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A RANGE PROPERTY RELATED TO NON-EXPANSIVE OPERATORS

BIAGIO RICCERI

Abstract. In this paper, we prove that if X is an infinite-dimensional real Hilbert space and $J : X \to \mathbb{R}$ is a sequentially weakly lower semicontinuous C^1 functional whose Gâteaux derivative is non-expansive, then there exists a closed ball B in X such that $(\mathrm{id} + J')(B)$ intersects every convex and dense subset of X.

Here and in what follows, $(X, \langle \cdot, \cdot \rangle)$ is an infinite-dimensional real Hilbert space and $T : X \to X$ is a non-expansive potential operator. This means that

$$\|T(x) - T(y)\| \leqslant \|x - y\|$$

for all $x, y \in X$ and that T is the Gâteaux derivative of a functional $J : X \to \mathbb{R}$.

For instance, any continuous symmetric linear operator from X into itself, with norm less than or equal to 1, is a non-expansive potential operator.

Another classical example of such operators is as follows. Let $f : X \to \mathbb{R}$ be a convex continuous function and, for each $x \in X$, let $\partial f(x)$ denote the subdifferential of f at x, i.e.

$$\partial f(x) = \left\{ z \in X : \inf_{y \in X} (f(y) - \langle z, y \rangle) \geqslant f(x) - \langle z, x \rangle \right\}.$$

Then $x \to x - (\mathrm{id} + \partial f)^{-1}(x)$ is a non-expansive potential operator [3].

In general, we refer to the monograph [1] for a thorough treatment of non-expansive mappings.

Now let $\Phi : X \to X$ be the operator defined by

$$\Phi(x) = x + T(x)$$

for all $x \in X$.

The aim of this short note is to establish the following result.

THEOREM 1. *If the functional J is sequentially weakly lower semicontinuous, then there exists a closed ball B in X such that $\Phi(B)$ intersects each convex and dense subset of X.*

Before proving Theorem 1, some remarks are in order.

When T is a contraction, Theorem 1 is immediate. Actually, in that case, thanks to the Banach fixed point principle, the operator Φ turns out to be a

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homeomorphism between X and itself. Hence, the really interesting case is when the Lipschitz constant of T is exactly 1.

Theorem 1 is no longer true if J is not sequentially weakly lower semicontinuous. In this connection, the simplest example is provided by $T(x) = -x$. Actually, since $\dim(X) = \infty$, the norm is not sequentially weakly upper semicontinuous.

A further remark is that, under the assumptions of Theorem 1, it may happen that the set $\Phi(B)$ has an empty interior for every ball B in X. In this connection, consider the case where T is a compact, symmetric, negative linear operator with norm 1. In such a case, by classical results, J is sequentially weakly continuous and $\Phi(X) \neq X$. Since $\Phi(X)$ is a linear subspace, this clearly implies that $\mathrm{int}(\Phi(X)) = \emptyset$.

Our proof of Theorem 1 comes out from combining some ideas from [4] with a very recent multiplicty result established in [5]. For the reader's convenience, we now recall it not in its general original form, but in a way which is enough for our current purpose.

THEOREM A [5, Theorem 3.2]. *Let $C \subset X$ be a weakly compact set and $Y \subseteq X$ a convex set. Moreover, let $f : C \times Y \to \mathbb{R}$ be a function such that, for each $y \in Y$, $f(\cdot, y)$ is weakly lower semicontinuous, while, for each $x \in C$, $f(x, \cdot)$ is concave and upper semicontinuous. Finally, assume that*

$$\sup_{y \in Y} \inf_{x \in C} f(x, y) < \inf_{x \in C} \sup_{y \in Y} f(x, y).$$

Then there exists $\hat{y} \in Y$ such that $f(\cdot, \hat{y})$ has at least two global minima in C.

Proof of Theorem 1. If the functional J is convex, then T is maximal monotone and, by a classical result of Minty [2], Φ turns out to be a homeomorphism between X and itself. So, in that case, we are done. Therefore, assume that J is not convex. As a consequence, there exist $x_1, x_2 \in X$ and $\lambda \in]0, 1[$ such that

$$\lambda J(x_1) + (1 - \lambda) J(x_2) < J(x_3),$$

where

$$x_3 = \lambda x_1 + (1 - \lambda) x_2.$$

Fix $z \in X$ so that

$$\langle x_1 - x_2, z \rangle = J(x_1) - J(x_2)$$

and put

$$\tilde{J}(x) = J(x_3 - x) - \langle x_3 - x, z \rangle$$

for all $x \in X$. Note that

$$\tilde{J}(\lambda(x_1 - x_2)) = \tilde{J}((1 - \lambda)(x_2 - x_1)) < \tilde{J}(0). \tag{1}$$

Now, put

$$r = \max\{\lambda, 1 - \lambda\} \|x_1 - x_2\|$$

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and denote by C the closed ball in X of radius r centered at 0. Fix a convex and dense set $V \subseteq X$ and put

$$Y = V - x_3 - z.$$

Hence, Y is convex and dense too. Consider the function $f : X \times Y \to \mathbb{R}$ defined by

$$f(x, y) = \tilde{J}(x) + \langle x, y \rangle$$

for all $(x, y) \in X \times Y$. Observing that, for each $y \in Y$, one has

$$\min\{\langle \lambda(x_1 - x_2), y \rangle, \langle (1 - \lambda)(x_2 - x_1), y \rangle\} \leqslant 0,$$

in view of the equality in (1), it follows that

$$\sup_{y \in Y} \inf_{x \in C} f(x, y) \leqslant \tilde{J}(\lambda(x_1 - x_2)). \tag{2}$$

On the other hand, by the density of Y, for each $x \in C \setminus \{0\}$, one has

$$\sup_{y \in Y} \langle x, y \rangle = +\infty,$$

and hence

$$\inf_{x \in C} \sup_{y \in Y} f(x, y) = \tilde{J}(0). \tag{3}$$

Thus, from (1)–(3) it follows that

$$\sup_{y \in Y} \inf_{x \in C} f(x, y) < \inf_{x \in C} \sup_{y \in Y} f(x, y).$$

Then, since $f(\cdot, y)_{|C}$ is weakly lower semicontinuous in C (thanks to the Eberlein–Smulyan theorem) and $f(x, \cdot)$ is concave and continuous in Y, we can apply Theorem A. Accordingly, there exists $\hat{y} \in Y$ such that $f(\cdot, \hat{y})_{|C}$ has at least two global minima u_1, u_2 in C. Now, consider the function $g : X \times [1, +\infty[\to \mathbb{R}$ defined by

$$g(x, \lambda) = \frac{\lambda}{2}(\|x\|^2 - r^2) + f(x, \hat{y})$$

for all $(x, \lambda) \in X \times [1, +\infty[$. Observe that the functional $g(\cdot, \lambda)$ (besides being continuous) is strictly convex and coercive if $\lambda > 1$, while it is convex if $\lambda = 1$. Indeed, let $\lambda \geqslant 1$. For each $x, y \in X$, we have

$$\begin{aligned}
&\langle \lambda x + T(x_3 - x) - \lambda y + T(x_3 - y), x - y \rangle \\
&\quad = \lambda \|x - y\|^2 - \langle T(x_3 - x) - T(x_3 - y), x - y \rangle \\
&\quad \geqslant \lambda \|x - y\|^2 - \|T(x_3 - x) - T(x_3 - y)\| \|x - y\| \geqslant (\lambda - 1)\|x - y\|^2.
\end{aligned}$$

From this, it follows that the Gâteaux derivative of the functional $g(\cdot, \lambda)$ (that is, the operator $x \to \lambda x - T(x_3 - x) + z + \hat{y}$) is monotone and that it is uniformly monotone if $\lambda > 1$. Now the claim follows from classical results [**3**, pp. 247–248]. Furthermore, for each $x \in X$, the function $g(x, \cdot)$ is concave and continuous, and $\lim_{\lambda \to +\infty} g(0, \lambda) = -\infty$. So, we are allowed to apply a

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classical saddle-point theorem [6, Theorem 49.A] to the function g. Accordingly, there exists $(\hat{x}, \hat{\lambda}) \in X \times [1, +\infty[$ such that

$$g(\hat{x}, \hat{\lambda}) = \inf_{x \in X} g(x, \hat{\lambda}) = \sup_{\lambda \geqslant 1} g(\hat{x}, \lambda).$$

This implies that $\hat{x} \in C$,

$$\frac{\hat{\lambda}}{2}\|\hat{x}\|^2 + f(\hat{x}, \hat{y}) = \inf_{x \in X}\left(\frac{\hat{\lambda}}{2}\|x\|^2 + f(x, \hat{y})\right) \tag{4}$$

and

$$\frac{\hat{\lambda}}{2}(\|\hat{x}\|^2 - r^2) = \frac{1}{2}(\|\hat{x}\|^2 - r^2). \tag{5}$$

We claim that $\hat{\lambda} = 1$. If $\|\hat{x}\| < r$, this follows directly from (5). So, assume that $\|\hat{x}\| = r$. In this case, for $i = 1, 2$, we have

$$\frac{\hat{\lambda}}{2}\|u_i\|^2 + f(u_i, \hat{y}) \leqslant \frac{\hat{\lambda}}{2}\|\hat{x}\|^2 + f(\hat{x}, \hat{y}),$$

and hence from (4) it follows that

$$\frac{\hat{\lambda}}{2}\|u_i\|^2 + f(u_i, \hat{y}) = \inf_{x \in X}\left(\frac{\hat{\lambda}}{2}\|x\|^2 + f(x, \hat{y})\right). \tag{6}$$

But, for $\lambda > 1$, the functional $x \to (\lambda/2)\|x\|^2 + f(x, \hat{y})$ has a unique global minimum in X because it is strictly convex. So, the equality $\hat{\lambda} = 1$ follows from (6). Therefore, by (4), the Gâteaux derivative of the functional $x \to \frac{1}{2}\|x\|^2 + f(x, \hat{y})$ vanishes at $\hat{x}$. This means that

$$T(x_3 - \hat{x}) - \hat{x} = z + \hat{y}.$$

Therefore, if B is the closed ball of radius r centered at x_3, we have $x_3 - \hat{x} \in B$ and $\Phi(x_3 - \hat{x}) \in V$, and the proof is complete. □

We conclude by proposing the following conjecture.

CONJECTURE. Assume that

$$\sup_{x, y \in X, x \neq y} \frac{\|T(x) - T(x)\|}{\|x - y\|} = 1$$

and that there exists a closed ball B in X such that $\Phi(B)$ intersects each convex and dense subset of X. Then J is sequentially weakly lower semicontinuous.

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Biagio Ricceri,
Department of Mathematics,
University of Catania,
Viale A. Doria 6,
95125 Catania,
Italy
E-mail: ricceri@dmi.unict.it

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