Solving the Cramer-Castillon problem using Pascal's theorem

ZIAN SHANG, DAVID TREEBY

1. *Introduction*

In 1742 the Swiss mathematician Gabriel Cramer gave the following seemingly innocuous problem to the young Italian mathematician, Jean de Castillon:

The Cramer-Castillon problem: Given a circle and three points A, B and C, inscribe in the circle a triangle DEF whose sides (extended) pass through each of these points.

Note that points A, B and C can be either inside or outside of the circle. The problem is illustrated for three points A, B and C located outside of a circle in Figure 1.

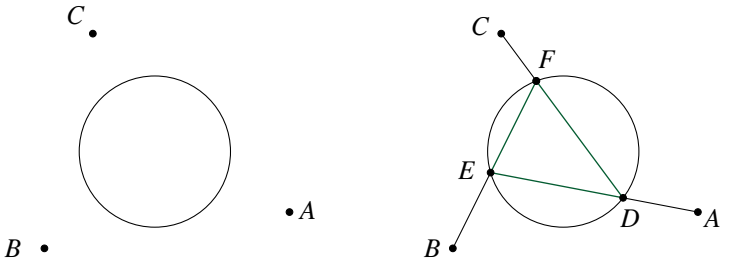


FIGURE 1

Castillon submitted his homework impressively late, when some thirty-four years later he presented a geometric solution to the Academy of Berlin. After listening to Castillon's presentation, Lagrange found an analytic solution just one day later [1]. Yes, some people really are that annoying. The problem even attracted the attention of Euler, whose solution was geometric in nature [2]. Analytic solutions tend to utilise Möbius transformations. Indeed, Wanner [1] demonstrated one such solution to the problem using maps of the form $z \to \dfrac{az + b}{cz + d}$ where $ad - bc \neq 0$.

In this paper, we present what we believe to be a new proof that spurns this approach. We instead look to Pascal's theorem as our primary tool.

2. *Existence of a solution*

Consider a circle $\mathscr{C}$. For any point A not on $\mathscr{C}$ we consider the map that takes point X on the circle to the other point of intersection of line AX with the circle, as indicated in Figure 2 below. We call this mapping a *projection*.

https://doi.org/10.1017/mag.2025.3 Published online by Cambridge University Press

We use the boldface symbol **A** to denote the projection corresponding to the point A. The composition of projections is denoted by juxtaposition. That is, $\mathbf{B} \circ \mathbf{A} = \mathbf{BA}$. Note that every projection is an involution. This means that $\mathbf{A}^2(X) = X$ for every $X \in \mathscr{C}$.

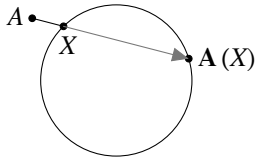


FIGURE 2

The Cramer-Castillon problem can now be recast as a fixed point problem. We need to construct a fixed point of the composition **CBA**; if we can find a point $X \in \mathscr{C}$ for which $\mathbf{CBA}(X) = X$, then the required triangle would have coordinates X, $\mathbf{A}(X)$ and $\mathbf{BA}(X)$, as shown below in Figure 3.

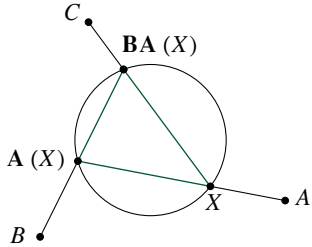


FIGURE 3

Note that if either two points, or no points, lie outside the circle, then **CBA** may not have fixed points. For example, consider the situation shown in Figure 4. Here A, B and C are collinear with B at the centre of the circle. If $|AB| = |BC|$, then A and C are both inside or both outside the circle. And in either case, X and $\mathbf{CBA}(X)$ are antipodal for all $X \in \mathscr{C}$. Therefore the mapping **CBA** has no fixed points.

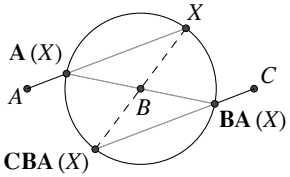


FIGURE 4

However, fixed points are guaranteed by requiring that points A, B and C satisfy the following simple condition.

Theorem 1 (Existence of fixed points): If an odd number of A, B and C lie outside of circle $\mathscr{C}$, then **CBA** has two fixed points.

https://doi.org/10.1017/mag.2025.3 Published online by Cambridge University Press

Proof: If point A lies outside of circle $\mathscr{C}$, then as the input point X turns through one revolution, the output point $\mathbf{A}(X)$ turns continuously one revolution in the opposite direction. On the other hand, if A lies inside circle $\mathscr{C}$, then, as the input point X turns through one revolution, the output point $\mathbf{A}(X)$ turns continuously one revolution in the same direction.

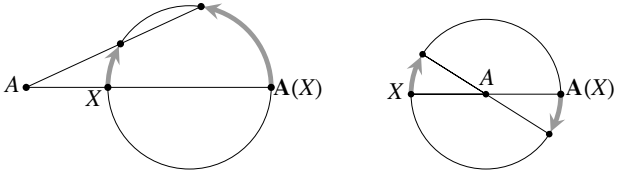


FIGURE 5

Therefore if an odd number of A, B or C lie outside $\mathscr{C}$, then, as the input point X turns through one revolution, the output point $\mathbf{CBA}(X)$ turns one revolution continuously in the opposite direction. So by the Intermediate Value Theorem we obtain two fixed points of $\mathbf{CBA}(X)$.

3. *Algebraic construction of a solution*

In this section we show how compositions of projections can be simplified. This will allow us to construct a solution to the Cramer-Castillon problem. Our main tool will be Pascal's theorem, which also goes by the pleasingly grandiose name, *hexagrammum mysticum* [3].

Theorem 2 (Pascal's Theorem): If a hexagon is inscribed in a conic, then the three points at which the pairs of opposite sides meet lie on a straight line.

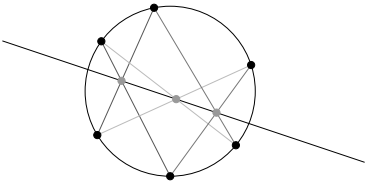


FIGURE 6

Lemma: If the three points A, B and C are on a line then there is a point D on such that $\mathbf{D} = \mathbf{CBA}$.

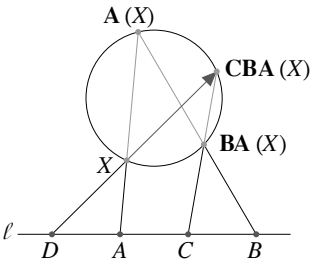


FIGURE 7

https://doi.org/10.1017/mag.2025.3 Published online by Cambridge University Press

Before supplying the proof of the Lemma, we note that Figure 7 illustrates the geometric construction of the point D. Pick any point X on the circle. Then D will be the point of intersection of line AB and the line containing points X and $\mathbf{CBA}(X)$.

Proof of the Lemma: Refer to Figure 8. Points A, B and C are located on line ℓ. Their order may be different to that shown. Let X_1 be any point on circle $\mathscr{C}$. Let $X_2 = \mathbf{A}(X_1)$, $X_3 = \mathbf{BA}(X_1)$ and $X_4 = \mathbf{CBA}(X_1)$, and suppose the line X_1X_4 intersects the line ℓ at D. We need to show that D is independent of the choice of X_1.

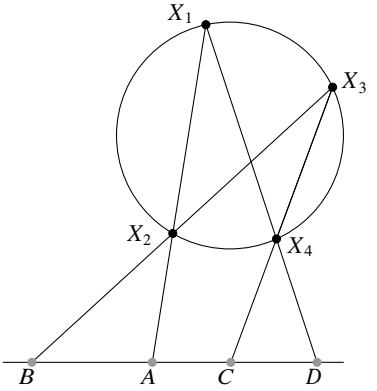


FIGURE 8

To that end, let Y_1 be some other point on $\mathscr{C}$. Let $Y_2 = \mathbf{A}(Y_1)$, $Y_3 = \mathbf{BA}(X_1)$ and $Y_4 = \mathbf{CBA}(X_1)$, and suppose X_1X_4 intersects Y_1Y_4 at E, as seen in Figure 9. It suffices to show that E is on ℓ, as this would mean that $D = E$.

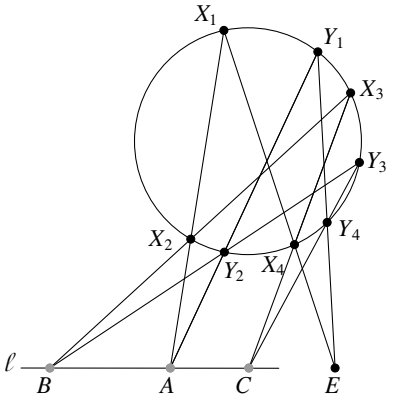


FIGURE 9

First, consider the hexagon $X_1X_2X_3Y_1Y_2Y_3$ shown below in Figure 10. Sides X_1Y_3 and Y_1X_3 (extended) intersect at the point F. Then, by applying Pascal's theorem to this hexagon, we find that the points A, B and F are collinear.

https://doi.org/10.1017/mag.2025.3 Published online by Cambridge University Press

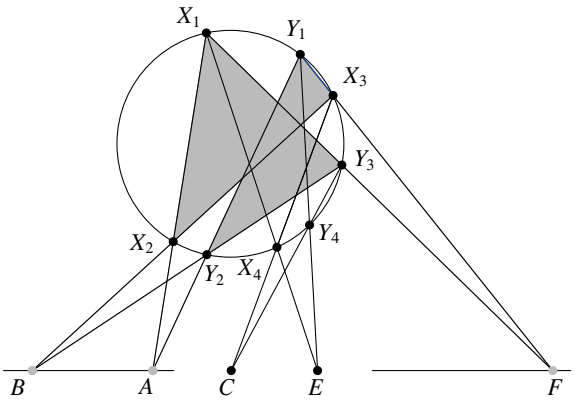


FIGURE 10

Now consider the hexagon $X_1X_4X_3Y_1Y_4Y_3$ shown below in Figure 11. By applying Pascal's theorem to this hexagon we see that the points C, E and F are also collinear. Therefore the five points A, B, C, E and F are collinear, which completes the proof.

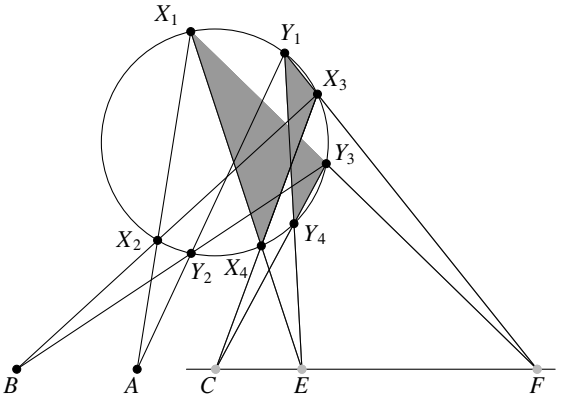


FIGURE 11

Although the collinearity requirement would seem to be unhelpfully restrictive, we now show that it allows us to simplify the composition of any four (or more) projections corresponding to any four (or more) points in the plane.

Theorem 3: For any four projections **A**, **B**, **C** and **D**, there are two projections **F** and **G** such that **DCBA** = **FG**.

Proof: Let E be the point of intersection (possibly at infinity) of lines BA and DC. Then, as E belongs to both lines BA and DC, by the Lemma in Theorem 2 there are two projections **F** and **G** such that **F** = **DCE** and **G** = **EBA**. Moreover, as projection **E** is an involution,

$$\mathbf{DCBA} = (\mathbf{DC})(\mathbf{BA}) = (\mathbf{FE})(\mathbf{EG}) = \mathbf{F}(\mathbf{EE})\mathbf{G} = \mathbf{FG}.$$

https://doi.org/10.1017/mag.2025.3 Published online by Cambridge University Press

The above theorem allows us to simplify the composition of any four (or more) projections. However, readers may have noted that the Cramer-Castillon problem requires that we somehow locate the fixed points of the composition of just *three* projections, **CBA**. This obstacle can be skirted by locating the fixed points of $(\mathbf{CBA})^2$ instead.

Theorem 4 (Cramer-Castillon): Consider circle $\mathscr{C}$ and three points A, B and C. If an odd number of these points are outside $\mathscr{C}$, then we can inscribe in the circle a triangle DEF whose sides (extended) pass through each of these points.

Proof: First note that **CBA** has two fixed points, by Theorem 1. These points will also be fixed by $(\mathbf{CBA})^2$. By Theorem 2, we can find two projections **D** and **E** for which $\mathbf{CBAC} = \mathbf{DE}$ so that

$$(\mathbf{CBA})^2 = (\mathbf{CBAC})\mathbf{BA} = \mathbf{DEBA}.$$

Using Theorem 2 once more, we can find two further projections **F** and **G** for which for which $\mathbf{DEBA} = \mathbf{FG}$. Therefore we now need to locate the fixed points of $(\mathbf{CBA})^2 = \mathbf{FG}$. Consider the points F and G corresponding to the projections **F** and **G**, respectively, as shown in Figure 12 below. The line FG intersects $\mathscr{C}$ at points X and Y. Since $\mathbf{FG}(X) = X$ and $\mathbf{FG}(Y) = Y$, these are the required points.

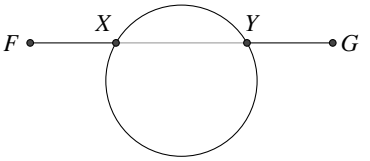


FIGURE 12

The algebra given in the above proof is simply shorthand for a geometric solution to the Cramer-Castillon problem. We leave it as a challenge to translate the solution we have given into a step-by-step geometric construction of the required fixed points. Finally, it is worth noting that the approach we have outlined can be extended inductively to produce the following generalisation.

Theorem 5: Consider a circle and n points $A_1, A_2, \ldots, A_n$. If an odd number of these points are outside $\mathscr{C}$, then we can inscribe an n-sided polygon $B_1, B_2, \ldots, B_n$ in the circle whose sides B_iB_{i+1} pass through A_i for $i = 1,2, \ldots, n$ (where $B_{n+1} = B_1$).

A solution for five points outside of a circle is shown below in Figure 13.

https://doi.org/10.1017/mag.2025.3 Published online by Cambridge University Press

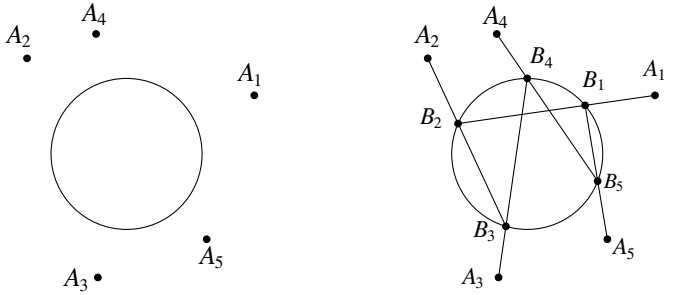


FIGURE 13

References

1. Gerhard Wanner, The Cramer-Castillon problem and Urquhart's 'most elementary' theorem, *Elem. Math.* **61**(2) (2006) pp. 58-64.
2. L. Euler, Problematis cuiusdam Pappi Alexandrini constructio, *Opera 26* (1783) pp. 237-242.
3. H. S. M. Coxeter, *Introduction to geometry* (2nd edn.), Wiley (1969).

10.1017/mag.2025.3 © The Authors, 2025
Published by Cambridge University Press on behalf of The Mathematical Association

ZIAN SHANG
Department of Mathematics,
Massachusetts Institute of Technology,
Cambridge, USA
e-mail: *zian_s@mit.edu*
DAVID TREEBY
Monash University,
Melbourne,
Australia
e-mail: *david.treeby@gmail.com*

https://doi.org/10.1017/mag.2025.3 Published online by Cambridge University Press