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The Intersection Exponent for Simple Random Walk

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The intersection exponent ξ for simple random walk in two and three dimensions gives a measure of the rate of decay of the probability that paths do not intersect. In this paper we show that the intersection exponent for random walks is the same as that for Brownian motion and show in fact that the probability of nonintersection up to distance n is comparable (equal up to multiplicative constants) to $n^{-\xi}$.

1. Introduction

The intersection exponent is a measure of the rate of decay of the probability of non-intersection of independent paths of simple random walks. Let $S_t, S_t^1, S_t^2, \cdots$ be independent simple random walks in $\mathbf{Z}^d$ $(d = 2, 3)$ starting at the origin. The intersection exponent $\xi = \xi_d(1, k)$ is defined by the relation

$$\mathbf{P}\{S[0, n^2] \cap (S^1(0, n^2] \cup \cdots \cup S^k(0, n^2]) = \emptyset\} \approx n^{-\xi}.$$

This exponent is only nontrivial for $d = 2, 3$, and that is why we restrict to those values. If $d = 1$, the gambler's ruin estimate for simple random walk shows that $\xi_1(1, k) = k$. For $d \geqslant 4$, the probability decays more slowly than a power law (and, in fact, for $d \geqslant 5$, does not even go to zero); see [12] and references therein. This exponent is closely related to a similar exponent for Brownian motion, and it has been shown [4] that the two exponents are the same. A more general intersection exponent can be defined as follows (we will be

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more precise below). Let

$$Y_n = Y_{n,k} = \mathbf{P}\{S[0,n^2] \cap (S^1(0,n^2] \cup \cdots \cup S^k(0,n^2]) = \emptyset \mid S[0,n^2]\}.$$

Then, for $\lambda > 0$, the exponent $\xi = \xi_d(\lambda, k)$ is defined by

$$\mathbf{E}[Y_n^\lambda] \approx n^{-\xi}.$$

If λ is an integer, this is the exponent for λ walks avoiding k walks. This definition also works for $\lambda = 0$ if we make the convention that $0^0 = 0$, that is,

$$\mathbf{P}[Y_n > 0] \approx n^{-\xi_d(0,k)}.$$

This more general intersection exponent arises in analysing the fine structure of the boundary of a path $S[0,n^2]$; see [19] for a discussion of the multifractal spectrum of harmonic measure of a simple random walk. In this paper we extend the results of [4, 20] to show that this more general intersection exponent is the same as the corresponding exponent for Brownian motion. The results in this paper are used in [19].

As well as being important for studying the fine behaviour of random walk paths, the intersection exponent is of interest in critical phenomena in statistical physics. There is a number of systems, for instance nonintersecting random walks, percolation, self-avoiding walks, loop-erased random walks, and the Ising model, that are relatively simple to define on the discrete level and have nontrivial exponents below the critical dimension. In two dimensions it is conjectured that the systems 'in the continuum limit at criticality' have conformal invariance properties. Ideas of conformal field theory have been used to give nonrigorous conjectures for the exponents (see, *e.g.*, [5, 7, 8, 9, 10, 11]); in particular, Duplantier and Kwon [8] were the first to use these ideas to conjecture values for the intersection exponent. Recent mathematical progress in trying to verify these conjectures has been focused on understanding the possible continuum limits (see, *e.g.*, [2, 22, 23, 24]). In particular (see [23]), it appears as if the exponents for nonintersecting random walks are closely related to exponents for percolation and self-avoiding walks. There is hope that mathematicians will be able to construct and analyse candidates for the continuum limits for the discrete models. Even if this is done, there will remain the hard question of showing that the discrete models have the same continuum limit. The importance of this paper is that it shows that this is the case for nonintersecting random walks – the continuum limit for nonintersecting random walks, that is, nonintersecting Brownian motions, does give the correct exponents for the discrete model.

We start by defining the exponents for Brownian motion. Let $B^1, \ldots, B^k$ be independent d-dimensional Brownian motions ($d = 2, 3$) defined on a probability space $(\Omega, \mathbf{P})$, and let B be another d-dimensional Brownian motion defined on another probability space $(\Omega_1, \mathbf{P}_1)$. Assume for now that the Brownian motions start at the origin. Define the stopping times

$$T_n^i = \inf\{t : |B^i(t)| = e^n\},$$

$$T_n = \inf\{t : |B(t)| = e^n\},$$

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and let

$$\Lambda(m,n) = B^1[T_m^1, T_n^1] \cup \cdots \cup B^k[T_m^k, T_n^k],$$
$$\Lambda_n = \Lambda(0,n),$$

and

$$Z_n = \mathbf{P}_1\{B[T_0, T_n] \cap \Lambda_n = \emptyset\}.$$

The intersection exponent $\xi = \xi_d(\lambda, k)$ is defined for $\lambda > 0$ by the relation

$$\mathbf{E}[Z_n^\lambda] \asymp e^{-n\xi}, \quad n \to \infty, \tag{1.1}$$

where the symbol $\asymp$ means that both sides are comparable, that is, bounded by a constant times the other side. It is not difficult, using Brownian scaling and subadditivity, to show that ξ is well defined in the sense that

$$\mathbf{E}[Z_n^\lambda] \approx e^{-n\xi}, \quad n \to \infty, \tag{1.2}$$

where $\approx$ denotes that the logarithms of the two sides are asymptotic. It takes more work (see [18]) to show that $\approx$ can be replaced with $\asymp$; in doing so one can show that the constants can be chosen uniformly on compact intervals, that is, for every $0 < \lambda_1 < \lambda_2 < \infty$, there exist positive constants c_1, c_2 such that, for all $n \geqslant 0$ and all $\lambda \in [\lambda_1, \lambda_2]$,

$$c_1 e^{-n\xi} \leqslant \mathbf{E}[Z_n^\lambda] \leqslant c_2 e^{-n\xi}.$$

If λ is an integer, we can interpret this result as saying that the probability that $k + \lambda$ Brownian motions starting on the unit sphere reach the sphere of radius n in a way so that the paths of the first k Brownian motions do not intersect the paths of the last λ Brownian motions is comparable to $n^{-\xi}$. The disconnection exponent in two dimensions is the exponent $\xi = \xi_2(0,k)$ defined by the relation

$$\mathbf{P}\{Z_n > 0\} \asymp e^{-n\xi}, \quad n \to \infty. \tag{1.3}$$

Again the existence of a number ξ satisfying

$$\mathbf{P}\{Z_n > 0\} \approx e^{-n\xi}, \quad n \to \infty, \tag{1.4}$$

can be shown by scaling and subadditivity. In [16] it was shown that (1.3) holds for $k = 2$; a similar argument works for all k. We will make the following convention: if X is a nonnegative random variable, then

$$X^0 = \begin{cases} 0, & X = 0, \\ 1, & X > 0. \end{cases}$$

Using this convention, we can write (1.3) as

$$\mathbf{E}[Z_n^0] \asymp e^{-n\xi}, \quad n \to \infty.$$

The exact values of the exponents are not known and determination of the values is an interesting open problem: see [18] for a discussion of these values. In $d = 2$, it is conjectured [8, 22] that

$$\xi(k,\lambda) = \frac{[\sqrt{24k+1} + \sqrt{24\lambda+1}][\sqrt{24k+1} + \sqrt{24\lambda+1} - 4]}{48}.$$

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Intersection exponents for simple random walks can be defined similarly. Let $S^1(t), \ldots, S^k(t)$ be simple d-dimensional random walks ($d = 2, 3$) with integer time t defined on the probability space $(\Omega, \mathbf{P})$ and let $S(t)$ be another simple random walk defined on $(\Omega_1, \mathbf{P}_1)$. Define the analogous stopping times

$$\tau_n^i = \inf\{t : |S^i(t)| \geqslant e^n\},$$
$$\tau_n = \inf\{t : |S(t)| \geqslant e^n\},$$

and

$$\Gamma(m, n) = S^1[\tau_m^1, \tau_n^1] \cup \cdots \cup S^k[\tau_m^k, \tau_n^k].$$

Let

$$Z^*(m, n) = \mathbf{P}_1\{S[\tau_m, \tau_n] \cap \Gamma(m, n) = \emptyset\}.$$

The goal is to show that the simple random walk has the same intersection exponent; in fact, we show that, for every $0 < \lambda_1 < \lambda_2 < \infty$, there exist constants c_1, c_2 such that, for all $0 \leqslant m \leqslant n < \infty$,

$$c_1 e^{-\xi(n-m)} \leqslant \mathbf{E}[Z^*(m, n)^\lambda] \leqslant c_2 e^{-\xi(n-m)},$$

where $\xi = \xi_d(\lambda, k)$ is the corresponding Brownian intersection exponent. Similarly, we show for $d = 2$ that

$$\mathbf{P}\{Z^*(m, n) > 0\} \asymp e^{-\xi(n-m)},$$

where $\xi = \xi_2(0, k)$ is the Brownian disconnection exponent.

This result extends results in [4, 20]: in the first paper the Brownian intersection exponent is shown to be the same as the random walk exponent for positive integer λ, and in the second paper a proof is given for the disconnection exponent with $k = 1$. Both of these papers are complicated by the fact that they only use (1.2) and (1.4) and consequently can only prove results for random walks with $\approx$ rather than $\asymp$. In [17] it was shown how to combine (1.1) and (1.3) with strong approximation to prove the random walk result for $k = 1, \lambda = 1$ with $\asymp$. This paper uses that basic framework to prove the main result. The results of this paper are used in [19] to analyse the multifractal spectrum of harmonic measure on a random walk path. The proofs in both [17] and [19] use the random walk result with $\asymp$ rather than just $\approx$; this stronger convergence allows 'second moment' techniques to be used. This is analogous to the Brownian motion case where the $\asymp$ results were used to prove results about Hausdorff dimension of the set of cut-points and the outer boundary [15, 16].

In [6] it was shown that the result of [4] holds for all mean-zero, finite variance walks. The methods used in this paper are not strong enough to conclude such a result. We only prove the result for simple random walks, but we expect that the results can be extended to random walks for which one can get a strong approximation as described in the next section. We do not know at this time which random walks this includes: there are certainly mean-zero, finite variance walks for which our method does not work. We do not know if the results will be true for all such walks or not. We will concentrate solely on simple random walks in this paper.

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The strong approximation that we need is basically the standard Skorohod approximation. By applying the Skorohod embedding to each component separately, one can define independent random walks $S^1,\dots,S^k$ and independent Brownian motions $B^1,\dots,B^k$ on the same probability space so that $|S^i(t)-B^i(t)|$ is of order $t^{1/4}$. More precisely, for every $\gamma > 1/2$, we can find c,β such that

$$\mathbf{P}\left\{\sup_{0\leqslant t\leqslant T_n^i\vee\tau_n^i}|B^i(t)-S^i(t)|\geqslant e^{\gamma n}\right\}\leqslant ce^{-n^\beta}.$$

The argument is complicated by the fact that we must use the strong approximation on both the random walks $S^1,\dots,S^k$ and on S. If there were only one random walk to approximate, the arguments would be similar but easier to follow; see [21] for such an argument using only one random walk in two dimensions.

The basic outline of the paper is as follows. In the next section we review the necessary preliminary facts about Brownian motions and random walks including the relevant results about the intersection and disconnection results. The proof follows in the following three sections. The first section shows that the intersection exponent for slightly fattened Brownian motions is the same as that for Brownian motions. The next section uses that fact and the strong approximation to give a lower bound for the nonintersection probability for random walks. The upper bound for random walks is handled in Section 5. The final section deals with a couple of straightforward extensions of the main result that are useful for applying the results in [19].

We will make some conventions about constants. For the remainder of this paper we fix positive integer k, $0<\lambda_1<\lambda_2<\infty$ and $\lambda\in[\lambda_1,\lambda_2]$. We let c,c_1,c_2,β denote arbitrary positive constants that depend only on k,λ_1,λ_2. The values of c,c_1,c_2,β may change from line to line. We will write

$$\begin{aligned}\xi &= \xi_d(\lambda,k),\\ \xi_2 &= \xi_d(\lambda_2,k),\\ \xi_0 &= \xi_2(0,k).\end{aligned}$$

If A is an event, we will also write A for the indicator function of the event.

2. Preliminaries

In this section we review the necessary facts from previous work. We start by outlining the strong approximation, which is basically a version of the Skorohod embedding; we review the necessary facts about the Brownian motion exponent; we discuss the results that rigorize the idea 'Brownian motions or random walks that get close are very likely to intersect'; and finally we discuss some standard convexity inequalities and prove a simple lemma about sequences of positive numbers.

Consider a d-dimensional Brownian motion

$$B(t)=B_t=(W_t^1,\dots,W_t^d),$$

where $W_t^1,\dots,W_t^d$ are independent one-dimensional Brownian motions. Let $X_t^1,\dots,X_t^d$ be the one-dimensional simple random walks with integer time t derived by the standard

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Skorohod embedding, which we now review. Set $\rho_0^i = 0$, and for $j > 0$,

$$\rho_j^i = \inf\{t > \rho_{j-1}^i : |W_t^i - W^i(\rho_{j-1}^i)| = 1\},$$
$$X_t^i = W^i(\rho_t^i).$$

Roughly we expect

$$\rho_t^i = t + O(t^{1/2}),$$

and hence

$$X_t^i = W_t^i + O(t^{1/4}).$$

Let L_t be a multinomial process in $\mathbf{Z}^d$ (sometimes called a directed random walk) with integer time, independent of the Brownian motions $W^1, \ldots, W^d$. By a multinomial process we mean

$$L_t = V_1 + \cdots + V_t,$$

where $V_1, V_2, \ldots$ are independent, identically distributed, with

$$\mathbf{P}\{V_i = (1,0,\ldots,0)\} = \mathbf{P}\{V_i = (0,1,\ldots,0)\} = \cdots = \mathbf{P}\{V_i = (0,0,\ldots,1)\} = \frac{1}{d}.$$

We write

$$L_t = (L_{t,1}, \ldots, L_{t,d}).$$

If we let

$$S_t = (X^1(L_{t,1}), \ldots, X^d(L_{t,d})),$$

then it is easy to verify that S_t is a simple random walk in $\mathbf{Z}^d$. Again we expect

$$L_{t,i} = \frac{1}{d}t + O(t^{1/2}),$$

and hence

$$S_t = (X^1(t/d), \ldots, X^d(t/d)) + O(t^{1/4}),$$

and hence

$$S_t = B_{t/d} + O(t^{1/4}).$$

Let us make things a little more precise. Suppose $Y_1, Y_2, \ldots$ are i.i.d., mean-zero random variables with $\mathrm{Var}(Y_i) = \sigma^2$ and $M(a) = \mathbf{E}[e^{aY_i}] < \infty$, for a in a neighbourhood of zero. Then

$$\mathbf{E}\left[\exp\left\{\frac{Y_1 + \cdots + Y_n}{\sqrt{n}}\right\}\right] = \left[1 - \frac{\sigma^2}{2n} + o\left(\frac{1}{n}\right)\right]^n \leqslant c < \infty,$$

for n sufficiently large, and hence

$$\mathbf{P}\{Y_1 + \cdots + Y_n \geqslant n^{(1/2)+\epsilon}\} \leqslant e^{-n^\epsilon}\mathbf{E}\left[\exp\left\{\frac{Y_1 + \cdots + Y_n}{\sqrt{n}}\right\}\right] \leqslant ce^{-n^\epsilon},$$

for all n sufficiently large (and hence is true for all n with a change in the constant c). Combining this estimate with the reflection principle, we can get the following standard estimate for Brownian motion: for every $\gamma, \epsilon > 0$, there exists $\delta > 0$ and a constant r such that

$$\mathbf{P}\left\{\sup\{|W_t^i - W_s^i| : 0 \leqslant s,t \leqslant n,\ |t-s| \leqslant n^{2\gamma}\} \geqslant n^{\gamma+\epsilon}\right\} \leqslant re^{-n^\delta}.$$

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Similarly, for each i,

$$\mathbf{P}\{\sup\{|\rho_s^i - s| : 0 \leqslant s \leqslant n^2\} \geqslant n^{1+\epsilon}\} \leqslant re^{-n^\delta},$$
$$\mathbf{P}\left\{\sup\left\{\left|L_{s,i} - \frac{s}{d}\right| : 0 \leqslant s \leqslant n^2\right\} \geqslant n^{1+\epsilon}\right\} \leqslant re^{-n^\delta}.$$

(The first inequality uses the facts that $\mathbf{E}[\rho_1^i] = 1$ and that, for sufficiently small positive a, $\mathbf{E}[\exp\{a\rho_1^i\}] < \infty$.) Using these estimates we can see that for every $\epsilon > 0$ there exist δ and r such that

$$\mathbf{P}\left\{\sup_{0\leqslant t\leqslant n^2} |S_t - B_{t/d}| \geqslant n^{(1/2)+\epsilon}\right\} \leqslant re^{-n^\delta}.$$

As before, let T_n be the first hitting time of the sphere e^n by B. Again, standard estimates give that

$$\mathbf{P}\{T_n \geqslant e^{2.1n}\} \leqslant \left[\sup_{|x|\leqslant e^n} \mathbf{P}^x\{T_n \geqslant e^{2n}\}\right]^{\lfloor \exp(.1n)\rfloor} \leqslant c\exp\{-e^{\beta n}\}. \tag{2.1}$$

Let U_n be the event

$$U_n = \left\{\sup\left\{\left|L_{t,i} - \frac{t}{d}\right| : 0 \leqslant t \leqslant e^{2.1n},\ i = 1\ldots,k\right\} \geqslant e^{1.1n}\right\}.$$

Then, as mentioned above,

$$\mathbf{P}[U_n] \leqslant c\exp\{-e^{\beta n}\}. \tag{2.2}$$

As before, let τ_n be the first hitting time of the sphere of radius e^n by S. An inequality similar to (2.1) holds for τ_n. Let

$$\Theta_n = \{x \in \mathbf{R}^d : \operatorname{dist}(x, B[0, T_n]) \leqslant e^{3n/4}\},$$
$$\theta_n = \{x \in \mathbf{R}^d : \operatorname{dist}(x, S[0, \tau_n]) \leqslant e^{3n/4}\}.$$

We can see that there exist c, β such that

$$\mathbf{P}\{S[0,\tau_n] \subset \Theta_{n+1}\} \geqslant 1 - c\exp\{-e^{\beta n}\},$$
$$\mathbf{P}\{B[0,T_n] \subset \theta_{n+1}\} \geqslant 1 - c\exp\{-e^{\beta n}\}.$$

There is a minor measurability question that will come up later. Let $\mathscr{H}_n$ denote the σ-algebra generated by

$$\{B_t^i, 0 \leqslant t \leqslant T_n^i, i = 1,\ldots,k\} \cup \{L_t^i, i = 1,\ldots,k, t = 0,1,\ldots\}.$$

The event

$$\{S[0,\tau_n] \subset \Theta_{n+1}\}$$

is not quite measurable with respect to $\mathscr{H}_{n+1}$. To see why this is true, consider the (unlikely) event that

$$L_t^1 = t, \quad 0 \leqslant t \leqslant e^{2n},$$
$$|B_t^1| \leqslant \tfrac{1}{4}e^n, \quad 0 \leqslant t \leqslant e^{2n},$$

and

$$\left|B^2\left(\frac{1}{4}e^{2n}\right)\right| \geqslant e^{n+1}.$$

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Then, on this event, we need information about B^1_t for $\frac{1}{4}e^{2n} \leqslant t \leqslant e^{2n}$ to determine $S[0,\tau_n]$, yet information about B^1_t for $t \geqslant \frac{1}{4}e^{2n}$ is not contained in $\mathscr{H}_{n+1}$. However, in the typical case, when L^i_t is about t/d for each i, we can determine $S[0,\tau_n]$ from the information in $\mathscr{H}_{n+1}$.

Let A_n be the event that

$$\sup\{|W^i_t - W^i_s| : 0 \leqslant s,t \leqslant T_{n+1},\ |t-s| \leqslant e^{5n/4},\ i = 1,\dots,d\} \leqslant e^{3n/4},$$

and

$$\sup\{|\rho^i_s - s| : 0 \leqslant s \leqslant T_{n+1},\ i = 1,\dots,d\} \leqslant e^{5n/4}.$$

Then A_n is measurable with respect to

$$\sigma\{B_t : 0 \leqslant t \leqslant T_{n+1}\},$$

there exist c,β such that

$$\mathbf{P}(A_n) \geqslant 1 - c\exp\{-e^{\beta n}\}, \tag{2.3}$$

and

$$A_n \cap (U_n)^c \subset \{S[0,\tau_n] \subset \Theta_{n+1}\}.$$

More generally, if Y is any random variable with $0 \leqslant Y \leqslant 1$ and such that the value of Y depends only on $S[0,\tau_n]$ and the σ-algebra $\mathscr{H}_{n+1}$, then there exists an $\mathscr{H}_{n+1}$-measurable random variable $\tilde{Y}$ such that

$$|Y - \tilde{Y}| \leqslant (1 - A_n) + U_n.$$

We now assume that we have independent Brownian motions $B^1_t,\dots,B^k_t$ and independent multinomial processes $L^1_t,\dots,L^k_t$ defined on the same probability space $(\Omega,\mathbf{P})$. By the Skorohod embedding as described above, this gives us simple random walks $S^1_t,\dots,S^k_t$ so that, if we define $\Theta^1,\dots,\Theta^k_n,\theta^1_n,\dots,\theta^k_n$ as above,

$$\mathbf{P}\{S^i[0,\tau^i_n] \subset \Theta^i_{n+1}\} \geqslant 1 - c\exp\{-e^{\beta n}\}, \tag{2.4}$$

$$\mathbf{P}\{B^i[0,T^i_n] \subset \theta^i_{n+1}\} \geqslant 1 - c\exp\{-e^{\beta n}\}. \tag{2.5}$$

We let U^i_n, A^i_n be as above.

Similarly we assume we have a Brownian motion B_t and an independent multinomial process L_t defined on the probability space $(\Omega_1,\mathbf{P}_1)$ that produce a simple random walk S_t with

$$\mathbf{P}\{S[0,\tau_n] \subset \Theta_{n+1}\} \geqslant 1 - c\exp\{-e^{\beta n}\}, \tag{2.6}$$

$$\mathbf{P}\{B[0,T_n] \subset \theta_{n+1}\} \geqslant 1 - c\exp\{-e^{\beta n}\}. \tag{2.7}$$

Again we let U_n and A_n be as above.

We now review the relevant results in [18, 16] about the intersection exponent and disconnection exponent for Brownian motion. It will be useful to set up some notation. Let $\mathscr{B}(x,r)$ be the closed ball of radius r about x, and let $\mathscr{S}_n$ be the sphere of radius e^n centred at the origin with $\mathscr{S} = \mathscr{S}_0$. Let $\mathscr{A}_r$ be the half-space

$$\mathscr{A}_r = \{(x^1,x^2) \in \mathbf{R}^2 : x^1 \geqslant r\} \text{ or } \{(x^1,x^2,x^3) \in \mathbf{R}^3 : x^1 \geqslant r\}.$$

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Assume we have Brownian motions $B^1,\dots,B^k$ defined on $(\Omega,\mathbf{P})$ and B defined on $(\Omega_1,\mathbf{P}_1)$ as above with stopping times $T_n^1,\dots,T_n^k,T_n$. As before, let

$$\Lambda(m,n)=B^1[T_m^1,T_n^1]\cup\cdots\cup B^k[T_m^k,T_n^k],$$
$$\Lambda_n=\Lambda(0,n),$$

and let $\mathscr{F}_n$ be the σ-algebra generated by

$$\{B_t^i:\ 0\leqslant t\leqslant T_n^i,\ i=1,\dots,k\}.$$

Let

$$\bar{Z}(m,n)=\sup_{x\in\mathscr{S}_m}\mathbf{P}_1^x\{B[0,T_n]\cap\Lambda(m,n)=\emptyset\}.$$

Note that, if $m\leqslant r\leqslant n$,

$$\bar{Z}(m,n)\leqslant\bar{Z}(m,r)\bar{Z}(r,n).$$

It was shown in [18] that there exist c_1,c_2 such that, for all $0\leqslant m\leqslant n$, all $\lambda\in[\lambda_1,\lambda_2]$, and all $\mathbf{x}\in\mathscr{S}_m^k$,

$$c_1e^{-\xi(n-m)}\leqslant\mathbf{E}^{\mathbf{x}}[\bar{Z}(m,n)^\lambda]\leqslant c_2e^{-\xi(n-m)},\tag{2.8}$$

where $\xi=\xi_d(\lambda,k)$. Similarly, if $d=2$ [16],

$$c_1e^{-\xi_0(n-m)}\leqslant\mathbf{P}^{\mathbf{x}}[\bar{Z}(m,n)>0]\leqslant c_2e^{-\xi_0(n-m)},\tag{2.9}$$

where ξ_0 is the disconnection exponent. (We note again that [16] actually only derived results for $k=2$. However, the proofs for other k are virtually the same so we will assume them in this paper.) It follows that, if $m\leqslant r\leqslant n$ and Y is an $\mathscr{F}_r$-measurable random variable,

$$\mathbf{E}[\bar{Z}(m,n)^\lambda\ Y\ \mid\mathscr{F}_r]\leqslant\bar{Z}(m,r)^\lambda\ Y\ \mathbf{E}[\bar{Z}(r,n)^\lambda\mid\mathscr{F}_r]\leqslant c\bar{Z}(m,r)^\lambda\ Y\ e^{-\xi(n-r)},$$

and hence, for each $\mathbf{x}\in\mathscr{S}_m^k$,

$$\mathbf{E}^{\mathbf{x}}[\bar{Z}(m,n)^\lambda Y]\leqslant ce^{-\xi(n-r)}\mathbf{E}^{\mathbf{x}}[\bar{Z}(m,r)^\lambda Y].\tag{2.10}$$

A similar inequality holds for the disconnection exponent ($\lambda=0$).

A stronger form of the lower bound was established in [16, 18]. This stronger version gives a precise statement of the plausible fact that Brownian motions conditioned to have no intersection have a reasonable chance of being somewhat far apart. Let u denote the unit vector whose first component is 1, and let $u_m=e^mu$. Let $0\leqslant m\leqslant n-2$ and

$$\tilde{Z}(m,n)=\mathbf{P}_1^{u_m}\big\{B[0,T_n]\cap\Lambda(m,n)=\emptyset;$$
$$B[0,T_n]\cap\mathscr{B}(0,e^{m+1})\subset\mathscr{A}_{e^m/2};\ B[T_{n-2},T_n]\subset\mathscr{A}_{e^{n-2}/2}\big\},$$

and let $J(m,n)$ be the event

$$J(m,n)=\big\{\Lambda(m,n)\cap\mathscr{B}(0,e^{m+1})\subset-\mathscr{A}_{e^m/2};\ \Lambda(n-2,n)\subset-\mathscr{A}_{e^{n-2}/2}\big\}.$$

Then there exists a constant c such that, for every $\lambda\in[\lambda_1,\lambda_2]$ and every $\mathbf{x}=(x^1,\dots,x^k)\in\mathscr{S}_m^k$ with $|x^i+u_m|\leqslant e^m/5$,

$$\mathbf{E}^{\mathbf{x}}[\tilde{Z}(m,n)^\lambda;J(m,n)]\ \geqslant\ ce^{-\xi(n-m)},$$
$$\mathbf{P}^{\mathbf{x}}[\tilde{Z}(m,n)>0;J(m,n)]\ \geqslant\ ce^{-\xi_0(n-m)}.\tag{2.11}$$

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We also need a result that quantifies the notion that Brownian motions conditioned not to intersect are transient. For a proof of the following lemma, see [18, Lemma 4.12]. This gives a proof for $\lambda > 0$, but the proof for the disconnection case $\lambda = 0$ is essentially the same.

Lemma 2.1. *There exist c, β such that the following is true. Assume $m \leqslant r \leqslant s \leqslant n$. Let*

$$J^i = J^i(r,s,n) = \{B^i[T^i_s, T^i_n] \cap \mathscr{B}(0,e^r) \neq \emptyset\},$$

$$\bar{Z}''(m,r,s,n) = \sup_{x \in \mathscr{S}_m} \mathbf{P}^x_1\{B[0,T_n] \cap \Lambda[m,n] = \emptyset;\ B[T_s,T_n] \cap \mathscr{B}(0,e^r) \neq \emptyset\}.$$

Then, for all $\mathbf{x} \in \mathscr{S}^k_m$,

$$\mathbf{E}^{\mathbf{x}}[\bar{Z}(m,n)^\lambda; J^1 \cup \cdots \cup J^k] \leqslant ce^{-\beta(s-r)}e^{-\xi(n-m)},$$
$$\mathbf{E}^{\mathbf{x}}[\bar{Z}''(m,r,s,n)^\lambda] \leqslant ce^{-\beta(s-r)}e^{-\xi(n-m)}.$$

We now formalize the notion that random walks and Brownian motions that get close to each other are very likely to intersect. For $d = 2$, there is a nice uniform bound given by the Buerling projection theorem, continuous and discrete versions (see, *e.g.*, [1, 3] for the continuous version and [12] and [13, Lemma 2.3] for the discrete). One way to phrase this theorem is as follows. Suppose C is a continuous curve (in the continuous case) or a connected path (in the discrete case) lying in the ball of radius n connecting the origin to the sphere of radius n. Suppose a Brownian motion B or random walk S starts within distance δ of C. Then the probability that the Brownian motion or random walk reaches the sphere of radius $2n$ without hitting C is bounded above by

$$c\left[\frac{\delta}{n}\right]^{1/2}.$$

where c is an absolute constant, independent of n and C. There is no such uniform bound in three dimensions (actually there is one for random walk where the power law is replaced with a power of the logarithm, but this is not strong enough for our purposes); however one can get good stochastic bounds on this quantity when C is a random walk or a Brownian motion path. Suppose C is a Brownian motion or random walk path going from the origin to the sphere of radius n. Consider the random variable $Y = Y_\delta$ defined as the supremum over all x within distance δ of C of the probability that another Brownian motion or random walk starting at x reaches the sphere of radius $2n$ without hitting C. Then it can be shown [15, 17] that, for every $M < \infty$, there exist K, α such that the probability that $Y \geqslant (\delta/n)^\alpha$ is less than or equal to Kn^{-M}. By a judicious choice of M, one can prove the following. We let B^1_t, S^1_t be defined on $(\Omega, \mathbf{P})$ and B_t, S_t be defined on $(\Omega_1, \mathbf{P}_1)$ as above. Assume in the next lemma that $B^1_0 = S^1_0 = 0$.

Lemma 2.2. *For every $0 < \delta_1 < \delta_2 \leqslant 1$ and every $M < \infty$, there exist $\alpha > 0$ and $K < \infty$ such that the following holds. Let*

$$\kappa_n = \kappa_n(\delta_2) = \inf\{t : |B(t) - B(0)| \geqslant e^{\delta_2 n}\},$$
$$\kappa^*_n = \kappa^*_n(\delta_2) = \inf\{t : |S(t) - S(0)| \geqslant e^{\delta_2 n}\}.$$

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Let F_n be the random variable on $(\Omega, \mathbf{P})$,

$$F_n = F_n(\delta_1, \delta_2) = \sup_x \mathbf{P}_1^x\{B[0, \kappa_n] \cap B^1[0, T_n^1] = \emptyset\},$$

where the supremum is over all x with

$$\operatorname{dist}(x, B^1[0, T_n^1]) \leqslant e^{\delta_1 n}.$$

Similarly, let F_n^* be the random variable on $(\Omega, \mathbf{P})$,

$$F_n^* = F_n^*(\delta_1, \delta_2) = \sup_x \mathbf{P}_1^x\{S[0, \kappa_n^*] \cap S^1[0, \tau_n^1] = \emptyset\},$$

where the supremum is over all x with

$$\operatorname{dist}(x, S^1[0, \tau_n^1]) \leqslant e^{\delta_1 n}.$$

Then

$$\mathbf{P}\{F_n \geqslant e^{-\alpha n}\} \leqslant K e^{-Mn},$$
$$\mathbf{P}\{F_n^* \geqslant e^{-\alpha n}\} \leqslant K e^{-Mn}.$$

Note that, if $d = 2$, the Beurling projection theorem allows us to find α such that, for all n sufficiently large,

$$\mathbf{P}\{F_n \geqslant e^{-\alpha n}\} = 0,$$
$$\mathbf{P}\{F_n^* \geqslant e^{-\alpha n}\} = 0.$$

We finish this section with some simple inequalities that will be used in the later sections. If X, Y are random variables, then, by concavity,

$$\mathbf{E}[|X + Y|^\lambda] \leqslant \mathbf{E}[|X|^\lambda] + \mathbf{E}[|Y|^\lambda], \quad 0 \leqslant \lambda \leqslant 1,$$

and by Minkowski's inequality,

$$\mathbf{E}[|X + Y|^\lambda]^{1/\lambda} \leqslant \mathbf{E}[|X|^\lambda]^{1/\lambda} + \mathbf{E}[|Y|^\lambda]^{1/\lambda}, \quad 1 \leqslant \lambda < \infty.$$

The particular case that will be important to us is as follows. Suppose for $0 \leqslant m \leqslant r \leqslant n$ there exist nonnegative random variables $X(m, n)$, $Y(m, r, n)$. Suppose there exist constants c_3, c_4, α such that, for all m, r, n,

$$\mathbf{E}[X(m, n)^\lambda] \geqslant c_3 e^{-\xi(n-m)},$$
$$\mathbf{E}[Y(m, r, n)^\lambda] \leqslant c_4 e^{-\alpha r} e^{-\xi(n-m)}.$$

Then, for all m sufficiently large and all $n \geqslant m$,

$$\mathbf{E}\left[\left[X(m, n) - \sum_{r=m}^{n} Y(m, r, n)\right]^\lambda\right] \geqslant \frac{1}{2} c_3 e^{-\xi(n-m)}. \tag{2.12}$$

We end this section with a simple lemma about sequences which we will need in Section 5.

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Lemma 2.3. *For every $K, \beta > 0$, $r < \infty$, there exists $M < \infty$ such that, if $1 \leqslant q \leqslant r$ and $b_1, b_2, \ldots$ is a sequence of positive numbers with*

$$b_{n+1} \leqslant \left[K \sum_{j=1}^{n} b_j^{1/q} e^{-\beta j} \right]^q, \quad n = 1, 2, 3, \ldots,$$

then

$$b_n \leqslant M b_1$$

for all n.

Proof. Without loss of generality we may assume that $b_1 = 1$ and

$$b_{n+1} = \left[K \sum_{j=1}^{n} b_j^{1/q} e^{-\beta j} \right]^q, \quad n = 1, 2, 3, \ldots.$$

Note that, if $n \geqslant 2$,

$$b_{n+1} = [b_n^{1/q} + K b_n^{1/q} e^{-\beta n}]^q = b_n [1 + K e^{-\beta n}]^q \leqslant b_n [1 + K e^{-\beta n}]^r.$$

In particular, $b_{n+1} \leqslant e^{\beta/2} b_n$ for all n sufficiently large. Hence

$$b_n \leqslant C_1 e^{\beta n/2},$$

for some $C_1 = C_1(K, r, \beta) < \infty$, and

$$b_n \leqslant \left[K \sum_{j=1}^{\infty} C_1^{1/q} e^{\beta j/2q} e^{-\beta j} \right]^q \leqslant C_1 \left[K \sum_{j=1}^{\infty} e^{-\beta j/2} \right]^r < \infty. \qquad \square$$

3. Bound for fattened Brownian motions

In the next two sections we show that we can fatten the Brownian motion paths in a way so that the following holds: they are sufficiently large that the random walk paths will lie in the fattened Brownian paths with high probability, yet they are sufficiently small that the intersection exponent for the fattened paths is the same as for the original paths. It turns out that fattening by $|B_t|^{1-\epsilon}$ works fine for any $0 < \epsilon < 1/2$. The condition $\epsilon < 1/2$ is what is needed so that the random walks lie within the fattened Brownian motions. If we tried to fatten the Brownian motions by as much as $\epsilon |B_t|$, then the intersection exponent would change. However, fattening by $|B_t|^{1-\epsilon}$ will not change the exponent. We will only need to do this for one value of ϵ, and, somewhat arbitrarily, we choose $\epsilon = 1/16$.

Let $B^1, \ldots, B^k, B$ be Brownian motions as before. For $m \leqslant r - 1 \leqslant n - 2$, let

$$X(m, r, n) = \sup_{x \in \mathcal{S}_m} \mathbf{P}_1^x \{ B[0, T_n] \cap \Lambda(m, n) = \emptyset;\ \operatorname{dist}[B[T_{r-1}, T_r], \Lambda(m, r)] \leqslant e^{15r/16} \}.$$

Lemma 3.1. *There exist c, β such that, for each $m \leqslant r - 1 \leqslant n - 2$, $\mathbf{x} \in \mathcal{S}_m^k$,*

$$\mathbf{E}^{\mathbf{x}} [X(m, r, n)^\lambda] \leqslant c e^{-\beta r} e^{-\xi(n-m)}.$$

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Proof. Fix $\mathbf{x} \in \mathscr{S}_m$ and write $\mathbf{E}$ for $\mathbf{E}^{\mathbf{x}}$. Note that

$$X(m,r,n) \leqslant X_1 \bar{Z}(r+1,n),$$

where $\bar{Z}(r+1,n)$ is as in the previous section and

$$X_1 = X_1(m,r) = \sup_{x \in \mathscr{S}_m} \mathbf{P}_1^x \big\{ B[T_m, T_{r+1}] \cap \Lambda(m,r+1) = \emptyset; \\ \operatorname{dist}[B[T_{r-1}, T_r], \Lambda(m,r)] \leqslant e^{15r/16} \big\}.$$

Since X_1 is $\mathscr{F}_{r+1}$-measurable we can use (2.10) to conclude that

$$\mathbf{E}[X(m,r,n)^\lambda] \leqslant c e^{-\xi(n-r)} \mathbf{E}[X_1^\lambda].$$

Let $F_r = F_r(c,\beta)$ be the $\mathscr{F}_r$-measurable event

$$F_r = \big\{ \sup \mathbf{P}_1^x \{ B[0, T_{r+1}] \cap \Lambda(m,r) = \emptyset \} \geqslant c e^{-\beta r} \big\},$$

where the supremum is over all x with

$$\operatorname{dist}[x, \Lambda(m,r)] \leqslant e^{15r/16}.$$

From Lemma 2.2 we know that we can find c, β so that

$$\mathbf{P}(F_r) \leqslant c e^{-2\xi_2 r},$$

where we recall that $\xi_2 = \xi_d(\lambda_2, k)$. Hence

$$\mathbf{E}[X_1^\lambda; F_r] \leqslant \mathbf{P}(F_r) \leqslant c e^{-2\xi_2 r} \leqslant c e^{-\xi r} e^{-\xi(r-m)}.$$

However, the strong Markov property applied to B tells us that, on the event F_r^c,

$$X_1 \leqslant c \bar{Z}(m, r-1) e^{-\beta r},$$

and hence

$$\mathbf{E}[X_1^\lambda; F_r^c] \leqslant c e^{-\beta r} \mathbf{E}[\bar{Z}(m,r-1)^\lambda] \leqslant c e^{-\xi(r-m)} e^{-\beta r}$$

(with a change in the values of c, β, of course). Combining these estimates, we have the lemma. □

The following lemma, which is similar, can be proved in a similar way. We omit the proof. Let

$$\tilde{X}(m,r,n) = \sup_{x \in \mathscr{S}_m} \mathbf{P}_1^x \big\{ B[0, T_n] \cap \Lambda(m,n) = \emptyset;\ \operatorname{dist}[B[0, T_r], \Lambda(r-1, r)] \leqslant e^{15r/16} \big\}.$$

Lemma 3.2. *There exists c, β such that for each $m \leqslant r-1 \leqslant n-2$, $\mathbf{x} \in \mathscr{S}_m^k$,*

$$\mathbf{E}^{\mathbf{x}}[\tilde{X}(m,r,n)^\lambda] \leqslant c e^{-\beta r} e^{-\xi(n-m)}. \qquad \square$$

We note that the above proof works equally well for $\lambda = 0$ in $d = 2$, that is, the disconnection exponent. This gives us the following lemma.

Lemma 3.3. *There exists c, β such that, for each $m \leqslant r-1 \leqslant n-2$, $\mathbf{x} \in \mathscr{S}_m^k$,*

$$\mathbf{P}^{\mathbf{x}}[X(m,r,n) > 0] + \mathbf{P}^{\mathbf{x}}[\tilde{X}(m,r,n) > 0] \leqslant c e^{-\beta r} e^{-\xi_0(n-m)}. \qquad \square$$

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For the remainder of this paper we will not explicitly discuss the disconnection exponent; instead we now make the comment that all the proofs work equally well for the $\lambda = 0$ case. We now improve on the estimate (2.11). Let

$$\begin{aligned}\hat{Z}(m,n) = \mathbf{P}_1^{u_m}\big\{ & B[0,T_n] \cap \Lambda(m,n) = \emptyset; \\ & B[0,T_n] \cap \mathscr{B}(0,e^{m+1}) \subset \mathscr{A}_{e^m/2};\ B[T_{n-2},T_n] \subset \mathscr{A}_{e^{n-2}/2}; \\ & \operatorname{dist}[B[T_{r-1},T_r],\Lambda_r] \geqslant e^{15r/16}, r = m+1,\ldots,n; \\ & \operatorname{dist}[B[T_0,T_r],\Lambda(r-1,r)] \geqslant e^{15r/16}, r = m+1,\ldots,n\big\}.\end{aligned}$$

Note that

$$\hat{Z}(m,n) \geqslant \tilde{Z}(m,n) - \sum_{r=m+1}^{n-2} [X(m,r,n) + \tilde{X}(m,r,n)].$$

Hence, by (2.11), Lemmas 3.1 and 3.2, and (2.12), we see that, for all sufficiently large m, if $\mathbf{x} = (x^1,\ldots,x^k) \in \mathscr{S}_m^k$ with $|x^i + u_m| \leqslant e^m/5$,

$$\mathbf{E}^{\mathbf{x}}[\hat{Z}(m,n)^{\lambda}; J(m,n)] \geqslant ce^{-\xi(n-m)}. \tag{3.1}$$

Once we have this for all sufficiently large m, we can easily conclude (3.1) for all $m \geqslant 0$ (with a change of constant) by attaching appropriate initial configurations. Similarly, for the disconnection exponent, for all $0 \leqslant m < n$,

$$\mathbf{P}^{\mathbf{x}}[\hat{Z}(m,n) > 0; J(m,n)] \geqslant ce^{-\xi_0(n-m)}. \tag{3.2}$$

4. Lower bound for the random walk

We now use the result of the last section to prove the lower bound for the simple random walk. The basic idea of the proof is simple. In the last section we showed that the probability that the Brownian motions stay reasonably far apart is comparable to the probability that they do not intersect. If the Brownian motions stay reasonably far apart, then the corresponding simple random walks should also stay reasonably far apart. It takes a little care to make this idea precise.

Let $S_t^1,\ldots,S_t^k,S_t$ be simple random walks as before and again let

$$\begin{aligned}\Gamma(m,n) &= S^1[\tau_m^1,\tau_n^1] \cup \cdots \cup S^k[\tau_m^k,\tau_n^k], \\ \Gamma_n &= S^1[0,\tau_n^1] \cup \cdots \cup S^k[0,\tau_n^k].\end{aligned}$$

Let $\mathscr{A}_r$ be as in Section 2 and

$$\begin{aligned}\hat{Z}^*(m,n) = \mathbf{P}_1^{u_m}\big\{ & S(0,\tau_n] \cap \Gamma_n = \emptyset; \\ & S(0,\tau_n] \cap \mathscr{B}(0,e^{m+1}) \subset \mathscr{A}_{e^m/3};\ S[\tau_{n-2},\tau_n] \subset \mathscr{A}_{e^{n-2}/3}; \\ & \operatorname{dist}[S[0,\tau_r],\Gamma_r] \geqslant \tfrac{1}{10}e^{15r/16}, r = m+1,\ldots,n\big\}.\end{aligned}$$

(In this definition and definitions below, when we write probabilities assuming that a random walk starts at $x = (x^1,\ldots,x^d) \in \mathbf{R}^d$, we will implicitly mean that the random walk starts at $(\lfloor x^1 \rfloor,\ldots,\lfloor x^d \rfloor)$.) Let

$$J^*(m,n) = \{\Gamma_n \cap \mathscr{B}(0,e^{m+1}) \subset -\mathscr{A}_{e^m/3};\ \Gamma(n-2,n) \subset -\mathscr{A}_{e^{n-2}/3}\}.$$

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Our goal is to prove the following.

Lemma 4.1. *There exists c such that, for all $\mathbf{x} = (x^1, \dots, x^k) \in \mathcal{S}_m^k$ with $|x^i + u_m| \leqslant e^m/10$,*

$$\mathbf{E}^{\mathbf{x}}[\hat{Z}^*(m,n)^\lambda; J^*(m,n)] \geqslant ce^{-\xi(n-m)}. \tag{4.1}$$

Let us first make some simplifying remarks. To determine the value of

$$\hat{Z}^*(m,n)^\lambda J^*(m,n),$$

one only needs to observe the random walks $S^1, \dots, S^k$ after they first reach the sphere of radius $e^m/5$ about $-u_m$. (Recall that we write $J^*(m,n)$ for the indicator function of $J^*(m,n)$.) Hence, by the discrete Harnack inequality (see [12]), there exists c such that, for all $\mathbf{x}, \mathbf{y}$ satisfying the conditions of Lemma 4.1,

$$\mathbf{E}^{\mathbf{x}}[\hat{Z}^*(m,n)^\lambda; J^*(m,n)] \leqslant c\mathbf{E}^{\mathbf{y}}[\hat{Z}^*(m,n)^\lambda; J^*(m,n)],$$

and hence it suffices to prove the lemma for $\mathbf{x} = (-u_m, \dots, -u_m)$. Similarly, if we replaced the $\mathbf{P}_1^{u_m}$ in the definition of $\hat{Z}^*(m,n)$ with the supremum of $\mathbf{P}_1^x$ over all $|x - u_m| \leqslant e^m/10$, we would change the value of $\hat{Z}^*(m,n)J^*(m,n)$ by at most a multiplicative constant. We also note that it suffices to prove the lemma for all m sufficiently large, for then we can always attach appropriate initial configurations to prove the result for all m (for example, start S with a straight line from u to u_m and start S^1 with a straight line from $-u$ to $-u_m$).

As in Section 2, we assume $B_t^1, \dots, B_t^k, L_t^1, \dots, L_t^k, S_t^1, \dots, S_t^k$ are defined on the same probability space $(\Omega, \mathbf{P})$ and B_t, L_t, S_t are defined on $(\Omega_1, \mathbf{P}_1)$. Here the random walks are obtained from the Brownian motion through the strong approximation. We let $U_n^1, \dots, U_n^k, A_n^1, \dots, A_n^k, U_n, A_n$ be the events as before. We let

$$\begin{aligned} \bar{U}_n &= U_n^1 \cup \cdots \cup U_n^k, \\ \bar{A}_n &= A_n^1 \cap \cdots \cap A_n^k. \end{aligned}$$

We recall that $\bar{A}_n$ is $\mathcal{F}_{n+1}$-measurable, that $\bar{U}_n$ is independent of the Brownian motions $B^1, \dots, B^k$, and that c, β can be found so that

$$\begin{aligned} \mathbf{P}[\bar{A}_n] &\geqslant 1 - c\exp\{-e^{\beta n}\}, \\ \mathbf{P}[\bar{U}_n] &\leqslant c\exp\{-e^{\beta n}\}. \end{aligned}$$

Also,

$$\bar{A}_n \setminus \bar{U}_n \subset \{\Gamma_n \subset \Theta_{n+1}^1 \cup \cdots \cup \Theta_{n+1}^k\}.$$

To prove Lemma 4.1 (for sufficiently large m) we will show that there is a sequence $\delta_m \to 0$ such that, if

$$Z'(m,n) = \sup_{x \in \mathcal{S}_m} \mathbf{P}_1^x\big\{B[0,T_n] \cap \Lambda(m,n) = \emptyset; S[0,\tau_r] \not\subset \Theta_{r+1} \text{ for some } r = m+1, \dots, n\big\},$$

then

$$\mathbf{E}^{\mathbf{x}}[Z'(m,n)^\lambda] \leqslant \delta_m e^{-\xi(n-m)}, \tag{4.2}$$

and

$$\mathbf{E}^{\mathbf{x}}\big\{\bar{Z}(m,n)^\lambda;\ S^i[0,\tau_r^i] \not\subset \Theta_{r+1} \text{ for some } r = m+1, \dots, n; i = 1, \dots, k\big\} \leqslant \delta_m e^{-\xi(n-m)}. \tag{4.3}$$

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To see that these inequalities suffice, let

$$\begin{aligned}\hat{Z}'(m,n) = \mathbf{P}^{u_m}\big\{ & B[0,T_n] \cap \Lambda(m,n) = \emptyset; \\ & B[0,T_n] \cap \mathscr{B}(0,e^{m+1}) \subset \mathscr{A}_{e^m/2}; B[T_{n-2},T_n] \subset \mathscr{A}_{e^{n-2}/2}, \\ & \operatorname{dist}[B[T_{r-1},T_r],\Lambda_r] \geqslant e^{15r/16}, r = m+1,\dots,n, \\ & \operatorname{dist}[B[T_0,T_r],\Lambda(r-1,r)] \geqslant e^{15r/16}, r = m+1,\dots,n, \\ & S[0,\tau_r] \subset \Theta_{r+1},\ r = m,m+1,\dots,n-1\big\}.\end{aligned}$$

and

$$J'(m,n) = J(m,n) \cap \big\{S^i[0,\tau_r^i] \subset \Theta_{r+1};\ r = m+1,\dots,n,\ i = 1,\dots,k\big\}.$$

Note that (3.1), (4.2), and (4.3) imply, for all m sufficiently large, if $\mathbf{x} = (x^1,\dots,x^k) \in \mathscr{S}_m^k$,

$$\mathbf{E}^{\mathbf{x}}[\hat{Z}'(m,n)^\lambda; J'(m,n)] \geqslant ce^{-\xi(n-m)}.$$

Also, note that, for m sufficiently large,

$$J'(m,n) \subset J^*(m,n),$$

and on the event $J'(m,n)$,

$$\hat{Z}'(m,n) \leqslant \hat{Z}^*(m,n).$$

Hence to prove Lemma 4.1 it suffices to prove (4.2) and (4.3).

For $m \leqslant r-1 \leqslant n-1$, let $\Lambda = \Lambda(m,n)$, and

$$Y(m,r,n) = \mathbf{P}_1^{u_m}\big\{S[0,\tau_r] \not\subset \Theta_{r+1};\ B[0,T_n] \cap \Lambda = \emptyset\big\}.$$

Then

$$Y(m,r,n) \leqslant \mathbf{P}_1^{u_m}\big\{A_r^c;\ B[0,T_n] \cap \Lambda = \emptyset\big\} + \mathbf{P}_1^{u_m}\big\{U_r;\ B[0,T_n] \cap \Lambda = \emptyset\big\}.$$

Since the event A_r depends only on $B_t, t \leqslant T_{r+1}$, the strong Markov property applied to B_t, together with (2.3), implies that

$$\mathbf{P}_1^{u_m}\big\{A_r^c;\ B[0,T_n] \cap \Lambda = \emptyset\big\} \leqslant \mathbf{P}_1^{u_m}(A_r^c)\bar{Z}(r+1,n) \leqslant ce^{-\beta r}e^{-\xi(r-m)}\bar{Z}(r+1,n).$$

(The last inequality follows since $\mathbf{P}_1(A_r^c)$ decays faster than $e^{-\alpha r}$ for any α.) Also, U_r is independent of the Brownian motion, and hence (2.2) implies

$$\mathbf{P}_1^{u_m}\big\{U_r; B[0,T_n] \cap \Lambda = \emptyset\big\} \leqslant \mathbf{P}^{u_m}(U_r)\bar{Z}(m,n) \leqslant ce^{-\beta r}e^{-\xi(r-m)}\bar{Z}(r+1,n).$$

But

$$\mathbf{E}^{\mathbf{x}}[\bar{Z}(r+1,n)^\lambda] \leqslant ce^{-\xi(n-r)}.$$

Hence,

$$\mathbf{E}^{\mathbf{x}}[Y(m,r,n)^\lambda)] \leqslant ce^{-\beta r}e^{-\xi n}.$$

A similar argument, which we omit, shows that

$$\mathbf{E}^{\mathbf{x}}[\bar{Z}(m,r,n)^\lambda;\ \bar{A}_r \cup \bar{U}_r] \leqslant ce^{-\xi n}e^{-\beta r}. \qquad \square$$

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5. Upper bound for the random walk

We return to the notation in the introduction. The purpose of this section is to show that

$$\mathbf{E}[Z^*(m,m+n)^\lambda] \leqslant ce^{-\xi n}. \tag{5.1}$$

Let

$$\tilde{b}_n = \sup e^{\xi n}\mathbf{E}[Z^*(m,m+n)^\lambda]$$

where the supremum is over all nonnegative m, and let

$$b_n = \max\{\tilde{b}_0, \tilde{b}_1, \ldots, \tilde{b}_n\}.$$

Note that $b_0 = 1$ and, for $n > 0$,

$$b_{n-1} \leqslant b_n \leqslant e^{\xi_2} b_{n-1}.$$

In particular, $b_n \leqslant e^{n\xi_2}$. Proving (5.1) is equivalent to proving that the b_n are bounded. We will derive a recursive inequality for b_n in terms of b_{n-1}.

We assume $B_t^1, \ldots, B_t^k, L_t^1, \ldots, L_t^k, S_t^1, \ldots, S_t^k$ are defined on the same probability space $(\Omega, \mathbf{P})$ and B_t, L_t, S_t are defined on $(\Omega_1, \mathbf{P}_1)$ as in Section 2. We assume for now that the Brownian motions, and hence the random walks, start at the origin. If the simple random walks do not intersect, then either the corresponding Brownian motions do not intersect, or there is a largest integer r such that there is an intersection of the Brownian motions after they reach the sphere $\mathscr{S}_r$. We use a decomposition like this to derive the inequality.

For integers $0 < r < n$ and nonnegative m, define the random variable $G(m,r,n)$ on $(\Omega, \mathbf{P})$ by

$$\begin{aligned} G(m,r,n) = \mathbf{P}_1\big\{ & S[\tau_m, \tau_{m+n}] \cap \Gamma(m, m+n) = \emptyset; \\ & B[T_{m+r}, T_{m+n-1}] \cap \Lambda(m+r, m+n-1) = \emptyset; \\ & B[T_{m+r-1}, T_{m+n-1}] \cap \Lambda(m+r-1, m+n-1) \neq \emptyset\big\}. \end{aligned}$$

We also define

$$\begin{aligned} G(m,0,n) = \mathbf{P}_1\big\{ & S[\tau_m, \tau_{m+n}] \cap \Gamma(m, m+n) = \emptyset; \\ & B[T_m, T_{m+n-1}] \cap \Lambda(m, m+n-1) = \emptyset\big\}, \end{aligned}$$

so that

$$Z^*(m,m+n) = \sum_{r=0}^{n-1} G(m,r,n).$$

We will show below that for some c, β,

$$\mathbf{E}[G(m,r,n)^\lambda] \leqslant cb_r e^{-\beta r} e^{-\xi n}. \tag{5.2}$$

Note that (5.2) and concavity (for $\lambda \leqslant 1$) or Minkowski's inequality (for $\lambda \geqslant 1$) imply that, if $q = \max\{1, \lambda\}$,

$$\begin{aligned} \mathbf{E}[Z^*(m,m+n)^\lambda] &\leqslant \left[\sum_{r=0}^{n-1} \mathbf{E}[G(m,r,n)^\lambda]^{1/q}\right]^q \\ &\leqslant e^{-\xi n}\left[c\sum_{r=0}^{n-1} b_r^{1/q} e^{-\beta r}\right]^q \end{aligned}$$

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(here the values of c, β have changed from (5.2)). Hence

$$b_n \leqslant \left[c \sum_{r=0}^{n-1} b_r^{1/q} e^{-\beta r} \right]^q .$$

By Lemma 2.3, this implies that the b_n are bounded and hence we get (5.1). Therefore, it suffices to derive the estimate (5.2).

First note that

$$G(m,r,n) \leqslant \mathbf{P}_1\big\{B[T_{m+r}, T_{m+n-1}] \cap \Lambda(m+r, m+n-1) = \emptyset\big\}.$$

Hence, by the upper bound for Brownian motions,

$$\mathbf{E}[G(m,r,n)^\lambda] \leqslant c e^{-\xi(n-r)}.$$

This shows that if we can prove (5.2) for all sufficiently large r, we also have it for all r. Without loss of generality we will assume $r \geqslant 3$. The basic idea of the estimate is as follows: requiring that the random walks not intersect up to the sphere of radius $m+r-2$ contributes a factor of order $b_r e^{-\xi r}$ (since the random walks come close to intersecting); requiring that the Brownian motions intersect between T_{m+r-1} and T_{m+r} and yet the random walks not intersect before τ_{m+r+1} contributes the factor of $e^{-\beta r}$ (since the random walks come close to intersecting); finally, forbidding the Brownian motions from intersecting from T_{m+r+2} to T_{m+n} contributes a factor of order $e^{-\xi(n-r)}$. It takes a little work to make this precise, but this is the basic idea.

Let

$$\begin{aligned} G_{1,i}(m,r,n) = \mathbf{P}_1\big\{ & S[\tau_m, \tau_{m+n}] \cap \Gamma(m, m+n) = \emptyset; \\ & B[T_{m+r}, T_{m+n-1}] \cap \Lambda(m+r, m+n-1) = \emptyset; \\ & B[T_{m+r-1}, T_{m+r}] \cap B^i[T^i_{m+r-1}, T^i_{m+n-1}] \neq \emptyset\big\}, \end{aligned}$$

and

$$\begin{aligned} G_{2,i}(m,r,n) = \mathbf{P}_1\big\{ & S[\tau_m, \tau_{m+n}] \cap \Gamma(m, m+n) = \emptyset; \\ & B[T_{m+r}, T_{m+n-1}] \cap \Lambda(m+r, m+n-1) = \emptyset; \\ & B[T_{m+r-1}, T_{m+n-1}] \cap B^i[T^i_{m+r-1}, T^i_{m+r}] \neq \emptyset\big\}. \end{aligned}$$

Then,

$$G(m,r,n) \leqslant \sum_{i=1}^{k} [G_{1,i}(m,r,n) + G_{2,i}(m,r,n)].$$

It suffices to show, for each i,

$$\begin{aligned} \mathbf{E}[G_{1,i}(m,r,n)^\lambda] &\leqslant c b_r e^{-\beta r} e^{-\xi n}, \\ \mathbf{E}[G_{2,i}(m,r,n)^\lambda] &\leqslant c b_r e^{-\beta r} e^{-\xi n}. \end{aligned}$$

By symmetry in i, we only need to show these results for $i = 1$. The arguments for $G_{1,1}$ and $G_{2,1}$ are similar, so we will only do the $G_{1,1}$ case, and we write

$$G' = G'(m,r,n) = G_{1,1}(m,r,n).$$

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Recall that $\mathscr{F}_s$ is the σ-algebra generated by

$$\{B_t^i : i = 1, \ldots, k,\ 0 \leqslant t \leqslant T_s^i\}.$$

Let $\tilde{\mathscr{F}}_s$ be the σ-algebra generated by

$$\mathscr{F}_s \cup \{L_t^1, \ldots, L_t^d : 0 \leqslant t < \infty\}.$$

Note that, since the L_t^i are independent of the Brownian motions $B^1, \ldots, B^t$, we can use the strong Markov property, that is, $\tilde{\mathscr{F}}_s$ is conditionally independent of the σ-algebra generated by

$$\{B_t^i : i = 1, \ldots, k,\ t \geqslant T_s^i\}$$

given

$$B^1(T_s^1), \ldots, B^k(T_s^k).$$

Let

$$H_1 = H_1(m, r) = \mathbf{P}_1\{S[\tau_m, \tau_{m+r-2}] \cap \Gamma(m, m + r - 2) = \emptyset\}.$$

Note that, by definition of b_r, $\mathbf{E}[H_1^\lambda] \leqslant b_r e^{-\xi(r-2)}$. We let $\bar{A}_{m+r-1}$ and U_{m+r} be as before. Although H_1 is not quite $\tilde{\mathscr{F}}_{m+r-1}$-measurable, there is an $\tilde{\mathscr{F}}_{m+r-1}$-measurable random variable $\tilde{H}_1 = \tilde{H}_1(m, r)$ such that

$$|H_1 - \tilde{H}_1| \leqslant \bar{A}_{m+r-1}^c + U_{m+r},$$

where as before we allow events to represent their indicator functions. In particular,

$$\mathbf{E}[\tilde{H}_1^\lambda] \leqslant c b_{r-2} e^{-\xi(r-2)} \leqslant c b_r e^{-\xi r}.$$

If $B[T_{m+r-1}, T_{m+r}] \cap B^1[T_{m+r-1}^1, T_{m+n-1}^1] \neq \emptyset$, the intersection must occur in the ball of radius e^{m+r} about the origin. If (s, t) is an intersection time, that is, $B(s) = B^1(t)$, then $s \leqslant T_{m+r}$, but all we can say about t is that $t \leqslant T_{m+n-1}^1$. We now use the transience of nonintersecting walks to show that the time t cannot be too large. Let

$$\begin{aligned} G_1' = G_1'(m, r, n) = \mathbf{P}_1\{&S[\tau_m, \tau_{m+n}] \cap \Gamma(m, m + n) = \emptyset; \\ &B[T_{m+r}, T_{m+n-1}] \cap \Lambda(m + r, m + n - 1) = \emptyset; \\ &B^1[T_{m+1.1r}^1, T_{m+n}^1] \cap \mathscr{B}(0, e^{m+r+3}) \neq \emptyset\}. \end{aligned}$$

By definition, $G_1' = 0$ if $n \leqslant 1.1r$. Let

$$\begin{aligned} H_2 = H_2(m, r, n) = \sup_{x \in \mathscr{S}_{m+r+1}} \mathbf{P}_1^x\{&B[0, T_{m+n}] \cap \Lambda(m + r + 1, m + n) = \emptyset; \\ &B^1[T_{m+1.1r}, T_{m+n}] \cap \mathscr{B}(0, e^{m+r+3}) \neq \emptyset\}. \end{aligned}$$

By Lemma 2.1, for all $\mathbf{x} \in \mathscr{S}_{m+r}^k$,

$$\mathbf{E}^{\mathbf{x}}[H_2^\lambda \mid \tilde{\mathscr{F}}_{m+r+1}] \leqslant c e^{-\beta r} e^{-\xi(n-r)}.$$

Note that

$$G_1' \leqslant [\tilde{H}_1 + \bar{A}_{m+r-1}^c + U_{m+r}] H_2.$$

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By the strong Markov property we then get

$$\begin{aligned}\mathbf{E}[(G_1')^{\lambda}] &\leqslant \mathbf{E}[\mathbf{E}[([\tilde{H}_1 + \bar{A}^c_{m+r-1} + U_{m+r}]H_2)^{\lambda} \mid \tilde{\mathscr{F}}_{m+r+1}]] \\ &\leqslant ce^{-\beta r}e^{-\xi(n-r)}\mathbf{E}[(\tilde{H}_1 + \bar{A}^c_{m+r-1} + U_{m+r})^{\lambda}] \\ &\leqslant cb_r e^{-\beta r}e^{-\xi n}.\end{aligned}$$

Since now we essentially need only worry about times up to $T^1_{m+1.1r+1}$, we need to consider the probability that the random walks and the Brownian motions are not close up to that time. Let

$$G_2' = G_2'(m,r,n) = \mathbf{P}_1\{B[T_{m+1.1r}, T_{m+n-1}] \cap \Lambda(m+1.1r, m+n-1) = \emptyset\},$$

and let $V = V(m,r)$ be the event

$$V = \{\text{there exists } t \leqslant T^1_{m+(1.05)r} \text{ with } |S^1(dt) - B^1(t)| \geqslant \tfrac{1}{2}e^{.8(m+r)}\}.$$

Note that V is not quite $\tilde{\mathscr{F}}_{m+(1.05)r+1}$-measurable (see the discussion in Section 2), but there exists an event $\tilde{V} = \tilde{V}(m,r)$ that is $\tilde{\mathscr{F}}_{m+(1.05)r+1}$-measurable so that

$$|V - \tilde{V}| \leqslant A^c_{m+(1.05)r} + U_{m+(1.05r)+1}.$$

Then the strong Markov property and estimates from Section 2 can be used to show

$$\mathbf{E}[(G_2')^{\lambda}; V] \leqslant c\exp\{-e^{\beta(m+r)}\}e^{-\xi(n-.9r)} \leqslant cb_r e^{-\beta r}e^{-\xi n}.$$

Similarly, if

$$\begin{aligned}G_3' = G_3'(m,r,n) = \mathbf{P}_1\{&B[T_{m+1.1r}, T_{m+n-1}] \cap \Lambda(m+1.1r, m+n-1) = \emptyset; \\ &\text{there exists } t \leqslant T_{m+1.1r} \text{ with } |S(dt) - B(t)| \geqslant \tfrac{1}{2}e^{.8(m+r)}\},\end{aligned}$$

we can show that

$$\mathbf{E}[(G_3')^{\lambda}] \leqslant cb_r e^{-\beta r}e^{-\xi n}.$$

We can now consider the dominant term of G'. Let

$$\begin{aligned}G_4' = \mathbf{P}_1\{&S[\tau_m, \tau_{m+r+2}] \cap \Gamma(m, m+r+2) = \emptyset; \\ &\operatorname{dist}(S[\tau_{m+r-.2}, \tau_{m+r+1.2}], S^1[0, \tau_{m+1.12r}]) \leqslant e^{.8(m+r)}; \\ &B[T_{m+r+3}, T_{m+n-1}] \cap \Lambda(m+r+3, m+n-1) = \emptyset\}.\end{aligned}$$

Note that

$$G' \leqslant G_1' + G_2'V + G_3' + G_4',$$

so it remains for us to prove

$$\mathbf{E}[(G_4')^{\lambda}] \leqslant cb_r e^{-\beta r}e^{-\xi n}.$$

Following the notation of Lemma 2.2, let

$$\kappa^* = \kappa^*(m,r) = \inf\{t : |S(t) - S(0)| \geqslant e^{.9(m+r)}\},$$

and

$$F = F(m,r) = \sup_x \mathbf{P}^x_1\{S[0,\kappa^*] \cap S^1[0, \tau_{m+1.12r}] = \emptyset\},$$

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where the supremum is over all x with

$$\operatorname{dist}(x, S^1[0, \tau_{m+1.12r}]) \leqslant e^{.8(m+r)}.$$

By Lemma 2.2, we can find α such that, for all sufficiently large r,

$$\mathbf{P}_1\{F \geqslant e^{-\alpha(m+r)}\} \leqslant e^{-4\xi_2(m+r)}.$$

Again the set $\{F \geqslant e^{-\alpha(m+r)}\}$ is not quite measurable with respect to $\tilde{\mathcal{F}}_{m+(1.12)r+1}$ but can be approximated sufficiently closely so that we can show that

$$\mathbf{E}[(G_4')^\lambda; F \geqslant e^{\alpha(m+r)}] \leqslant ce^{-4\xi_2(m+r)}e^{-\xi n} \leqslant cb_r e^{-\beta r}e^{-\xi n}.$$

Finally the strong Markov property applied to S allows us to conclude

$$\mathbf{E}[(G_4')^\lambda; F \leqslant e^{-\alpha(m+r)}] \leqslant cb_r e^{-r\beta}e^{-\xi n}.$$

6. Extensions

Here we derive some useful extensions of the main estimate, one having to do with supremums and the others having to do with random walks run for a fixed length of time. If $m < n$, let

$$\bar{Z}^*(m,n) = \sup_{|y| \leqslant e^m} \mathbf{P}_1^y\{S[0,\tau_n] \cap \Gamma(0,n) = \emptyset\}.$$

Note that

$$Z^*(m,n) \leqslant \bar{Z}^*(m,n).$$

However, by the discrete Harnack inequality applied to S,

$$\bar{Z}^*(m,n) \leqslant cZ^*(m+1,n).$$

Also, if $\mathbf{x} = (x^1, \ldots, x^k)$ with $|x^i| \leqslant e^m$, the discrete Harnack inequality applied to $S^1, \ldots, S^k$ gives

$$\mathbf{E}^{\mathbf{x}}[Z^*(m+1,n)^\lambda] \leqslant c\mathbf{E}[Z^*(m+1,n)^\lambda] \leqslant ce^{-\xi(n-m)},$$

where the expectation on the right assumes that the random walks start at the origin. Hence we have the following corollary.

Corollary 6.1. *There exists c such that the following holds. If $\mathbf{x} = (x^1, \ldots, x^k)$ with $|x^i| \leqslant e^m$,*

$$\mathbf{E}^{\mathbf{x}}[\bar{Z}^*(m,n)^\lambda] \leqslant ce^{-\xi(n-m)}. \qquad \square$$

Let

$$\tau_{n,+} = \max\{\tau_n, e^{2n}\}, \quad \tau_{n,-} = \min\{\tau_n, e^{2n}\},$$

and define $\tau^i_{n,+}, \tau^i_{n,-}$ similarly. Let

$$\begin{aligned}
\Gamma_{n,+} &= S^1[0,\tau^1_{n,+}] \cup \cdots \cup S^k[0,\tau^k_{n,+}],\\
\Gamma_{n,-} &= S^1[0,\tau^1_{n,-}] \cup \cdots \cup S^k[0,\tau^k_{n,-}],\\
Z^*_{n,+} &= \mathbf{P}_1\{S(0,\tau_{n,+}] \cap \Gamma_{n,+} = \emptyset\},\\
Z^*_{n,-} &= \mathbf{P}_1\{S(0,\tau_{n,-}] \cap \Gamma_{n,-} = \emptyset\}.
\end{aligned}$$

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Clearly,

$$Z^*_{n,+} \leqslant Z^*_n \leqslant Z^*_{n,-},$$

where $Z^*_n = Z^*(0,n)$. Using Lemma 4.1 (and, if necessary, conditioning the random walks so that after reaching $\mathscr{S}_n$ they go distance at most $e^n/20$ in the next e^{2n} steps), we can see that

$$\mathbf{E}[(Z^*_{n,+})^\lambda] \geqslant ce^{-\xi n}.$$

The goal of the remainder of this section is to prove the other bound. The argument follows that in [15].

Proposition 6.2. *There exists c such that*

$$\mathbf{E}[(Z^*_{n,-})^\lambda] \leqslant ce^{-\xi n}.$$

Proof. It is easy to see using the invariance principle that there exist α and c_4 such that, for all $|x| \leqslant e^n/2$ and all $s > 0$,

$$\mathbf{P}^x_1\{\tau_n \geqslant se^{2n}\} \leqslant c_4 e^{-\alpha s}.$$

Let $V_{n,s}$ be the event

$$V_{n,s} = \{\tau^i_n \geqslant se^{2n} \text{ for some } i = 1,\dots,k\}.$$

Our first goal is to show that

$$\mathbf{E}[(Z^*_n)^\lambda; V_{n,s}] \leqslant ce^{-\xi n}e^{-\alpha s}. \tag{6.1}$$

Let

$$\tilde{V}_{n,s} = \{\tau^i_n - \tau^i_{n-1} \geqslant se^{2n} \text{ for some } i = 1,\dots,k\}.$$

Let $\mathscr{G}_n$ denote the σ-algebra generated by

$$\{S^i_t : i = 1,\dots,k,\ t \leqslant \tau^i_n\}.$$

Then

$$\begin{aligned}\mathbf{E}[(Z^*_n)^\lambda; \tilde{V}_{n,s}] &\leqslant \mathbf{E}[(Z^*_{n-1})^\lambda; \tilde{V}_{n,s}] \\ &\leqslant \mathbf{E}[\mathbf{E}[(Z^*_{n-1})^\lambda; \tilde{V}_{n,s} \mid \mathscr{G}_{n-1}]] \\ &\leqslant ce^{-\alpha s}e^{-\xi n}.\end{aligned}$$

If $m < n$, we can use the inequality

$$Z^*_n \leqslant Z^*_m \bar{Z}^*(m,n),$$

the strong Markov property and Corollary 6.1 to conclude

$$\mathbf{E}[(Z^*_n)^\lambda; V_{m,s}] \leqslant ce^{-\alpha s}e^{-\xi n}.$$

Assume $\tau^i_n \geqslant se^{2n}$ for some integer n. Then there exists an integer $m \leqslant n$ such that

$$\tau^i_m - \tau^i_{m-1} \geqslant 4s\left(\tfrac{3}{4}\right)^{n-m}e^{2n} \geqslant 4s2^{n-m}e^{2m}.$$

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By the above estimate,

$$\mathbf{E}[(Z_n^*)^\lambda; \tau_m^i - \tau_{m-1}^i \geqslant 4s2^{n-m}e^{2m}] \leqslant ce^{-\xi n} \exp\{-\alpha 4s2^{n-m}\}.$$

By summing over all m we get (6.1).

Similarly, if

$$\tilde{Z}_{n,s}^* = \mathbf{P}^1\{S(0,\tau_n] \cap \Gamma_n = \emptyset; \tau_n \geqslant se^{2n}\},$$

we can show

$$\mathbf{E}[(\tilde{Z}_{n,s}^*)^\lambda] \leqslant ce^{-\xi n}e^{-\alpha s}. \tag{6.2}$$

But,

$$\mathbf{E}[(Z_{n,-}^*)^\lambda] \leqslant \mathbf{E}[(Z_n^*)^\lambda] + \sum_{m=0}^{n-1}[\mathbf{E}[(Z_m^*)^\lambda; V_{m,s_m}] + \sum_{m=0}^{n-1}\mathbf{E}[(\tilde{Z}_{m,s}^*)^\lambda],$$

where $s_m = e^{2(n-m)}$. The result then follows from (6.1) and (6.2). □

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