements x and y of S $\lambda x+\mu y \in S$. (Such a set is often called 'convex and circled'.) It is obvious that any polar closed set is absolutely convex.

The weak topology of $[X, L]$ will be called the $w(X, L)$-*topology*. If t is the name of a topology, the terms t-closure, t-convergent, t-compact (= bi-compact), t-totally-bounded (= précompact), etc., will be used in the obvious sense. X^t will denote the set of all t-continuous linear functionals on X. All topologies of vector spaces will be assumed to be such that all the vector operations are continuous mappings of the appropriate product space into the vector space.

LEMMA 2. *The polar closure of S with respect to $[X, L]$ is the $w(X, L)$-closure of the absolutely convex hull of S. This is equal to the t-closure of the absolutely convex hull of S providing $X^t = L$ and t is a locally convex topology.*

Proof. The first part of this lemma is (4), Proposition 1. The second is a consequence of (2), Theorem 7.

A set of subsets $\mathscr{B}$ of X is said to define a *notion of boundedness*, if it satisfies the following axioms:

B 1. If $B_1 \in \mathscr{B}$ and $B_2 \subseteq B_1$, $B_2 \in \mathscr{B}$.

B 2. If B_1 and B_2 are in $\mathscr{B}$, $B_1 \cup B_2 \in \mathscr{B}$.

B 3. If $B \in \mathscr{B}$, the absolutely convex hull of B is in $\mathscr{B}$.

B 4. $\bigcup \{B: B \in \mathscr{B}\} = X$.

B 5. If $B \in \mathscr{B}$ and λ is any scalar, $\lambda B \in \mathscr{B}$.

If a set of subsets $\mathscr{B}'$ of X is such that the set of all subsets of sets of $\mathscr{B}'$ defines a notion of boundedness, we say that $\mathscr{B}'$ is a *base* for this notion of boundedness. The principal example of a notion of boundedness is given by taking $\mathscr{B}$ to be the set of all t-bounded sets, where t is some locally convex topology of X. Other examples are the set of all t-totally-bounded sets and those given by bases consisting of all absolutely convex t-compact sets or the absolutely convex hulls of all finite sets. The conditions B 1 to B 5 are easily verified for all these notions of boundedness except the axiom B 3 for t-totally-bounded sets.

LEMMA 3. *If N is an absolutely convex neighbourhood of the origin and S has a finite $\frac{1}{2}N$-covering, the absolutely convex hull of S has a finite N-covering.*

Proof. Let $\{x_j + \frac{1}{2}N : 1 \leqslant j \leqslant n\}$ be the finite $\frac{1}{2}N$-covering of S. Now we can find a positive integer p_j such that $x_j \in \frac{1}{4}p_j N$ for each j. It is easy to verify that the sets

$$\frac{s_{ju}+it_{ju}}{p_j}x_j+\frac{s'_{kv}+it'_{kv}}{p_k}+N,$$

where j and k run from 1 to n, s_{ju} and t_{ju} run through all integers such that $s_{ju}^2+t_{ju}^2 \leqslant p_j^2$ and s'_{kv} and t'_{kv} run through all integers such that $s'_{kv}+t'_{kv} \leqslant p_k^2$ form a finite N-covering of the absolutely convex hull of S, if the scalar field is complex, and that, if the scalars are real, the same formula without the t_{ju} and t'_{kv} serves.

https://doi.org/10.1017/S0305004100028267 Published online by Cambridge University Press

COROLLARY. *If t is a locally convex topology, the set of all t-totally-bounded sets satisfies* B 3.

If there is a notion of p-boundedness on X, we will denote the set of all elements of X^* which map all p-bounded sets into bounded sets by X^P. When it is desirable to emphasize the space X on which a notion of p-boundedness is formed we shall call it the notion of $p(X)$-boundedness. If there is a notion of $p(X)$-boundedness and L is a subspace of X^P, we can form a locally convex topology on L by taking the polars of all $p(X)$-bounded sets (or a base of the notion of $p(X)$-boundedness) as a base of neighbourhoods of the origin. Call this topology the $p_1(L, X)$-topology. If Y is a subset of $L^{P_1(L,X)}$, call the topology of Y given by a base of neighbourhoods of the origin consisting of the polars of $p_1(X, L)$-bounded sets the $p_2(Y, L, X)$-topology.

3. THEOREM 1. *If $p(X)$ is a notion of boundedness and L a subspace of X^P, the $p_1(L, X)$-completion of L is the subspace M of X^*, such that each $f \in M$ is $w(X, L)$-continuous on each $p(X)$-bounded set.*

Proof. This result is due to Grothendieck ((5), Proposition 2, Corollaire 3). We will, however, give a different proof.

If $g \in M$ and P is a $p(X)$-bounded set, there is a functional f in L such that

$$|\langle g, x\rangle| \leqslant |\langle f, x\rangle|$$

for all $x \in P$, since g is $w(X, L)$-continuous on P. This inequality still holds on the linear span of P. Thus by the Hahn-Banach theorem we can extend the restriction of g to the span of P to a functional g_P on the whole of X so that $|\langle g_P, x\rangle| \leqslant |\langle f, x\rangle|$ for all $x \in X$. This inequality implies that g_P is $w(X, L)$-continuous on the whole of X and so $g_P \in L$. The axioms B 1 and B 2 imply that the complements of the $p(X)$-bounded sets form a filter. If the complement of P lies in the complement of P_0,

$$\sup |\langle P_0, g_P - g\rangle| = 0,$$

since $g_P = g$ on P and so also on P_0. Hence $g_P - g \in P_0^0$ and g_P $p_1(M, X)$-converges to g as the complement of P runs through the filter of complements of $p(X)$-bounded sets. Thus L is $p_1(M, X)$-dense in M.

If $\mathscr{F}$ is any $p_1(M, X)$-Cauchy filter, $\langle \mathscr{F}, x\rangle$ is a uniformly Cauchy filter of continuous functions on each $p(X)$-bounded set and so is uniformly convergent to some continuous function on each $p(X)$-bounded set. Thus $\mathscr{F}$ is $p_1(M, X)$-convergent to an element of M and M is $p_1(M, X)$-complete.

If $\mathscr{F}$ is a filter on X, $p(X)$ a notion of boundedness and L a subspace of X^P, we say that $\mathscr{F}$ is *$p(X)$-boundedly-$w(X, L)$-convergent* if $\mathscr{F}$ is $w(X, L)$-convergent and contains a $p(X)$-bounded set. A subset S of a subspace M of the $p_1(L, X)$-completion of L such that $M \supseteq L$ is said to be *$p_1(M, X, L)$-limited* if and only if $\sup |\langle S, \mathscr{F}\rangle| \to 0$ for all filters $\mathscr{F}$ which are $p(X)$-boundedly-$w(X, L)$-convergent to zero. Since by Theorem 1 the $w(X, L)$- and $w(X, M)$-topologies induce the same topology on each $p(X)$-bounded set, a set is $p_1(M, X, L)$-limited if and only if it is $p_1(M, X, M)$-limited. Accordingly, we shall simplify the notation to $p_1(M, X)$-limited. It is evident that, if M is the $p_1(L, X)$-completion of L, a set is $p_1(L, X)$-limited if and only if it is a subset of L which is $p_1(M, X)$-limited.

https://doi.org/10.1017/S0305004100028267 Published online by Cambridge University Press

The set of all $p_1(L, X)$-limited sets gives a notion of boundedness in L. The conditions B 1, B 2, B 4 and B 5 are obviously satisfied. If S is $p_1(L, X)$-limited, it is evident from the mode of definition that the polar closure of S with respect to $[X, L]$ is $p_1(L, X)$-limited. Hence by Lemma 2 the absolutely convex hull of S is $p_1(L, X)$-limited and the condition B 3 is satisfied.

Define the $q(Y, L, X)$-*topology* as that given by a base of neighbourhoods of the origin consisting of all polar sets in Y of $p_1(L, X)$-limited sets, where Y is a subspace of L^* such that every $p_1(L, X)$-limited set is mapped into a bounded set by each functional in Y.

THEOREM 2. *The $q(X, L, X)$-topology is the strongest topology of X given by a base of neighbourhoods of the origin consisting of polars of sets in L that induces the same topology on each $p(X)$-bounded set as that induced by the $w(X, L)$-topology.*

Proof. Clearly a filter on a $p(X)$-bounded set is $w(X, L)$-convergent if and only if it is $q(X, L, X)$-convergent. Hence the $w(X, L)$- and $q(X, L, X)$-topologies induce the same topology on each $p(X)$-bounded set.

Conversely, if t is a stronger topology of X with a base of polars of sets in L, there is a set S in L such that S^0 is a t-nucleus but S is not $p_1(L, X)$-limited. Thus there is a $p(X)$-bounded set P and a filter $\mathscr{F}$ on P which is $w(X, L)$-convergent to the origin, such that $\sup |\langle S, \mathscr{F}\rangle|$ does not tend to zero. Then for some $\epsilon > 0$ and some refinement $\mathscr{G}$ of $\mathscr{F}$ $\sup |\langle S, G\rangle| \leqslant \epsilon$ for all $G \in \mathscr{G}$. Hence no set $G \in \mathscr{G}$ is included in the t-neighbourhood of the origin ϵS^0, and so $\mathscr{F}$ is not t-convergent to the origin. Thus the t-topology induces a stronger topology on P than that induced by the $w(X, L)$-topology.

In the next theorem it is necessary to restrict L to those subspaces of X^* such that the $w(X, L)$-closure of any $p(X)$-bounded set is $p(X)$-bounded. This condition is satisfied by any L which includes X^t, when the notion of $p(X)$-boundedness is that of t-boundedness or t-total-boundedness or is given by a base of all t-compact absolutely convex sets. The condition is not satisfied if the notion of $p(X)$-boundedness is given by a base consisting of the absolutely convex hulls of finite sets, unless $L = X^*$. This condition does not impose any restriction on the locally convex topologies that can be obtained as the $p_1(L, X)$-topology, since it is always satisfied by taking $X = L^t$ and the notion of $p(X)$-boundedness to be that given by t-equicontinuous sets of functionals, where t is any locally convex topology of L. The $p_1(L, X)$-topology is then equivalent to the t-topology.

THEOREM 3. *If L is any subspace of X^P such that the $w(X, L)$-closure of any $p(X)$-bounded set is $p(X)$-bounded, a set is $p_1(L, X)$-limited if and only if it is $p_1(L, X)$-totally-bounded.*

Proof. We will first prove that all $p_1(L, X)$-compact sets are $p_1(L, X)$-limited. Let K be a $p_1(L, X)$-compact set which is not $p_1(L, X)$-limited. Then there is a filter $\mathscr{F}$ on some $p(X)$-bounded set which is $w(X, L)$-convergent to the origin such that $\sup |\langle K, \mathscr{F}\rangle|$ does not converge to zero. By refining this filter and multiplying it by a scalar we can obtain a filter $\mathscr{G}$ on a $p(X)$-bounded set which is $w(X, L)$-convergent to the origin and such that $\sup |\langle K, G\rangle| > 1$ for each $G \in \mathscr{G}$. Thus K is not included in

https://doi.org/10.1017/S0305004100028267 Published online by Cambridge University Press

G^0 and so $K \cap \mathscr{C}(G^0)$ is not empty for any $G \in \mathscr{G}$ ($\mathscr{C}(G^0)$ denotes the complement of G^0). Now, if G_1 and G_2 are members of $\mathscr{G}$,

$$\mathscr{C}(G_1^0) \cap \mathscr{C}(G_2^0) = \mathscr{C}(G_1^0 \cup G_2^0) \supseteq \mathscr{C}((G_1^0 \cup G_2^0)^{00}) = \mathscr{C}((G_1^{00} \cap G_2^{00})^0) \supseteq \mathscr{C}((G_1 \cap G_2)^0),$$

where the second identity is a consequence of Lemma 1 (iii). Hence $\{\mathscr{C}(G) : G \in \mathscr{G}\}$ is a filter base on L and so $\{K \cap \mathscr{C}(G) : G \in \mathscr{G}\}$ is a filter base on K. Since K is $p_1(L, X)$-compact this filter must have a $p_1(L, X)$-adherent point f. Since every set $G \in \mathscr{G}$ is a $p(X)$-bounded subset of X, $\frac{1}{2}G^0$ is a $p_1(L, X)$-neighbourhood of the origin and so we can find a functional g in $K \cap \mathscr{C}(G^0)$ such that $\sup |\langle f-g, G\rangle| \leqslant \frac{1}{2}$. Now $\sup |\langle g, G\rangle| > 1$, for if not $g \in G^0$. Hence $\sup |\langle f, G\rangle| > \frac{1}{2}$, which is contrary to the assumption that $\mathscr{F}$ is $w(X, L)$-convergent to the origin. Thus every $p_1(L, X)$-compact set is $p_1(L, X)$-limited.

Let M denote the $p_1(L, X)$-completion of L. Then any $p_1(L, X)$-totally-bounded set is a subset of a $p_1(M, X)$-compact set, and so by what we have just proved is a subset of a $p_1(M, X)$-limited set. Thus every $p_1(L, X)$-totally-bounded set is $p_1(L, X)$-limited.

We will now prove the converse. If S is any $p_1(L, X)$-limited set, S^0 is a $q(X, L, X)$-neighbourhood of the origin. Now the $q(X, L, X)$-topology induces a topology not stronger than that induced by the $w(X, L)$-topology on each $p(X)$-bounded set. Hence $S^0 \supseteq F^0 \cap 2P$ for some finite subset F of L for each $p(X)$-bounded polar closed set P. Hence

$$S \subseteq S^{00} \subseteq (F^0 \cap 2P)^0 = (F^{000} \cap 2P^{00})^0 = (F^{00} \cup \tfrac{1}{2}P^0)^{00},$$

by Lemma 1. Now, if a set has a finite $\frac{1}{2}P^0$-covering, by Lemma 3 its absolutely convex hull has a finite P^0-covering. As this covering consists of $w(L, X)$-closed sets, it also covers the $w(L, X)$-closure of the absolutely convex hull of the set, which by Lemma 2 is its polar closure. Thus, since F trivially has a finite $\frac{1}{4}P^0$-covering and $\frac{1}{2}P^0$ has a finite $\frac{1}{2}P^0$-covering, $(F^{00} \cup \frac{1}{2}P^0)^{00}$ has a finite P^0-covering. Now the sets P^0 where P runs through all polar closed $p(X)$-bounded sets form a base of neighbourhoods of the origin for the $p_1(L, X)$-topology, since the polar closed $p(X)$-bounded sets form a base for the notion of $p(X)$-boundedness. Thus S has a finite N-covering for each $p_1(L, X)$-neighbourhood of the origin N and so is $p_1(L, X)$-totally-bounded.

COROLLARY. *If every $q(X, L, X)$-convergent sequence and the $w(X, L)$-closure of every $p(X)$-bounded set is $p(X)$-bounded, the $p_1(L, X)$-completion of L lies in $X^{p_2(X, L, X)}$.*

Proof. Since the continuous functionals are determined by the convergent sequences, the $q(X, L, X)$-continuous linear functionals are those that are $q(X, L, X)$-continuous on each $p(X)$-bounded set. By Theorem 2 these are just those that are $w(X, L)$-continuous and so by Theorem 1 $X^{q(X, L, X)}$ is the $p_1(L, X)$-completion of L. On the other hand, since every totally-bounded set is bounded Theorem 3 implies that $X^{q(X, L, X)} \subseteq X^{p_2(X, L, X)}$†.

Since every convergent sequence is compact, it is sufficient for the validity of this corollary that every $q(X, L, X)$-compact set is $p(X)$-bounded or, since the $q(X, L, X)$-

† The hypothesis that every $q(X, L, X)$-convergent sequence is $p(X)$-bounded was erroneously omitted from the original draft of this paper in this and subsequent results. It was added in November 1952.

https://doi.org/10.1017/S0305004100028267 Published online by Cambridge University Press

topology is stronger than the $w(X, L)$-topology, that every $w(X, L)$-compact set is $p(X)$-bounded.

LEMMA 4. *$X^{p_1(L,X)}$ is the union of the $w(L^*, L)$-closures of all the $p(X)$-bounded sets.*

Proof. If $y \in L^*$ is $p_1(L, X)$-continuous, $|\langle y, f\rangle| \leqslant 1$ for all $f \in P^0$ for some $p(X)$-bounded set P. Thus $y \in P^{00}$ where the polarity is with respect to $[L, L^*]$. Now the absolutely convex hull of P is $p(X)$-bounded by B 3 and so P^{00} is the $w(L^*, L)$-closure of some $p(X)$-bounded set by Lemma 2.

Conversely, if y is in the $w(L^*, L)$-closure of a $p(X)$-bounded set P, $y \in P^{00}$. Thus $|\langle y, f\rangle| \leqslant \epsilon$ for all $f \in \left(\frac{1}{\epsilon}P\right)^0$. Now $\frac{1}{\epsilon}P$ is $p(X)$-bounded by B 5 and so $\left(\frac{1}{\epsilon}P\right)^0$ is a $p_1(L, X)$-neighbourhood of the origin. Thus y is $p_1(L, X)$-continuous.

THEOREM 4. *If the $w(L, X)$-closure of every $p(X)$-bounded set and every $q(X, L, X)$-convergent sequence is $p(X)$-bounded, the $p_1(L, X)$-completion of L is the union of the $w(X^*, X)$-closures of all $p_1(L, X)$-totally-bounded sets.*

Proof. As in the corollary to Theorem 3 the $p_1(L, X)$-completion of L is $X^{q(X,L,X)}$. By Lemma 4 and Theorem 3 this is the union of the $w(X^*, X)$-closures of all $p_1(L, X)$-totally-bounded sets.

COROLLARY. *If the $w(X, L)$-closure of every $p(X)$-bounded set and every $q(X, L, X)$ convergent sequence is $p(X)$-bounded and $L^{p_1(L,X)} = X$, L is $p_1(L, X)$-complete if and only if every $p_1(L, X)$-totally-bounded set is $p_1(L, X)$-relatively-compact.*

Proof. Let M denote the $p_1(L, X)$-completion of L. If T is any $p_1(L, X)$-totally-bounded absolutely convex set, the $p_1(L, X)$-closure of T is $p_1(L, X)$-compact and so $p_1(M, X)$-closed. It is absolutely convex and so by Lemma 2 is $w(M, M^{p_1(M,X)})$-closed. Now $M^{p_1(M,X)} = L^{p_1(L,X)} = X$, since L is $p_1(M, X)$-dense in M. Thus the $w(M, X)$-closure of every $p_1(L, X)$-totally-bounded set lies in L and so by the theorem L is $p_1(L, X)$-complete. The converse is trivial.

4. Let us call the $p_1(L, X)$-topology obtained by taking the notion of $p(X)$-boundedness to be that given by the base consisting of all absolutely convex $w(L, X)$-compact sets the $k(L, X)$-*topology*. By the Mackey-Arens Theorem ((1), Theorem 2) this is the strongest locally convex topology of L whose continuous linear functionals are contained in X. Let the $b(M, X, L)$- and $f(Y, M, X, L)$-*topologies* be the $p_1(M, X)$- and $p_2(Y, M, X)$-topologies respectively when $p(X)$-boundedness is $w(X, L)$-boundedness. Then, since a set is $k(L, X)$-bounded if and only if it is $w(L, X)$-bounded, the $k(L, X)$-completion of L lies in $X^{b(X,L,X)}$ by the corollary to Theorem 3, and by the corollary to Theorem 4 L is $k(L, X)$-complete if and only if every $k(L, X)$-totally-bounded set is $k(L, X)$-relatively compact. The corollary to Theorem 3 also tells us that the $b(L, X, L)$-completion of L lies in $X^{f(X,L,X,L)}$. This result is stronger than that whereby the $b(L, X, L)$-completion of L lies in $X^{w(L,X)}$ (which is what is called in (7) the bounded closure of L), since it is easy to see that a set is $f(X, L, X, L)$-bounded if and only if it is $w(X, L)$-bounded and so $X^{f(X,L,X,L)} = X^{w(X,L)}$. If t is a topology of X and $L = X^t$, this result implies that the completion of the strong conjugate of X consists of functionals on X that are continuous in the strong topology of the second conjugate of X, since this is the $f(X^{b(X^t,X,X^t)}, X^t, X, X^t)$-topology.

https://doi.org/10.1017/S0305004100028267 Published online by Cambridge University Press

REFERENCES

(1) Arens, R. *Duke math. J.* 14 (1947), 787–94.
(2) Bourgin, D. C. *Amer. J. Math.* 65 (1943), 637–59.
(3) Dieudonné, J. *Proc. Amer. math. Soc.* 1 (1950), 54–9.
(4) Dieudonné, J. and Schwartz, L. *Ann. Inst. Fourier*, 1 (1949), 61–101.
(5) Grothendieck, A. *C.R. Acad. Sci., Paris*, 230 (1950), 605–6.
(6) Koethe, G. *Math. Nachr.* 4 (1950), 70–80.
(7) Mackey, G. W. *Trans. Amer. math. Soc.* 57 (1945), 155–207.

King's College
Cambridge

https://doi.org/10.1017/S0305004100028267 Published online by Cambridge University Press