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ARTICLE

Hypergraphs without non-trivial intersecting subgraphs

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Abstract

A hypergraph $\mathcal{F}$ is non-trivial intersecting if every pair of edges in it have a nonempty intersection, but no vertex is contained in all edges of $\mathcal{F}$. Mubayi and Verstraëte showed that for every $k \geq d+1 \geq 3$ and $n \geq (d+1)k/d$ every k-graph $\mathcal{H}$ on n vertices without a non-trivial intersecting subgraph of size $d+1$ contains at most $\binom{n-1}{k-1}$ edges. They conjectured that the same conclusion holds for all $d \geq k \geq 4$ and sufficiently large n. We confirm their conjecture by proving a stronger statement.

They also conjectured that for $m \geq 4$ and sufficiently large n the maximum size of a 3-graph on n vertices without a non-trivial intersecting subgraph of size $3m+1$ is achieved by certain Steiner triple systems. We give a construction with more edges showing that their conjecture is not true in general.

Keywords: hypergraphs; extremal set theory; intersecting family

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1. Introduction

For an integer $k \geq 2$, a k-uniform hypergraph (henceforth k-graph) $\mathcal{H}$ is a collection of k-subsets of some finite set V. The set V is called the vertex set of $\mathcal{H}$, and k-subsets in $\mathcal{H}$ are called edges of $\mathcal{H}$. We use $|\mathcal{H}|$ to denote the size of $\mathcal{H}$, that is, the number of edges in $\mathcal{H}$.

As one of the earliest results in extremal combinatorics, Mantel [20] proved that every n-vertex graph that does not contain a copy of triangle has at most $\lfloor n^2/4 \rfloor$ edges. Motivated by Mantel's result, Erdős considered the following extension in hypergraphs.

A collection of three k-sets A, B, C is called a triangle if $A \cap B$, $A \cap C$, $B \cap C$ are each nonempty, but $A \cap B \cap C$ is empty. Erdős (see e.g. [2, 5, 7]) conjectured that for $k \geq 3$ and $n \geq 3k/2$ every n-vertex k-graph that does not contain a copy of triangle has at most $\binom{n-1}{k-1}$ edges. Later, Chvátal [2] confirmed Erdős' conjecture for $k = 3$. In fact, he proved the following stronger result.

A collection of $d+1$ k-sets $\{A_1, \ldots, A_{d+1}\}$ is called a d-simplex if $\bigcap_{i \neq j} A_i \neq \emptyset$ for all $j \in [d+1]$, but $\bigcap_{i \in [d+1]} A_i = \emptyset$.

Theorem 1.1. (Chvátal [2]) *Suppose that $n \geq k+2 \geq 5$ are integers. Then every n-vertex k-graph $\mathcal{H}$ that does not contain a copy of $(k-1)$-simplex has size at most $\binom{n-1}{k-1}$.*

In the same paper, Chvátal posed the following conjecture.

Conjecture 1.2. (Chvátal's Simplex Conjecture [2]) *Suppose that $k \geq d+1 \geq 3$ and $n \geq d(k+1)/k$ are integers. Then every n-vertex k-graph $\mathcal{H}$ that does not contain a copy of d-simplex has size at most $\binom{n-1}{k-1}$. Moreover, equality holds only if every edge in $\mathcal{H}$ contains a fixed vertex.*

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It is clear that for $d = 2$ Chvátal's Simplex Conjecture is the same as Erdős's conjecture. In [21], Mubayi and Verstraëte considered the case $d = 2$ and proved the following stronger statement.

For an integer $\ell \geq 2$ a hypergraph $\mathcal{F}$ is called ℓ-wise-intersecting if $\bigcap_{i\in[\ell]} E_i \neq \emptyset$ for all $E_1, \ldots, E_d \in \mathcal{F}$, and $\mathcal{F}$ is called non-trivial ℓ-wise-intersecting if it is ℓ-wise-intersecting but $\bigcap_{E\in\mathcal{F}} E = \emptyset$. If $\ell = 2$, we simply call $\mathcal{F}$ intersecting and non-trivial intersecting, respectively.

Theorem 1.3. (Mubayi–Verstraëte [21]) *Let $k \geq d+1 \geq 3$ and $n \geq (d+1)k/d$ be integers. Suppose that $\mathcal{H}$ is an n-vertex k-graph that does not contain non-trivial intersecting subgraph of size $d+1$. Then $|\mathcal{H}| \leq \binom{n-1}{k-1}$, with equality holds only if every edge in $\mathcal{H}$ contains a fixed vertex.*

Since a k-graph of size three is non-trivial intersecting iff it is a triangle, Theorem 1.3 confirms the conjecture of Erdős.

Recently, Currier [4] proved Conjecture 1.2 for all $k \geq d+1 \geq 3$ and $n \geq 2k$. It is still open in general for $n < 2k$ and $3 \leq d \leq k-2$, and we refer the reader to [1, 3, 6, 7, 9, 14, 15, 18] and their references for more results related to Conjecture 1.2.

In [21], Mubayi and Verstraëte remarked that their proof of Theorem 1.3 actually works for d slightly greater than k as well, and they conjectured that the same conclusion should be true for all $d > k \geq 4$.

Conjecture 1.4. (Mubayi–Verstraëte [21]) *Let $d \geq k \geq 4$ be integers and let n be sufficiently large. Suppose that $\mathcal{H}$ is an n-vertex k-graph that does not contain non-trivial intersecting subgraph of size $d+1$. Then $|\mathcal{H}| \leq \binom{n-1}{k-1}$, with equality holds only if every edge in $\mathcal{H}$ contains a fixed vertex.*

The situation when $k = 3$ is different from other cases, and we need the following definition to state their conjecture.

Let $m \geq 2$ be an integer. A Steiner $(n, 3, m-1)$-system is a 3-graph $\mathcal{S}$ on n vertices such that every pair of vertices is contained in exactly $m-1$ edges of $\mathcal{S}$. It follows from Keevash's result [13] that if n is sufficiently large and is a multiple of 3, then there exists a Steiner $(n, 3, m-1)$-system.

Mubayi and Verstraëte observed in [21] that a Steiner $(n, 3, m-1)$-system does not contain non-trivial intersecting subgraph of size $3m+1$, and it has size $\frac{m-1}{3}\binom{n}{2}$, which is greater than $\binom{n-1}{2}$ when $m \geq 4$. Therefore, they made the following conjecture for 3-graphs.

Conjecture 1.5. (Mubayi–Verstraëte [21]) *Let $m \geq 4$ and n be sufficiently large. Suppose that $\mathcal{H}$ is an n-vertex 3-graph that does not contain non-trivial intersecting subgraph of size $3m+1$. Then $|\mathcal{H}| \leq \frac{m-1}{3}\binom{n}{2}$, with equality holds iff $\mathcal{H}$ is a Steiner $(n, 3, m-1)$-system.*

In this paper, we consider these two conjectures of Mubayi and Verstraëte. First, we prove a stronger statement that implies Conjecture 1.4. Due to technical reasons, we postpone the statement of our main result (Theorem 3.3) to Section 4. Instead, we state the following corollary of it.

Corollary 1.6. *Let $k > p \geq 3$ and $d \geq p$ be integers. Suppose that $n \geq n_0(k, d)$ is sufficiently large, and $\mathcal{H}$ is an n-vertex k-graph that does not contain non-trivial $(p-1)$-wise-intersecting subgraph of size $d+1$. Then $|\mathcal{H}| \leq \binom{n-1}{k-1}$, with equality holds only if every edge in $\mathcal{H}$ contains a fixed vertex.*

Remark. Taking $p = 3$ in Theorem 1.6, we confirm Conjecture 1.4.

For 3-graphs, we have the following result that disproves Conjecture 1.5.

Theorem 1.7. *Let $m \geq 4$ be an integer and let n be a sufficiently large integer that is a multiple of 3. Then there exists a 3-graph $\widehat{\mathcal{S}}$ on n vertices with $\frac{m-1}{3}\binom{n}{2} + \frac{n}{3}$ edges and contains no non-trivial intersecting subgraph of size $3m+1$.*

In Section 2, we present a construction that proves Theorem 1.7. In Section 3, we state our main results (Theorems 3.3 and 3.4) and present some preliminary definitions and results. In

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Section 4, we prove some technical lemmas that will be used in the proofs of Theorems 3.3 and 3.4. In Section 5, we prove Theorems 3.3 and 3.4.

2. Constructions

In this section, we give a construction to show that Conjecture 1.5 is not true in general.

Our construction will be obtained from a Steiner $(n, 3, m-1)$-system by adding a perfect matching. In order to show that it is does not contain non-trivial intersecting subgraph of size $3m+1$, we need the following structural theorem of intersecting 3-graphs due to Kostochka and Mubayi [16].

Define

- $EKR(n) = \left\{A \in \binom{[n]}{3} \colon 1 \in A\right\}$.
- $H_0(n) = \left\{A \in \binom{[n]}{3} \colon |A \cap [3]| \geq 2\right\}$.
- $H_1(n) = \left\{A \in \binom{[n]}{3} \colon 1 \in A \text{ and } |A \cap \{2,3,4\}| \geq 1\right\} \cup \{234\}$.
- $H_2(n) = \left\{A \in \binom{[n]}{3} \colon 1 \in A \text{ and } |A \cap \{2,3\}| \geq 1\right\} \cup \{234, 235, 145\}$.
- $H_3(n) = \left\{A \in \binom{[n]}{3} \colon \{1,2\} \in A\right\} \cup \{134, 135, 145, 234, 235, 245\}$.
- $H_4(n) = \left\{A \in \binom{[n]}{3} \colon \{1,2\} \in A\right\} \cup \{134, 156, 235, 236, 245, 246\}$.
- $H_5(n) = \left\{A \in \binom{[n]}{3} \colon \{1,2\} \in A\right\} \cup \{134, 156, 136, 235, 236, 246\}$.

Theorem 2.1. (Kostochka–Mubayi [16]) *Every intersecting 3-graph with at least 11 edges is contained in one of $EKR(n), H_0(n), H_1(n), \ldots, H_5(n)$.*

For a 3-graph $\mathcal{H}$ and $\{u, v\} \subset V(\mathcal{H})$ the codegree $\deg_{\mathcal{H}}(uv)$ of $\{u, v\}$ is the number of edges in $\mathcal{H}$ that contain both u and v. Let $\Delta_2(\mathcal{H}) = \max\{\deg_{\mathcal{H}}(uv) \colon \{u, v\} \subset V(\mathcal{H})\}$ denote the maximum codegree of $\mathcal{H}$. The following observation about $H_0(n), H_1(n), \ldots, H_5(n)$ is easy to confirm.

Observation 2.2. *Let $\mathcal{H}$ be a 3-graph with e edges.*

1. *If $\mathcal{H} \subset H_0(n)$, then $\Delta_2(\mathcal{H}) \geq \lceil \frac{e}{3} \rceil$.*
2. *If $\mathcal{H} \subset H_2(n)$, then $\Delta_2(\mathcal{H}) \geq \lceil \frac{e-3}{2} \rceil$.*
3. *If $\mathcal{H}$ is contained in $H_3(n)$, $H_4(n)$, or $H_5(n)$, then $\Delta_2(\mathcal{H}) \geq e - 6$.*

Now we define our construction. Let n be a sufficiently large integer that is a multiple of 3. Let $\mathcal{S} \subset \binom{[n]}{3}$ be a Steiner $(n, 3, m-1)$-system. Notice that the complement of $\mathcal{S}$, which is $\bar{\mathcal{S}} := \binom{[n]}{3} \setminus \mathcal{S}$, satisfies that $d_{\bar{\mathcal{S}}}(uv) = n - 2 - (m-1) = n - m - 1 = n - o(n)$ for all $\{u, v\} \subset V(\mathcal{S})$. Therefore, by a theorem of Rödl, Ruciński, and Szemerédi (e.g. see Theorem 1.1 in [22]), $\bar{\mathcal{S}}$ contains a perfect matching $\mathcal{M}$. Let $\widehat{\mathcal{S}} = \mathcal{S} \cup \mathcal{M}$ and note that

$$|\widehat{\mathcal{S}}| = |\mathcal{S}| + |\mathcal{M}| = \frac{m-1}{3}\binom{n}{2} + \frac{n}{3}.$$

Next we prove that $\widehat{\mathcal{S}}$ does not contain a non-trivial intersecting subgraph of size $3m+1$.

The idea in the proof below is to show that the maximum codegree of every subgraph of size $3m+1$ in $\widehat{\mathcal{S}}$ is less than the maximum codegree of every subgraph of the same size in $H_i(n)$ for $i \in \{0, \ldots, 5\} \setminus \{1\}$. On the other hand, using an easy structural property of $H_1(n)$, we can show that

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every non-trivial intersecting subgraph of size $3m+1$ in $H_1(n)$ cannot appear in $\widehat{\mathcal{S}}$ as a subgraph. So, it follows from Theorem 2.1 that every intersecting subgraph of size $3m+1$ in $\widehat{\mathcal{S}}$ is trivial.

Proposition 2.3. *Suppose that $m \geq 4$ is an integer. Then $\widehat{\mathcal{S}}$ does not contain a non-trivial intersecting subgraph of size $3m+1$.*

Proof. Suppose to the contrary that there exists a non-trivial intersecting subgraph $\mathcal{F} \subset \widehat{\mathcal{S}}$ with $3m+1 \geq 11$ edges. It follows from Theorem 2.1 that $\mathcal{F}$ is contained in $H_i(n)$ for some $i \in \{0,\ldots,5\}$. Notice that $\Delta_2(\mathcal{F}) \leq \Delta_2(\widehat{\mathcal{S}}) = m$. If $\mathcal{F}$ is contained in $H_i(n)$ for some $i \in \{0,\ldots,5\} \setminus \{1\}$, then by Observation 2.2, we would have $\Delta_2(\mathcal{F}) \geq \min\left\{\lceil\frac{3m+1}{3}\rceil, \lceil\frac{3}{2}m-1\rceil, 3m-5\right\} > m$, a contradiction. Therefore, $\mathcal{F} \subset H_1(n)$. Then it is easy to see from the structure of $H_1(n)$ that $\mathcal{F}$ contains four vertices v_1, v_2, v_3, v_4 such that

$$\deg_{\widehat{\mathcal{S}}}(v_1v_2) + \deg_{\widehat{\mathcal{S}}}(v_1v_3) + \deg_{\widehat{\mathcal{S}}}(v_1v_4) \geq 3m,$$

This implies that $\deg_{\widehat{\mathcal{S}}}(v_1v_2) = \deg_{\widehat{\mathcal{S}}}(v_1v_3) = \deg_{\widehat{\mathcal{S}}}(v_1v_4) = m$ since each of them is at most m. However, this is impossible since the set $\left\{\{u,v\} \subset V(\mathcal{S}) \colon \deg_{\mathcal{S}}(uv) = m\right\}$ consists of $n/3$ copies of pairwise vertex-disjoint triangles. □

Remark. It seems that our proof of Proposition 2.3 cannot show that adding one more edge into $\widehat{\mathcal{S}}$ does not create a non-trivial intersecting subgraph of size $3m+1$. On the other hand, we do not see how to show that adding one more edge into $\widehat{\mathcal{S}}$ will create a non-trivial intersecting subgraph of size $3m+1$. It would be interesting to see if there is a fast way to determine whether $\widehat{\mathcal{S}}$ is maximal.

3. Main results and preliminaries

In this section, we state our main results and present some definitions and preliminary results.

3.1 Main results

Let us first introduce some necessary definitions before stating our main results.

A k-graph $\mathcal{H}$ is called a star if there exists a vertex that is contained in every edge of $\mathcal{H}$. It is easy to see that the maximum size of a k-uniform star on n vertices is $\binom{n-1}{k-1}$.

Let $s \geq 2$ be an integer. A family $\mathcal{D} = \{D_1,\ldots,D_s\}$ is called a Δ-system (or a sunflower) if there exists a vertex set C (might be the empty set) such that $D_i \cap D_j = C$ for all $\{i,j\} \subset [s]$. The set C is called the centre of $\mathcal{D}$.

Definition 3.1. Let $k, d \geq p \geq 2$ be integers and $\vec{a} = (a_1,\ldots,a_p)$, $\vec{b} = (b_1,\ldots,b_p)$ be two sequences of positive integers with $\sum_{i=1}^{p} a_i = k$.

1. A partition $E = \bigcup_{i\in[p]} A_i$ of E is called an $\vec{a}$-partition if $|A_i| = a_i$ for all $i \in [p]$.
2. A family $\{E_0, E_1^1,\ldots,E_1^{b_1},\ldots,E_p^1,\ldots,E_p^{b_p}\}$ is called a semi-$(\vec{a},\vec{b})$-Δ-system if there exists an $\vec{a}$-partition $E_0 = \bigcup_{i\in[p]} A_i$, such that for each $i \in [p]$ the family $\{E_0, E_i^1,\ldots,E_i^{b_i}\}$ is a Δ-system with centre $E_0 \setminus A_i$. The set E_0 is called the host of this semi-$(\vec{a},\vec{b})$-Δ-system.
3. An $(\vec{a},\vec{b})$-Δ-system is a semi-$(\vec{a},\vec{b})$-Δ-system $\{E_0, E_1^1,\ldots,E_1^{b_1},\ldots,E_p^1,\ldots,E_p^{b_p}\}$ such that sets $E_1^1 \setminus E_0,\ldots,E_1^{b_1} \setminus E_0,\ldots,E_p^1 \setminus E_0,\ldots,E_p^{b_p} \setminus E_0$ are pairwise disjoint.
4. An $(\vec{a},d)$-Δ-system is an $(\vec{a},\vec{b})$-Δ-system for some $\vec{b}$ such that $\sum_{i=1}^{p} b_i = d$.

From the definitions one can easily obtain the following observation.

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Observation 3.2. *Suppose that $k, d \geq p \geq 3$ are integers and $\vec{a} = (a_1, \ldots, a_p)$ is a sequence of positive integers with $\sum_{i=1}^{p} a_i = k$. Then an $(\vec{a}, d)$-Δ-system is a nontrivial $(p-1)$-wise-intersecting hypergraph with $d+1$ edges.*

An $(\vec{a}, d)$-Δ-system in which $d = p$, i.e. $b_1 = \cdots = b_p = 1$ was studied by Füredi and Özkahya in [11]. In this paper, we employ a machinery (a complicated version of the delta-system method) developed by them and even earlier by Frankl and Füredi [8], to obtain the following tight bound for the size of an $(\vec{a}, d)$-Δ-system-free hypergraph for all $d \geq p \geq 2$. Here a hypergraph is $(\vec{a}, d)$-Δ-system-free means it does not contain any $(\vec{a}, d)$-Δ-system as a subgraph.

Theorem 3.3. *Let $k > p \geq 2$, $d \geq p$ be integers and $\vec{a} = (a_1, \ldots, a_p)$ be a sequence of positive integers with $\sum_{i=1}^{p} a_i = k$. Let $n_0(k, d)$ be a sufficiently large constant. Suppose that $n \geq n_0(k, d)$ is an integer and $\mathcal{H}$ is an $(\vec{a}, d)$-Δ-system-free k-graph on n vertices. Then $|\mathcal{H}| \leq \binom{n-1}{k-1}$, and equality holds only if $\mathcal{H}$ is a star.*

Remarks.

1. Our proof of Theorem 3.3 uses the delta-system method and a theorem (Theorem 3.7) of Füredi, so the lower bound we obtained for $n_0(k, d)$ is double exponential in k and d. It would be interesting to find the minimum value of $n_0(k, d)$ such that the statement in Theorem 3.3 holds.
2. It is easy to see that Corollary 1.6 is an immediate consequence of Theorem 3.3 and Observation 3.2.

In order to prove Theorem 3.3, we will first prove the following stability result about $(\vec{a}, d)$-Δ-system-free k-graphs.

Theorem 3.4. *Let $k > p \geq 2$, $d \geq p$ be integers and $\vec{a} = (a_1, \ldots, a_p)$ be a sequence of positive integers with $\sum_{i=1}^{p} a_i = k$. For every $\delta > 0$ there exists $\epsilon > 0$ and $n_0(k, d, \delta)$ such that the following holds for all $n \geq n_0(k, d, \delta)$. Suppose that $\mathcal{H}$ is an $(\vec{a}, d)$-Δ-system-free k-graph on n vertices and $|\mathcal{H}| \geq (1-\epsilon)\binom{n-1}{k-1}$. Then there exists a vertex in $\mathcal{H}$ that is contained in all but at most δn^{k-1} edges in $\mathcal{H}$.*

3.2 Preliminaries

In this section, we introduce some theorems and prove some lemmas that are useful for the proofs of Theorems 3.3 and 3.4.

First, we show that every sufficiently large semi-$(\vec{a}, \vec{c})$-Δ-system contains an $(\vec{a}, \vec{b})$-Δ-system.

Lemma 3.5. *Let $k, d \geq p \geq 2$ be integers and let $\vec{a} = (a_1, \ldots, a_p)$, $\vec{b} = (b_1, \ldots, b_p)$, $\vec{c} = (c_1, \ldots, c_p)$ be sequences of positive integers with $\sum_{i=1}^{p} a_i = k$. Suppose that $c_i \geq b_i + \sum_{j=1}^{i-1} a_j b_j$ for all $i \in [p]$. Then every semi-$(\vec{a}, \vec{c})$-Δ-system contains an $(\vec{a}, \vec{b})$-Δ-system. In particular, if $c_1 \geq 1$ and $c_i \geq kd$ for $2 \leq i \leq p$, then every semi-$(\vec{a}, \vec{c})$-Δ-system contains an $(\vec{a}, d)$-Δ-system.*

Proof. Let $\mathcal{F} = \{E_0, E_1^1, \ldots, E_1^{c_1}, \ldots, E_p^1, \ldots, E_p^{c_p}\}$ be a semi-$(\vec{a}, \vec{c})$-Δ-system. We will show that there exists a subset $\{F_i^1, \ldots, F_i^{b_i}\} \subset \{E_i^1, \ldots, E_i^{c_i}\}$ for all $i \in [p]$ such that the family $\{E_0, F_1^1, \ldots, F_1^{b_1}, \ldots, F_p^1, \ldots, F_p^{b_p}\}$ is an $(\vec{a}, \vec{b})$-Δ-system.

First, since $c_1 \geq b_1$, we can simply let $F_1^j = E_1^j$ for $j \in [b_1]$. Now suppose that we have defined the family $\{F_1^1, \ldots, F_1^{b_1}, \ldots, F_i^1, \ldots, F_i^{b_i}\}$ for some $i \in [p-1]$. We are going to define

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sets $F_{i+1}^1, \dots, F_{i+1}^{b_{i+1}}$. Note that for every $j \in [i]$ and $\ell \in [b_j]$ the set $F_j^\ell \setminus E_0$ can have nonempty intersection with at most a_j sets in $\{E_{i+1}^1, \dots, E_{i+1}^{c_{i+1}}\}$. Since $c_{i+1} \geq b_{i+1} + \sum_{j=1}^{i} a_j b_j$, there exist at least b_{i+1} sets in $\{E_{i+1}^1, \dots, E_{i+1}^{c_{i+1}}\}$ that have empty intersection with all sets in $\{F_1^1 \setminus E_0, \dots, F_1^{b_1} \setminus E_0, \dots, F_i^1 \setminus E_0, \dots, F_i^{b_i} \setminus E_0\}$. We may choose arbitrary b_{i+1} sets from them to form $\{F_{i+1}^1, \dots, F_{i+1}^{b_{i+1}}\}$. We terminate the process when $i = p$. It is clear that the family $\{E_0, F_1^1, \dots, F_1^{b_1}, \dots, F_p^1, \dots, F_p^{b_p}\}$ is an $(\vec{a}, \vec{b})$-Δ-system. □

For a hypergraph $\mathcal{H}$ and an edge $E \in \mathcal{H}$. The intersection structure of E with respect to $\mathcal{H}$ is

$$\mathcal{I}(E, \mathcal{H}) := \{E \cap E' : E' \in \mathcal{H} \setminus \{E\}\}.$$

Given a family $\mathcal{F}$ (not necessarily k-uniform), we say it is k-partite if there exists a partition $V(\mathcal{F}) = V_1 \cup \cdots \cup V_k$ such that $|E \cap V_i| \leq 1$ for all $E \in \mathcal{F}$ and $i \in [k]$. Suppose that $\mathcal{F}$ is k-partite with k parts $V_1, \dots, V_k$. For every set $E \in \mathcal{F}$, let the projection of E be $\Pi(E) := \{i : E \cap V_i \neq \emptyset\}$. Let the projection of $\mathcal{F}$ be $\Pi(\mathcal{F}) := \{\Pi(E) : E \in \mathcal{F}\}$[1].

Definition 3.6. Let $s \geq 2$ be an integer. A k-graph $\mathcal{H}$ is s-homogeneous if it satisfies the following conditions.

1. The k-graph $\mathcal{H}$ is k-partite.
2. There exists a family $\mathcal{J} \subset 2^{[k]} \setminus \{[k]\}$ such that $\Pi(\mathcal{I}(E, \mathcal{H})) = \mathcal{J}$ for all $E \in \mathcal{H}$, where $\mathcal{J}$ is called the intersection pattern of $\mathcal{H}$.
3. The family $\mathcal{J}$ is closed under intersection, i.e. if $A, B \in \mathcal{J}$, then $A \cap B \in \mathcal{J}$.
4. For every edge $E \in \mathcal{H}$ and every set $S \in \mathcal{I}(E, \mathcal{H})$ there exists a Δ-system $\mathcal{D}$ of size s with centre S such that $E \in \mathcal{D} \subset \mathcal{H}$.

We call $\mathcal{H}$ homogeneous if it is 2-homogeneous.

A powerful result of Füredi [10] shows that for every integer $s \geq 2$ every hypergraph contains a large s-homogeneous subgraph.

Theorem 3.7. (Füredi [10]) *For fixed integers $s, k \geq 2$, there exists a constant $c(k, s) > 0$ such that every k-graph $\mathcal{H}$ contains an s-homogeneous subgraph $\mathcal{H}^*$ with $|\mathcal{H}^*| \geq c(k, s)|\mathcal{H}|$.*

For a family $\mathcal{J} \subset 2^{[k]} \setminus \{[k]\}$, let the rank of $\mathcal{J}$ be

$$r(\mathcal{J}) := \min\{|A| : A \subset [k], A \notin \mathcal{J} \text{ and } \nexists B \in \mathcal{J} \text{ such that } A \subset B\}.$$

It is easy to see from the definition that if $\mathcal{J}$ is closed under intersection, then $r(\mathcal{J}) = k$ iff $\mathcal{J} = 2^{[k]} \setminus \{[k]\}$.

For a k-graph $\mathcal{H}$ and integer $i \in [k-1]$, let the ith shadow of $\mathcal{H}$ be

$$\partial_i \mathcal{H} := \left\{ A \in \binom{[n]}{k-i} : \exists E \in \mathcal{H} \text{ such that } A \subset E \right\}.$$

For convenience, we let $\partial \mathcal{H} = \partial_1 \mathcal{H}$.

The following lemma gives an upper bound for the size of a homogeneous hypergraph $\mathcal{H}$ in terms of its shadow and the rank of its intersection pattern.

Lemma 3.8. *Let $\mathcal{H}$ be a homogeneous k-graph with intersection pattern $\mathcal{J} \subset 2^{[k]} \setminus \{[k]\}$. Then $|\mathcal{H}| \leq |\partial_{k-r} \mathcal{H}|$ for every integer r that satisfies $r(\mathcal{J}) \leq r \leq k-1$.*

[1] Later, when we talk about a k-partite k-graph, we will assume that there is a fixed k-partition of its vertex set. Differen k-partite k-graphs on the same vertex set could have different k-partitions.

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Proof. Let r be a positive integer that satisfies $r(\mathcal{J}) \le r \le k-1$. By the definition of rank, there exists an $r(\mathcal{J})$-set $S \subset [k]$ that is not contained in $\mathcal{J}$, and moreover, every set $T \subset [k]$ that contains S is also not contained in $\mathcal{J}$. In particular, every set $T \subset [k]$ of size r that contains S is not contained in $\mathcal{J}$. Since $\Pi(\mathcal{I}(E,\mathcal{H})) = \mathcal{J}$ for all $E \in \mathcal{H}$, there exists an r-set in every $E \in \mathcal{H}$ that is not contained in any other edges in $\mathcal{H}$. Therefore, $|\mathcal{H}| \le |\partial_{k-r}\mathcal{H}|$. □

4. Lemmas

In this section, we prove some technical lemmas that will be used later in the proofs of Theorems 3.3 and 3.4.

First, we show that the rank of the intersection pattern of such a k-graph is at most $k-1$.

Lemma 4.1. *Let $d \ge p \ge 2$, $k > p$, $s \ge kd+1$ be integers, and $\vec{a} = (a_1, \dots, a_p)$ be a sequence of positive integers with $\sum_{i=1}^{p} a_i = k$. Let $\mathcal{H}$ be an s-homogeneous k-graph with intersection pattern $\mathcal{J} \subset 2^{[k]} \setminus \{[k]\}$. If $r(\mathcal{J}) = k$, then $\mathcal{H}$ contains an $(\vec{a}, d)$-Δ-system.*

Proof. Since $r(\mathcal{J}) = k$ and $\mathcal{J}$ is closed under intersection, we have $\mathcal{J} = 2^{[k]} \setminus \{[k]\}$. Let $E \in \mathcal{H}$ and let $\bigcup_{i=1}^{p} A_i = E$ be an $\vec{a}$-partition of E. Since $\Pi(\mathcal{I}(E,\mathcal{H})) = \mathcal{J} = 2^{[k]} \setminus \{[k]\}$, we have $E \setminus A_i \in \mathcal{I}(E,\mathcal{H})$ for all $i \in [p]$. Since $\mathcal{H}$ is s-homogeneous, for every $i \in [p]$ there exists a Δ-system $\mathcal{D}_i$ of size s with centre $E \setminus A_i$. Since $s \ge kd+1$, it follows from Lemma 3.5 that $\mathcal{H}$ contains an $(\vec{a}, d)$-Δ-system. □

Lemmas 4.1 and 3.8, and Theorem 3.7 imply that following proposition.

Proposition 4.2. *Let $d \ge p \ge 2$, $k > p$ be integers, and $\vec{a} = (a_1, \dots, a_p)$ be a sequence of positive integers with $\sum_{i=1}^{p} a_i = k$. Then there exists a constant $c(k,d) > 0$ such that every $(\vec{a}, d)$-Δ-systems-free k-graph $\mathcal{H}$ satisfies $|\partial\mathcal{H}| \ge c(k,d)|\mathcal{H}|$.*

Proof. Let $s = kd+1$ and $\mathcal{H}^*$ be a maximum s-homogeneous subgraph of $\mathcal{H}$ with intersection pattern $\mathcal{J}$. Then by Theorem 3.7, $|\mathcal{H}^*| \ge c(k,d)|\mathcal{H}|$ for some constant $c(k,d) > 0$. Since $\mathcal{H}^*$ contains no $(\vec{a}, d)$-Δ-systems, it follows from Lemma 4.1 that $r(\mathcal{J}) \le k-1$. Then it follows from Lemma 3.8 that $|\mathcal{H}^*| \le |\partial\mathcal{H}^*|$. Therefore,

$$|\partial\mathcal{H}| \ge |\partial\mathcal{H}^*| \ge |\mathcal{H}^*| \ge c(k,d)|\mathcal{H}|.$$ □

The next lemma gives another condition that guarantees a hypergraph to contain an $(\vec{a}, d)$-Δ-system as a subgraph.

Lemma 4.3. *Let $d \ge p \ge 2$, $k > p$, $s \ge kd+1$ be integers and $\vec{a} = (a_1, \dots, a_p)$ be a sequence of positive integers with $\sum_{i=1}^{p} a_i = k$. Let $\mathcal{H}$ be a k-graph and $\mathcal{H}^*$ be an s-homogeneous subgraph of $\mathcal{H}$. Let $E_0 \in \mathcal{H}^*$ and $\bigcup_{i\in[p]} A_i = E_0$ be an $\vec{a}$-partition. Suppose that there exists $i_0 \in [p]$ and $E \in \mathcal{H}$ such that $E_0 \setminus A_i \in \mathcal{I}(E,\mathcal{H}^*)$ for all $i \in [p] \setminus \{i_0\}$, and $E \cap E_0 = E_0 \setminus A_{i_0}$. Then $\mathcal{H}$ contains an $(\vec{a}, d)$-Δ-system.*

Proof. Without loss of generality, we may assume that $i_0 = 1$. By assumption, $E_0 \setminus A_i$ is the centre of a Δ-system of size $s \ge kd+1$ in $\mathcal{H}^* \subset \mathcal{H}$ for $2 \le i \le k$, and $E_0 \setminus A_1$ is the centre of a Δ-system of size 2 in $\mathcal{H}$, i.e. $\{E_0, E\}$. So it follows from Lemma 3.5 that $\mathcal{H}$ contains an $(\vec{a}, d)$-Δ-system. □

Lemma 4.4. *Let $d \ge p \ge 2$, $k > p$, $s \ge kd+1$ be integers, and $\vec{a} = (a_1, \dots, a_p)$ be a sequence of positive integers with $\sum_{i=1}^{p} a_i = k$. Let $\mathcal{H}$ be an $(\vec{a}, d)$-Δ-system-free k-graph and $\mathcal{H}^*$ be an s-homogeneous subgraph of $\mathcal{H}$ with intersection pattern $\mathcal{J}$. Suppose that $r(\mathcal{J}) = k-1$ and the number of $(k-1)$-set in $\mathcal{J}$ is exactly $k-1$. For every $E \in \mathcal{H}^*$ if $v \in E$ is the vertex that is contained in all $(k-1)$-sets in $\mathcal{I}(E,\mathcal{H}^*)$, then $v \in F$ for every $F \in \mathcal{H}$ that satisfies $|F \cap E| \ge k - a_1$*

Proof. Let $E = \{v_1, \dots, v_k\} \in \mathcal{H}^*$ and suppose that v_1 is contained in all $(k-1)$-sets in $\mathcal{I}(E,\mathcal{H}^*)$. Note that since $\mathcal{I}(E,\mathcal{H}^*)$ is closed under intersection, all proper subsets of E that contain v_1 are

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contained in E. Let $F \in \mathcal{H}$ and suppose that $|E \cap F| = k - t$ for some $1 \le t \le a_1$. Suppose to the contrary that $v_1 \notin F$.

If $t = a_1$, then let $\bigcup_{i\in[p]} A_i = E$ be an $\vec{a}$-partition such that $A_1 = E \setminus F$. For $2 \le i \le p$ since $v_1 \in E \setminus A_i$, we have and it follows from the assumption that $E \setminus A_i$ is the centre of a Δ-system of size s in $\mathcal{H}^*$. On the other hand, notice that $E \setminus A_1$ is the centre of a Δ-system of size 2, i.e. $\{E, F\}$. So it follows from Lemma 3.5 that $\mathcal{H}$ contains an $(\vec{a}, d)$-Δ-system, a contradiction. Therefore, $t < a_1$.

Let $M \subset E$ be a set of size $k - a_1 + t$ such that $E \setminus F \subset M$. Since $|M| \le k - 1$ and $v_1 \in M$, we have $M \in \mathcal{I}(E, \mathcal{H}^*)$. By assumption, M is the center of a Δ-system $\mathcal{D} \subset \mathcal{H}^*$ of size s in $\mathcal{H}^*$. Since $F \setminus E$ can have nonempty intersection with at most $k < s$ edges in $\mathcal{D}$, there exists an edge $E_1 \in \mathcal{D}$ such that $(E_1 \setminus E) \cap F = \emptyset$. Notice that E_1 satisfies $E_1 \cap F = M \setminus (E \setminus F)$ and $|E_1 \cap F| = k - a_1$. Since $\Pi(\mathcal{I}(E_1, \mathcal{H}^*)) = \Pi(\mathcal{I}(E, \mathcal{H}^*))$, applying the same argument as above to E_1 and F we obtain that $\mathcal{H}$ contains an $(\vec{a}, d)$-Δ-system, a contradiction. □

For a hypergraph $\mathcal{H}$ and an edge $E \in \mathcal{H}$ let the weight of E in $\mathcal{H}$ be

$$\omega_{\mathcal{H}}(E) := \sum_{\substack{E' \subset E \\ |E'|=k-1}} \frac{1}{\deg_{\mathcal{H}}(E')},$$

where $\deg_{\mathcal{H}}(E')$ is the number of edges in $\mathcal{H}$ containing E'.

From the definition we have the following identity:

$$\sum_{E\in\mathcal{H}} \omega_{\mathcal{H}}(E) = \sum_{E\in\mathcal{H}} \sum_{\substack{E' \subset E \\ |E'|=k-1}} \frac{1}{\deg_{\mathcal{H}}(E')} = \sum_{E'\in\partial\mathcal{H}} \sum_{\substack{E\in\mathcal{H} \\ E' \subset E}} \frac{1}{\deg_{\mathcal{H}}(E')} = |\partial\mathcal{H}|. \tag{1}$$

The following lemma gives a lower bound for $\omega_{\mathcal{H}}(E)$ regarding the intersection structure of E in a homogeneous subgraph of $\mathcal{H}$. This lemma together with Equation (1) will be crucial later to give an upper bound for the size of a homogeneous subgraph of an $(\vec{a}, d)$-Δ-system-free k-graph.

Lemma 4.5. *Let $d \ge p \ge 2$, $k > p$, $s \ge kd + 1$ be integers, and $\vec{a} = (a_1, \dots, a_p)$ be a sequence of positive integers with $\sum_{i=1}^{p} a_i = k$. Let $\mathcal{H}$ be an $(\vec{a}, d)$-Δ-system-free k-graph and $\mathcal{H}^*$ be an s-homogeneous subgraph of $\mathcal{H}$ with intersection pattern $\mathcal{J}$. Suppose that $r(\mathcal{J}) = k - 1$. Then the following statements hold.*

1. *If the number of $(k-1)$-set in $\mathcal{J}$ is exactly $k-1$, then every $E \in \mathcal{H}^*$ contains a $(k-1)$-subset that is not contained in any other edges in $\mathcal{H}$. More specifically, if $v \in E$ is the vertex such that all $(k-1)$-sets but $E \setminus \{v\}$ are contained in $\mathcal{I}(E, \mathcal{H}^*)$, then $E \setminus \{v\}$ is not contained in any other edge of $\mathcal{H}$. Consequently, $\omega_{\mathcal{H}}(E) \ge 1$ for all $E \in \mathcal{H}^*$.*
2. *If the number of $(k-1)$-set in $\mathcal{J}$ is at most $k-2$, then $\omega_{\mathcal{H}}(E) \ge \frac{k}{k-1}$ for all $E \in \mathcal{H}^*$.*

Proof. We may assume that $a_1 \ge \cdots \ge a_p$. Note that since $\sum_{i=1}^{p} a_i = k > p$, we have $a_1 \ge 2$.

Let us begin with the proof for the first statement. Let $E = \{v_1, \dots, v_k\} \in \mathcal{H}^*$. By assumption, the number of $(k-1)$-set in $\mathcal{I}(E, \mathcal{H}^*)$ is exactly $k-1$. Without loss of generality, we may assume that $E \setminus \{v_i\} \in \mathcal{I}(E, \mathcal{H}^*)$ for $2 \le i \le k$. We claim that $\{v_2, \dots, v_k\}$ is not contained in any set in $\mathcal{H} \setminus \{E\}$. Indeed, if there exists $E_1 \in \mathcal{H} \setminus \{E\}$ such that $\{v_2, \dots, v_k\} \subset E_1$, then we have $|E_1 \cap E| \ge k - 1$, but it follows from Lemma 4.4 that $v_1 \in E_1$ as well. This means that $E_1 = E$, a contradiction.

Now we prove the second statement. Suppose that the number of $(k-1)$-set in $\mathcal{J}$ is $k - t$ for some integer t with $2 \le t \le k$. Let $E = \{v_1, \dots, v_k\} \in \mathcal{H}^*$ be an edge. Without loss of generality, we may assume that $E \setminus \{v_i\} \in \mathcal{I}(E, \mathcal{H}^*)$ for $t + 1 \le i \le k$ and $E \setminus \{v_j\} \notin \mathcal{I}(E, \mathcal{H}^*)$ for $j \in [t]$.

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Our goal is to find a set $S \subset \{v_1, \dots, v_t\}$ of size ℓ for some integer $\ell \le t$ such that $d_{\mathcal{H}}(E \setminus \{v\})$ is at most $\ell - 1$ for every $v \in S$. Then it would imply that

$$\omega_{\mathcal{H}}(E) = \sum_{\substack{E' \subset E \\ |E'|=k-1}} \frac{1}{\deg_{\mathcal{H}}(E')} > \sum_{v \in L \setminus S} \frac{1}{\deg_{\mathcal{H}}(E \setminus \{v\})} \ge \frac{\ell}{\ell - 1} \ge \frac{k}{k-1}.$$

First, we need to prove the following claim.

Claim 4.6. *There do not exist a $(t-1)$-set $I \subset [t]$ and $t-1$ distinct vertices $\{u_i : i \in I\}$, such that $(E \setminus \{v_i\}) \cup \{u_i\} \in \mathcal{H}$ for all $i \in I$.*

Proof. Suppose to the contrary that this statement is not true. Without loss of generality, we may assume that $I = [t] \setminus \{1\}$ and there exist distinct vertices $u_2, \dots, u_t$ such that $F_i := (E \setminus \{v_i\}) \cup \{u_i\} \in \mathcal{H}$ for $2 \le i \le t$.

By assumption $\mathcal{I}(E, \mathcal{H}^*)$ contains all $(k-1)$-sets that contain $\{v_1, \dots, v_t\}$. Since $\mathcal{I}(E, \mathcal{H}^*)$ is closed under intersection, $\mathcal{I}(E, \mathcal{H}^*)$ contains all proper subsets of E that contain $\{v_1, \dots, v_t\}$. In other words, if $A \subset \{v_{t+1}, \dots, v_k\}$ and $A \neq \emptyset$, then $E \setminus A \in \mathcal{I}(E, \mathcal{H}^*)$.

We claim that for all $\{i, j\} \subset [t]$ we have $E \setminus \{v_i, v_j\} \in \mathcal{I}(E, \mathcal{H}^*)$. This is true because otherwise it would follow from the assumption $E \setminus \{v_i\}, E \setminus \{v_j\} \notin \mathcal{I}(E, \mathcal{H}^*)$ that $r(\mathcal{I}(E, \mathcal{H}^*)) \le k-2$, which contradicts the fact that $r(\mathcal{I}(E, \mathcal{H}^*)) = k-1$. Now by the argument above and the property that $\mathcal{I}(E, \mathcal{H}^*)$ is closed under intersection, we know that if a set $A \subset E$ satisfies $|A \cap \{v_1, \dots, v_t\}| \ge 2$, then $E \setminus A \in \mathcal{I}(E, \mathcal{H}^*)$.

Let $i_0 \in [p]$ such that $\sum_{i=1}^{i_0-1} a_i < t \le \sum_{i=1}^{i_0} a_i$, and let $\ell = t - \sum_{i=1}^{i_0-1} a_i$. Note that $i_0 \le t$. Recall that $a_1 \ge \cdots \ge a_p \ge 1$ and $a_1 \ge 2$.

Suppose that $\ell \ge 2$. Then there exists an $\vec{a}$-partition $E = \bigcup_{i \in [p]} A_i$ such that $A_1, \dots, A_{i_0-1} \subset \{v_1, \dots, v_t\}$, $|A_{i_0} \cap \{v_1, \dots, v_t\}| = \ell \ge 2$, and $A_{i_0+1}, \dots, A_p \subset \{v_{t+1}, \dots, v_k\}$. Since $a_1 \ge \cdots \ge a_{i_0-1} \ge a_{i_0} \ge 2$, by the argument above, $E \setminus A_i \in \mathcal{I}(E, \mathcal{H}^*)$ for all $i \in [p]$. Therefore, $E \setminus A_i$ is the centre of a Δ-system of size $s \ge kd + 1$ in $\mathcal{H}^*$ for $i \in [p]$, and it follows from Lemma 3.5 that $\mathcal{H}$ contains an $(\vec{a}, d)$-Δ-system, a contradiction. Therefore, $\ell = 1$.

Suppose that $a_{i_0} = 1$. Then we let $E = \bigcup_{i \in [p]} A_i$ be an $\vec{a}$-partition such that $\bigcup_{i \in [i_0]} A_i = \{v_1, \dots, v_t\}$ and $v_1 \in A_1$. Since $A_1 \subset \{v_1, \dots, v_t\}$ and $|A_1| \ge 2$, we have $E \setminus A_1 \in \mathcal{I}(E, \mathcal{H}^*)$. So $E \setminus A_1$ is the centre of a Δ-system $\mathcal{D}_1 \subset \mathcal{H}^*$ of size s. We will assume that $a_2 = \cdots = a_{i_0} = 1$. Other cases follow analogously. For $i_0 + 1 \le i \le p$ since $A_i \subset \{v_{t+1}, \dots, v_k\}$, we have $E \setminus A_i \in \mathcal{I}(E, \mathcal{H}^*)$ and $E \setminus A_i$ is the centre of a Δ-system $\mathcal{D}_i \subset \mathcal{H}^*$ of size s. Notice that by assumption, for every $2 \le i \le i_0$ we have $F_i \in \mathcal{H}$ such that $F_i \cap E = E \setminus \{v_i\}$, and moreover, sets in $\{F_i \setminus E : 2 \le i \le i_0\}$ are disjoint. Now, fix F_i for $2 \le i \le i_0$ and choose greedily a set $E_j \in \mathcal{D}_j$ for $i_0 < j \le p$ such that sets in $\{F_i \setminus E : 2 \le i \le i_0\} \cup \{E_j \setminus E : i_0 < j \le p\}$ are disjoint (since $s \ge kd + 1$ such E_j's always exist). Finally, choose $d - p + 1$ sets from $\mathcal{D}_1$ such that they have empty intersection with all sets in $\{F_i \setminus E : 2 \le i \le i_0\} \cup \{E_j \setminus E : i_0 < j \le p\}$ (since $s \ge kd + 1$ we can always find such $d - p + 1$ sets). This shows that $\mathcal{H}$ contains an $(\vec{a}, d)$-Δ-system, a contradiction. Therefore, $a_{i_0} \ge 2$.

Suppose that $a_1 \ge 3$. Then we let $E = \bigcup_{i \in [p]} A_i$ be an $\vec{a}$-partition such that $A_2 \cup \cdots \cup A_{i_0-1} \subset \{v_1, \dots, v_t\}$, $v_{t+1} \in A_1$ and $A_1 \setminus \{v_{t+1}\} \subset \{v_1, \dots, v_t\}$, and $\{v_1, \dots, v_t\} \setminus \left(\bigcup_{i \in [i_0-1]} A_i\right) \subset A_{i_0}$. By the argument in the last paragraph, we have $|A_i \cap \{v_1, \dots, v_t\}| \ge 2$ for all $i \in [i_0]$. The partition $E = \bigcup_{i \in [p]} A_i$ we chose ensures that $A_j \subset \{v_{t+1}, \dots, v_k\}$ for all $i_0 + 1 \le j \le p$. Therefore, for every $i \in [p]$ the set $E \setminus A_i$ is contained in $\mathcal{I}(E, \mathcal{H}^*)$ (this follows from the second and the third paragraphs in the proof of Claim 4.6), and hence, $E \setminus A_i$ is the center of a Δ-system of size $s \ge kd + 1$ in $\mathcal{H}^*$. So, it follows from Lemma 3.5 that $\mathcal{H}$ contains an $(\vec{a}, d)$-Δ-system, a contradiction. Therefore, $a_1 = 2$.

Suppose that $a_p = 1$. Then we let $E = \bigcup_{i \in [p]} A_i$ be an $\vec{a}$-partition such that $A_1 \cup \cdots \cup A_{i_0-1} = \{v_1, \dots, v_{t-1}\}$ and $A_p = \{v_t\}$. For every $i \in [i_0 - 1]$ since $|A_i \cap E| \ge 2$, we have $E \setminus A_i \in \mathcal{I}(E, \mathcal{H}^*)$.

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For every $i_0 \le j \le p-1$ since $A_j \subset \{v_{t+1}, \dots, v_k\}$, we have $E \setminus A_j \in \mathcal{I}(E, \mathcal{H}^*)$. Therefore, $E \setminus A_i$ is the center of a Δ-system of size $s \ge kd+1$ in $\mathcal{H}^*$ for $i \in [p-1]$. Note that $E \setminus A_p$ is the center of a Δ-system of size 2 in $\mathcal{H}$, i.e. $\{E, F_t\}$. So it follows from Lemma 3.5 that $\mathcal{H}$ contains an $(\vec{a}, d)$-Δ-system, a contradiction. Therefore, $a_p = 2$.

Now we have $a_1 = \cdots = a_p = 2$ and $\ell = 1$. This implies that t is odd, $t \ge 3$, k is even, and $k > t$. Since $E \setminus \{v_k\} \in \mathcal{I}(E, \mathcal{H}^*)$, we know that $E \setminus \{v_k\}$ is the center of a Δ-system $\mathcal{D}$ of size s in $\mathcal{H}^*$. So we can choose a set F_k from $\mathcal{D}$ such that $F_k \setminus E$ has empty intersection with $\{u_2, \dots, u_t\}$. In other words, there exists a vertex u_k that is not contained in $\{u_2, \dots, u_t\}$ such that $F_k = (E \setminus \{v_k\}) \cup \{u_k\}$. Let $F_k = \bigcup_{i\in[p]} A_i$ be an $\vec{a}$-partition such that $A_1 = \{v_2, u_k\}$, $A_2 = \{v_1, v_3\}$, and $A_i = \{v_{2i-2}, v_{2i-1}\}$ for $3 \le i \le p$. Then for every $i \in [p] \setminus \{1\}$, either $A_i \subset \{v_1, \dots, v_t\}$ or $A_i \subset \{v_{t+1}, \dots, v_{k-1}\}$. Since $\Pi(\mathcal{I}(F_k, \mathcal{H}^*)) = \Pi(\mathcal{I}(E, \mathcal{H}^*))$, the set $F_k \setminus A_i$ is the center of a Δ-system of size $s \ge kd+1$ in $\mathcal{H}^*$ for $i \in [p] \setminus \{1\}$. Since $F_k \setminus A_1$ is the center of a Δ-system of size 2 in $\mathcal{H}$, i.e. $\{F_k, F_2\}$. So it follows from Lemma 3.5 that $\mathcal{H}$ contains an $(\vec{a}, d)$-Δ-system, a contradiction. This completes the proof of Claim 4.6. □

Define a bipartite graph G with two parts $L = \{v_1, \dots, v_t\}$ and $R = [n] \setminus E$, and for every $v_i \in L$ and $u \in R$, $v_i u$ is an edge in G iff $(E \setminus \{v_i\}) \cup \{u\} \in \mathcal{H}$. Claim 4.6 implies that there are at most $t-2$ pairwise disjoint edges in G. Therefore, by the König–Hall theorem, G contains a vertex cover S with $|S| \le t-2$. Let $\ell = |L \setminus S| \ge 2$. Then $|S \cap R| \le \ell - 2$. For every $v \in L \setminus S$ since $N_G(v) \subset S \cap R$, we obtain

$$\deg_{\mathcal{H}}(E \setminus \{v\}) = \deg_G(v) + 1 \le \ell - 1.$$

This implies that

$$\omega_{\mathcal{H}}(E) = \sum_{\substack{E' \subset E \\ |E'|=k-1}} \frac{1}{\deg_{\mathcal{H}}(E')} > \sum_{v \in L \setminus S} \frac{1}{\deg_{\mathcal{H}}(E \setminus \{v\})} \ge \frac{\ell}{\ell - 1} \ge \frac{k}{k-1}.$$

This completes the proof of Lemma 4.5. □

5. Proofs of Theorems 3.3 and 3.4

In this section, we prove Theorems 3.3 and 3.4.

First, let us prove Theorem 3.4. The proof can be divided into two steps. In the first step, we will decompose the edge set of an $(\vec{a}, d)$-Δ-system-free k-graph $\mathcal{H}$ into many disjoint s-homogeneous subgraphs, and then using Lemma 3.8, Lemma 4.3, and Equation (1), we will show that most edges (let us call them good edges) are contained in s-homogeneous subgraphs whose intersection patterns contain exactly $k-1$ $(k-1)$-sets (see Claim 5.1). In the second step, we will partition the set of good edges into stars with different centers and prove some property about them (see Claim 5.2). Then using the classical Kruskal–Katona theorem and some simple inequalities, we will show that most edges are contained in one star.

Proof of Theorem 3.4. Let $k > p \ge 2$, $d \ge p$ be integers, and $\vec{a} = (a_1, \dots, a_p)$ be a sequence of integers with $a_1 \ge \cdots \ge a_p \ge 1$ and $\sum_{i\in[p]} a_i = k$. Let $\epsilon > 0$ be a sufficiently small constant and n be sufficiently large integer. Let $\mathcal{H} \subset \binom{[n]}{k}$ be an $(\vec{a}, d)$-Δ-system-free hypergraph with at least $(1-\epsilon)\binom{n-1}{k-1}$ edges.

Let $s = kd+1$ and $\mathcal{H}_1$ be a maximum s-homogeneous subgraph of $\mathcal{H}$. Suppose that we have defined $\mathcal{H}_1, \dots, \mathcal{H}_i$[2] for some $i \ge 1$. Let $\mathcal{H}_{i+1}$ be the maximum s-homogeneous subgraph of $\mathcal{H} \setminus \left(\bigcup_{j=1}^{i} \mathcal{H}_j\right)$. This process terminates after m steps for some positive integer m if

[2]Here for distinct i_1 and i_2, the vertex sets $V(\mathcal{H}_{i_1})$ and $V(\mathcal{H}_{i_2})$ are allowed to have different k-partitions.

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$\mathcal{H} \setminus \left(\bigcup_{j=1}^{m} \mathcal{H}_j\right) = \emptyset$ or the intersection pattern of $\mathcal{H}_{m+1}$ has rank at most $k-2$. Let $\mathcal{J}_i$ denote the intersection pattern of $\mathcal{H}_i$ for $i \in [m+1]$. It follows from the definition and Lemma 4.1 that $r(\mathcal{J}_i) = k-1$ for $i \in [m]$. Let

$$\widehat{\mathcal{H}}_1 = \bigcup \left\{\mathcal{H}_i \colon i \in [m] \text{ and the number of } (k-1)\text{-set in } \mathcal{J}_i \text{ is exactly } k-1\right\},$$

$$\widehat{\mathcal{H}}_2 = \bigcup \left\{\mathcal{H}_i \colon i \in [m] \text{ and the number of } (k-1)\text{-set in } \mathcal{J}_i \text{ is at most } k-2\right\}, \quad \text{and}$$

$$\widehat{\mathcal{H}}_3 = \mathcal{H} \setminus \left(\widehat{\mathcal{H}}_1 \cup \widehat{\mathcal{H}}_2\right) = \mathcal{H} \setminus \left(\bigcup_{i \in [m]} \mathcal{H}_i\right).$$

Our first step is to show that $|\widehat{\mathcal{H}}_2|$ and $|\widehat{\mathcal{H}}_3|$ are small.

Claim 5.1. $|\widehat{\mathcal{H}}_2| + |\widehat{\mathcal{H}}_3| < 3\epsilon k\binom{n-1}{k-1}$.

Proof. First, we show that $|\widehat{\mathcal{H}}_3| = O(n^{k-2})$. We may assume that $\widehat{\mathcal{H}}_3 \neq \emptyset$. Recall that $\mathcal{H}_{m+1}$ is a maximum s-homogeneous subgraph of $\widehat{\mathcal{H}}_3$ with intersection pattern $\mathcal{J}_{m+1}$ of rank at most $k-2$. It follows from Theorem 3.7 that there exists a constant $c(k,s) > 0$ such that $|\mathcal{H}_{m+1}| \geq c(k,s)|\widehat{\mathcal{H}}_3|$. Since $r(\mathcal{J}_{m+1}) \leq k-2$, it follows from Lemma 3.8 that $|\mathcal{H}_{m+1}| \leq |\partial_2 \mathcal{H}_{m+1}| \leq \binom{n}{k-2}$. Therefore, $|\widehat{\mathcal{H}}_3| \leq \frac{1}{c(k,s)}|\mathcal{H}_{m+1}| \leq \frac{1}{c(k,s)}\binom{n}{k-2}$.

Next we show that $\widehat{\mathcal{H}}_2 \leq 2\epsilon k\binom{n-1}{k-1}$. By Equation (1) and Lemma 4.5, we have

$$|\partial \mathcal{H}| = \sum_{E \in \mathcal{H}} \omega_{\mathcal{H}}(E) \geq \sum_{E \in \widehat{\mathcal{H}}_1} \omega_{\mathcal{H}}(E) + \sum_{E \in \widehat{\mathcal{H}}_2} \omega_{\mathcal{H}}(E) \geq |\widehat{\mathcal{H}}_1| + \frac{k}{k-1}|\widehat{\mathcal{H}}_2|.$$

This implies that $|\widehat{\mathcal{H}}_1| + \frac{k}{k-1}|\widehat{\mathcal{H}}_2| \leq \binom{n}{k-1}$, which further gives

$$\begin{aligned} |\widehat{\mathcal{H}}_2| &= (k-1)\left(|\widehat{\mathcal{H}}_1| + \frac{k}{k-1}|\widehat{\mathcal{H}}_2| + |\widehat{\mathcal{H}}_3| - |\mathcal{H}|\right) \\ &\leq (k-1)\left(\binom{n}{k-1} + |\widehat{\mathcal{H}}_3| - (1-\epsilon)\binom{n-1}{k-1}\right) < 2\epsilon k\binom{n-1}{k-1}. \end{aligned}$$

This completes the proof of Claim 5.1. □

Notice that in the proof of Claim 5.1, we showed that $|\widehat{\mathcal{H}}_3| \leq \frac{1}{c(k,s)}\binom{n}{k-2}$ and $|\widehat{\mathcal{H}}_1| + \frac{k}{k-1}|\widehat{\mathcal{H}}_2| \leq \binom{n}{k-1}$. Consequently,

$$\begin{aligned} |\mathcal{H}| \leq |\widehat{\mathcal{H}}_1| + \frac{k}{k-1}|\widehat{\mathcal{H}}_2| + |\widehat{\mathcal{H}}_3| &\leq \binom{n}{k-1} + \frac{1}{c(k,s)}\binom{n}{k-2} \\ &\leq \binom{n-1}{k-1} + \epsilon\binom{n-1}{k-1}, \end{aligned} \tag{2}$$

where the last inequality holds because $n \geq n_0(k,d)$ is large.

Claim 5.1 implies that

$$|\widehat{\mathcal{H}}_1| = |\mathcal{H}| - \left(|\widehat{\mathcal{H}}_2| + |\widehat{\mathcal{H}}_3|\right) > (1 - 4\epsilon k)\binom{n-1}{k-1}. \tag{3}$$

By definition, for every $E \in \widehat{\mathcal{H}}_1$, there exists a unique s-homogeneous hypergraph $\mathcal{H}_i$ for some $i \in [m]$ such that $E \in \mathcal{H}_i$, and moreover, $r(\mathcal{J}_i) = k-1$ and the number of $(k-1)$-set in $\mathcal{J}_i$ is exactly $k-1$. This implies that there is a unique vertex $c \in E$ such that every $(k-1)$-subset of E that contains c is contained in $\mathcal{I}(E, \mathcal{H}_i)$. For every edge $E \in \widehat{\mathcal{H}}_1$ we use $c(E)$ to denote this unique vertex. Define $\mathcal{G}_i = \left\{E \in \widehat{\mathcal{H}}_1 \colon c(E) = i\right\}$ for $i \in [n]$, and notice that $\bigcup_{i \in [n]} \mathcal{G}_i = \widehat{\mathcal{H}}_1$ is a partition. Let $\mathcal{G}_i' = \{E \setminus \{i\} \colon E \in \mathcal{G}_i\}$ for $i \in [n]$. From the first statement of Lemma 4.5, we know that for every $i \in [n]$

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and every $E \in \mathcal{G}_i$ the set $E \setminus \{i\}$ is not contained in any set in $\mathcal{H} \setminus \{E\}$. Therefore, $\mathcal{G}'_i \cap \mathcal{G}'_j = \emptyset$ for all $\{i,j\} \subset [n]$. Using Lemma 4.4, we can prove a even stronger statement.

Claim 5.2. *$\partial\mathcal{G}'_i \cap \partial\mathcal{G}'_j = \emptyset$ for all $\{i,j\} \subset [n]$.*

Proof. Suppose to the contrary that this is not true. Without loss of generality, we may assume that there exists $A \in \partial\mathcal{G}'_1 \cap \partial\mathcal{G}'_2$. Then this means that there exists $E_1 \in \mathcal{G}_1$ and $E_2 \in \mathcal{G}_2$ such that $E_1 = \{1,u\} \cup A$ and $E_2 = \{2,v\} \cup A$ for some $u, v \in [n]$. Since $\mathcal{G}_1$ is s-homogeneous and $|E_2 \cap E_1| \geq k-2 \geq k-a_1$, it follows from Lemma 4.4 that $1 \in E_2$. Similarly, we obtain $2 \in E_1$. Therefore, $E_1 = E_2 = \{1,2\} \cup A$, which implies that $\{1,2\} \cup A \in \mathcal{G}_1 \cap \mathcal{G}_2$, a contradiction. □

For every $i \in [n]$ let $x_i \in \mathbb{R}^+$ be a positive real number such that $|\mathcal{G}_i| = |\mathcal{G}'_i| = \binom{x_i}{k-1}$. Without loss of generality we may assume that $x_1 \geq \cdots \geq x_n \geq 0$. By the well-known Kruskal–Katona theorem (see e.g. [12, 17, 19]), for every $i \in [n]$ we have

$$|\mathcal{G}'_i| \leq \frac{\binom{x_i}{k-1}}{\binom{x_i}{k-2}} |\partial\mathcal{G}'_i| = \frac{x_i - k + 2}{k-1} |\partial\mathcal{G}'_i|.$$

Therefore, by Equation (3) and Claim 5.2, we have

$$\begin{aligned}(1-4\epsilon k)\binom{n-1}{k-1} < |\widehat{\mathcal{H}}_1| = \sum_{i\in[n]} |\mathcal{G}_i| = \sum_{i\in[n]} |\mathcal{G}'_i| &\leq \sum_{i\in\mathcal{H}} \frac{x_i - k + 2}{k-1} |\partial\mathcal{G}'_i| \\ &\leq \frac{x_1 - k + 2}{k-1} \sum_{i\in\mathcal{H}} |\partial\mathcal{G}'_i| \leq \frac{x_1 - k + 2}{k-1} \binom{n}{k-2},\end{aligned}$$

which implies that

$$x_1 \geq (k-1)\frac{(1-4\epsilon k)\binom{n-1}{k-1}}{\binom{n}{k-2}} + k - 2 > (1-5\epsilon k)n.$$

Therefore,

$$|\mathcal{G}_1| = \binom{x_1}{k-1} > \binom{(1-5\epsilon k)n}{k-1} > \frac{\big((1-6\epsilon k)n\big)^{k-1}}{(k-1)!} > (1-7\epsilon k^2)\binom{n-1}{k-1}.$$

This inequality together with Equation (2) imply that all but at most $7\epsilon k^2\binom{n-1}{k-1} + \epsilon\binom{n-1}{k-1} \leq 30\epsilon n^{k-1}$ edges in $\mathcal{H}$ contain the vertex 1. This completes the proof of Theorem 3.4. □

Based on Theorem 3.4, we are now ready to prove Theorem 3.3.

Proof of Theorem 3.3. Let $d \geq p \geq 2$, $k > p$, $s = kd+1$ be integers, and $\vec{a} = (a_1, \ldots, a_p)$ be a sequence of positive integers with $a_1 \geq \cdots \geq a_p$ and $\sum_{i\in[p]} a_i = k$. Let $c(k,d) > 0$ be the constant guaranteed by Proposition 4.2 and let $\delta = \big(c(k,s)/100\big)^{k-1}/k$. Let $n_0(k,d,\delta)$ the constant guaranteed by Theorem 3.4 and let $n \geq n_0(k,d,\delta)$ be an integer. Note that since δ only depends on k and d, the constant $n_0(k,d,\delta)$ also only depends on k and d. Let $\mathcal{H} \subset \binom{[n]}{k}$ be an $(\vec{a},d)$-Δ-system-free hypergraph with $|\mathcal{H}| = \binom{n-1}{k-1}$. It suffices to show that all edges in $\mathcal{H}$ contain a fixed vertex.

From the proof of Theorem 3.4, we know that $\mathcal{H}$ contains a subgraph $\mathcal{G}_1$ of size at least $\binom{n-1}{k-1} - \delta n^{k-1}$ such that all edges in $\mathcal{G}_1$ contain a fixed vertex (we may assume that this vertex is 1), and moreover, $\mathcal{G}_1$ consists of pairwise edge-disjoint s-homogeneous hypergraphs whose intersection patterns have rank $k-1$ and contain all $(k-1)$-subsets of $[k]$ that contain 1.

Define

$$\mathcal{B}_0 = \{E \in \mathcal{H} \colon 1 \notin E\},$$

$$\mathcal{B}_1 = \{E \in \mathcal{H} \colon 1 \in E \text{ and } |E \cap B| \geq k - a_1 \text{ for some } B \in \mathcal{B}_0\},$$

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$$\mathcal{G} = \{E \in \mathcal{H} \setminus (\mathcal{B}_1 \cup \mathcal{B}_0) \colon \forall S \subset E \text{ with } 1 \in S, S \text{ is the center of a } \Delta\text{-system of size } s \text{ in } \mathcal{H}\},$$

$$\mathcal{B}_2 = \{E \in \mathcal{H} \colon 1 \in E\} \setminus (\mathcal{B}_1 \cup \mathcal{G}).$$

Note that $\mathcal{G}_1 \subset \mathcal{G}$. Therefore, $|\mathcal{B}_0| + |\mathcal{B}_1| + |\mathcal{B}_2| \le \delta n^{k-1}$.

Our aim is to prove that $|\mathcal{B}_0| + |\mathcal{B}_1| + |\mathcal{B}_2| = 0$. We will do this by first showing that every set in $\mathcal{B}_i$ for $i \in \{0, 1, 2\}$ has a $(k-2)$-subset that is not contained in other sets in $\mathcal{G}$ and $\mathcal{B}_i$. Then using the Kruskal–Katona theorem and some simple inequalities, we will show that $|\mathcal{G}| + 4|\mathcal{B}_i| \le \binom{n-1}{k-1}$ for $i \in \{0, 1, 2\}$. Adding these three inequalities up and using the assumption that $|\mathcal{H}| = \binom{n-1}{k-1}$ we will get $|\mathcal{B}_0| + |\mathcal{B}_1| + |\mathcal{B}_2| = 0$.

Let

$$\mathcal{B}_1' = \{E \setminus 1 \colon E \in \mathcal{B}_1\}, \quad \mathcal{G}' = \{E \setminus 1 \colon E \in \mathcal{G}\}, \quad \text{and} \quad \mathcal{B}_2' = \{E \setminus 1 \colon E \in \mathcal{B}_2\}.$$

Let $\mathcal{B}_1^*, \mathcal{B}_2^*$ be maximum s-homogeneous subgraphs of $\mathcal{B}_1', \mathcal{B}_2'$, respectively. It follows from Theorem 3.7 that $|\mathcal{B}_i^*| \ge c(k,s)|\mathcal{B}_i'|$ for $i = 1, 2$. Recall that for every $E \in \partial\mathcal{G}'$, $\deg_{\mathcal{G}'}(E)$ is the number of edges in $\mathcal{G}'$ that contain E. It follows from $\sum_{E \in \partial\mathcal{G}'} \deg_{\mathcal{G}'}(E) = (k-1)|\mathcal{G}'|$ and $\deg_{\mathcal{G}'}(E) \le n - k + 1$ that

$$|\partial\mathcal{G}'| \ge \frac{k-1}{n-k+1}|\mathcal{G}'|. \tag{4}$$

Claim 5.3. $|\mathcal{G}| + 4|\mathcal{B}_0| \le \binom{n-1}{k-1}$.

Proof. It is easy to see from the definition of $\mathcal{G}$ and $\mathcal{B}_0$ that $|E \cap B| \le k - a_1 - 1 \le k - 3$ for all $E \in \mathcal{G}'$ and $B \in \mathcal{B}_0$. Therefore, $\partial\mathcal{G}' \cap \partial_2\mathcal{B}_0 = \emptyset$, and hence $|\partial\mathcal{G}'| + |\partial_2\mathcal{B}_0| \le \binom{n-1}{k-2}$. Let $x \in \mathbb{R}^+$ be a positive integer such that $|\partial\mathcal{B}_0| = \binom{x}{k-1}$. It follows from the Kruskal–Katona theorem and Proposition 4.2 that

$$|\partial_2\mathcal{B}_0| \ge \frac{k-1}{x-k+1}|\partial\mathcal{B}_0| \ge \frac{k-1}{x-k+1}c(k,s)|\mathcal{B}_0|.$$

This together with Equation (4) and $|\partial\mathcal{G}'| + |\partial_2\mathcal{B}_0| \le \binom{n-1}{k-2}$ imply that

$$\frac{k-1}{n-k+1}|\mathcal{G}'| + \frac{k-1}{x-k+1}c(k,s)|\mathcal{B}_0| \le \binom{n-1}{k-2}.$$

Consequently,

$$\begin{aligned}
|\mathcal{G}| + c(k,s)\frac{n-k+1}{x-k+1}|\mathcal{B}_0| &= |\mathcal{G}'| + c(k,s)\frac{n-k+1}{x-k+1}|\mathcal{B}_0| \\
&= \frac{n-k+1}{k-1}\left(\frac{k-1}{n-k+1}|\mathcal{G}'| + \frac{k-1}{x-k+1}c(k,s)|\mathcal{B}_0|\right) \\
&\le \frac{n-k+1}{k-1}\binom{n-1}{k-2} = \binom{n-1}{k-1}.
\end{aligned}$$

By Theorem 3.4, we have $\binom{x}{k-1} = |\partial\mathcal{B}_0| \le k|\mathcal{B}_0| \le \delta k n^{k-1}$. It follows that $x \le (k\delta)^{\frac{1}{k-1}} n \le c(k,s)n/100$. Since $n \ge n_0(k,d,\delta)$ is large, we have

$$c(k,s)\frac{n-k+1}{x-k+1} \ge c(k,s)\frac{n-k+1}{c(k,s)n/100 - k + 1} > 4.$$

Therefore, the inequality above gives $|\mathcal{G}| + 4|\mathcal{B}_0| \le \binom{n-1}{k-1}$. □

Claim 5.4. *Every $E \in \mathcal{B}_1^*$ contains a $(k-2)$-subset that is not contained in any other set in $\mathcal{B}_1^* \cup \mathcal{G}'$.*

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Proof. Suppose to the contrary that there exists an edge $E=\{v_1,\dots,v_{k-1}\}\in\mathcal{B}_1^*$ such that for every $i\in[k-1]$ the set $E\setminus\{v_i\}$ is contained in some other set in $\mathcal{B}_1^*\cup\mathcal{G}'$. Without loss of generality, we may assume that for some integer ℓ the set $E\setminus\{v_i\}$ is contained in $\mathcal{I}(E,\mathcal{G}')$ for all $i\in[\ell]$, and the set $E\setminus\{v_j\}$ is contained in $\mathcal{I}(E,\mathcal{B}_1^*)$ for all $\ell+1\le j\le k-1$.

Let $\mathcal{J}_{\mathcal{B}_1^*}$ be the intersection pattern of $\mathcal{B}_1^*$. Let $\widehat{\mathcal{B}_1^*}=\{E\cup\{1\}\colon E\in\mathcal{B}_1^*\}$, and notice that $\widehat{\mathcal{B}_1^*}$ is s-homogeneous with intersection pattern $\mathcal{J}_{\widehat{\mathcal{B}_1^*}}:=\{A\cup\{1\}\colon A\in\mathcal{J}_{\mathcal{B}_1^*}\}$. Let $\widehat{E}=E\cup\{1\}$ and note that $\widehat{E}$ is an edge in $\widehat{\mathcal{B}_1^*}$.

If $\ell=0$, then $\mathcal{J}_{\mathcal{B}_1^*}=2^{[k-1]}\setminus\{[k-1]\}$. Consequently, $r(\mathcal{I}(\widehat{E},\widehat{\mathcal{B}_1^*}))=k-1$ and $\mathcal{I}(\widehat{E},\widehat{\mathcal{B}_1^*})$ contains all $(k-1)$-subsets of $\widehat{E}$ that contain 1. By the definition of $\mathcal{B}_1$, there exists $B\in\mathcal{B}_0$ such that $|B\cap\widehat{E}|\ge k-a_1$. It follows from Lemma 4.4 that $1\in B$, a contradiction. Therefore, $\ell\ge1$.

For every $i\in[\ell]$ let $E_i\in\mathcal{G}$ be an edge such that $E_i\cap\widehat{E}=\widehat{E}\setminus\{v_i\}$. Let $B\in\mathcal{B}_0$ be an edge such that $|B\cap\widehat{E}|\ge k-a_1$. Suppose that $|B\cap\widehat{E}|=k-t$, where t is an integer that satisfies $1\le t\le a_1$. Then for every $i\in[\ell]$ we have $|B\cap E_i|\ge k-t-1$. On the other hand, by the definition of $\mathcal{G}$, for every $i\in[\ell]$ we have $|B\cap E_i|\le k-a_1-1\le k-t-1$. Therefore, we must have $t=a_1$, $|B\cap\widehat{E}|=k-a_1$, and $v_i\in B$ for all $i\in[\ell]$. Let $\bigcup_{i\in[p]}A_i=\widehat{E}$ be an $\vec{a}$-partition such that $A_1=\widehat{E}\setminus B$. Note that for $2\le i\le p$, either there exists some $j\in[\ell]$ such that $\widehat{E}\setminus A_i\subset E_j$ or $\{1,v_1,\dots,v_\ell\}\subset\widehat{E}\setminus A_i$. In the former case, since $1\in\widehat{E}\setminus A_i$, it follows from the definition of $\mathcal{G}$ that $\widehat{E}\setminus A_i$ is the center of some Δ-system of size s in $\mathcal{H}$. In the latter case, we have $\widehat{E}\setminus A_i\in\mathcal{I}(\widehat{E},\widehat{\mathcal{B}_1^*})$, which implies that $\widehat{E}\setminus A_i$ is the center of some Δ-system of size s in $\widehat{\mathcal{B}_1^*}$. Note that $E\setminus A_1$ is the center of a Δ-system of size 2, i.e. $\{\widehat{E},B\}$. So, it follows from Lemma 3.5 that $\mathcal{H}$ contains an $(\vec{a},d)$-Δ-system, a contradiction. □

By Claim 5.4, we obtain $|\partial\mathcal{G}'|+|\mathcal{B}_1^*|\le\binom{n-1}{k-2}$, which together with Equation (4) and $|\mathcal{B}_1^*|\ge c(k,s)|\mathcal{B}'_1|=c(k,s)|\mathcal{B}_1|$ imply that

$$\begin{aligned}|\mathcal{G}|+c(k,s)\frac{n-k+1}{k-1}|\mathcal{B}_1|&=|\mathcal{G}'|+c(k,s)\frac{n-k+1}{k-1}|\mathcal{B}_1|\\&\le\frac{n-k+1}{k-1}\left(|\partial\mathcal{G}'|+|\mathcal{B}_1^*|\right)\le\binom{n-1}{k-1}.\end{aligned}$$

Since $n\ge n_0(k,d,\delta)$ is large, we have $c(k,s)\frac{n-k+1}{k-1}>4$. Therefore,

$$|\mathcal{G}|+4|\mathcal{B}_1|\le\binom{n-1}{k-1}.$$

Claim 5.5. *Every $E\in\mathcal{B}_2^*$ has a $(k-2)$-subset that is not contained in any other set in $\mathcal{B}_2^*\cup\mathcal{G}'$.*

Proof. Suppose to the contrary that there exists an edge $E=\{v_1,\dots,v_{k-1}\}\in\mathcal{B}_2^*$ such that for every $i\in[k-1]$ the set $E\setminus\{v_i\}$ is contained in some other set in $\mathcal{B}_2^*\cup\mathcal{G}'$. Without loss of generality we may assume that there exists an integer $[\ell]$ such that $E\setminus\{v_i\}\in\mathcal{I}(E,\mathcal{G}')$ for $i\in[\ell]$, and $E\setminus\{v_j\}\in\mathcal{I}(E,\mathcal{B}_2^*)$ for $\ell+1\le j\le k-1$.

Let $\mathcal{J}_{\mathcal{B}_2^*}$ be the intersection pattern of $\mathcal{B}_2^*$. Let $\widehat{\mathcal{B}_2^*}=\{E\cup\{1\}\colon E\in\mathcal{B}_2^*\}$. Note that $\widehat{\mathcal{B}_2^*}$ is also s-homogeneous with intersection pattern $\mathcal{J}_{\widehat{\mathcal{B}_2^*}}:=\{A\cup\{1\}\colon A\in\mathcal{J}_{\mathcal{B}_2^*}\}$. Let $\widehat{E}=E\cup\{1\}$ and note that $\widehat{E}$ is an edge in $\widehat{\mathcal{B}_2^*}$.

If $\ell=0$, then $\mathcal{J}_{\mathcal{B}_2^*}=2^{[k-1]}\setminus\{[k-1]\}$. Consequently, $r(\mathcal{I}(\widehat{E},\widehat{\mathcal{B}_2^*}))=k-1$, and hence $\mathcal{I}(\widehat{E},\widehat{\mathcal{B}_2^*})$ contains all $(k-1)$-subsets of $\widehat{E}$ that contain 1. Since $\mathcal{I}(\widehat{E},\widehat{\mathcal{B}_2^*})$ is closed under intersection, all proper subsets of $\widehat{E}$ that contain 1 are contained in $\mathcal{I}(\widehat{E},\widehat{\mathcal{B}_2^*})$. However, by the definition of $\mathcal{G}$, this implies that $\widehat{E}\in\mathcal{G}$, a contradiction. Therefore, $\ell\ge1$.

For every $i\in[\ell]$ let $E_i\in\mathcal{G}$ be an edge such that $E_i\cap\widehat{E}=\widehat{E}\setminus\{v_i\}$. Let S be a proper subset of $\widehat{E}$ with $1\in S$. If $v_i\notin S$ for some $i\in[\ell]$, then $S\subset E_i$ and it follows from the definition of $\mathcal{G}$ that S is the

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center of some Δ-system of size s in $\mathcal{H}$. If $\{v_1, \ldots, v_\ell\} \subset S$, then $S \in \mathcal{I}(\widehat{E}, \widehat{\mathcal{B}_2^*})$ and it follows that S is the center of some Δ-system of size s in $\mathcal{B}_2^*$. Therefore, every proper subset of $\widehat{E}$ that contains 1 is the center of some Δ-system of size s in $\mathcal{H}$. However, it follows from the definition of $\mathcal{G}$ that $\widehat{E} \in \mathcal{G}$, a contradiction. □

Claim 5.5 implies that $|\partial\mathcal{G}'| + |\mathcal{B}_2^*| \le \binom{n-1}{k-2}$. Similarly to the argument above Claim 5.5 we obtain

$$|\mathcal{G}| + 4|\mathcal{B}_2| \le \binom{n-1}{k-1}.$$

It follows from $|\mathcal{H}| = \binom{n-1}{k-1}$ that

$$\begin{aligned} 3\binom{n-1}{k-1} &\le 3|\mathcal{H}| + |\mathcal{B}_0| + |\mathcal{B}_1| + |\mathcal{B}_2| \\ &= 3\left(|\mathcal{G}| + |\mathcal{B}_0| + |\mathcal{B}_1| + |\mathcal{B}_2|\right) + |\mathcal{B}_0| + |\mathcal{B}_1| + |\mathcal{B}_2| \\ &= |\mathcal{G}| + 4|\mathcal{B}_0| + |\mathcal{G}| + 4|\mathcal{B}_1| + |\mathcal{G}| + 4|\mathcal{B}_2| \le 3\binom{n-1}{k-1}. \end{aligned}$$

This implies that $\mathcal{B}_0 = \mathcal{B}_1 = \mathcal{B}_2 = \emptyset$ and $|\mathcal{G}| = \binom{n-1}{k-1}$. Therefore, $\mathcal{H} = \mathcal{G}$, and hence every edge in $\mathcal{H}$ contains the vertex 1. This completes the proof of Theorem 3.3. □

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References

[1] Bermond, J.-C. and Frankl, P. (1977) On a conjecture of Chvátal on m-intersecting hypergraphs. *Bull. London Math. Soc.* **9**(3) 310–312.
[2] Chvátal, V. (1974/75) An extremal set-intersection theorem. *J. London Math. Soc. (2)* **9** 355–359.
[3] Csákány, R. and Kahn, J. (1999) A homological approach to two problems on finite sets. *J. Algebraic Combin.* **9**(2) 141–149.
[4] Currier, G. (2021) New results on simplex-clusters in set systems. *Combinatorica* **41**(4) 495–506.
[5] Erdős, P. (1971) Topics in combinatorial analysis. In *Proceedings of the Second Louisiana Conference on Combinatorics, Graph Theory and Computing*, pp.2–20.
[6] Frankl, P. (1976) On Sperner families satisfying an additional condition. *J. Combin. Theory Ser. A* **20**(1) 1–11.
[7] Frankl, P. (1981) On a problem of Chvátal and Erdős on hypergraphs containing no generalized simplex. *J. Combin. Theory Ser. A* **30**(2) 169–182.
[8] Frankl, P. and Füredi, Z. (1985) Forbidding just one intersection. *J. Combin. Theory Ser. A* **39**(2) 160–176.
[9] Frankl, P. and Füredi, Z. (1987) Exact solution of some Turán-type problems. *J. Combin. Theory Ser. A* **45**(2) 226–262.
[10] Füredi, Z. (1983) On finite set-systems whose every intersection is a kernel of a star. *Discrete Math.* **47**(1) 129–132.
[11] Füredi, Z. and Özkahya, L. (2011) Unavoidable subhypergraphs: **a**-clusters. *J. Combin. Theory Ser. A* **118**(8) 2246–2256.
[12] Katona, G. (1968) A theorem of finite sets. In *Theory of Graphs (Proc. Colloq., Tihany, 1966)*, pp.187–207.
[13] Keevash, P. (2018) The existence of designs II. arXiv preprint arXiv: 1802. 05900.
[14] Keevash, P. and Mubayi, D. (2010) Set systems without a simplex or a cluster. *Combinatorica* **30**(2) 175–200.
[15] Keller, N. and Lifshitz, N. (2021) The junta method for hypergraphs and the Erdős-Chvátal simplex conjecture. *Adv. Math* **392**(Paper No. 107991) 95.
[16] Kostochka, A. and Mubayi, D. (2017) The structure of large intersecting families. *Proc. Amer. Math. Soc.* **145**(6) 2311–2321.
[17] Kruskal, J. B. (1963) The number of simplices in a complex. *In Mathematical Optimization Techniques*. Univ. California Press, pp. 251–278.
[18] Lifshitz, N. (2020) On set systems without a simplex-cluster and the junta method. *J. Combin. Theory Ser. A* **170**(105139) 20.
[19] Lovász, L. (1993) *Combinatorial Problems and Exercises, Second edition*. North-Holland Publishing Co.

https://doi.org/10.1017/S096354832200013X Published online by Cambridge University Press

[20] Mantel, W. (1907) Problem 28. *Wiskundige Opgaven* **10**(60-61) 320.
[21] Mubayi, D. and Verstraëte, J. (2005) Proof of a conjecture of Erdős on triangles in set-systems. *Combinatorica* **25**(5) 599–614.
[22] Rödl, V., Ruciński, A. and Szemerédi, E. (2006) Perfect matchings in uniform hypergraphs with large minimum degree. *European J. Combin.* **27**(8) 1333–1349.

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