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Abstract

David Vogan proved that if G is a real reductive group, and if K is a maximal compact subgroup of G, then every irreducible representation of K is included as a minimal K-type in precisely one tempered, irreducible unitary representation of G with real infinitesimal character, and that moreover it is included there with multiplicity one, and is the unique minimal K-type in that representation. We shall prove that the Connes–Kasparov isomorphism in operator K-theory is equivalent to a K-theoretic version of Vogan's result.

1. Introduction

In the early 1980s, Alain Connes and Gennadi Kasparov separately conjectured a formula for the K-theory of the reduced C^*-algebra of a connected Lie group, G. The conjecture was formulated using index theory for Dirac-type operators on the homogeneous space G/K, where K is a maximal compact subgroup in G, drawing inspiration from Parthasarathy's construction of the discrete series representations for real reductive groups using the Dirac operator [Par72], and the work of Atiyah and Schmid in this direction [AS77].

The case of real reductive groups, which is the case of the conjecture that we shall consider here, was first settled using detailed results from tempered representation theory. See the announcement [Was87], and see [CHS+24, CHS24] for a comprehensive treatment.

In a remarkable and completely separate development, Vincent Lafforgue gave a proof of the Connes–Kasparov conjecture using index-theoretic methods [Laf02b]. Subsequently, he proved that Harish-Chandra's classification of the discrete series may be recovered from the Connes–Kasparov isomorphism [Laf02a, § 2].

The purpose of this paper is to go beyond the discrete series, and prove, as a consequence of the Connes–Kasparov isomorphism, a representation-theoretic result that applies to all components of the tempered dual. A unitary representation of a real reductive group G is *tempiric* if it is tempered, irreducible, and has real infinitesimal character (this terminology was suggested to us by Alexandre Afgoustidis; see § 2 for a discussion of the notion of real infinitesimal character). Every discrete series representation is tempiric, but not *vice versa*, and, as Vogan has pointed out, the whole of the tempered dual of a real reductive group may be classified rather easily using tempiric representations; see § 2.4 below for a short account of this. Moreover, the following celebrated result of Vogan, which in turn classifies the tempiric representations, makes the parameters in that classification very explicit.

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Theorem (Vogan). *If G is a real reductive group, and if K is a maximal compact subgroup of G, then every irreducible representation of K is included as a minimal K-type in precisely one tempiric representation of G, and moreover it is included there with multiplicity one, and is the unique minimal K-type in that representation.*

See [Vog81] for the representation-theoretic ideas involved, and [Vog07, Theorem 1.2] for the precise statement. The main result in this paper is a K-theoretic version of Vogan's theorem, which we shall prove using the Connes–Kasparov isomorphism (and so, in particular, using methods that are completely different from those of Vogan). Our theorem involves the free abelian group on the set of unitary equivalence classes of tempiric representations of G, which we shall denote by $R(G)_{\text{tempiric}}$.

Theorem. *Let G be a real reductive group, and let K be a maximal compact subgroup of G. There is an isomorphism of abelian groups from the representation ring*

$$R(K) \xrightarrow{\cong} R(G)_{\text{tempiric}}$$

that maps an irreducible representation $\tau \in \widehat{K}$ to the combination $\sum_\pi \text{mult}(\tau, \pi) \cdot \pi$, whose coefficients are the multiplicities with which τ occurs in the tempiric representation π.

In fact, we shall also show that, in the reverse direction, our K-theoretic version of Vogan's theorem, for all G, implies the Connes–Kasparov isomorphism, for all G (we need to consider all G at once because the argument involves a study of all the Levi subgroups of G). So, in some sense, our theorem presents the full representation-theoretic content of the Connes–Kasparov isomorphism.

The proof of our theorem is based on the notion of the imaginary part of the infinitesimal character of an irreducible unitary representation, reviewed in § 2. The importance of this concept has been emphasized by Vogan [Vog00]. We shall use it in § 3 to filter the reduced C^*-algebra of a real reductive group by ideals whose associated subquotients have an especially simple structure. Among other things, the study of the reduced group C^*-algebra through these subquotients obviates any need to study intertwining operators, which is otherwise by far the most difficult aspect of the C^*-algebra.

In § 4 we shall do essentially the same for the Cartan motion group associated to a real reductive group. This is much easier. Then, in § 5, we shall repeat the construction for all the groups in the smooth family that interpolates between a reductive group and its Cartan motion group (the deformation to the normal cone construction). By good fortune, perhaps, the K-theory groups here can be computed in a very simple and direct manner using the abelian group morphisms defined in the theorem above. Our result follows from this and from Alain Connes' reformulation of the Connes–Kasparov isomorphism as an assertion about the operator K-theory of the smooth family [Con94, Proposition 9, p. 141].

Our main theorem shares much in common with the so-called Mackey bijection between irreducible tempered unitary representations of a real reductive group and the irreducible unitary representations of its Cartan motion group. In particular, the work of Afgoustidis on the Mackey bijection [Afg19, Afg21] is extremely closely related, and was an inspiration for the present work. Other very relevant works are [DH26] and [HR20]; the latter introduces the essential rescaling technique that we use in § 5.4.

2. Imaginary part of the infinitesimal character

In this section, we shall gather most of the information from Lie theory and representation theory that we shall need in the paper.

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2.1 Real reductive groups

Throughout the paper, we shall use the term *real reductive group* in the sense that it is used in Vogan's monograph [Vog81, Chapter 0, § 1]. This means, among other things, that a real reductive group G is required to be linear, but it need not be connected, although there are restrictions in that regard. Examples include the group of real points of any connected reductive algebraic group defined over $\mathbb{R}$, including, for instance, $SL(n,\mathbb{R})$, $Sp(2n,\mathbb{R})$, etc.

We shall use the following structure, as discussed in [Vog81, Chapter 0, § 1]. Every real reductive group G is equipped with a *Cartan involution*

$$\theta\colon G \longrightarrow G$$

whose fixed subgroup is precisely K. We shall write the *Cartan decomposition* of the Lie algebra $\mathfrak{g}$ into ± 1 eigenspaces of θ as $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}$ (we shall not follow the custom of indicating real Lie algebras by a subscript 0; instead, we shall indicate complexifications by a subscript $\mathbb{C}$). We can and will equip $\mathfrak{g}$ with an invariant nondegenerate symmetric bilinear form

$$B\colon \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathbb{R} \tag{2.1}$$

that is positive-definite on $\mathfrak{s}$ and negative-definite on $\mathfrak{k}$, and that is invariant under θ. When we speak of inner products on subspaces of $\mathfrak{k}$ or of $\mathfrak{s}$, we shall always mean the positive-definite inner products on these subspaces obtained from B.

We shall fix throughout a maximal abelian subspace $\mathfrak{a}_{\min}$ of $\mathfrak{s}$. We shall fix a system of positive restricted roots

$$\Delta^+(\mathfrak{g}, \mathfrak{a}_{\min}) \subseteq \Delta(\mathfrak{g}, \mathfrak{a}_{\min}), \tag{2.2}$$

along with the associated *Iwasawa decomposition* $G = KA_{\min}N_{\min}$ and *minimal parabolic subgroup*

$$P_{\min} = M_{\min}A_{\min}N_{\min} \tag{2.3}$$

(see e.g. [Kna02, Chapter VI]). Here, $M_{\min}$ is the centralizer of $\mathfrak{a}_{\min}$ in K.

We shall always denote by $S \subseteq \Delta^+(\mathfrak{g}, \mathfrak{a}_{\min})$ the set of simple restricted roots. The *standard parabolic subgroups* of G are the parabolic subgroups that include the chosen minimal parabolic subgroup $P_{\min}$, and they are in bijection with subsets $I \subseteq S$, as follows:

$$P_I = M_I A_I N_I, \quad \mathfrak{a}_I = \bigcap \{\ker(\alpha) : \alpha \in I\}. \tag{2.4}$$

Here, the Levi subgroup $M_I A_I$ is the centralizer of $\mathfrak{a}_I$ in G, and N_I is determined by the requirement that P_I be standard. See, for instance, [Kna02, § VII.7].

Finally, if $P = MAN$ is a standard parabolic subgroup, then we shall always regard the dual vector space $\mathfrak{a}^*$ as a vector subspace of $\mathfrak{a}^*_{\min}$ by means of the map

$$\mathfrak{a}^* \longrightarrow \mathfrak{a}^*_{\min} \tag{2.5}$$

that takes a linear functional on $\mathfrak{a}$ to its unique extension to a linear functional on $\mathfrak{a}_{\min}$ that vanishes on the orthogonal complement of $\mathfrak{a}$ in $\mathfrak{a}_{\min}$. The orthogonal complement may also be described as the intersection of $\mathfrak{a}_{\min}$ with $\mathfrak{m}_I$. When $\mathfrak{a} = \mathfrak{a}_I$, the image of (2.5) is the space

$$\mathfrak{a}^*_I \cong \{\gamma \in \mathfrak{a}^*_{\min} : \langle \alpha, \gamma \rangle = 0 \ \forall \alpha \in I\}. \tag{2.6}$$

2.2 Imaginary part of the infinitesimal character

Definition 2.1. Denote by $\mathcal{Z}(\mathfrak{g}_{\mathbb{C}})$ the center of the enveloping algebra of $\mathfrak{g}_{\mathbb{C}}$, the complexification of the real Lie algebra $\mathfrak{g}$. If $\mathfrak{h}_{\mathbb{C}}$ is any Cartan subalgebra of $\mathfrak{g}_{\mathbb{C}}$, then we shall

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denote by

$$\xi_{\mathfrak{h}} \colon \mathcal{Z}(\mathfrak{g}_{\mathbb{C}}) \xrightarrow{\cong} \mathrm{Sym}(\mathfrak{h}_{\mathbb{C}})^{W(\mathfrak{g}_{\mathbb{C}},\mathfrak{h}_{\mathbb{C}})} \tag{2.7}$$

the Harish-Chandra isomorphism (see e.g. [Kna02, § V.5]).

On the right-hand side of the Harish-Chandra isomorphism is the Weyl group-invariant part of the symmetric algebra of $\mathfrak{h}_{\mathbb{C}}$, or equivalently, the algebra of Weyl group-invariant polynomial functions on ${\mathfrak{h}_{\mathbb{C}}}^*$. Every character of $\mathrm{Sym}(\mathfrak{h}_{\mathbb{C}})^{W(\mathfrak{g}_{\mathbb{C}},\mathfrak{h}_{\mathbb{C}})}$, or in other words every algebra homomorphism

$$\mathrm{Sym}(\mathfrak{h}_{\mathbb{C}})^{W(\mathfrak{g}_{\mathbb{C}},\mathfrak{h}_{\mathbb{C}})} \longrightarrow \mathbb{C},$$

is given by evaluation at some weight $\lambda \in \mathfrak{h}_{\mathbb{C}}^*$. Two weights correspond to the same character if and only if they lie in the same $W(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$-orbit.

It follows that every character

$$\chi \colon \mathcal{Z}(\mathfrak{g}_{\mathbb{C}}) \longrightarrow \mathbb{C}$$

may be written as a composition

$$\mathcal{Z}(\mathfrak{g}_{\mathbb{C}}) \xrightarrow{\xi_{\mathfrak{h}}} \mathrm{Sym}(\mathfrak{h}_{\mathbb{C}})^{W(\mathfrak{g}_{\mathbb{C}},\mathfrak{h}_{\mathbb{C}})} \xrightarrow{\mathrm{eval}_{\lambda}} \mathbb{C} \tag{2.8}$$

of the Harish-Chandra isomorphism $\xi_{\mathfrak{h}}$ and evaluation at some $\lambda \in \mathfrak{h}_{\mathbb{C}}^*$, which is unique up to the action of $W(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$. We shall write such a composition as

$$\chi = \xi_{\mathfrak{h},\lambda} \colon \mathcal{Z}(\mathfrak{g}_{\mathbb{C}}) \longrightarrow \mathbb{C}.$$

Following Vogan [Vog00, § 2], we wish to identify some of these characters as *real* and some as *imaginary*.

Definition 2.2. Let $\mathfrak{h} = \mathfrak{t} \oplus \mathfrak{a}$ be a θ-stable Cartan subalgebra of $\mathfrak{g}$, decomposed into its compact and noncompact parts (i.e. into the $+1$ and -1 eigenspaces of θ, respectively). The *real part* of its complexification, $\mathfrak{h}_{\mathbb{C}}$, is the real vector subspace

$$\mathrm{Re}(\mathfrak{h}_{\mathbb{C}}) = i\mathfrak{t} \oplus \mathfrak{a} \subseteq \mathfrak{h}_{\mathbb{C}}.$$

If the group G is realized as a group of real matrices in such a way that the Cartan involution is $\theta(X) = -X^{\mathrm{transp.}}$, then these are the matrices in $\mathfrak{h}_{\mathbb{C}}$ with real eigenvalues. A weight $\lambda \in \mathfrak{h}_{\mathbb{C}}^*$ is *real* if it is real-valued on $\mathrm{Re}(\mathfrak{h}_{\mathbb{C}})$. Similarly, it is *imaginary* if it is imaginary-valued on $\mathrm{Re}(\mathfrak{h}_{\mathbb{C}})$.

Lemma 2.3 [Vog00, Lemma 2.6]. *The action of the Weyl group $W(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$ on $\mathfrak{h}_{\mathbb{C}}$ preserves the real subspace $\mathrm{Re}(\mathfrak{h}_{\mathbb{C}})$. More generally, if Ad_g is any inner automorphism of the complex Lie algebra $\mathfrak{g}_{\mathbb{C}}$ that conjugates a Cartan subalgebra $\mathfrak{h}_{1,\mathbb{C}}$ to a second, $\mathfrak{h}_{2,\mathbb{C}}$, then Ad_g conjugates $\mathrm{Re}(\mathfrak{h}_{1,\mathbb{C}})$ and $\mathrm{Im}(\mathfrak{h}_{1,\mathbb{C}})$ to $\mathrm{Re}(\mathfrak{h}_{2,\mathbb{C}})$ and $\mathrm{Im}(\mathfrak{h}_{2,\mathbb{C}})$, respectively.*

Definition 2.4. If $\lambda \in \mathfrak{h}_{\mathbb{C}}^*$, then we shall denote by

$$\lambda = \mathrm{Re}(\lambda) + \mathrm{Im}(\lambda)$$

the (unique) decomposition of a weight λ as a sum of real and imaginary weights.

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From Lemma 2.3, we have the following.

Lemma 2.5 [Vog00, Corollary 2.7]. *Let $\mathfrak{h}$ be a θ-stable Cartan subalgebra of $\mathfrak{g}$. Let $\chi\colon \mathcal{Z}(\mathfrak{g}_{\mathbb{C}}) \to \mathbb{C}$ be any character. If χ is written in the form*

$$\chi = \xi_{\mathfrak{h},\lambda}\colon \mathcal{Z}(\mathfrak{g}_{\mathbb{C}}) \longrightarrow \mathbb{C}$$

for some $\lambda \in \mathfrak{h}^*_{\mathbb{C}}$, then the characters

$$\xi_{\mathfrak{h},\mathrm{Re}(\lambda)}\colon \mathcal{Z}(\mathfrak{g}_{\mathbb{C}}) \to \mathbb{C} \quad \textit{and} \quad \xi_{\mathfrak{h},\mathrm{Im}(\lambda)}\colon \mathcal{Z}(\mathfrak{g}_{\mathbb{C}}) \to \mathbb{C}$$

depend only on χ, and not on the choices of $\mathfrak{h}$ and λ.

Definition 2.6. Given a character $\chi\colon \mathcal{Z}(\mathfrak{g}_{\mathbb{C}}) \to \mathbb{C}$ as in Lemma 2.5, we shall write

$$\mathrm{Re}(\chi) = \xi_{\mathfrak{h},\mathrm{Re}(\lambda)}\colon \mathcal{Z}(\mathfrak{g}_{\mathbb{C}}) \to \mathbb{C} \quad \text{and} \quad \mathrm{Im}(\chi) = \xi_{\mathfrak{h},\mathrm{Im}(\lambda)}\colon \mathcal{Z}(\mathfrak{g}_{\mathbb{C}}) \to \mathbb{C}.$$

These are the *real and imaginary parts* of χ.

Now, let π be an irreducible unitary representation of G. Recall that, by Schur's Lemma, π has an associated infinitesimal character

$$\mathrm{Inf\,Ch}(\pi) : \mathcal{Z}(\mathfrak{g}_{\mathbb{C}}) \to \mathbb{C}.$$

From the above, we can form the real and imaginary parts of the infinitesimal character of π:

$$\mathrm{Re\,Inf\,Ch}(\pi)\colon \mathcal{Z}(\mathfrak{g}_{\mathbb{C}}) \to \mathbb{C} \quad \text{and} \quad \mathrm{Im\,Inf\,Ch}(\pi)\colon \mathcal{Z}(\mathfrak{g}_{\mathbb{C}}) \to \mathbb{C}.$$

We shall need some information about the real and imaginary parts of the infinitesimal character of an irreducible, tempered unitary representation, starting with the following result.

Theorem 2.7. *If π is an irreducible, square-integrable representation of a real reductive group, then* $\mathrm{Im\,Inf\,Ch}(\pi) = 0$.

This follows from Harish-Chandra's classification of the discrete series via Harish-Chandra parameters (which determine the infinitesimal characters of the discrete series). See Remark A.1 in the appendix.

We shall use Theorem 2.7 and the following well-known principle of Harish-Chandra and Langlands [Lan89] to say something about the infinitesimal character of a general irreducible tempered unitary representation.

Theorem 2.8 (See e.g. [Kna86, Theorem 8.5.3]). *Every irreducible tempered unitary representation of G embeds into a parabolically induced representation $\mathrm{Ind}_P^G \sigma \otimes \exp(i\nu)$, for some parabolic subgroup $P = MAN$ of G, some discrete series representation σ of M, and some $\nu \in \mathfrak{a}^*$.*

It follows from Theorem 2.8 that every irreducible tempered unitary representation has the same infinitesimal character as some cuspidal principal series representation of the form $\mathrm{Ind}_P^G \sigma \otimes \exp(i\nu)$. But the infinitesimal character of the latter may be readily computed as follows:

$$\begin{cases} \mathrm{Im\,Inf\,Ch}(\mathrm{Ind}_P^G \sigma \otimes \exp(i\nu)) = i\nu \in i\mathfrak{a}^*, \\ \mathrm{Re\,Inf\,Ch}(\mathrm{Ind}_P^G \sigma \otimes \exp(i\nu)) = \mathrm{Inf\,Ch}(\sigma) \in i\mathfrak{t}^*. \end{cases} \tag{2.9}$$

For instance, see [Kna86, Proposition 8.22]. The infinitesimal character in (2.9) is computed using a θ-stable Cartan subalgebra of the form $\mathfrak{h} = \mathfrak{t} \oplus \mathfrak{a}$ that is constructed from the parabolic subgroup $P = MAN$ in Theorem 2.8 by defining $\mathfrak{t}$ to be the Lie algebra of a maximal torus in $K \cap M$ (which, by another well-known result of Harish-Chandra, is a Cartan subalgebra of $\mathfrak{m}$) and, of course, defining $\mathfrak{a}$ to be the Lie algebra of A. The infinitesimal character of the discrete series representation σ lies in $i\mathfrak{t}^*$ thanks to Theorem 2.7.

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We want to use a single θ-stable Cartan subalgebra of $\mathfrak{g}$ to describe the infinitesimal characters of *all* the irreducible tempered unitary representations of G. To this end, we shall use the minimal parabolic subgroup that we fixed in (2.3). Form the 'minimally compact' Cartan subalgebra

$$\mathfrak{h}_{\min} = \mathfrak{t}_{\min} \oplus \mathfrak{a}_{\min}, \tag{2.10}$$

where $\mathfrak{t}_{\min}$ is the Lie algebra of a maximal torus in $M_{\min}$. It follows from computations with Cayley transforms (see e.g. [Kna02, § VI.7]) that there is an inner automorphism of the complex Lie algebra $\mathfrak{g}_{\mathbb{C}}$ that conjugates the Cartan subalgebra $\mathfrak{h}_{\mathbb{C}}$ used in (2.9) to $\mathfrak{h}_{\min,\mathbb{C}}$, and that conjugates $\mathfrak{a}$ into $\mathfrak{a}_{\min}$, and $\mathfrak{t}$ to $\mathfrak{t}_{\min} \oplus i(\mathfrak{m} \cap \mathfrak{a}_{\min})$. This inner automorphism is not unique, but the possible images of ν lie within a single orbit of the restricted Weyl group.

Lemma 2.9 (See [Vog00, Lemma 3.4] and [Hel78, Corollary VII.8.9]). *Two elements of* $\mathfrak{a}_{\min}$ *are conjugate in* $\mathfrak{g}_{\mathbb{C}}$ *by an element of* $\mathrm{Inn}(\mathfrak{g}_{\mathbb{C}})$ *if and only if they are conjugate by an element of* $W(\mathfrak{g}, \mathfrak{a}_{\min})$.

We arrive at the following result.

Theorem 2.10 [Vog00, Corollary 3.5(1)]. *The imaginary part of the infinitesimal character of any irreducible tempered unitary representation* π *may be written in the following form, using the* θ*-stable Cartan subalgebra* $\mathfrak{h}_{\min} = \mathfrak{t}_{\min} \oplus \mathfrak{a}_{\min}$*:*

$$\mathrm{Im\,Inf\,Ch}(\pi) = [(0, \nu)] \in (\mathfrak{t}^*_{\min} \oplus i\mathfrak{a}^*_{\min})/W(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\min,\mathbb{C}}).$$

The $W(\mathfrak{g}, \mathfrak{a}_{\min})$*-orbit of* ν *in* $i\mathfrak{a}^*_{\min}$ *is uniquely determined by* $\mathrm{Im\,Inf\,Ch}(\pi)$.

From Lemma 2.9, the inclusion of $i\mathfrak{a}^*_{\min}$ into $\mathrm{Im}(\mathfrak{h}_{\min,\mathbb{C}})^*$ determines a continuous map

$$i\mathfrak{a}^*_{\min}/W(\mathfrak{g}, \mathfrak{a}_{\min}) \longrightarrow \mathrm{Im}(\mathfrak{h}_{\min,\mathbb{C}})^*/W(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\min,\mathbb{C}})$$

that is a homeomorphism onto its image. From Theorem 2.10, there is a diagram of continuous maps.

$$\begin{array}{ccc} \widehat{G}_{\text{tempered}} & \xrightarrow{\text{Im Inf Ch}} & \mathrm{Im}(\mathfrak{h}_{\min,\mathbb{C}})^*/W(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\min,\mathbb{C}}) \\ \| & & \uparrow \\ \widehat{G}_{\text{tempered}} & \xrightarrow[\text{Im Inf Ch}]{} & i\mathfrak{a}^*_{\min}/W(\mathfrak{g}, \mathfrak{a}_{\min}) \end{array} \tag{2.11}$$

We have labeled the bottom map, like the top map, as the imaginary part of the infinitesimal character, and we shall use the bottom version of the imaginary part of the infinitesimal character from now on. According to (2.9), this means, for example, that if $P = MAN$ is a standard parabolic subgroup of G, as above, and if π is a direct summand of the parabolically induced representation $\mathrm{Ind}_P^G \sigma \otimes \exp(i\nu)$, as in Theorem 2.8, then

$$\mathrm{Im\,Inf\,Ch}(\pi) = [i\nu] \in i\mathfrak{a}^*_{\min}/W(\mathfrak{g}, \mathfrak{a}_{\min}),$$

where $\mathfrak{a}^*$ is embedded into $\mathfrak{a}^*_{\min}$ using (2.5).

2.3 Stratification of the space of imaginary values of the infinitesimal character

Recall that we fixed a system (2.2) of positive restricted roots and that we are denoting by $S \subseteq \Delta^+(\mathfrak{g}, \mathfrak{a}_{\min})$ the subset of simple roots.

Definition 2.11. We shall write

$$\mathfrak{a}^*_{\mathrm{dom}} = \{\nu \in \mathfrak{a}^*_{\min} : \langle \nu, \alpha \rangle \geqslant 0 \ \ \forall \alpha \in S\}.$$

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This is the dominant Weyl chamber in $\mathfrak{a}^*_{\min}$ that is associated to the given choice of system of positive roots.

The natural map

$$\mathfrak{a}^*_{\mathrm{dom}} \xrightarrow{\cong} \mathfrak{a}^*_{\min}/W(\mathfrak{g}, \mathfrak{a}_{\min})$$

induced from the inclusion of $\mathfrak{a}^*_{\mathrm{dom}}$ into $\mathfrak{a}^*_{\min}$ is a homeomorphism, and from now on we shall use this homeomorphism to identify imaginary parts of infinitesimal characters with points in $\mathfrak{a}^*_{\mathrm{dom}}$.

Definition 2.12. For every $I \subseteq S$, define

$$\mathfrak{a}^*_{I,+} = \{\nu \in \mathfrak{a}^*_{\mathrm{dom}} : \langle \nu, \alpha \rangle = 0 \ \forall \alpha \in I \text{ and } \langle \nu, \beta \rangle \neq 0 \ \forall \beta \notin I\}.$$

Note that $\mathfrak{a}^*_{I,+} \subseteq \mathfrak{a}^*_I$ as in (2.6). Explicitly, we have

$$\mathfrak{a}^*_{I,+} = \mathfrak{a}^*_{\mathrm{dom}} \cap \mathfrak{a}^*_I \setminus \bigcup_{I \subsetneq J} \mathfrak{a}^*_J. \tag{2.12}$$

The sets $\mathfrak{a}^*_{I,+}$ are locally closed subsets of $\mathfrak{a}^*_{\mathrm{dom}}$, and the full chamber $\mathfrak{a}^*_{\mathrm{dom}}$ is the disjoint union of all of them:

$$\mathfrak{a}^*_{\mathrm{dom}} = \bigsqcup_{I \subseteq S} \mathfrak{a}^*_{I,+}. \tag{2.13}$$

At one extreme, when $I = \emptyset$, $\mathfrak{a}^*_{I,+}$ is the interior of the dominant chamber; at the other extreme, when $I = S$, $\mathfrak{a}^*_{I,+}$ is the intersection of $\mathfrak{a}_{\min}$ with the center of $\mathfrak{g}$ (and so, for instance, it is a point if the center of G is compact).

2.4 Classification of representations via the imaginary part of the infinitesimal character

In this section, we shall describe Vogan's characterization of the fibers of the map Im Inf Ch in (2.11) over the various components in (2.13).

Definition 2.13. Let G be a real reductive group.

(i) An irreducible unitary representation π of a real reductive group is said to have *real infinitesimal character* if

$$\mathrm{Im\,Inf\,Ch}(\pi) = 0 \in i\mathfrak{a}^*_{\mathrm{dom}}.$$

(ii) An irreducible, tempered unitary representation of G is said to be *tempiric* (terminology of Afgoustidis) if it has real infinitesimal character. We shall denote by $\widehat{G}_{\mathrm{tempiric}}$ the set of unitary equivalence classes of tempiric representations of G.

Theorem 2.14 [Vog00, Corollary 3.5(2)]. *Let I be a subset of the set of simple roots $S \subseteq \Delta^+(\mathfrak{g}, \mathfrak{a}_{\min})$, let $\nu \in \mathfrak{a}^*_{I,+}$, and denote by P_I the associated standard parabolic subgroup of G, as in (2.3). The correspondence*

$$\sigma \longmapsto \mathrm{Ind}_{P_I}^G \sigma \otimes \exp(i\nu)$$

determines a bijection from the set of unitary equivalence classes of tempiric representations σ of M_I to the set of unitary equivalence classes of irreducible tempered unitary representations π of G for which $\mathrm{Im\,Inf\,Ch}(\pi) = i\nu$.

Remark 2.15. Actually, Vogan considered general unitary representations in [Vog00], rather than tempered representations, and for these the result is much more substantial. Vogan credits

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the characterization in this case to Knapp [Kna86, Theorem 16.10]. The proof of the tempered case that is stated above as Theorem 2.14 is relatively straightforward; see § A.2.

Theorem 2.14 determines a bijection

$$\widehat{G}_{\text{tempered}} \cong \bigsqcup_{I \subseteq S} \widehat{M}_{I,\text{tempiric}} \times i\mathfrak{a}_{I,+}^{*}. \tag{2.14}$$

This will be central to what follows.

3. A decomposition of the reduced group C^*-algebra via the imaginary part of the infinitesimal character

The purpose of this section is to obtain, from the decomposition of the tempered dual in (2.14) above, a filtration of the reduced C^*-algebra of the real reductive group G by ideals whose subquotients correspond to the individual parts in (2.14), indexed by $I \subseteq S$.

3.1 The reduced group C^*-algebra and the infinitesimal character

Let G be a real reductive group and denote by $C_r^*(G)$ the reduced group C^*-algebra of G. In this section, we shall use the theory of the imaginary part of the infinitesimal character that we reviewed in the last section to fiber the spectrum $\widehat{G}_{\text{tempered}}$ of $C_r^*(G)$ over $i\mathfrak{a}_{\text{dom}}^{*}$. In order to do this, we must prove that the imaginary part of the infinitesimal character defines a continuous map.

Theorem 3.1. *The map*

$$\operatorname{Im}\operatorname{Inf}\operatorname{Ch} : \widehat{G}_{\text{tempered}} \to i\mathfrak{a}_{\text{dom}}^{*}$$

is continuous.

Remark 3.2. The full infinitesimal character is also a continuous map, of course (see Theorem 3.6), but it is the imaginary part which is essential to our view of the Connes–Kasparov isomorphism.

Definition 3.3. Let G be any Lie group and let $T \in \mathcal{Z}(\mathfrak{g}_{\mathbb{C}})$. Define a function

$$\operatorname{Inf}\operatorname{Ch}_T : \widehat{G} \longrightarrow \mathbb{C} \tag{3.1}$$

on the unitary dual of G by

$$\operatorname{Inf}\operatorname{Ch}_T(\pi) \cdot I = \pi(T)$$

(this is an identity between operators acting on the smooth vectors in the Hilbert space of the representation π).

We shall use the following simple and well-known fact. We include a proof because we shall use the same argument to prove a less familiar fact later on, in Lemma 5.10.

Lemma 3.4. *Let G be any Lie group. For each $T \in \mathcal{Z}(\mathfrak{g}_{\mathbb{C}})$, the function* $\operatorname{Inf}\operatorname{Ch}_T$ *is a continuous function on the unitary dual of G (and in particular, in the reductive case, on the tempered dual).*

In this paper, we shall mostly understand the topology on the dual through the bijection between closed sets in the spectrum of a C^*-algebra and closed, two-sided ideals in that C^*-algebra (in which the closed set in the spectrum associated to an ideal is the set of representations that vanish on the ideal). See [Dix77, Chapters 3 and 13] for information about

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the topology of the unitary dual, and for information about the topology on the spectra of C^*-algebras in general.

But for the proof of the lemma, it will be better to use the following characterization of convergence in the unitary dual in terms of matrix coefficients: a net $\{\pi_\alpha\}$ of irreducible unitary representations of G converges to an irreducible unitary representation π in the dual if and only if there are unit vectors v_α in the Hilbert spaces of the representations π_α, and a unit vector v in the Hilbert space of π, such that

$$\lim_{\alpha\to\infty} \langle v_\alpha, \pi_\alpha(a)v_\alpha\rangle = \langle v, \pi(a)v\rangle, \quad \forall a \in C_c^\infty(G). \tag{3.2}$$

See again [Dix77, Chapters 3 and 13].

Proof of Lemma 3.4. Let $T \in \mathcal{Z}(\mathfrak{g}_\mathbb{C})$. Given (3.2) above, we wish to prove that

$$\lim_{\alpha\to\infty} \operatorname{Inf}\operatorname{Ch}_T(\pi_\alpha) = \operatorname{Inf}\operatorname{Ch}_T(\pi). \tag{3.3}$$

Choose $a \in C_c^\infty(G)$ such that $\langle v, \pi(a)v\rangle \neq 0$ and then write

$$\operatorname{Inf}\operatorname{Ch}_T(\pi_\alpha)\langle v_\alpha, \pi_\alpha(a)v_\alpha\rangle = \langle v_\alpha, \pi_\alpha(T)\pi_\alpha(a)v_\alpha\rangle = \langle v_\alpha, \pi_\alpha(Ta)v_\alpha\rangle,$$

where, in the last inner product, we view T as a differential operator on G. We find that

$$\lim_{\alpha\to\infty} \operatorname{Inf}\operatorname{Ch}_T(\pi_\alpha)\langle v_\alpha, \pi_\alpha(a)v_\alpha\rangle = \langle v, \pi(Ta)v\rangle = \operatorname{Inf}\operatorname{Ch}_T(\pi)\langle v, \pi(a)v\rangle,$$

from which (3.3) follows. □

Definition 3.5. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$. Define the map

$$\operatorname{Inf}\operatorname{Ch}_\mathfrak{h} : \widehat{G}_{\text{tempered}} \longrightarrow \mathfrak{h}_\mathbb{C}^*/W(\mathfrak{g}_\mathbb{C}, \mathfrak{h}_\mathbb{C})$$

by means of the formula

$$\operatorname{Inf}\operatorname{Ch}_\mathfrak{h}(\pi) = [\lambda] \quad \text{when} \quad \operatorname{Inf}\operatorname{Ch}(\pi) = \xi_{\mathfrak{h},\lambda}.$$

When $\mathfrak{h} = \mathfrak{h}_{\min}$, we shall often write simply Inf Ch in place of $\operatorname{Inf}\operatorname{Ch}_{\mathfrak{h}_{\min}}$.

Theorem 3.6. *Let G be a real reductive group and let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$. The map*

$$\operatorname{Inf}\operatorname{Ch}_\mathfrak{h} : \widehat{G}_{\text{tempered}} \to \mathfrak{h}_\mathbb{C}^*/W(\mathfrak{g}_\mathbb{C}, \mathfrak{h}_\mathbb{C})$$

is continuous.

Proof. It suffices to prove that if $f \in \operatorname{Sym}(\mathfrak{h}_\mathbb{C})^{W(\mathfrak{g}_\mathbb{C},\mathfrak{h}_\mathbb{C})}$, then the composition $f \circ \operatorname{Inf}\operatorname{Ch}_\mathfrak{h}$ is continuous. This follows from Lemma 3.4 and the Harish-Chandra homomorphism. □

Proof of Theorem 3.1. The inclusion

$$i\mathfrak{a}_{\text{dom}}^* \xrightarrow{\cong} i\mathfrak{a}_{\min}^*/W(\mathfrak{g}, \mathfrak{a}_{\min}) \to \operatorname{Im}(\mathfrak{h}_{\min,\mathbb{C}})^*/W(\mathfrak{g}_\mathbb{C}, \mathfrak{h}_{\min,\mathbb{C}})$$

is a homeomorphism onto its image. Therefore, it suffices to prove that the composition of Im Inf Ch with the above inclusion is continuous. This composition is simply Inf Ch composed with the (continuous) projection

$$\mathfrak{h}_{\min,\mathbb{C}}^*/W(\mathfrak{g}_\mathbb{C}, \mathfrak{h}_{\min,\mathbb{C}}) \longrightarrow \operatorname{Im}(\mathfrak{h}_{\min,\mathbb{C}}^*)/W(\mathfrak{g}_\mathbb{C}, \mathfrak{h}_{\min,\mathbb{C}}),$$

so continuity follows from Theorem 3.6. □

Remark 3.7. It follows from the Dauns–Hofmann theorem (see [DH68] or [Ped79, § 4.4] for an exposition) that the continuity of Im Inf Ch allows us to embed $C_0(i\mathfrak{a}_{\text{dom}}^*)$ into the center of

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the multiplier algebra of $C_r^*(G)$. More precisely, there is a unique inclusion of the C^*-algebra $C_0(i\mathfrak{a}_{\mathrm{dom}}^*)$ into the center of the multiplier algebra of $C_r^*(G)$ such that

$$\pi(f \cdot a) = f(\operatorname{Im}\operatorname{Inf}\operatorname{Ch}(\pi)) \cdot \pi(a)$$

for every tempered irreducible representation π, every $f \in C_0(i\mathfrak{a}_{\mathrm{dom}}^*)$, and every $a \in C_r^*(G)$.

3.2 Definition of the subquotients

We shall use the imaginary part of the infinitesimal character to construct a (finite) filtration of $C_r^*(G)$ by ideals, whose subquotients have a particularly simple structure.

DEFINITION 3.8. For every $I \subseteq S$, define

$$U_I = \{i\nu \in i\mathfrak{a}_{\mathrm{dom}}^* : \langle \nu, \beta \rangle \neq 0 \ \forall \beta \notin I\} = \bigsqcup_{J \subseteq I} i\mathfrak{a}_{J,+}^*.$$

Thanks to the continuity of $\operatorname{Im}\operatorname{Inf}\operatorname{Ch}$, the tempered representations for which the imaginary part of the infinitesimal character lies in U_I form an open subset of $\widehat{G}_{\mathrm{tempered}}$. We shall denote by

$$C_r^*(G; U_I) \triangleleft C_r^*(G)$$

the C^*-algebra ideal in the reduced group C^*-algebra associated to this open subset of $\widehat{G}_{\mathrm{tempered}}$. That is, $C_r^*(G, U_I)$ is the mutual kernel of all the irreducible representations π of $C_r^*(G)$ for which $\operatorname{Im}\operatorname{Inf}\operatorname{Ch}(\pi)$ lies in the complement of U_I.

It follows from the formula (2.12) that $i\mathfrak{a}_{I,+}^*$ is a closed subset of U_I. We shall denote by

$$\pi_I \colon C_r^*(G; U_I) \longrightarrow C_r^*(G; I)$$

the quotient associated to this closed subset. That is, $C_r^*(G; I)$ is the quotient of $C_r^*(G; U_I)$ by the ideal consisting of all elements that vanish in every representation π with $\operatorname{Im}\operatorname{Inf}\operatorname{Ch}(\pi) \in U_I \setminus i\mathfrak{a}_{I,+}^*$.

By elementary C^*-algebra theory [Dix77, Chapter 2], every irreducible representation π of $C_r^*(G)$ for which $\operatorname{Im}\operatorname{Inf}\operatorname{Ch}(\pi) \in U_I$ restricts to an irreducible representation of $C_r^*(G; U_I)$, and moreover, every irreducible representation of $C_r^*(G; U_I)$ extends uniquely to an irreducible representation π of $C_r^*(G)$ for which $\operatorname{Im}\operatorname{Inf}\operatorname{Ch}(\pi) \in U_I$. Furthermore, an irreducible representation π of $C_r^*(G; U_I)$ factors through the quotient $C_r^*(G; I)$ if and only if $\operatorname{Im}\operatorname{Inf}\operatorname{Ch}(\pi) \in i\mathfrak{a}_{\mathrm{min},I}^*$. More precisely, we have the following.

LEMMA 3.9. *The procedure that associates to each irreducible representation π' of $C_r^*(G; I)$ the unique extension to $C_r^*(G)$ of the composition of π' with the quotient map $C_r^*(G; U_I) \to C_r^*(G; I)$ in Definition 3.8 determines a homeomorphism*

$$\widehat{C_r^*(G; I)} \xrightarrow{\cong} \{[\pi] \in \widehat{G}_{\mathrm{tempered}} : \operatorname{Im}\operatorname{Inf}\operatorname{Ch}(\pi) \in i\mathfrak{a}_{I,+}^*\}.$$

3.3 Discrete series representations and tempiric representations

We shall use the following substantial results of Harish-Chandra about the discrete series representations of real reductive groups (in the statement, by a K-type of a representation π of G, we mean an irreducible constituent of $\pi|_K$).

THEOREM 3.10. *Let G be a real reductive group and let K be a maximal compact subgroup of G. Every irreducible representation of K occurs as a K-type in at most finitely many discrete series representations of G.*

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Remark 3.11. In view of the aims of this paper, it is interesting to note that this result follows from the Connes–Kasparov isomorphism. See § A.1.

Theorem 3.10 has the following consequence.

Theorem 3.12. *Let G be a real reductive group and let K be a maximal compact subgroup of G. Every irreducible representation of K occurs as a K-type in at most finitely many tempiric representations of G.*

Proof. By Theorem 2.8 and (2.9), for every tempiric representation π of G there is a standard parabolic subgroup $P = MAN$ of G and a discrete series representation (σ, H_σ) of M such that π embeds as a subrepresentation of $\mathrm{Ind}_P^G \sigma \otimes 1$. By Frobenius reciprocity, for a given irreducible representation (γ, V_γ) of K,

$$\mathrm{Hom}_K(V_\gamma, \mathrm{Ind}_P^G \sigma \otimes 1) \cong \mathrm{Hom}_{M\cap K}(V_\gamma, H_\sigma).$$

Therefore, V_γ occurs in $\mathrm{Ind}_P^G \sigma \otimes 1$ only if an $(M \cap K)$-type occurring in V_γ also occurs in H_σ. By Theorem 3.10, there are finitely many discrete series representations σ of M which contain one of the (finitely many) $(M \cap K)$-types occurring in V_γ. We conclude that V_γ is contained in finitely many representations of the form $\mathrm{Ind}_P^G \sigma \otimes 1$, and in particular it is contained in finitely many tempiric representations. □

Corollary 3.13. *The topological subspace $\widehat{G}_{\mathrm{tempiric}} \subseteq \widehat{G}$ is a closed subspace, and the topology that it inherits from $\widehat{G}$ is the discrete topology (in which every subset is an open set).*

Proof. The subspace $\widehat{G}_{\mathrm{tempiric}} \subseteq \widehat{G}_{\mathrm{tempered}}$ is the preimage of 0 under Im Inf Ch, and is therefore closed. Now, for any given irreducible representation γ of K, the subset $\widehat{G}_{\mathrm{tempered}}$ comprised of those representations that include γ as a K-type is open, while its intersection with $\widehat{G}_{\mathrm{tempiric}}$ is finite by Theorem 3.12. Therefore, $\widehat{G}_{\mathrm{tempiric}}$ has a covering consisting of finite open sets. But points in the unitary dual of a real reductive group are closed in the topology of the unitary dual (see e.g. [Dix77, § 15.5]). The result follows. □

3.4 Structure of the subquotients

In this section, we shall fix a subset $I \subseteq S$ and determine $C_r^*(G; I)$ up to isomorphism. For this and later purposes, we shall need to recall some details about the compact model of a parabolically induced representation. For a full account, see [Kna86, § VII.1].

Let $P = MAN$ be a parabolic subgroup of G, let (σ, H_σ) be a unitary representation of M, and let $\nu \in \mathfrak{a}^*$. The unitary representation

$$\pi_{\sigma,\nu} = \mathrm{Ind}_P^G \sigma \otimes \exp(i\nu)$$

may be realized on the Hilbert space

$$\mathrm{Ind}_P^G H_\sigma = L^2(K, H_\sigma)^{M\cap K},$$

where $M \cap K$ acts on $L^2(K, H_\sigma)$ by $(m \cdot \psi)(k) = \sigma(m)\psi(km)$. The unitary operator

$$\pi_{\sigma,\nu}(g) : L^2(K, H_\sigma)^{M\cap K} \longrightarrow L^2(K, H_\sigma)^{M\cap K}$$

is defined by

$$(\pi_{\sigma,\nu}(g)\psi)(k) = e^{-(i\nu+\rho)(\log a(g^{-1}k))}\sigma(m(g^{-1}k))^{-1}\psi(k(g^{-1}k)), \tag{3.4}$$

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where $\rho(H) = \frac{1}{2}\operatorname{tr}(\operatorname{ad}(H)|_{\mathfrak{n}})$ for $H \in \mathfrak{a}$, and where a factorization

$$g = k(g)m(g)a(g)n(g) \in KMAN = G$$

is used; the factorization is not unique, but the operator $\pi_{\sigma,\nu}(g)$ is nevertheless well-defined.

Theorem 3.14. *For every $\sigma \in \widehat{M}_{I,\mathrm{tempiric}}$, there is a unique surjective C^*-algebra morphism*

$$\pi_\sigma \colon C_r^*(G; I) \longrightarrow C_0(\mathfrak{a}_{I,+}^*, \mathfrak{K}(\operatorname{Ind}_{P_I}^G H_\sigma))$$

such that

$$\pi_\sigma(f)(\nu) = \pi_{\sigma,\nu}(f)$$

for every $\nu \in \mathfrak{a}_{I,+}^*$ and every $f \in C_r^*(G; I)$.

Proof. Of course, we would like to *define* π_σ by the formula

$$\pi_\sigma(f)(\nu) = \pi_{\sigma,\nu}(f)$$

that appears in the statement of the theorem. That we may do so follows from the considerations in [CCH16, Lemma 4.10], which show that $\pi_{\sigma,\nu}(f)$ is a continuous function of ν, and indeed a C_0-function of ν (this is essentially the Riemann–Lebesgue lemma; cf. the proof of Lemma 5.19 below, which deals with the same issue for a different group). It remains to prove that π_σ is surjective; this follows from Dixmier's Stone–Weierstrass theorem [Dix77, Theorem 11.1.8]. □

Theorem 3.15. *There is a unique C^*-algebra isomorphism*

$$C_r^*(G; I) \xrightarrow{\cong} \bigoplus_{\sigma \in \widehat{M}_{I,\mathrm{tempiric}}} C_0(\mathfrak{a}_{I,+}^*, \mathfrak{K}(\operatorname{Ind}_{P_I}^G H_\sigma))$$

whose composition with the projection onto the σ-summand in the direct sum is π_σ.

Proof. This follows from Theorem 3.14 and the fact that the supports of the representations π_σ, for $\sigma \in \widehat{M}_{I,\mathrm{tempiric}}$, are pairwise disjoint. Compare [CCH16, Lemma 5.14]. □

4. The Cartan motion group

Let G be a real reductive group and let K be a maximal compact subgroup. The associated *Cartan motion group* is the semidirect product group

$$G_0 = K \ltimes (\mathfrak{g}/\mathfrak{k})$$

that is constructed from the adjoint action of K on $\mathfrak{g}/\mathfrak{k}$. In this section, we shall review the unitary representation theory of G_0 and related matters, and construct a theory of the imaginary part of the infinitesimal character for the irreducible unitary representations of G_0.

4.1 Irreducible unitary representations of the Cartan motion group

In what follows, we shall use the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}$ to identify the quotient vector space $\mathfrak{g}/\mathfrak{k}$ with $\mathfrak{s}$, and so write the Cartan motion group as

$$G_0 = K \ltimes \mathfrak{s}. \tag{4.1}$$

Let ξ be a linear functional on the real vector space $\mathfrak{s}$, and let K_ξ be its isotropy subgroup in K for the coadjoint action. Mackey [Mac49] proved that if τ is any irreducible unitary representation

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of K_ξ, then the induced unitary representation

$$\rho_{\tau,\xi} = \mathrm{Ind}_{K_\xi \ltimes \mathfrak{s}}^{K \ltimes \mathfrak{s}} \tau \otimes \exp(i\xi) \tag{4.2}$$

is an irreducible unitary representation of G_0. He further proved that every irreducible unitary representation of G_0 is equivalent to a representation of this form, and that the only equivalences among these representations are the obvious ones coming from K-conjugacies, so that from (4.2) we obtain a bijection

$$\widehat{G}_0 \cong \Big(\bigsqcup_{\xi \in \mathfrak{s}^*} \widehat{K}_\xi \Big) / K. \tag{4.3}$$

For a review of these results from the C^*-algebraic point of view of this paper, see [Hig08, § 3.1].

The bijection in (4.3) may be adapted to our current purposes by using the inclusion of $\mathfrak{a}_{\min}$ into $\mathfrak{s}$, along with the following terminology and conventions (for our later purposes, we shall give more details here than are immediately needed). We identify $\mathfrak{a}_{\min}^*$ with a subspace of $\mathfrak{s}^*$ by extending each $\nu \in \mathfrak{a}_{\min}^*$ by zero on the orthogonal complement of $\mathfrak{a}_{\min}$ in $\mathfrak{s}$ to obtain an element of $\mathfrak{s}^*$. Having done so, every functional in $\mathfrak{s}^*$ may be conjugated by K to a functional in $\mathfrak{a}_{\min}^*$, and moreover to a unique functional in $\mathfrak{a}_{\mathrm{dom}}^*$. Indeed, the inclusion of $\mathfrak{a}^*$ into $\mathfrak{s}^*$ induces a homeomorphism

$$\mathfrak{a}_{\min}^* / W(\mathfrak{g}, \mathfrak{a}_{\min}) \xrightarrow{\cong} \mathfrak{s}^* / K.$$

Definition 4.1. Let I be a subset of the set of simple roots in $\Delta^+(\mathfrak{g}, \mathfrak{a}_{\min})$ and let $\nu \in \mathfrak{a}_{I,+}^*$ (notation from Definition 2.12). Let K_I be the centralizer of $\mathfrak{a}_I$ in K (this may also be described as the isotropy group of the single element ν, as well as the intersection $K \cap M_I$), and let σ be a unitary representation of K_I. We shall denote by $\rho_{\sigma,\nu}$ the unitary representation

$$\rho_{\sigma,\nu} = \mathrm{Ind}_{K_I \ltimes \mathfrak{s}}^{K \ltimes \mathfrak{s}} \sigma \otimes \exp(i\nu)$$

of $G_0 = K \ltimes \mathfrak{s}$. Here, ν is extended to a linear functional on $\mathfrak{s}$ that is zero on the orthogonal complement of $\mathfrak{a}_I$ in $\mathfrak{s}$. We shall realize the representation as follows. We let K_I act on $L^2(K, H_\sigma)$ by $(k_I \cdot f)(k) = \sigma(k_I) f(k k_I)$. The Hilbert space of the representation is $\mathrm{Ind}_{K_I}^K H_\sigma = L^2(K, H_\sigma)^{K_I}$ and the action of G_0 on an element $f \in L^2(K, H_\sigma)^{K_I}$ is determined by

$$\begin{cases} (k \cdot f)(k_1) = f(k^{-1} k_1), & \forall k, k_1 \in K, \\ (X \cdot f)(k_1) = e^{i\nu(\mathrm{Ad}_{k_1}^{-1}(X))} f(k_1), & \forall X \in \mathfrak{s}\ \forall k_1 \in K. \end{cases}$$

Remark 4.2. For the time being, we are interested in the cases of the definition where σ is an irreducible unitary representation of K_I, but later we shall consider some unitary but reducible representations, too.

With this notation established, the bijection (4.3) can be recast as follows.

Theorem 4.3. *The representations $\rho_{\tau,\nu}$ of G_0, as I ranges over the finite subsets of the set S of simple restricted roots, as τ ranges over the unitary dual of K_I, and as ν ranges over $\mathfrak{a}_{I,+}^*$, are irreducible and pairwise inequivalent. They exhaust the unitary dual of G_0, and determine a bijection*

$$\widehat{G}_0 \cong \bigsqcup_{I \subseteq S} \widehat{K}_I \times \mathfrak{a}_{I,+}^*.$$

Remark 4.4. Of course, this is to be compared with the similar bijection for the tempered dual of G in (2.14). The two bijections are the foundation of Afgoustidis's Mackey bijection [Afg21].

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4.2 Imaginary part of the infinitesimal character for the motion group

The vector subspace $\mathfrak{s} \subseteq \mathfrak{g}$, considered as an abelian Lie algebra, is a Lie subalgebra of the Lie algebra of the Cartan motion group G_0, and there is an inclusion

$$\mathrm{Sym}(\mathfrak{s}_{\mathbb{C}})^K \longrightarrow \mathcal{Z}(\mathfrak{g}_{0,\mathbb{C}})^K.$$

Often, but not always, this is an isomorphism. In any case, each $T \in \mathrm{Sym}(\mathfrak{s}_{\mathbb{C}})^K$ determines a continuous map

$$\mathrm{Inf\,Ch}_T \colon \widehat{G}_0 \longrightarrow \mathbb{C} \tag{4.4}$$

by the formula $\mathrm{Inf\,Ch}_T(\pi) \cdot I = \pi(T)$ (action on smooth vectors).

The map (4.4) is easy to compute: identifying the element $T \in \mathrm{Sym}(\mathfrak{s}_{\mathbb{C}})^K$ with a K-invariant polynomial function on $\mathfrak{s}^*$, we have

$$\mathrm{Inf\,Ch}_T(\rho_{\tau,\nu}) = T(i\nu).$$

Lemma 4.5. *The map*

$$\begin{aligned} &\mathrm{Inf\,Ch} \colon \widehat{G}_0 \longrightarrow i\mathfrak{s}^*/K, \\ &\mathrm{Inf\,Ch}(\rho_{\tau,\nu}) = i\nu \in i\mathfrak{s}^*/K \end{aligned} \tag{4.5}$$

is continuous.

Proof. This follows from Lemma 3.4. □

Remark 4.6. As in Remark 3.7, the Dauns–Hofmann theorem allows us to embed $C_0(i\mathfrak{s}^*/K)$ into the center of the multiplier algebra of $C^*(G_0)$. However, the Dauns–Hofmann theorem is not necessary as Fourier theory [Hig08, Theorem 3.2] provides an isomorphism

$$C^*(G_0) \cong C(i\mathfrak{s}^*, \mathfrak{K}(L^2(K)))^K$$

under which the action of $C_0(i\mathfrak{s}^*/K)$ is simply pointwise multiplication.

Since the inclusion of $i\mathfrak{a}^*_{\min}$ into $i\mathfrak{s}^*$ as imaginary-valued linear functionals vanishing on the orthogonal complement of $\mathfrak{a}_{\min}$ induces a homeomorphism

$$i\mathfrak{a}^*_{\min}/W(\mathfrak{g}, \mathfrak{a}_{\min}) \xrightarrow{\cong} i\mathfrak{s}^*/K,$$

we can view $\mathrm{Inf\,Ch}(\pi)$ as an element of $i\mathfrak{a}^*_{\min}/W(\mathfrak{g}, \mathfrak{a}_{\min})$, and so obtain from (4.5) a continuous map

$$\mathrm{Inf\,Ch} \colon \widehat{G}_0 \longrightarrow i\mathfrak{a}^*_{\min}/W(\mathfrak{g}, \mathfrak{a}_{\min}). \tag{4.6}$$

A comparison between this and the map $\mathrm{Im\,Inf\,Ch}$ that we defined in the reductive case, see (2.11), makes it reasonable to also use the notation $\mathrm{Im\,Inf\,Ch}$ for the map in (4.6), which we shall do from now on. In other words, we are effectively declaring the real part of the infinitesimal character to be always zero on $\widehat{G}_0$. Thus,

$$\mathrm{Im\,Inf\,Ch}(\rho_{\tau,\nu}) := i\nu \in i\mathfrak{a}^*_{\min}/W(\mathfrak{g}, \mathfrak{a}_{\min}) \cong i\mathfrak{a}^*_{\mathrm{dom}}. \tag{4.7}$$

4.3 Subquotients of the motion group C^*-algebra

Using the continuous map $\mathrm{Im\,Inf\,Ch}$ just defined on $\widehat{G}_0$, we can now repeat the constructions of § 3.2 more or less word for word, with G replaced by G_0.

Definition 4.7. We shall denote by

$$C^*(G_0; U_I) \triangleleft C^*(G_0)$$

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the C^*-algebra ideal in $C^*(G_0)$ associated to the open subset of the unitary dual consisting of representations whose infinitesimal character belongs to the open subset $U_I \subseteq i\mathfrak{a}^*_{\mathrm{dom}}$ in Definition 3.8. We shall denote by $C^*(G_0; I)$ the quotient C^*-algebra associated to the closed subset $i\mathfrak{a}^*_{\mathrm{min},I} \subseteq U_I$, and by

$$\rho_I \colon C^*(G_0; U_I) \longrightarrow C^*(G_0; I)$$

the quotient mapping.

Theorem 4.8. *For every $\tau \in \widehat{K}_I$, there is a unique surjective C^*-algebra morphism*

$$\rho_\tau \colon C^*(G_0; I) \longrightarrow C_0(\mathfrak{a}^*_{I,+}, \mathfrak{K}(\mathrm{Ind}_{K_I}^K H_\tau))$$

such that

$$\rho_\tau(\rho_I(a))(\nu) = \rho_{\tau,\nu}(a)$$

for every $\nu \in \mathfrak{a}^*_{I,+}$ and every $a \in C^*(G_0; U_I)$.

Theorem 4.9. *There is a unique C^*-algebra isomorphism*

$$C^*(G_0; I) \xrightarrow{\cong} \bigoplus_{\tau \in \widehat{K}_I} C_0(\mathfrak{a}^*_{I,+}, \mathfrak{K}(\mathrm{Ind}_{K_I}^K H_\tau))$$

whose composition with the projection onto the τ-summand in the direct sum is the morphism ρ_τ from Theorem 4.8.

5. Scaling the imaginary part of the infinitesimal character

The Mackey deformation family [Hig08, Afg19] is a smooth one-parameter family of Lie groups that links a real reductive group G to its Cartan motion group. In this section, we shall extend the constructions of the previous sections – the imaginary part of the infinitesimal character and the associated filtration of the reduced group C^*-algebra by ideals – to the context of the Mackey family.

5.1 Deformation families of Lie groups

Let G be a Lie group and let K be a compact subgroup of G (or, for nearly everything that follows, any closed subgroup). The deformation to the normal cone construction, applied to $K \subseteq G$, yields a smooth family of Lie groups $\{G_t\}_{t\in[0,\infty)}$ with

$$G_t = \begin{cases} G, & t > 0, \\ K \ltimes (\mathfrak{g}/\mathfrak{k}), & t = 0. \end{cases} \tag{5.1}$$

See [Hig08, § 6.2] for this particular instance of the deformation to the normal cone construction. We shall use the following features.

(i) The disjoint union $\boldsymbol{G} = \bigsqcup\{G_t : t \geqslant 0\}$ is equipped with a smooth manifold (with boundary) structure for which the map

$$\boldsymbol{G} \to [0, \infty), \quad G_t \ni g \mapsto t \in [0, \infty)$$

is a submersion.

(ii) The fiberwise multiplication, inverse, and unit maps $\boldsymbol{G} \times_{[0,\infty)} \boldsymbol{G} \to \boldsymbol{G}$, $\boldsymbol{G} \to \boldsymbol{G}$, and $[0, \infty) \to \boldsymbol{G}$ are smooth maps.

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(iii) If V is a finite-dimensional real vector space, and if $E\colon V \to G$ is a smooth map for which $E(0) = e$, and for which the map

$$K \times V \to G, \quad (k, v) \mapsto k \cdot E(v)$$

is a diffeomorphism, then the map

$$K \times V \times [0, \infty) \to \boldsymbol{G}; \quad (k, v, t) \mapsto \begin{cases} k \cdot E(tv) \in G_t, & t > 0, \\ (k, dE(v)) \in G_0, & t = 0 \end{cases}$$

is a diffeomorphism, where $dE\colon V \to \mathfrak{g}/\mathfrak{k}$, is the composition of the derivative of E at $0 \in V$ with the projection $\mathfrak{g} \to \mathfrak{g}/\mathfrak{k}$ (for this, we use $G_0 = K \ltimes \mathfrak{g}/\mathfrak{k}$).

Remark 5.1. A diffeomorphism as in point (iii) exists in all cases of interest to us. Specifically, when G is a real reductive group and K a maximal compact subgroup, we have the Cartan decomposition. Likewise, when P is a parabolic subgroup and K is a maximal parabolic of P, we can combine the Langlands decomposition of the Levi factor with the exponential map on the nilpotent radical.

Equip the groups G_t with Haar measures μ_t, as follows. Fix a Haar measure $\mu = \mu_1$ on $G = G_t$, and then define

$$\mu_t = t^{-d}\mu \quad \text{for } t > 0, \tag{5.2}$$

where $d = \dim(G/K)$. As for μ_0, the measure μ with which we started determines a Lebesgue measure on $\mathfrak{g}$, which may be written as the product of a Lebesgue measure $\mu_{\mathfrak{k}}$ on $\mathfrak{k}$ and a Lebesgue measure $\mu_{\mathfrak{s}}$ on $\mathfrak{s}$ (the summands in the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}$). The measure $\mu_{\mathfrak{k}}$ determines a Haar measure μ_K on K, and we define μ_0 to be the product of μ_K and $\mu_{\mathfrak{s}}$ on $G \cong K \ltimes \mathfrak{s}$.

The measures μ_t vary smoothly with t in the sense that if f is a smooth and compactly supported function on $\boldsymbol{G}$, then $\int_{G_t} f(g_t)\, d\mu_t g_t$ is a smooth function of t. Moreover, the operations of fiberwise convolution and adjoint define a $*$-algebra structure on the space $C_c^\infty(\boldsymbol{G})$ of all smooth and compactly supported functions on $\boldsymbol{G}$.

Definition 5.2. We shall denote by $C_r^*(\boldsymbol{G})$ the C^*-algebra completion of the $*$-algebra $C_c^\infty(\boldsymbol{G})$ in the norm

$$\|f\| = \sup\{\|f_t\|_{C_r^*(G_t)} : t \geqslant 0\},$$

where f_t is the restriction of f to $G_t \subseteq \boldsymbol{G}$.

Remark 5.3. It is shown in [Hig08, Lemma 6.13], using $C^*(G_0) = C_r^*(G_0)$, that for $f \in C_c^\infty(\boldsymbol{G})$, the norms $\|f_t\|_{C_r^*(G_t)}$ vary continuously with t, so that we obtain from the constructions above a continuous field of C^*-algebras $\{C_r^*(G_t)\}_{t \geqslant 0}$, in the sense of [Dix77, Chapter 10].

The spectrum of $C_r^*(\boldsymbol{G})$ is the disjoint union

$$\widehat{\boldsymbol{G}}_{\text{tempered}} := \widehat{C_r^*(\boldsymbol{G})} = \bigsqcup_{t \geqslant 0} \widehat{G}_{t,\text{tempered}}.$$

We shall equip it with the usual hull-kernel topology it carries as the spectrum of a C^*-algebra. But, for the most part, we shall only need the following simpler notion of convergence among representations of the groups G_t.

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Definition 5.4. By a *continuous family of unitary representations* of $\boldsymbol{G}$ on a fixed Hilbert space H, we shall mean a strongly continuous map $\pi : \boldsymbol{G} \to U(H)$ whose restriction π_t to each G_t is a unitary representation.

Lemma 5.5. *If $\{\pi_t\}$ is a continuous family of representations in the above sense, and if π_t is irreducible for all $t > 0$, then every irreducible component of π_0 is a limit of $\{\pi_t\}$ in $\widehat{\boldsymbol{G}}$ as $t \to 0$.*

Example 5.6. Let $\pi : G \to U(H)$ be a unitary representation. Define

$$\begin{cases} \pi_t(g) = \pi(g), & t > 0, \\ \pi_0(k, X) = \pi(k), & t = 0. \end{cases}$$

Then, $\{\pi_t\}_{t\in[0,\infty)}$ is a continuous family of representations.

Example 5.7. Let $P = MAN$ be a parabolic subgroup of a real reductive group. Form the deformation to the normal cone $\boldsymbol{P}$ associated to P and the compact subgroup $K \cap M$. Let σ be an irreducible unitary representation of M and let $\nu \colon \mathfrak{a} \to \mathbb{R}$ be a linear functional. Define unitary representations λ_t of P_t on the Hilbert space of σ by

$$\begin{cases} \lambda_t(man) = e^{it^{-1}\nu(\log(a))} \cdot \sigma(m), & t > 0, \\ \lambda_0(k, X) = e^{i\nu(X)} \cdot \sigma(k), & t = 0, \end{cases}$$

where ν is transferred to a linear functional on $\mathfrak{p}/(\mathfrak{k} \cap \mathfrak{m})$ by setting it equal to zero on the image of $\mathfrak{m} \oplus \mathfrak{n}$. This is a continuous family of representations of $\boldsymbol{P}$.

The following proposition is a special case of a more general principle that unitary induction takes continuous families to continuous families.

Proposition 5.8. *Let $P = MAN$ be a parabolic subgroup of a real reductive group G. Let σ be a unitary representation of M and let $\nu \in \mathfrak{a}^*$. The representations*

$$\pi_t = \begin{cases} \pi_{\sigma, t^{-1}\nu}, & t > 0, \\ \rho_{\sigma,\nu}, & t = 0 \end{cases}$$

form a continuous family of unitary representations of $\boldsymbol{G}$.

Proof. Note that $\boldsymbol{P}$ embeds as a smooth closed submanifold of $\boldsymbol{G}$. For $g \in \boldsymbol{G}$, we have a (non-unique) decomposition

$$g = \kappa(g) \cdot \varphi(g)$$

with $\kappa(g) \in K$ and $\varphi(g) \in \boldsymbol{P}$. It is not usually possible to choose κ and φ to be globally smooth functions, but we can at least choose them to be smooth in a neighborhood of any point $g \in \boldsymbol{G}$.

Now, for $\xi \in L^2(K, H_\sigma)^{M\cap K}$, we may write

$$(\pi(g)\xi)(k') = \delta^{-1/2}(\varphi(g^{-1}k')) \cdot \lambda(\varphi(g^{-1}k'))^{-1}\xi(\kappa(g^{-1}k')),$$

where δ is the modular function for P (a necessary ingredient in unitary induction) and where λ is as in Example 5.7. This, plus the formula above, shows that $g \mapsto \pi(g)\xi$ is continuous from $\boldsymbol{G}$ to $L^2(K, H_\sigma)$, as claimed. □

5.2 The imaginary part of the infinitesimal character for the Mackey deformation

We turn now to infinitesimal characters in the context of the deformation to the normal cone family. The fundamental observation underlying all of this is the following lemma. For the groups of interest to us, it is a consequence of condition (iii) on $\boldsymbol{G}$ in § 5.1.

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LEMMA 5.9. *Let G be any Lie group, let K be a compact subgroup of G (or indeed any closed subgroup), and let X be a right-invariant vector field on G. The family of vector fields $X_t = tX$ on G_t, for $t > 0$, extends to a smooth family of vector fields on the fibers G_t of the deformation family $\boldsymbol{G}$. If we consider X as an element in $\mathfrak{g}$, then the vector field X_0 on G_0 is the right-invariant vector field associated to the image of X in $\mathfrak{g}/\mathfrak{k}$.*

Let us apply this to elements in the enveloping algebra of G. If $T \in \mathcal{U}(\mathfrak{g}_{\mathbb{C}})$, and if T has order k for the usual increasing filtration on the enveloping algebra (so that T is a sum of products of k or fewer elements from $\mathfrak{g}$), then the family of right-invariant differential operators

$$T_t = t^k T \colon C_c^\infty(G_t) \longrightarrow C_c^\infty(G_t) \quad (t > 0)$$

extends to a smooth family of right-invariant differential operators defined on all the fibers of $\boldsymbol{G}$. We obtain from this a families version of Lemma 3.4; we shall formulate it for real reductive groups and the tempered dual only because we have not defined the full unitary dual of the family $\boldsymbol{G}$.

LEMMA 5.10. *Let G be a real reductive group and let $T \in \mathcal{Z}(\mathfrak{g}_{\mathbb{C}})$. If T has order k, and if $T_t = t^k T$ for $t \neq 0$, then the function*

$$\widehat{G}_t \ni \pi_t \longmapsto \mathrm{Inf\ Ch}_{T_t}(\pi_t) \in \mathbb{C} \quad (t > 0)$$

extends at $t = 0$ to a continuous function on $\widehat{\boldsymbol{G}}_{\mathrm{tempered}}$.

Proof. The argument that was used for the proof of Lemma 3.4 may be applied here, too. □

THEOREM 5.11. *Let G be a real reductive group and let $\mathfrak{h}_{\mathbb{C}}$ be a Cartan subalgebra of $\mathfrak{g}_{\mathbb{C}}$. The family of maps*

$$t \cdot \mathrm{Inf\ Ch} : \widehat{G}_{t,\mathrm{tempered}} \longrightarrow \mathfrak{h}_{\mathbb{C}}^*/W(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}}) \quad (t > 0)$$

extends to a continuous map

$$\widehat{\boldsymbol{G}}_{\mathrm{tempered}} \longrightarrow \mathfrak{h}_{\mathbb{C}}^*/W(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}}).$$

Specifically, taking $\mathfrak{h} = \mathfrak{h}_{\min}$, the continuous extension to $t = 0$ is given by the infinitesimal character of G_0, from (4.7),

$$\mathrm{Inf\ Ch} : \widehat{G}_0 \to i\mathfrak{a}_{\min}^*/W(\mathfrak{g}, \mathfrak{a}_{\min}) \subseteq \mathfrak{h}_{\min,\mathbb{C}}^*/W(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}}).$$

Proof. According to Chevalley's theorem [Che55], there are homogeneous, Weyl group-invariant polynomial functions $p_1, \ldots, p_r$ on $\mathfrak{h}_{\mathbb{C}}^*$, say of degrees $k_1, \ldots, k_r$, that freely generate the algebra of all Weyl group-invariant polynomial functions on $\mathfrak{h}_{\mathbb{C}}^*$. The function

$$\mathfrak{h}_{\mathbb{C}}^*/W(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}}) \longrightarrow \mathbb{C}^r, \quad \lambda \longmapsto (p_1(\lambda), \ldots, p_r(\lambda))$$

is then a homeomorphism (indeed, an isomorphism of affine varieties). The compositions of this homeomorphism with the maps in the statement of the theorem are the maps

$$\widehat{G}_t \ni \pi_t \longmapsto (\mathrm{Inf\ Ch}_{T_{1,t}}(\pi_t), \ldots, \mathrm{Inf\ Ch}_{T_{r,t}}(\pi_t)),$$

where $T_j \in \mathcal{Z}(\mathfrak{g}_{\mathbb{C}})$ corresponds to $p_j \in \mathbb{C}[\mathfrak{h}_{\mathbb{C}}^*]$ under the Harish-Chandra isomorphism, and where $T_{j,t} = t^{k_j} T_j$. Since T_j has order k_j, it follows from Lemma 5.10 that the composition gives a continuous map from $\widehat{\boldsymbol{G}}_{\mathrm{tempered}}$ to $\mathbb{C}^r$. This proves the existence of the continuous extension.

In order to determine the value of this extension at $t = 0$, we make an explicit computation using the minimal principal series of G. We have seen in (2.9) that for the minimal parabolic subgroup $P = P_{\min}$ of G,

$$\mathrm{Inf\ Ch}(\pi_{\sigma,t^{-1}\nu}) = \mathrm{Inf\ Ch}(\sigma) \oplus it^{-1}\nu \in i\mathfrak{t}_{\min}^* \oplus i\mathfrak{a}_{\min}^*,$$

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so that

$$t \cdot \operatorname{Inf} \operatorname{Ch}(\pi_{\sigma,t^{-1}\nu}) = t \cdot \operatorname{Inf} \operatorname{Ch}(\sigma) \oplus i\nu \in i\mathfrak{t}^*_{\min} \oplus i\mathfrak{a}^*_{\min}. \tag{5.3}$$

The representation $\pi_{\sigma,\nu}$ will be irreducible if ν is fixed by no nontrivial element of $W(\mathfrak{g}, \mathfrak{a}_{\min})$, but in any case it always has an infinitesimal character. According to Proposition 5.8, the representations $\pi_{\sigma,t^{-1}\nu}$ of G_t extend to a continuous family of representations with limit $\rho_{\sigma,\nu}$ on G_0. The continuous extension is therefore given on $\widehat{G}_0$ by

$$\rho_{\tau,\nu} \longmapsto 0 \oplus i\nu \in i\mathfrak{t}^*_{\min} \oplus i\mathfrak{a}^*_{\min},$$

for every irreducible constituent $\rho_{\tau,\nu}$ of $\rho_{\sigma,\nu}$ Since every $\rho_{\tau,\nu}$ occurs in some $\rho_{\sigma,\nu}$, this completely determines the continuous extension to $t = 0$, as claimed. □

The imaginary part of the infinitesimal character always belongs to the image of the inclusion of $i\mathfrak{a}^*_{\mathrm{dom}}$ into $\mathfrak{h}^*_{\mathbb{C}}/W(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$; see Theorem 3.1. Therefore, composing the map of Theorem 5.11 with the projection

$$\mathfrak{h}^*_{\mathbb{C}}/W(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}}) \longrightarrow \operatorname{Im}(\mathfrak{h}^*_{\mathbb{C}})/W(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}}),$$

we obtain the following.

Theorem 5.12. *Let G be a real reductive group. There is a continuous map*

$$\widehat{\boldsymbol{G}}_{\mathrm{tempered}} \longrightarrow i\mathfrak{a}^*_{\mathrm{dom}}$$

such that

$$\widehat{G}_{t,\mathrm{tempered}} \ni \pi_t \longmapsto t \cdot \operatorname{Im} \operatorname{Inf} \operatorname{Ch}(\pi_t) \quad (t > 0)$$

and

$$\widehat{G}_{0,\mathrm{tempered}} \ni \pi_0 \longmapsto \operatorname{Im} \operatorname{Inf} \operatorname{Ch}(\pi_0) \quad (t = 0).$$

Definition 5.13. Given an irreducible representation π of $C^*_r(\boldsymbol{G})$, we shall call the element of $i\mathfrak{a}^*_{\mathrm{dom}}$ obtained by applying the map in Theorem 5.12 to π the *scaled imaginary part of the infinitesimal character* of π.

Thus, for $t > 0$, the scaled imaginary part of the infinitesimal character of the irreducible representation $\pi_t = \pi_{\sigma,t^{-1}\nu}$ of $G_t = G$ is $i\nu$, while the scaled imaginary part of the infinitesimal character of the irreducible representation $\rho_{\tau,\nu}$ of G_0 is $i\nu$.

Remark 5.14. Similarly to G and G_0, by the Dauns–Hofmann theorem there is a unique inclusion of $C_0(i\mathfrak{a}^*_{\mathrm{dom}})$ into the center of the multiplier algebra of $C^*_r(\boldsymbol{G})$ such that

$$\pi_t((f \cdot a)(t)) = f(t \cdot \operatorname{Im} \operatorname{Inf} \operatorname{Ch}(\pi_t)) \cdot \pi_t(a)$$

for all $t > 0$ and

$$\pi_0((f \cdot a)(0)) = f(\operatorname{Im} \operatorname{Inf} \operatorname{Ch}(\pi_0)) \cdot \pi_0(a)$$

for all irreducible tempered unitary representations π_t of G_t and π_0 of G_0, all $f \in C_0(i\mathfrak{a}^*_{\mathrm{dom}})$, and all $a \in C^*_r(\boldsymbol{G})$.

5.3 Subquotients of the C^*-algebra of continuous sections

Definition 5.15. We shall denote by

$$C^*_r(\boldsymbol{G}; U_I) \triangleleft C^*_r(\boldsymbol{G})$$

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the C^*-algebra ideal associated to the open subset of the tempered dual consisting of representations whose scaled infinitesimal character belongs to the open subset $U_I \subseteq i\mathfrak{a}^*_{\mathrm{dom}}$. We shall denote by

$$C^*_r(\boldsymbol{G}; U_I) \to C^*_r(\boldsymbol{G}; I)$$

the quotient of this ideal associated to the closed subset $i\mathfrak{a}^*_{I,+} \subseteq U_I$.

Remark 5.16. If $\{A_t\}_{t\in T}$ is any continuous field of C^*-algebras, and if, for every $t \in T$, J_t is a C^*-algebra ideal in A_t, then $\{J_t\}_{t\in T}$ becomes a continuous field of C^*-algebras whose continuous sections are those of the original field that take values in the ideals J_t, provided that for every t_0 and for every $a \in J_{t_0}$, there is a continuous section of the original continuous field that takes values in the ideals J_t and takes the value a at t_0. This latter condition is not automatic, but it is satisfied for the fields of ideals $C^*_r(G_t; U_I)$.

Remark 5.17. We may also ask whether or not the field of quotients $\{A_t/J_t\}_{t\in T}$ becomes a continuous field of C^*-algebras, if one deems the continuous sections to be those that lift to continuous sections of the original field. This is not necessarily so, but the field is at least upper semicontinuous, in the sense that for every liftable section s, and every $t_0 \in T$,

$$\limsup_{t\to t_0} \|s(t)\|_{A_t/J_t} \leqslant \|s(t_0)\|_{A_{t_0}/J_{t_0}}.$$

In fact, the fields of quotients introduced in this section *are* continuous fields, and not just upper-semicontinuous fields. We shall not need this fact, but it follows from a minor supplementary argument; see Remark 5.20 below.

5.4 Structure of the subquotients

We shall now determine the structure of the subquotients $C^*_r(\boldsymbol{G}, I)$. The formulation of the result and the method of proof are borrowed from [HR20]; see also [CHR26].

Lemma 5.18. *Let G be a real reductive group and let I be a subset of the set S of simple roots in $\Delta^+(\mathfrak{g}, \mathfrak{a}_{\min})$. There is a unique continuous action of the multiplicative group $\mathbb{R}^\times_+$ on $C^*_r(G; I)$ by C^*-algebra automorphisms,*

$$\alpha_t \colon C^*_r(G, I) \longrightarrow C^*_r(G, I) \quad (t > 0),$$

such that

$$\pi_{\sigma,\nu}(\alpha_t(f)) = \pi_{\sigma,t^{-1}\nu}(f)$$

for all $t > 0$, all $f \in C^*_r(G; I)$, all $\sigma \in \widehat{M}_{I,\mathrm{tempiric}}$, and all $\nu \in \mathfrak{a}^*_{I,+}$.

Proof. The uniqueness assertion is clear, since the representations $\pi_{\sigma,\nu}$ separate the points of $C^*_r(G; I)$. Existence follows from the isomorphism

$$C^*_r(G; I) \xrightarrow{\cong} \bigoplus_{\sigma\in\widehat{M}_{I,\mathrm{tempiric}}} C_0(\mathfrak{a}^*_{I,+}, \mathfrak{K}(\mathrm{Ind}_{P_I}^G H_\sigma)) \tag{5.4}$$

in Theorem 3.15, using which we may define α_t by requiring it to correspond with the automorphisms

$$\alpha_{\sigma,t} \colon C_0(\mathfrak{a}^*_{I,+}, \mathfrak{K}(\mathrm{Ind}_{P_I}^G H_\sigma)) \longrightarrow C_0(\mathfrak{a}^*_{I,+}, \mathfrak{K}(\mathrm{Ind}_{P_I}^G H_\sigma))$$

of the summands on the right-hand side above, defined by

$$\alpha_{\sigma,t}(h)(\nu) = h(t^{-1}\nu). \qquad \square$$

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Lemma 5.19. *Let G be a real reductive group and let I be a subset of the set S of simple roots in $\Delta^+(\mathfrak{g}, \mathfrak{a}_{\min})$. There is a unique C^*-algebra homomorphism*

$$\alpha_0 \colon C_r^*(G_0; I) \longrightarrow C_r^*(G; I)$$

such that

$$\pi_{\sigma,\nu}(\alpha_0(f_0)) = \rho_{\sigma,\nu}(f_0)$$

for all $f_0 \in C^*(G_0; I)$, all $\sigma \in \widehat{M}_{I,\text{tempiric}}$, and all $\nu \in \mathfrak{a}_{I,+}^*$.

Proof. As with Lemma 5.18, the uniqueness assertion is clear, and to prove existence, we can again use the isomorphism (5.4), using which it suffices to construct a suitable C^*-homomorphism

$$\alpha_0 \colon C_r^*(G_0; I) \longrightarrow \bigoplus_{\sigma \in \widehat{M}_{I,\text{tempiric}}} C_0(\mathfrak{a}_{I,+}^*, \mathfrak{K}(\operatorname{Ind}_{P_I}^G H_\sigma)). \tag{5.5}$$

We define

$$\alpha_0 \colon f_0 \longmapsto \oplus_\sigma [\nu \mapsto \rho_{\sigma,\nu}(f_0)]. \tag{5.6}$$

Considering an individual summand, if f_0 is a smooth and compactly supported function on G_0, and if $\varphi, \psi \in L^2(K, H_\sigma)^{K_I}$, then

$$\langle \varphi, \rho_{\sigma,\nu}(f_0)\psi \rangle = \int_K \int_K \int_{\mathfrak{s}} f_0(k'k^{-1}, \operatorname{Ad}_k X) e^{-i\nu(X)} \langle \varphi(k'), \psi(k) \rangle \, dk' \, dk \, dX, \tag{5.7}$$

and the Riemann–Lebesgue lemma shows that the above is a C_0-function on ν. If, moreover, f_0 is K-bi-finite, then the operator $\rho_{\sigma,\nu}(f_0)$ is identically zero on a subspace of finite codimension in $L^2(K, H_\sigma)^{K_I}$, and on the finite-dimensional orthogonal complement it is unitarily equivalent to a finite matrix with the above matrix coefficients (5.7) as entries. Therefore, $\rho_{\sigma,\nu}(f_0)$ is a compact operator-valued C_0 function of ν, for every σ. But, in addition, K-bi-finiteness implies that only finitely many summands on the right of (5.6) are nonzero.

It follows that the formula (5.6) does indeed define an element of the right-hand side in (5.5), not just for a smooth, compactly supported, and K-bi-finite function f_0 on G_0, but, by an approximation argument, for any $f_0 \in C_r^*(G_0)$, since the functions we have considered up to now are dense in $C_r^*(G_0)$. It is evident that the homomorphism α_0 so defined has the required property. □

Remarks 5.20. The morphism α_0 is injective. This follows from the definition (5.6) above and the fact that the representations $\rho_{\sigma,\nu}$ include, as subrepresentations, all the irreducible representations of $C^*(G_0; I)$ (in fact, by Frobenius reciprocity, every irreducible representation of K_I occurs as a K_I-type in some minimal principal series representation of M_I, and the base principal series representations, with $\nu = 0$, are tempiric).

In view of the descriptions of $C_r^*(G; I)$ and $C_r^*(G_0; I)$ given in Theorems 3.15 and 4.9, respectively, the embedding α_0 in Lemma 5.19 determines the embedding of C^*-algebras that is represented by the dotted arrow in the following commuting diagram.

$$\begin{array}{ccc} C_r^*(G_0; I) & \xrightarrow{\quad\alpha_0\quad} & C_r^*(G; I) \\ \text{Theorem 4.9} \downarrow \cong & & \cong \downarrow \text{Theorem 3.15} \\ \displaystyle\bigoplus_{\tau \in \widehat{K}_I} C_0(\mathfrak{a}_{I,+}^*, \mathfrak{K}(\operatorname{Ind}_{K_I}^K H_\tau)) & \dashrightarrow & \displaystyle\bigoplus_{\sigma \in \widehat{M}_{I,\text{tempiric}}} C_0(\mathfrak{a}_{I,+}^*, \mathfrak{K}(\operatorname{Ind}_{P_I}^G H_\sigma)) \end{array}$$

We shall give a full account of this embedding in Theorem 6.12.

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Theorem 5.21. *Let G be a real reductive group and let I be a subset of the set S of simple roots in $\Delta^+(\mathfrak{g}, \mathfrak{a}_{\min})$. If $\{f_t\}$ is a continuous section of the continuous field of C^*-algebras $\{C_r^*(G_t; U_I)\}$, then*

$$\lim_{t \searrow 0} \alpha_t(\pi_I(f_t)) = \alpha_0(\rho_I(f_0)),$$

where the limit is taken in the norm topology of $C_r^*(G, I)$, and where $\pi_I : C_r^*(G, U_I) \to C_r^*(G, I)$ and $\rho_I : C_r^*(G_0, U_I) \to C_r^*(G_0, I)$ are the quotient maps that were introduced in Definitions 3.8 and 4.7.

Remark 5.22. If we apply the representation $\pi_{\sigma,\nu}$ to both sides of the limit formula in the theorem, and use the characteristic properties of α_t and α_0, then we obtain the formula

$$\lim_{t \searrow 0} \pi_{\sigma, t^{-1}\nu}(f_t) = \rho_{\sigma,\nu}(f_0).$$

This is a consequence of Example 5.7 and Proposition 5.8. In effect, Theorem 5.21 promotes this pointwise limit formula (valid for σ and one value of ν at a time) to a uniform limit formula over all σ and ν. This is a technical matter: the main conceptual content of the theorem is already present in Proposition 5.8.

Proof of Theorem 5.21. Let $P_I = MAN$ be the Langlands decomposition of P_I. Let $\{f_t\}$ be a continuous section of the continuous field of C^*-algebras $\{C_r^*(G_t; U_I)\}$. The same argument that we used in the proof of Lemma 5.19 shows that it suffices to prove that for every tempiric representation σ, and for every pair φ, ψ of smooth functions in $L^2(K, H_\sigma)^{K \cap M}$, the limit formula

$$\lim_{t \searrow 0} \langle \varphi, \pi_{\sigma, t^{-1}\nu}(f_t)\psi \rangle = \langle \varphi, \rho_{\sigma,\nu}(f_0)\psi \rangle$$

holds uniformly in ν.

It follows from the formula (3.4) for $\pi_{\sigma,\nu}(g)$, plus a change of variables, that if $t > 0$, and if f_t is a smooth and compactly supported function on $G_t = G$, then

$$(\pi_{\sigma, t^{-1}\nu}(f_t)\psi)(\ell) = t^{-d} \int_G f_t(\ell g^{-1}) e^{-it^{-1}\nu(\log(a(g)))} e^{-\rho(\log(a(g)))} \sigma(m(g))^{-1} \psi(k(g))\, dg, \tag{5.8}$$

where $d = \dim(G/K)$. Here, we are using the smooth family of Haar measures $t^{-d} dg$ on $G_t = G$ as in (5.2).

Let us now fix an Iwasawa decomposition $M = K'A'N'$, with $K' = K \cap M$ and $A' \subseteq A_{\min}$. The integral in (5.8) may then be written as a multiple integral

$$\begin{aligned} t^{-d} \int_K \int_{A'} \int_{N'} \int_A \int_N & f_t(\ell(ka'n'an)^{-1}) e^{-it^{-1}\nu(\log(a))} e^{-\rho(\log(a))} \\ & \times \sigma(a'n')^{-1} \psi(k) e^{+2(\rho'(\log(a'))+\rho(\log(a)))}\, dk\, da'\, dn'\, da\, dn, \end{aligned}$$

for suitable choices of Haar measure on the factor groups. See, for instance, [Kna86, § V.6] for this change-of-variables formula. It follows that the inner product $\langle \varphi, \pi_{\sigma, t^{-1}\nu}(f_t)\psi \rangle$ is equal to

$$\begin{aligned} t^{-d} \int_K \int_K \int_{A'} \int_{N'} \int_A \int_N & f_t(\ell(ka'n'an)^{-1}) e^{-it^{-1}\nu(\log(a))} \\ & \times \langle \varphi(\ell), \sigma(a'n')^{-1}\psi(k) \rangle e^{2\rho'(\log(a'))+\rho(\log(a))}\, d\ell\, dk\, da'\, dn'\, da\, dn. \end{aligned} \tag{5.9}$$

Let us make some adjustments to (5.9). The exponential maps for the groups A', N', A, and N are measure-preserving diffeomorphisms (for suitable multiples of Lebesgue measure on the Lie algebras). So, we may switch to Lie algebra coordinates in the integrals over these groups. Then, after scaling all four Lie algebra coordinates by a factor of t, we obtain the reformulation

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of the integral in (5.9):

$$\int_K \int_K \int_{\mathfrak{a}'} \int_{\mathfrak{n}'} \int_{\mathfrak{a}} \int_{\mathfrak{n}} f_t(\ell(k\exp(tX')\exp(tY')\exp(tX)\exp(tY))^{-1}) \\ \times e^{-i\nu(X)}\langle\varphi(\ell), \sigma(\exp(tX')\exp(tY'))^{-1}\psi(k)\rangle e^{+t(2\rho'(X')+\rho(X))} d\ell\, dk\, dX'\, dY'\, dX\, dY. \quad (5.10)$$

The point about this reformulation is that the expression

$$k\exp(tX')\exp(tY')\exp(tX)\exp(tY) \in G_t$$

that is included in the integrand above determines a diffeomorphism

$$K \times \mathfrak{a}' \times \mathfrak{n}' \times \mathfrak{a} \times \mathfrak{n} \times \mathbb{R} \xrightarrow{\cong} \boldsymbol{G} \quad (5.11)$$

for which

$$(k, X', Y', X, Y, 0) \longmapsto (k, X'+Y'+X+Y) \in G_0 = K \ltimes (\mathfrak{g}/\mathfrak{k}).$$

So, if the functions f_t on the individual G_t $(t>0)$ assemble and extend to a smooth, compactly supported function on $\boldsymbol{G}$, then the integrand in (5.10) is uniformly compactly supported on $K \times \mathfrak{a}' \times \mathfrak{n}' \times \mathfrak{a} \times \mathfrak{n}$ as $t>0$ varies, and moreover, as t tends to 0, it converges uniformly to

$$f_0(\ell(k, X'+Y'+X+Y)^{-1})e^{-i\nu(X)}\langle\varphi(\ell), \psi(k)\rangle$$

(note that extending the functional ν, defined on $\mathfrak{a}$, to a linear functional on $\mathfrak{g}/\mathfrak{k}$ that vanishes on $\mathfrak{a}'$, $\mathfrak{n}'$, and $\mathfrak{n}$ is the same as extending it to a linear functional on $\mathfrak{s}$ that vanishes on the orthogonal complement of $\mathfrak{a}$, and then identifying $\mathfrak{s}$ with $\mathfrak{g}/\mathfrak{k}$). This gives convergence of the integral in (5.10) to

$$\int_K \int_K \int_{\mathfrak{a}'+\mathfrak{n}'+\mathfrak{a}+\mathfrak{n}} f_0(\ell(k, Z)^{-1})e^{-i\nu(Z)}\langle\varphi(\ell), \psi(k)\rangle\, d\ell\, dk\, dZ, \quad (5.12)$$

which, as we have seen in (5.7), is $\langle\varphi, \rho_{\sigma,\nu}(f_0)\psi\rangle$, and the convergence is uniform in ν, as required. □

With this rather technical computation completed, we can readily determine the structure of the C^*-algebra $C_r^*(\boldsymbol{G}; I)$ in a way that will be useful for the K-theory computations of the next section.

Definition 5.23. Let $\alpha : A_0 \to A$ be a C^*-algebra homomorphism. The *mapping cone* C^*-algebra is

$$\mathrm{Cone}(\alpha) = \{(f_0, F) \in A_0 \oplus C_0([0,\infty), A) : \alpha(f) = F(0)\}.$$

Theorem 5.24 (Cf. [HR20, Theorem 5.2.2]). *Let $\alpha_0 \colon C_r^*(G_0; I) \to C_r^*(G; I)$ be the C^*-algebra morphism from Lemma 5.19. The formula*

$$\alpha_I(f) = (f_0, t \mapsto \alpha_t(f_t)) \quad (f \in C_r^*(\boldsymbol{G}; I))$$

defines a C^*-algebra isomorphism

$$\alpha_I \colon C_r^*(\boldsymbol{G}; I) \xrightarrow{\cong} \mathrm{Cone}(\alpha_0).$$

Proof. Theorem 5.21 tells us that the morphism α_I is well-defined, in that $t \mapsto \alpha_t(f_t)$ really is a C_0-function on $[0,\infty)$ with values in $C_r^*(G; I)$. That α_I is an isomorphism follows from the below commuting diagram with exact rows, in which $\{\alpha_t\}$ indicates the morphism that maps a

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function $t \mapsto f_t$ to the function $t \mapsto \alpha_t(f_t)$.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C_0((0,\infty), C_r^*(G;I)) & \longrightarrow & C_r^*(\boldsymbol{G};I) & \longrightarrow & C_r^*(G_0;I) & \longrightarrow & 0 \\ & & \{\alpha_t\}\downarrow\cong & & \alpha_I\downarrow & & \| & & \\ 0 & \longrightarrow & C_0((0,\infty), C_r^*(G;I)) & \longrightarrow & \mathrm{Cone}(\alpha_0) & \longrightarrow & C_r^*(G_0;I) & \longrightarrow & 0 \end{array}$$

□

6. Vogan's theorem in K-theory

In this section, we shall formulate and prove our main theorem, which frames the Connes–Kasparov isomorphism in operator K-theory in purely representation-theoretic terms.

6.1 Vogan's theorem on minimal K-types

Definition 6.1 (See [Vog81, Definition 5.4.18]). Let G be a real reductive group with maximal compact subgroup K. If τ is an irreducible unitary representation of K, then its *norm* is

$$\|\tau\| = \|\mu + 2\rho_K\|,$$

where μ is a highest weight for some choice of maximal torus $T \subseteq K$ and some choice of a system of positive roots in $\Delta(\mathfrak{k}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$; the choices do not affect the norm.

Definition 6.2. Let π be an irreducible unitary (or simply admissible) representation of G. A *minimal K-type* of G is any K-type of π whose norm is minimal among the norms of all K-types of π.

Remark 6.3. Since every norm-bounded subset of $\widehat{K}$ is finite, every admissible representation has at least one, and at most finitely many, minimal K-types.

The simplicity of these definitions belies their importance. We shall be concerned in this paper with the following fundamental result about minimal K-types, due to Vogan, which, as we shall see, completely determines the tempered dual.

Theorem 6.4 (See [Vog07, Theorem 1.2]). *Let G be a real reductive group with maximal compact subgroup K. Every tempiric representation has a unique minimal K-type, which occurs in the representation with multiplicity one. The map that associates to a tempiric representation its minimal K-type is a bijection*

$$\widehat{G}_{\mathrm{tempiric}} \xrightarrow{\cong} \widehat{K}.$$

Among other things, this theorem immediately gives a parametrization of the tempered dual of G in terms of irreducible representations of compact subgroups, for we have, from (2.14) and Vogan's bijection,

$$\widehat{G}_{\mathrm{tempered}} \cong \bigsqcup_{I \subseteq S} \widehat{M}_{I,\mathrm{tempiric}} \times i\mathfrak{a}_{I,+}^* \cong \bigsqcup_{I \subseteq S} \widehat{K}_I \times i\mathfrak{a}_{I,+}^* \cong \widehat{G}_0.$$

This is the Mackey bijection of Afgoustidis [Afg21].

In the next section, we shall prove that the Connes–Kasparov isomorphism in operator K-theory is equivalent to a K-theoretic version of Vogan's theorem that involves the following concepts.

Definition 6.5. Let G be a real reductive group. Denote by $R_{\mathrm{tempiric}}(G)$ the free abelian group on the set of equivalence classes of tempiric representations of G.

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Definition 6.6. Denote by

$$\mathrm{mult}\colon R(K) \longrightarrow R_{\mathrm{tempiric}}(G)$$

the homomorphism of abelian groups defined by

$$[\tau] \longmapsto \sum_{\sigma \text{ tempiric}} \mathrm{mult}(\tau : \sigma) \cdot [\sigma] \quad \forall \tau \in \widehat{K},$$

where $\mathrm{mult}(\tau : \sigma)$ is the multiplicity with which τ occurs in the restriction of the tempiric representation σ to K.

Remark 6.7. The multiplicity homomorphism is well-defined in view of the fact, noted in Theorem 3.12, that each irreducible representation of K occurs as a K-type in at most finitely many pairwise inequivalent tempiric representations. Formally, mult is the transpose of the map induced by restriction of tempiric G-representations to K, although that map is not well-defined to $R(K)$ since its image contains infinite sums.

It is a simple matter of solving linear equations to prove, using Vogan's minimal K-type theorem (Theorem 6.4), that the multiplicity homomorphism in Definition 6.6 is an isomorphism. The main goal of this paper is to show that it is also possible to prove that mult is an isomorphism without using Vogan's minimal K-type theorem but instead using the Connes–Kasparov isomorphism.

This is very close to a new proof of Vogan's minimal K-type theorem. For instance, if we were to assume that no $\tau \in \widehat{K}$ may occur as a minimal K-type of two inequivalent tempiric representations, then elementary linear algebra would allow us to deduce Vogan's theorem from the isomorphism of the multiplicity map.

Lemma 6.8. *Suppose that the multiplicity map is an isomorphism. Suppose, in addition, that each K-type occurs in at most one tempiric representation of G as a minimal K-type. Then every tempiric representation has a unique minimal K-type, which occurs in the representation with multiplicity one, and the map that sends a tempiric representation to its minimal K-type is a bijection.*

6.2 The Connes–Kasparov isomorphism

We turn, finally, to the Connes–Kasparov isomorphism. In view of the aims of this paper, it is certainly important to note that Lafforgue [Laf02b] has given a proof of the Connes–Kasparov isomorphism using techniques from K-theory and index theory, rather than representation theory. On the other hand, representation-theoretic treatments of the Connes–Kasparov isomorphism can be found in [Was87, Hig08, Afg19, CHS24, Bra25].

We shall use the following formulation of the Connes–Kasparov isomorphism, which is due to Connes (see [Con94, Proposition 9, p. 141]).

Theorem 6.9 (Connes–Kasparov isomorphism). *Let G be a real reductive group. The C^*-algebra $C_r^*(\boldsymbol{G})$ in Definition 5.2 has vanishing K-theory.*

Remark 6.10. The morphism $C_r^*(\boldsymbol{G}) \to C_r^*(G_0)$ given by evaluation at $t = 0$ leads to an extension of C^*-algebras

$$0 \longrightarrow C_0((0,\infty), C_r^*(G)) \longrightarrow C_r^*(\boldsymbol{G}) \longrightarrow C^*(G_0) \longrightarrow 0$$

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and hence (using Bott periodicity) a six-term exact sequence in K-theory.

$$\begin{array}{ccccc} K_1(C_r^*(G)) & \longrightarrow & K_0(C_r^*(\boldsymbol{G})) & \longrightarrow & K_0(C^*(G_0)) \\ \uparrow & & & & \downarrow \\ K_1(C^*(G_0)) & \longleftarrow & K_1(C_r^*(\boldsymbol{G})) & \longleftarrow & K_0(C_r^*(G)) \end{array}$$

The assertion that the K-theory of $C_r^*(\boldsymbol{G})$ vanishes is therefore equivalent to the assertion that the vertical maps in the six-term sequence are isomorphisms, and it is these that Connes considers in [Con94].

6.3 K-theory of the mapping cone

The main theorem of this paper is the following consequence of the Connes–Kasparov isomorphism for real reductive groups. In fact, this statement is equivalent to the Connes–Kasparov isomorphism for real reductive groups.

Theorem 6.11. *Let G be a real reductive group and let K be a maximal compact subgroup of G. The morphism of abelian groups*

$$\begin{aligned} \mathrm{mult} : R(K) &\longrightarrow R_{\mathrm{tempiric}}(G), \\ [\tau] &\longmapsto \sum_{\sigma} \mathrm{mult}(\tau : \sigma) \cdot [\sigma], \quad \forall \tau \in \widehat{K}, \end{aligned}$$

is an isomorphism.

To prove the theorem, we shall relate the morphism in the statement to the induced maps in K-theory associated to the C^*-algebra homomorphisms

$$\alpha_0 \colon C_r^*(G_0; I) \longrightarrow C_r^*(G; I) \quad (I \subseteq S)$$

from Lemma 5.19. The point is that this is a morphism between C^*-algebras whose K-theory groups are easily determined in representation-theoretic terms. First, from the isomorphism

$$C_r^*(G_0; I) \xrightarrow{\cong} \bigoplus_{\tau \in \widehat{K}_I} C_0(\mathfrak{a}_{I,+}^*, \mathfrak{K}(\mathrm{Ind}_{K_I}^K H_\tau))$$

in Theorem 4.9, we obtain isomorphisms

$$\begin{aligned} K_*(C_r^*(G_0; I)) &\xrightarrow{\cong} \bigoplus_{\tau \in \widehat{K}_I} K_*(C_0(\mathfrak{a}_{I,+}^*, \mathfrak{K}(\mathrm{Ind}_{K_I}^K H_\tau))) \\ &\cong \bigoplus_{\tau \in \widehat{K}_I} K_*(C_0(\mathfrak{a}_{I,+}^*)) \\ &\cong \bigoplus_{\tau \in \widehat{K}_I} \mathbb{Z}, \end{aligned}$$

using first the stability property of K-theory and then the Bott periodicity theorem, which implies that the K-theory of $C_0(\mathfrak{a}_{I,+}^*)$ is the infinite cyclic group $\mathbb{Z}$. We fix a choice of generator for each I. Second, using Theorem 3.15 in a similar fashion, we have isomorphisms

$$K_*(C_r^*(G; I)) \xrightarrow{\cong} \bigoplus_{\sigma \in \widehat{M}_{I,\mathrm{tempiric}}} \mathbb{Z}.$$

Theorem 6.12. *There is a commutative diagram as below in which the vertical isomorphisms are the ones described above, and the bottom morphism is the multiplicity map for the*

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group M_I:

$$\mathrm{mult}_I : [\tau] \mapsto \sum_{\sigma \in \widehat{M}_{I,\,tempiric}} \mathrm{mult}(\tau : \sigma)[\sigma].$$

$$\begin{array}{ccc} K_*(C_r^*(G_0; I)) & \xrightarrow{\alpha_{0*}} & K_*(C_r^*(G; I)) \\ \cong\downarrow & & \downarrow\cong \\ R(K_I) & \xrightarrow{\mathrm{mult}_I} & R_{\mathrm{tempiric}}(M_I) \end{array}$$

Proof. Let $\tau \in \widehat{K}_I$ and let $\sigma \in \widehat{M}_{I,\mathrm{tempiric}}$. We need to compute that the bottom arrow in the following diagram is multiplication by $\mathrm{mult}(\tau : \sigma)$.

$$\begin{array}{ccc} K_*(C_r^*(G_0; I)) & \xrightarrow{\alpha_{0,*}} & K_*(C_r^*(G; I)) \\ {\scriptstyle \rho_{\tau *}}\downarrow & & \downarrow{\scriptstyle \pi_{\sigma *}} \\ K_*(C_0(\mathfrak{a}_{I,+}^*, \mathfrak{K}(\mathrm{Ind}_{K_I}^K H_\tau))) & \longrightarrow & K_*(C_0(\mathfrak{a}_{I,+}^*, \mathfrak{K}(\mathrm{Ind}_{P_I}^G H_\sigma))) \\ \cong\downarrow & & \downarrow\cong \\ \mathbb{Z}\cdot[\tau] & \longrightarrow & \mathbb{Z}\cdot[\sigma] \end{array}$$

For this, we shall calculate the map

$$C_0(\mathfrak{a}_{I,+}^*, \mathfrak{K}(\mathrm{Ind}_{K_I}^K H_\tau)) \longrightarrow C_0(\mathfrak{a}_{I,+}^*, \mathfrak{K}(\mathrm{Ind}_{P_I}^G H_\sigma))$$

which induces the middle arrow in K-theory. Let $F \in C_0(\mathfrak{a}_{I,+}^*, \mathfrak{K}(\mathrm{Ind}_{K_I}^K H_\tau))$, and let f_0 be the element of $C_r^*(G_0; I)$ such that $\rho_{\tau,\nu}(f) = F(\nu)$ and $\rho_{\tau',\nu}(f) = 0$ when $[\tau'] \neq [\tau]$. By definition, the composition

$$C_r^*(G_0; I) \to C_r^*(G; I) \to C_0(\mathfrak{a}_{I,+}^*, \mathfrak{K}(\mathrm{Ind}_{P_I}^G H_\sigma))$$

sends f_0 to the function $\rho_\sigma(f_0)$ defined by $\rho_\sigma(f_0)(\nu) = \rho_{\sigma,\nu}(f_0)$.

Now, we have the decomposition

$$L^2(K, H_\sigma) = \bigoplus_{\tau' \in \widehat{K}_I} \mathrm{mult}(\tau' : \sigma) \cdot L^2(K, H_{\tau'}),$$

$$\rho_{\sigma,\nu}(f_0) = \bigoplus_{\tau' \in \widehat{K}_I} \mathrm{mult}(\tau' : \sigma) \cdot \rho_{\tau',\nu}(f_0),$$

where $n \cdot T$ denotes the direct sum of T with itself n times. It follows that the middle arrow in the above diagram is induced by the map

$$F \mapsto \mathrm{mult}(\tau : \sigma) \cdot F,$$

sending a matrix to the direct sum of that matrix with itself $\mathrm{mult}(\tau : \sigma)$ times. In terms of K-theory, this corresponds to multiplication by $\mathrm{mult}(\tau : \sigma)$, once appropriate generators corresponding to $[\tau]$ and $[\sigma]$ are fixed. □

We now turn to the proof of Theorem 6.11. We will prove it simultaneously with the following theorem, by using the Connes–Kasparov isomorphism and induction on the real rank of the group G (note that $G = K$ when the real rank is 0). We recall that the real rank of G is the dimension of $\mathfrak{a}_{\min}$.

Theorem 6.13. *Let G be a real reductive group, and let S be a set of simple roots for the restricted root system $\Delta(\mathfrak{g}, \mathfrak{a}_{\min})$. For every $I \subseteq S$, $K_*(C_r^*(\boldsymbol{G}; I)) = 0$.*

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Proof of Theorems 6.11 and 6.13. Assume that Theorem 6.11 holds for groups with real rank strictly less than r, and suppose that G has real rank r. First, suppose that $I \subset S$ is a proper subset of S. As the group M_I has lower real rank, the morphism

$$R(K_I) \to R_{\text{tempiric}}(M_I)$$

is an isomorphism by the inductive hypothesis. By Theorem 6.12, this means that the C^*-homomorphism $\alpha_0 : C^*(G_0; I) \to C_r^*(G; I)$ induces an isomorphism on K-theory, so by Theorem 5.24, we obtain that $K_*(C_r^*(\boldsymbol{G}; I)) = 0$ for $I \subset S$ proper.

Now, for each $0 \leqslant k \leqslant r$, define

$$U_k = \bigcup_{|I|=k} U_I,$$

where $|I|$ denotes the cardinality of I. Define $C_r^*(\boldsymbol{G}; U_k)$ to be the C^*-algebra corresponding to the open set U_k. Then, $U_k \setminus U_{k-1}$ is the disjoint union of closed subsets

$$U_k \setminus U_{k-1} = \bigsqcup_{|I|=k} i\mathfrak{a}_{I,+}^*,$$

and so we have the exact sequence

$$0 \to C_r^*(\boldsymbol{G}; U_{k-1}) \to C_r^*(\boldsymbol{G}; U_k) \to \bigoplus_{|I|=k} C_r^*(\boldsymbol{G}; I) \to 0.$$

It therefore follows from the six-term exact sequence in K-theory, and from induction on k, that $K_*(C_r^*(\boldsymbol{G}; U_k)) = 0$ for $k < r$.

Now consider $I = S$. In this case, we note that

$$U_r \setminus U_{r-1} = i\mathfrak{a}_{S,+}^*,$$

and $C_r^*(\boldsymbol{G}; U_r) = C_r^*(\boldsymbol{G})$, so we have an exact sequence

$$0 \to C_r^*(\boldsymbol{G}; U_{r-1}) \to C_r^*(\boldsymbol{G}) \to C_r^*(\boldsymbol{G}; S) \to 0.$$

As both $C_r^*(\boldsymbol{G}; U_{r-1})$ and $C_r^*(\boldsymbol{G})$ vanish in K-theory (this is where we use Connes–Kasparov isomorphism), so must $C_r^*(\boldsymbol{G}; S)$, which proves Theorem 6.13 for G. By Theorem 6.12, this also proves Theorem 6.11 for G. □

We have used the Connes–Kasparov isomorphism to prove Theorem 6.11. Conversely, by arguing exactly as in the above proof, we can prove that $C_r^*(\boldsymbol{G})$ has vanishing K-theory if we are given Theorem 6.11. That is, we have the following.

Theorem 6.14. *The assertion that the multiplicity map in Definition 6.6 is an isomorphism, for every real reductive group, is equivalent to the Connes–Kasparov isomorphism for every real reductive group.*

Appendix A.

A.1 Consequences of the Connes–Kasparov isomorphism

Lafforgue used the Connes–Kasparov isomorphism to obtain a new proof in [Laf02a] of Harish-Chandra's equal-rank criterion for the existence of discrete series representations of a connected real reductive group with compact center [Har66, Theorem 13], along with a new proof of Harish-Chandra's parametrization of the discrete series representations [Har66, Theorem 16], when they exist. The purpose of this section is simply to recall that the following result also follows

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from Harish-Chandra's classification of the discrete series, and therefore may also be viewed as consequences of the Connes–Kasparov isomorphism.

Theorem (Theorem 3.10 in this text). *Let G be a real reductive group with compact center that admits discrete series representations and let K be a maximal compact subgroup of G. Each irreducible representation of K occurs as a K-type in at most finitely many discrete series representations of G, up to unitary equivalence.*

Proof. Assume first that G is a connected group that admits discrete series. Let T be a maximal torus in K. According to Harish-Chandra, the discrete series are parametrized by their *Harish-Chandra parameters*, which are certain classes

$$[\lambda] \in i\mathfrak{t}^*/W(\mathfrak{k}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$$

such that $\lambda + \rho_G$ is analytically integral. The character of the discrete series representation π_λ associated to λ is the class

$$[\lambda] \in \mathfrak{t}^*_{\mathbb{C}}/W(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}}) \tag{A.1}$$

(according to Harish-Chandra's criterion, discrete series representations exist for G if and only if $\mathfrak{t}$ by itself is a Cartan subalgebra of $\mathfrak{g}$). It therefore follows from a well-known calculation (cf. [Kna02, Proposition 5.28]) that if

$$\Omega_{\mathfrak{g}} = \sum X^a X_a$$

is the Casimir element in the center of the enveloping algebra for $\mathfrak{g}$ (where $\{X_a\}$ is a basis for $\mathfrak{g}$, and $\{X^a\}$ is a dual basis for the invariant bilinear form B in (2.1)), and if $\operatorname{Inf} \operatorname{Ch}_{\Omega_{\mathfrak{g}}}$ is the function on $\widehat{G}$ from Definition 3.3, then

$$\operatorname{Inf} \operatorname{Ch}_{\Omega_{\mathfrak{g}}}(\pi_\lambda) = \|\lambda\|^2 - \|\rho_G\|^2.$$

Now, write the Casimir as

$$\Omega_{\mathfrak{g}} = \Delta_{\mathfrak{s}} + \Omega_{\mathfrak{k}} = \sum_b Y_b^2 - \sum_c Z_c^2,$$

where $\{Y_b\}$ is an orthonormal basis for $\mathfrak{s}$ and $\{Z_c\}$ is an orthonormal basis for $\mathfrak{k}$ in the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}$. It follows from the unitarity of π_λ that if τ is an irreducible unitary representation of K with infinitesimal character $\mu \in i\mathfrak{t}^*/W(\mathfrak{k}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$, and if v is a unit vector in the τ-isotypical part of H_π, then

$$\|\lambda\|^2 = \langle v, \Omega_{\mathfrak{g}} v\rangle = \langle v, \Delta_{\mathfrak{s}} v\rangle + \|\mu\|^2 - \|\rho_K\|^2 \leqslant \|\mu\|^2 - \|\rho_K\|^2.$$

It follows that only finitely many π_λ can include τ as a K-type, which proves Theorem 3.10 in the connected case.

The general case follows from the connected case by Frobenius reciprocity, since the restriction of a discrete series representation of G to the connected component of G is a direct sum of discrete series representations, all of them sharing the same infinitesimal character. □

Remark A.1. The formula (A.1) shows immediately that every discrete series representation has real infinitesimal character: recall from Definition 2.2 that if $\mathfrak{h} = \mathfrak{t} \oplus \mathfrak{a}$ is any θ-stable Cartan subalgebra of $\mathfrak{g}$, then a weight $\lambda \in \mathfrak{h}^*_{\mathbb{C}}$ is real if it is imaginary-valued on $\mathfrak{t}$ and real-valued on $\mathfrak{a}$, and this is the case here, where in fact $\mathfrak{a} = 0$.

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A.2 Classification of tempered irreducible unitary representations by tempiric representations

We shall sketch a proof of Vogan's Theorem 2.14, which classifies tempered irreducible unitary representations in terms of the imaginary part of the infinitesimal character and tempiric representations of Levi subgroups. For convenience, we restate the result here. Recall that we had fixed an Iwasawa decomposition $G = KA_{\min}N$ of a real reductive group and that we are denoting by $S \subseteq \Delta(\mathfrak{g}, \mathfrak{a}_{\min})$ the associated set of simple restricted roots. Here, again, is the classification theorem.

Theorem (Theorem 2.14 in this text). *Let $I \subseteq S$ and let $\nu \in \mathfrak{a}_{I,+}^*$. The correspondence*

$$\sigma \longmapsto \operatorname{Ind}_{P_I}^G \sigma \otimes \exp(i\nu)$$

determines a bijection from the set of unitary equivalence classes of tempiric representations σ of M_I to the set of unitary equivalence classes of irreducible tempered unitary representations π of G for which $\operatorname{Im}\operatorname{Inf}\operatorname{Ch}(\pi) = i\nu$.

We shall use Harish-Chandra's Theorem 2.8 in the proof, along with the following result, which is also due to Harish-Chandra.

Theorem A.2. *Let $P = MAN$ be a parabolic subgroup of G. Let $\nu \in \mathfrak{a}^*$, and assume that the Levi subgroup $L = MA$ is the full centralizer of ν in G. If σ is an irreducible unitary representation of M with real infinitesimal character, then the parabolically induced representation* $\operatorname{Ind}_P^G \sigma \otimes \exp(i\nu)$ *is irreducible. Moreover, if σ_1 and σ_2 are irreducible unitary representations of M with real infinitesimal character, and if the representations* $\operatorname{Ind}_P^G \sigma_1 \otimes \exp(i\nu)$ *and* $\operatorname{Ind}_P^G \sigma_2 \otimes \exp(i\nu)$ *are equivalent, then σ_1 and σ_2 are equivalent.*

Remark A.3. In the case of the minimal parabolic subgroup, Theorem A.2 was proved by Bruhat [Bru56]. Harish-Chandra developed Bruhat's method to yield the general result above, but did not publish his result. See [KV96] for further historical remarks, and for an exposition of most of the proof. See [vdBS05, Theorem 10.7] for the remaining details.

Proof of Theorem 2.14. It follows from Theorem A.2 that the representations $\operatorname{Ind}_P^G \sigma_1 \otimes \exp(i\nu)$ listed in the statement of the theorem are irreducible, and it follows from (2.9) that they all have imaginary part of the infinitesimal character equal to $i\nu$.

Let π be a tempered, irreducible unitary representation of G. According to Theorem 2.8, the representation π may be embedded as a subrepresentation of some parabolically induced representation $\operatorname{Ind}_P^G \theta \otimes \exp(i\nu)$, where $P = MAN$ is a standard parabolic subgroup of G, θ is a discrete series representation of M, and $\nu \in \mathfrak{a}^*$.

If we conjugate by a representative in K of an appropriate element of the group $W(\mathfrak{g}, \mathfrak{a}_{\min})$, then we may arrange that $\nu \in \mathfrak{a}_{\mathrm{dom}}^*$ (recall our convention that we extend ν to $\mathfrak{a}_{\min}$ by requiring it to be zero on the orthogonal complement of $\mathfrak{a} \subseteq \mathfrak{a}_{\min}$). The conjugation may take P to a nonstandard parabolic subgroup, but we may replace the conjugated parabolic subgroup by a standard parabolic subgroup in the same associate class without changing the equivalence class of the parabolically induced representation.

In summary, we may embed π as a subrepresentation of some parabolically induced representation $\operatorname{Ind}_P^G \theta \otimes \exp(i\nu)$, where $P = MAN$ is a standard parabolic subgroup of G, θ is a discrete series representation of M, and $\nu \in \mathfrak{a}^* \cap \mathfrak{a}_{\mathrm{dom}}^*$. Of course, $\operatorname{Im}\operatorname{Inf}\operatorname{Ch}(\pi) = i\nu$.

Now, let $I \subseteq S$ be such that $\nu \in \mathfrak{a}_{I,+}^*$. The group $L = MA$ is a Levi subgroup of $L_I = M_I A_I$, and if we parabolically induce $\theta \otimes \exp(i\nu)$ from L to L_I, we obtain a representation of the form $\sigma \otimes \exp(i\nu)$. The representation σ is not necessarily irreducible, but it is a direct sum of tempiric

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representations of M_I,

$$\sigma = \sigma_1 \oplus \cdots \oplus \sigma_d.$$

It follows from induction in stages that π embeds in some parabolically induced representation $\mathrm{Ind}_{P_I}^G \sigma_k \otimes \exp(i\nu)$. It follows from Theorem A.2 that the parabolically induced representation is irreducible, and therefore the embedding of π into it must be a unitary equivalence. It also follows from Theorem A.2 that the unitary equivalence class of π determines the unitary equivalence class of σ_k. This completes the proof of the theorem. □

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Conflicts of interest

None.

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