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L_2-SMALL DEVIATIONS FOR WEIGHTED STATIONARY PROCESSES

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Abstract. We find logarithmic asymptotics of L_2-small deviation probabilities for weighted stationary Gaussian processes (both for real and complex-valued) having a power-type discrete or continuous spectrum. Our results are based on the spectral theory of pseudo-differential operators developed by Birman and Solomyak.

§1. *Introduction.* Let $(Y(t))_{t\in T}$ be a random process defined on some parametric measure space $(T, \mathfrak{m})$. Many studies have been devoted to the asymptotic behavior of its L_2-small deviation probabilities

$$\mathbb{P}\left(\|Y\|_2^2 = \int_T |Y(t)|^2 \mathfrak{m}(dt) \leqslant \varepsilon^2\right) \quad \text{as } \varepsilon \to 0,$$

see e.g. [**7**–**9**, **18**–**21**, **24**], to mention just a small sample. The importance of small deviation probabilities in a broader context and a large number of their applications are described in the surveys [**14**, **15**]; for an extensive up-to-date bibliography see [**16**].

In this work, we explore L_2-small deviation probabilities for weighted stationary Gaussian processes having power-type spectrum. Our goal is to relate the asymptotics of small deviation probabilities with that of the spectrum. From the historical point of view our results are closely related to those on fractional Brownian motion and its relatives, see e.g. [**6**, **10**, **11**, **17**]. In terms of such processes with stationary increments our message is that the spectral asymptotics is relevant to the small deviation behavior but the self-similarity is not.

In §2 we consider periodic processes that correspond to a discrete spectrum, while §4 handles continuous time processes with spectral density. The final results of two sections are quite similar, although intermediate technical details differ.

Our results are based on the spectral theory of pseudo-differential operators developed by Birman and Solomyak [**3**, **4**]. This approach was initiated in [**12**], where a similar problem was considered in the discrete time setting. In passing, in §3 we prove a slightly stronger version of one result from [**12**].

The spectral results that we use are not sensible to the symmetry of the spectral measure. Therefore, it is very natural to apply them to the complex-valued processes. In this context *proper* Gaussian processes are particularly convenient because their distributions are determined by the

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spectra of the corresponding covariance operators. Our main results are logarithmic asymptotics of L_2-small deviation probabilities for weighted stationary Gaussian processes having a power-type spectrum in Theorem 2.2 (real-valued periodic process), in Theorem 2.4 (complex-valued proper periodic process), in Theorem 4.2 (real-valued process with continuous spectrum), and in Theorem 4.3 (complex-valued proper process with continuous spectrum).

For the reader's convenience, in Appendix A we formulate some particular cases of deep results of [**2–4**] used in our proofs.

We denote by $\mathcal{F}$ the Fourier transform

$$\mathcal{F}u(\xi) = \int_{\mathbb{R}} \exp(-\mathrm{i}\xi x) u(x)\, dx.$$

For any two sequences a_k, b_k the standard notation $a_k \sim b_k$ means that $\lim_{k\to\infty} a_k/b_k = 1$.

§2. *Periodic stationary processes.*

2.1. *Spectral representations.* We first recall the necessary information on the spectral representations of stationary periodic processes.

Let $X = \{X(t), t \in \mathbb{R}\}$ be a complex-valued 2π-periodic centered second order mean-square continuous stationary process. Then its covariance function admits a spectral representation

$$K_X(s-t) := \operatorname{cov}(X(s), X(t)) = \sum_{k\in\mathbb{Z}} \mu_k \mathrm{e}^{\mathrm{i}k(s-t)}, \quad s, t \in \mathbb{R},$$

where $\mu := (\mu_k)_{k\in\mathbb{Z}}$ is a finite non-negative measure on $\mathbb{Z}$ called the spectral measure of X.

The spectral representation of X itself is written as

$$X(t) = \sum_{k\in\mathbb{Z}} \sqrt{\mu_k}\, \xi_k \mathrm{e}^{\mathrm{i}kt}, \tag{2.1}$$

where ξ_k are centered uncorrelated complex random variables with $\mathbb{E}\,|\xi_k|^2 = 1$.

Just for completeness, recall a straightforward reformulation for real-valued processes. Let denote $\xi_k := \xi_k^{(\mathrm{re})} + \mathrm{i}\xi_k^{(\mathrm{im})}$. The process X is real-valued if and only if:

- ξ_0 is real;
- $\mu_{-k} = \mu_k$ for all $k > 0$;
- $\xi_{-k} = \overline{\xi_k}$ for all $k > 0$;
- $\mathbb{E}\,|\xi_k^{(\mathrm{re})}|^2 = \mathbb{E}\,|\xi_k^{(\mathrm{im})}|^2 = 1/2$ for all $k \in \mathbb{Z}$;
- the real random variables $(\xi_0, (\xi_k^{(\mathrm{re})}, \xi_k^{(\mathrm{im})})_{k>0})$ are uncorrelated.

In this case (2.1) is written as

$$X(t) = \sqrt{\mu_0}\,\xi_0 + \sum_{k=1}^{\infty} \sqrt{\mu_k}\,(\sqrt{2}\xi_k^{(\mathrm{re})}[\sqrt{2}\cos(kt)] - \sqrt{2}\xi_k^{(\mathrm{im})}[\sqrt{2}\sin(kt)]), \tag{2.2}$$

where the random variables $\sqrt{2}\xi_k^{(\mathrm{re})}$, $\sqrt{2}\xi_k^{(\mathrm{im})}$ have unit variance.

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2.2. *Covariance operators and their factorization.* Let $\nu(du) := du/2\pi$ be the normalized Lebesgue measure on $[0, 2\pi]$. In the following, we will consider X as a random element of $L_2([0, 2\pi], \nu)$. From this point of view, equations (2.1) and (2.2) represent the orthogonal expansions of X with respect to the orthonormal bases $(e^{ikt})_{k\in\mathbb{Z}}$ and $\{1, (\sqrt{2}\cos(kt), \sqrt{2}\sin(kt))_{k\geqslant 1}\}$, respectively. The elements of these bases are eigenvectors of the corresponding covariance operator $\mathcal{K}_X$ in $L_2([0, 2\pi], \nu)$ and the corresponding eigenvalues are μ_k.

The orthogonal expansions generate natural decompositions of $\mathcal{K}_X$. Let $e_k := \exp(ik\cdot)$, $k \in \mathbb{Z}$. Then the operator square root of $\mathcal{K}_X$ is defined by the formula $\mathcal{D}e_k := \sqrt{\mu_k}\, e_k$, $k \in \mathbb{Z}$. Operator $\mathcal{D}$ is bounded, self-adjoint, satisfies $\mathcal{D}\mathcal{D} = \mathcal{D}\mathcal{D}^* = \mathcal{K}_X$, and can be interpreted as a convolution operator with the kernel

$$\mathcal{D}(s) := \sum_{\ell\in\mathbb{Z}} \sqrt{\mu_\ell}\, e_\ell(s).$$

Indeed, for every $k \in \mathbb{Z}$ and $s \in [0, 2\pi]$ we have

$$\begin{aligned}\int_0^{2\pi} \mathcal{D}(s-t)\, e_k(t)\, \nu(dt) &= \sum_{\ell\in\mathbb{Z}} \sqrt{\mu_\ell} \int_0^{2\pi} e_\ell(s-t)\, e_k(t)\, \nu(dt)\\ &= \sum_{\ell\in\mathbb{Z}} \sqrt{\mu_\ell}\, e_\ell(s) \int_0^{2\pi} e_{k-\ell}(t)\, \nu(dt) = \sqrt{\mu_k}\, e_k(s).\end{aligned}$$

In the following we are interested in the small ball behavior of the *weighted* L_2-norm

$$\int_0^{2\pi} q(t)|X(t)|^2\, dt = 2\pi \|\sqrt{q}X\|_{2,\nu}^2 \tag{2.3}$$

with some weight $q \in L_1[0, 2\pi]$.

We have a decomposition for covariance operator

$$\mathcal{K}_{\sqrt{q}X} = \mathcal{Q}\mathcal{K}_X\mathcal{Q} = \mathcal{Q}\mathcal{D}\mathcal{D}\mathcal{Q} =: \mathcal{T}^*\mathcal{T}, \qquad \mathcal{T} = \mathcal{D}\,\mathcal{Q}, \tag{2.4}$$

where $\mathcal{Q}$ stands for the self-adjoint multiplication operator related to the function $\sqrt{q} \in L_2[0, 2\pi]$. We claim that $\mathcal{T}$ is the Hilbert–Schmidt operator although $\mathcal{Q}$ need not be even bounded. Indeed, since $\sqrt{q} \in L_2[0, 2\pi]$, it admits a Fourier series expansion

$$\sqrt{q}(t) := \sum_{m\in\mathbb{Z}} q_m e^{imt},$$

with $(q_m) \in \ell_2(\mathbb{Z})$. Then we have

$$\mathcal{T}e_\ell = \mathcal{D}\Big[\sum_{k\in\mathbb{Z}} q_{k-\ell}e_k\Big] = \sum_{k\in\mathbb{Z}} \sqrt{\mu_k}\, q_{k-\ell}\, g_\ell e_k,$$

and therefore

$$\sum_{\ell\in\mathbb{Z}} \|\mathcal{T}e_\ell\|_{2,\nu}^2 = \sum_{\ell\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} \mu_k\, |q_{k-\ell}|^2 = \mu(\mathbb{Z})\, \|q\|_{1,\nu}.$$

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For the study of logarithmic asymptotics of small deviation probabilities, we need to know the main term of the eigenvalue asymptotics for $\mathcal{K}_{\sqrt{q}X}$, see [18]. Since $K_{\sqrt{q}X}$ is non-negative, its eigenvalues $\lambda_n(\mathcal{K}_{\sqrt{q}X})$ coincide with its singular values $s_n(\mathcal{K}_{\sqrt{q}X})$. We always label the eigenvalues and singular values in non-increasing order counting multiplicity.

We study the *distribution function of singular values*

$$N(\lambda; \mathcal{K}_{\sqrt{q}X}) := \#\{n : s_n(\mathcal{K}_{\sqrt{q}X}) \geqslant \lambda\}, \quad \lambda > 0,$$

and its asymptotics as $\lambda \to 0$. For a compact operator T we introduce the notation

$$\Delta_\theta(T) = \lim_{\lambda \to 0_+} \lambda^\theta N(\lambda; T).$$

The following relation is important in what follows:

$$\Delta_\theta(T) = \Delta \quad \Longleftrightarrow \quad \lim_{n\to\infty} n^{1/\theta} s_n(T) = \Delta^{1/\theta}. \tag{2.5}$$

2.3. *Spectral asymptotics.* From now on we assume that the spectral measure has a power-like decay

$$\lim_{k \to \pm\infty} |k|^r \mu_k = M_\pm, \tag{2.6}$$

with some $r > 1$ and $M_\pm \geqslant 0$, $M_+ + M_- > 0$. Assumption (2.6) is typical of the literature on small deviations of Gaussian processes; see for example [7].

LEMMA 2.1. *Let the spectral measure of X satisfy (2.6), and let $q \in L_1[0, 2\pi]$. Then*

$$\lambda_n(\mathcal{K}_{\sqrt{q}X}) = s_n(\mathcal{K}_{\sqrt{q}X}) \sim \left(\frac{M_-^{1/r} + M_+^{1/r}}{2\pi} \int_0^{2\pi} q(t)^{1/r}\, dt\right)^r n^{-r} \quad \text{as } n \to \infty. \tag{2.7}$$

Proof. We proceed in a similar way to [12] where we used quite general results of Birman and Solomyak [3, 4].

We can consider $\mathcal{K}_{\sqrt{q}X}$ as an operator in $L_2(\mathbb{R})$:

$$(\mathcal{K}_{\sqrt{q}X} u)(s) = b(s) \int_{\mathbb{R}} \frac{1}{2\pi} K_X(s-t) b(t) u(t)\, dt,$$

where $b = \sqrt{q} \cdot \mathbf{1}_{[0,2\pi]}$. Notice that since we are working on the interval of length 2π, it is sufficient to consider only the restriction of our periodic function K_X to $[-2\pi, 2\pi]$.

Let h be a smooth cut-off function such that $h(t) = 1$ if $t \in [3\pi/2, 2\pi]$ and $h(t) = 0$ if $t \in [-2\pi, \pi]$. Then it follows that the function

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$h_0(s) := 1 - h(s) - h(-s)$ equals one on $[-\pi, \pi]$ and vanishes outside of the interval $[-3\pi/2, 3\pi/2]$. We decompose the kernel K_X as follows:

$$\frac{1}{2\pi} K_X(s) = \frac{1}{2\pi} K_X(s)[h(s) + h(-s) + h_0(s)] =: K_+(s) + K_-(s) + K^{(0)}(s), \tag{2.8}$$

and claim that the function $K^{(0)}$ satisfies

$$\lim_{\xi \to \pm\infty} |\xi|^r \cdot \mathcal{F}K^{(0)}(\xi) = M(\operatorname{sgn}(\xi)), \tag{2.9}$$

where $M(\pm 1) = M_\pm$. Indeed, we have

$$\frac{1}{2\pi} \mathcal{F}(K_X \cdot h_0)(\xi) = \frac{1}{2\pi} \sum_{k \in \mathbb{Z}} \mu_k \mathcal{F}h_0(\xi - k),$$

and then by splitting the series into two sums, we obtain

$$\mathcal{F}K^{(0)}(\xi) = \Sigma_1 + \Sigma_2 := \left(\sum_{|k-\xi| \leqslant \sqrt{\xi}} + \sum_{|k-\xi| > \sqrt{\xi}} \right) \frac{\mu_k}{2\pi} \mathcal{F}h_0(\xi - k).$$

Since $\mathcal{F}h_0$ rapidly decays at infinity, we have $\Sigma_2 = o(|\xi|^{-r})$ as $|\xi| \to \infty$. Furthermore, (2.6) implies that

$$\begin{aligned} \Sigma_1 &= \frac{M(\operatorname{sgn}(\xi))}{2\pi} |\xi|^{-r} \sum_{|k-\xi| \leqslant \sqrt{\xi}} \mathcal{F}h_0(\xi - k) + o(|\xi|^{-r}) \\ &= \frac{M(\operatorname{sgn}(\xi))}{2\pi} |\xi|^{-r} \sum_{k} \mathcal{F}h_0(\xi - k) + o(|\xi|^{-r}) \\ &= M(\operatorname{sgn}(\xi))|\xi|^{-r} + o(|\xi|^{-r}) \end{aligned}$$

by the Poisson summation formula (see, e.g., [27, Ch. II, §13]), so that (2.9) follows.

Now we introduce a model operator

$$(\mathcal{A}u)(s) = b(s)\mathcal{F}^{-1}(a(\xi)\mathcal{F}(bu)(\xi)),$$

with

$$a(\xi) = \zeta(\xi) M(\operatorname{sgn}(\xi))|\xi|^{-r},$$

where ζ is a smooth cut-off function vanishing in a neighborhood of the origin. Since $b \in L_2$, Proposition A.1 can be applied to the operator $\mathcal{A}$. This gives

$$\begin{aligned} \Delta_{1/r}(\mathcal{A}) &= \frac{1}{2\pi} \int_0^{2\pi} \int_{\mathbb{R}\setminus\{0\}} \mathbf{1}_{\{|b(t)|^2 \, |M(\operatorname{sgn}(\xi))| \, |\xi|^{-r} \geqslant 1\}} \, d\xi \, dt \\ &= \frac{1}{2\pi} \int_0^{2\pi} q(t)^{1/r} \, dt \, (M(-1)^{1/r} + M(1)^{1/r}). \end{aligned}$$

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Furthermore, the decomposition (2.8) generates the corresponding operator decomposition

$$\mathcal{K}_{\sqrt{q}X} = \mathcal{K}_+ + \mathcal{K}_- + \mathcal{K}^{(0)}.$$

Since the relation (2.9) implies $\mathcal{F}K^{(0)}(\xi) - a(\xi) = o(|\xi|^{-r})$ as $\xi \to \infty$, part (1) of Proposition A.2 gives $\Delta_{1/r}(\mathcal{K}^{(0)} - \mathcal{A}) = 0$. Moreover, since K_X is 2π-periodic, the singular values of $\mathcal{K}_+$ coincide with the singular values of the operator

$$b(s+\pi)\mathbf{1}_{[0,\pi]}(s)\int_{\mathbb{R}} \frac{1}{2\pi} K_X(s-t)h(s+2\pi-t)b(t)\mathbf{1}_{[\pi,2\pi]}(t)u(t)\,dt.$$

For this operator, the support of the "left" weight is $[0,\pi]$, and the support of the "right" weight is $[\pi, 2\pi]$. Part (2) in Proposition A.2 gives $\Delta_{1/r}(\mathcal{K}_+) = 0$. Similarly, $\Delta_{1/r}(\mathcal{K}_-) = 0$. By Proposition A.4 we obtain

$$\Delta_{1/r}(\mathcal{K}_{\sqrt{q}X}) = \Delta_{1/r}(\mathcal{K}^{(0)}) = \Delta_{1/r}(\mathcal{A}),$$

and the equivalence in (2.5) gives (2.7). □

2.4. *Gaussian small deviations.* Now we transform the information about the eigenvalues into that on small deviation asymptotic behavior. This can be done for real processes and also for an important class of complex processes. We handle two cases separately because the constants appearing in the results are slightly different.

2.4.1. *Real processes.* Recall that if we have a centered Gaussian random vector Z, in a real Hilbert space, and $\mathcal{K}_Z$ stands for its covariance operator, then, by the Karhunen–Loève expansion (see [**1**, §1.4]),

$$\|Z\|^2 = \sum_{n=1}^{\infty} \lambda_n \xi_n^2,$$

where $(\xi_n)_{n\in\mathbb{N}}$ is a sequence of independent standard normal random variables and $(\lambda_n)_{n\in\mathbb{N}}$ are the eigenvalues of $\mathcal{K}_Z$. Therefore, the sequence $(\lambda_n)_{n\in\mathbb{N}}$ determines the distribution of $\|Z\|$. In particular, if

$$\lambda_n \sim C\, n^{-r} \quad \text{as } n \to \infty, \tag{2.10}$$

then it is well known from [**7**, p. 67] or [**26**], that

$$\ln \mathbb{P}(\|Z\| \leqslant \varepsilon) \sim -B_r\, (C/\varepsilon^2)^{1/(r-1)} \tag{2.11}$$

with C from (2.10) and $B_r := ((r-1)/2)(\pi/r\sin(\pi/r))^{r/(r-1)}$. If our process X is real, we can apply the formula (2.11) to $\sqrt{q}X$ considered as an element of $L_2([0,2\pi],\nu)$ and using eigenvalue asymptotics (2.7) as (2.10). Notice that for real processes the spectral measure is symmetric, i.e. we have $M_+ = M_- := M$. Taking into account (2.3) we immediately obtain the following result.

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THEOREM 2.2. *Let $\{X(t), t \in \mathbb{R}\}$ be a 2π-periodic real centered mean-square continuous stationary Gaussian process. Assume that its spectral measure satisfies the asymptotic condition*

$$\mu_k \sim M|k|^{-r}, \quad as\ |k| \to \infty,$$

with some $r > 1$, $M > 0$. Let q be a summable weight.

Then we have, as $\varepsilon \to 0$,

$$\ln \mathbb{P}\left(\int_0^{2\pi} q(t)|X(t)|^2\,dt \leqslant \varepsilon^2\right) \sim -\left(\frac{M^{1/r}}{r\sin(\pi/r)}\int_0^{2\pi} q(t)^{1/r}\,dt\right)^{r/(r-1)} \times \frac{(r-1)(2\pi)^{1/(r-1)}}{2\,\varepsilon^{2/(r-1)}}.$$

2.4.2. *Examples.* Consider the Bogoliubov process [24, 25] – a 1-periodic centered stationary Gaussian process (with parameter $\omega > 0$) defined by

$$\beta^{(\omega)}(s) := \sqrt{\mu_0}\,\xi_0 + \sum_{k=1}^{\infty} \sqrt{\mu_k}(\xi_k[\sqrt{2}\cos(2\pi ks)] + \zeta_k[\sqrt{2}\sin(2\pi ks)]), \quad s \in \mathbb{R},$$

with independent standard normal random variables $((\xi_k)_{k\geqslant 0}, (\zeta_k)_{k>0})$ and $\mu_k = 1/(\omega^2 + (2\pi k)^2)$. Define a 2π-periodic process $X(t) := \beta^{(\omega)}(t/2\pi)$, $t \in \mathbb{R}$. In our notation, for the spectrum of X we have $r = 2$, $M = 1/(2\pi)^2$. By applying Theorem 2.2 we obtain for $q \in L_1[0, 1]$ that

$$\begin{aligned}
&\ln \mathbb{P}\left(\int_0^1 q(s)|\beta^{(\omega)}(s)|^2\,ds \leqslant \varepsilon^2\right) \\
&\quad = \ln \mathbb{P}\left(\int_0^{2\pi} q(t/2\pi)|X(t)|^2\,dt \leqslant (\varepsilon\sqrt{2\pi})^2\right) \\
&\quad \sim -\frac{1}{8}\left(\int_0^1 \sqrt{q(s)}\,ds\right)^2 \varepsilon^{-2}.
\end{aligned}$$

In the simplest case $q(s) \equiv 1$ we have

$$\ln \mathbb{P}\left(\int_0^1 |\beta^{(\omega)}(s)|^2\,ds \leqslant \varepsilon^2\right) \sim -\frac{1}{8}\,\varepsilon^{-2},$$

cf. [24, Theorem 1].

For $q(s) = \mathrm{e}^{2as}$, $a \neq 0$, our result gives

$$\ln \mathbb{P}\left(\int_0^1 \mathrm{e}^{2as}|\beta^{(\omega)}(s)|^2\,ds \leqslant \varepsilon^2\right) \sim -\frac{1}{8}\left[\frac{\mathrm{e}^a - 1}{a}\right]^2 \varepsilon^{-2},$$

as proved in [24, Theorem 2].

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Our next example is the so-called m-times integrated-centered Brownian bridge. Let $B_0(\tau)$ be standard Brownian bridge on $[0, 1]$. We define the sequence of Gaussian processes

$$B_{\{m\}}(s) = B_{m-1}(s) - \int_0^1 B_{m-1}(\tau)\, d\tau; \qquad B_m(\tau) = \int_0^\tau B_{\{m\}}(s)\, ds, \quad m \in \mathbb{N}.$$

It was shown in [19, §3] that

$$B_{\{m\}}(s) = \sum_{k=1}^{\infty} (2\pi k)^{-m} (\xi_k[\sqrt{2}\cos(2\pi k s)] + \zeta_k[\sqrt{2}\sin(2\pi k s)]), \quad s \in [0, 1],$$

with independent standard normal random variables $((\xi_k)_{k>0}, (\zeta_k)_{k>0})$. This formula obviously defines a 1-periodic centered stationary Gaussian process on $\mathbb{R}$. Define a 2π-periodic process $X_m(t) = B_{\{m\}}(t/2\pi)$. Then for the spectrum of X_m we have $r = 2m$, $M = (2\pi k)^{-2m}$. By applying Theorem 2.2 we obtain for $q \in L_1[0, 1]$ that

$$\begin{aligned}
&\ln \mathbb{P}\left(\int_0^1 q(s) |B_{\{m\}}(s)|^2\, ds \leqslant \varepsilon^2 \right) \\
&\quad = \ln \mathbb{P}\left(\int_0^{2\pi} q(t/2\pi) |X_m(t)|^2\, dt \leqslant (\varepsilon\sqrt{2\pi})^2 \right) \\
&\quad \sim -\frac{2m-1}{2} \left(\frac{1}{2m \sin(\pi/2m)} \int_0^1 q(s)^{1/2m}\, ds \right)^{2m/(2m-1)} \varepsilon^{-2/(2m-1)}.
\end{aligned}$$

For $q(s) \equiv 1$ this result agrees with [19, Theorem 3.2].

Remark 2.3. In fact, the sharp small ball asymptotics for these processes were obtained in [19] and [24], see also [21] for more general weights. However, this is strongly connected with the fact that $\beta^{(\omega)}$ and $B_{\{m\}}$ are *the Green Gaussian processes*, i.e. their covariances are the Green functions for ordinary differential operators. In general, this seems to be a much harder problem.

2.4.3. *Proper complex processes.* If we have a centered Gaussian random vector Z in a complex Hilbert space, and $\mathcal{K}_Z$ stands for its covariance operator, then Karhunen–Loève expansion yields

$$Z = \sum_{n=1}^{\infty} \lambda_n \xi_n e_n, \tag{2.12}$$

where $(\xi_n)_{n\in\mathbb{N}}$ are uncorrelated complex jointly Gaussian random variables satisfying $\mathbb{E}|\xi_n|^2 = 1$ and $(\lambda_n, e_n)_{n\in\mathbb{N}}$ are the eigenpairs of $\mathcal{K}_Z$. We still have

$$\|Z\|^2 = \sum_{n=1}^{\infty} \lambda_n |\xi_n|^2, \tag{2.13}$$

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but, unfortunately, unlike the real case, the variables ξ_n need not be independent, although they are uncorrelated. Indeed, the independence of two centered complex Gaussian random variables η_1 and η_2 is equivalent to the *pair* of relations

$$\begin{cases} \operatorname{cov}(\eta_1, \eta_2) = \mathbb{E}\,\eta_1\overline{\eta_2} = 0; \\ \mathbb{E}\,\eta_1\eta_2 = 0. \end{cases}$$

Therefore, the sequence $(\lambda_n)_{n\in\mathbb{N}}$ *does not* determine the distribution of $\|Z\|$ in general. For this reason, we need to restrict the consideration to an important subclass of the variables and processes where uncorrelated variables are independent, cf. [22, 23].

A complex-valued random process $(X(t))_{t\in T}$ is called centered *proper* (or *circularly*) Gaussian if:

- for any $t_1, \dots, t_n \in T$ the coordinate vector $(X^{(\mathrm{re})}(t_1), X^{(\mathrm{im})}(t_1), \dots, X^{(\mathrm{im})}(t_n))$ is a centered Gaussian vector in $\mathbb{R}^{2n}$;
- $\mathbb{E}\,X(t_1)X(t_2) = 0$ for all $t_1, t_2 \in T$.

We clearly have $\mathbb{E}\,X(t) = 0$, for all $t \in T$. Moreover, the property $\mathbb{E}\,X(t)^2 = 0$ yields that the distribution of $X(t)$ in the complex plane $\mathbb{C}$ is spherically symmetric.

These properties extend to the span of X. Let us denote $\mathcal{X} := \overline{\operatorname{span}\{X(t), t \in T\}}$. For every $Y \in \mathcal{X}$ we have $\mathbb{E}\,Y = 0$, $\mathbb{E}\,Y^2 = 0$, hence its distribution in $\mathbb{C}$ is spherically symmetric Gaussian. Moreover, for any Y_1, $Y_2 \in \mathcal{X}$ we have $\mathbb{E}\,Y_1Y_2 = 0$ and Y_1, Y_2 are independent if and only if they are uncorrelated, i.e. $\mathbb{E}\,Y_1\overline{Y_2} = 0$. This can be easily verified by checking that their coordinates are uncorrelated.

By applying these facts to the expansion (2.12) of a proper Gaussian process Z, we see that the variables $(\xi_n)_{n\in\mathbb{N}}$ are independent and spherically symmetric. Therefore, (2.13) becomes

$$\|Z\|^2 = \sum_{n=1}^{\infty} \frac{\lambda_n}{2}(\xi_{n,1}^2 + \xi_{n,2}^2),$$

where $(\xi_{n,j})_{n\in\mathbb{N}, j\in\{1,2\}}$ are independent real standard Gaussian random variables. This formula can be rewritten as

$$\|Z\|^2 = \sum_{n=1}^{\infty} \lambda_n^* \xi_n^{*2},$$

where

$$\lambda_{2n-1}^* := \frac{\lambda_n}{2}; \qquad \xi_{2n-1}^* := \xi_{n,1},$$
$$\lambda_{2n}^* := \frac{\lambda_n}{2}; \qquad \xi_{2n}^* := \xi_{n,2},$$

for all $n \geqslant 1$.

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A straightforward calculation shows that $\lambda_n \sim Cn^{-r}$ yields $\lambda_n^* \sim 2^{r-1}Cn^{-r}$, as $n \to \infty$. By applying (2.11) with $2^{r-1}C$ instead of C we obtain the following result.

THEOREM 2.4. *Let $\{X(t), t \in \mathbb{R}\}$ be a 2π-periodic complex centered mean-square continuous stationary proper Gaussian process. Assume that its spectral measure satisfies the asymptotic condition (2.6) with some $r > 1$. Let q be a summable weight.*

Then we have, as $\varepsilon \to 0$,

$$\ln \mathbb{P}\left(\int_0^{2\pi} q(t)|X(t)|^2\,dt \leqslant \varepsilon^2\right) \sim -\left(\frac{M_-^{1/r} + M_+^{1/r}}{2r\sin(\pi/r)}\int_0^{2\pi} q(t)^{1/r}\,dt\right)^{r/(r-1)} \times \frac{(r-1)(2\pi)^{1/(r-1)}}{\varepsilon^{2/(r-1)}}.$$

§3. *Stationary sequences.* Let a real stationary centered Gaussian sequence $(U_k)_{k\in\mathbb{Z}}$ admit a representation

$$U_k = \sum_{m=-\infty}^{\infty} a_m X_{k-m}, \tag{3.1}$$

where $(a_m) \in \ell_2(\mathbb{Z})$, and X_m are independent standard Gaussian random variables (this representation exists if and only if (U_k) has a spectral density).

The following result was essentially obtained in [12].

THEOREM 3.1. *Let a real stationary centered Gaussian sequence $(U_k)_{k\in\mathbb{Z}}$ admit a representation (3.1) and let the coefficients $(d_k)_{k\in\mathbb{Z}}$ have the asymptotics*

$$\lim_{k\to\pm\infty} |k|^p d_k = d_\pm \quad \text{for some } p > \tfrac{1}{2},$$

where at least one of the numbers $d_\pm$ is strictly positive. Then, as $\varepsilon \to 0$,

$$\ln \mathbb{P}\left(\sum_{k\in\mathbb{Z}} d_k^2 U_k^2 \leqslant \varepsilon^2\right) \sim -\left(\frac{d_-^{1/p} + d_+^{1/p}}{4\,p\sin(\pi/2p)}\int_0^{2\pi} |\mathfrak{a}(t)|^{1/p}\,dt\right)^{2p/(2p-1)} \times \frac{2p-1}{2\,\varepsilon^{2/(2p-1)}},$$

where $\mathfrak{a}(t) = \sum_{k\in\mathbb{Z}} a_k \mathrm{e}^{\mathrm{i}kt}$.

However, in [12], for $p < 1$ an additional assumption was imposed. Now we show that it was not necessary, answering the question raised in [12, Remark 1.2].

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Sketch of the proof. We have to study the norm of the random vector $Z \in \ell_2(\mathbb{Z})$ defined by its coordinates $Z_k = d_k U_k$, $k \in \mathbb{Z}$. It was proved in [12] that the corresponding covariance operator $\mathcal{K}_Z$ admits a representation

$$\mathcal{K}_Z = \mathbf{DAA}^*\mathbf{D}, \tag{3.2}$$

where $\mathbf{D}$ is the convolution operator with the kernel $\sum_{k\in\mathbb{Z}} d_k e^{ikt}$ while $\mathbf{A}$ is the multiplication operator related to the function $\mathfrak{a}(t)$.

We see that the elements of decomposition in (3.2) are the same as in (2.4) but the order of use of operators is different. However, a well-known theorem in operator theory, see, e.g., [5, §2.10, Theorem 5], implies the coincidence of non-zero eigenvalues for operators $\mathcal{T}\mathcal{T}^*$ and $\mathcal{T}^*\mathcal{T}$ for any bounded linear operator $\mathcal{T}$. Thus, Lemma 2.1 implies that spectral asymptotics of (2.7) type holds for the operator $\mathcal{K}_Z$ (with the natural replacement $r \to 2p$, $M_\pm \to d_\pm^2$, $\sqrt{q} \to \mathfrak{a}$). Using formula (2.11) we obtain the claimed small deviation asymptotics. □

§4. *Stationary processes with continuous spectra.*

4.1. *Spectral representations.* Now we consider general aperiodic stationary processes. Let $X(t), t \in \mathbb{R}$, be a centered second order complex stationary process on $\mathbb{R}$.

The analogue of spectral representation (2.1) is more involved and is written as follows:

$$X(t) = \int_{\mathbb{R}} e^{itu}\xi(du), \quad t \in \mathbb{R},$$

where $\xi(du)$ is an uncorrelated white noise with a control measure μ called spectral measure of X.

The only information about white noise integrals, that we need here is that the random variable $\int_{\mathbb{R}} g(u)\xi(du)$ is well defined and centered if and only if $g \in L_2(\mathbb{R}, \mu)$, while for the covariances we have the expression

$$\operatorname{cov}\left(\int_{\mathbb{R}} g_1(u)\,\xi(du), \int_{\mathbb{R}} g_2(u)\,\xi(du)\right) = \int_{\mathbb{R}} g_1(u)\,\overline{g_2(u)}\,\mu(du).$$

In particular,

$$K_X(s-t) := \operatorname{cov}(X(s), X(t)) = \int_{\mathbb{R}} e^{iu(s-t)}\,\mu(du), \quad s, t \in \mathbb{R}.$$

We are interested in the small ball behavior of the weighted L_2-norm

$$\int_{\mathbb{R}} q(t)|X(t)|^2\,dt = \|\sqrt{q}\,X\|_2^2,$$

where $q \in L_1(\mathbb{R})$ is a non-negative weight.

Assume that the spectral measure μ has a density $m \in L_1(\mathbb{R})$. Then it is easy to see that

$$\begin{aligned}(\mathcal{K}_{\sqrt{q}X}u)(s) &= \sqrt{q(s)}\int_{\mathbb{R}} K_X(s-t)\sqrt{q(t)}u(t)\,dt \\ &= 2\pi\sqrt{q(s)}\,\mathcal{F}^{-1}(m(\xi)\mathcal{F}(\sqrt{q}\,u)(\xi))\end{aligned}$$

(we recall that $\mathcal{F}$ stands for the Fourier transform).

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4.2. *Spectral asymptotics.* From now on we assume that the spectral density m has a power-like decay analogous to (2.6),

$$\lim_{u\to\pm\infty} |u|^r m(u) = M_\pm, \tag{4.1}$$

with some $r > 1$ and $M_\pm \geqslant 0$, $M_+ + M_- > 0$.

LEMMA 4.1. *Let the spectral density of X satisfy (4.1). Assume that $q \in L_1(\mathbb{R})$, and*

$$|q|_r := \sum_{j\in\mathbb{Z}} \|q\|^{1/r}_{L_1(j,j+1)} < \infty. \tag{4.2}$$

Then

$$\lambda_n(\mathcal{K}_{\sqrt{q}X}) = s_n(\mathcal{K}_{\sqrt{q}X}) \sim 2\pi \left(\frac{M_-^{1/r} + M_+^{1/r}}{2\pi} \int_{\mathbb{R}} q(t)^{1/r}\, dt \right)^r n^{-r}, \quad \text{as } n \to \infty. \tag{4.3}$$

Proof. We cannot apply Proposition A.1 directly since it requires boundedness of the weights supports. Therefore, we use subtle estimates of [**2**, §5], see Proposition A.3. We introduce a decomposition similar to (2.4):

$$\mathcal{K}_{\sqrt{q}X} = \widetilde{\mathcal{T}}^*\widetilde{\mathcal{T}}, \quad \widetilde{\mathcal{T}} = \mathcal{M}\mathcal{F}\mathcal{Q},$$

where $\mathcal{M}$ and $\mathcal{Q}$ stand for the multiplication by $\sqrt{m} \in L_2(\mathbb{R})$ and $\sqrt{q} \in L_2(\mathbb{R})$, respectively.

Following [**2**], for $f \in L_2(\mathbb{R})$ we define the numerical sequence

$$v(f) = \{v_j(f)\}_{j\in\mathbb{Z}}; \qquad v_j(f) := \|f\|_{L_2(j,j+1)}. \tag{4.4}$$

Using the notation in Appendix A we can write the assumption (4.1) as follows:

$$\|\!| v(\sqrt{m}) |\!\|_{2/r,\infty} < \infty.$$

Further, the assumption (4.2) is equivalent to $v(\sqrt{q}) \in \ell_{2/r}$, and the (quasi)-norm $\|v(\sqrt{q})\|_{2/r}$ coincides with $|q|_r$.

Now we consider the sequence of operators $\widetilde{\mathcal{T}}_k = \mathcal{M}\mathcal{F}\mathcal{Q}_k$, $k \in \mathbb{N}$, where $\mathcal{Q}_k$ is multiplication by compactly supported weight

$$b_k(t) = \sqrt{q(t)} \cdot \mathbf{1}_{[-k,k]}(t).$$

Obviously, $v(b_k) \to v(\sqrt{q})$ in $\ell_{2/r}$.

Since $2/r < 2$, we can apply Proposition A.3 to the operator $\widetilde{\mathcal{T}}^* - \widetilde{\mathcal{T}}_k^*$. This gives

$$\sup_n (n^{r/2} s_n(\widetilde{\mathcal{T}}^* - \widetilde{\mathcal{T}}_k^*)) \leqslant \text{const} \cdot \|\!| v(\sqrt{m}) |\!\|_{2/r,\infty} \cdot \|v(\sqrt{q}) - v(b_k)\|_{2/r} \to 0 \quad \text{as } k \to \infty.$$

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By (2.5) and Proposition A.4 we infer

$$\Delta_{2/r}(\widetilde{\mathcal{T}}_k^*) \to \Delta_{2/r}(\widetilde{\mathcal{T}}^*) \quad \text{as } k \to \infty.$$

Since $\lambda_n(\mathcal{K}_{\sqrt{q}X}) = s_n^2(\widetilde{\mathcal{T}}^*)$, this implies

$$\Delta_{1/r}(\mathcal{K}_k) \to \Delta_{1/r}(\mathcal{K}_{\sqrt{q}X}) \quad \text{as } k \to \infty,$$

where

$$(\mathcal{K}_k u)(s) = (\widetilde{\mathcal{T}}_k^* \widetilde{\mathcal{T}}_k u)(s) = b_k(s) \int_{\mathbb{R}} K_X(s-t) b_k(t) u(t)\, dt.$$

The weights b_k satisfy the assumptions of Proposition A.1. Using Proposition A.1, part (1) in Proposition A.2 and the last statement in Proposition A.4, we obtain

$$\begin{aligned}\Delta_{1/r}(\mathcal{K}_k) &= \frac{1}{2\pi} \int_{\mathbb{R}} \int_{\mathbb{R}\setminus\{0\}} \mathbf{1}_{\{2\pi |b_k(t)|^2 \, |M(\operatorname{sgn}(\xi))| \, |\xi|^{-r} \geqslant 1\}}\, d\xi\, dt \\ &= \frac{1}{2\pi} \int_{-k}^{k} (2\pi q(t))^{1/r}\, dt \, (M(-1)^{1/r} + M(1)^{1/r})\end{aligned}$$

(recall that $M(\pm 1) = M_{\pm}$). We pass to the limit as $k \to \infty$, and the equivalence in (2.5) yields (4.3).

4.3. *Gaussian small deviations.*

4.3.1. *Real processes.* By combining spectral asymptotics (4.3) with small deviation asymptotics (2.11) we immediately obtain the following result.

THEOREM 4.2. *Let $\{X(t), t \in \mathbb{R}\}$ be a real centered mean-square continuous stationary Gaussian process. Assume that it has a spectral density satisfying asymptotical condition*

$$m(u) \sim M|u|^{-r} \quad \text{as } |u| \to \infty,$$

with some $r > 1$, $M > 0$. Let q be a summable weight satisfying condition (4.2).

Then we have, as $\varepsilon \to 0$,

$$\ln \mathbb{P}\left(\int_{\mathbb{R}} q(t) |X(t)|^2\, dt \leqslant \varepsilon^2 \right) \sim -\left(\frac{M^{1/r}}{r \sin(\pi/r)} \int_{\mathbb{R}} q(t)^{1/r}\, dt \right)^{r/(r-1)} \times \frac{(r-1)(2\pi)^{1/(r-1)}}{2\, \varepsilon^{2/(r-1)}}.$$

Apart from the weight integration domain, the constant in the limit is exactly the same as in Theorem 2.2.

This result has an intersection with that of Gengembre [**10**] who considered the non-weighted L_p-norm, $1 \leqslant p \leqslant +\infty$, on a bounded interval and the range $1 < r < 3$ that enables comparison with fractional Ornstein–Uhlenbeck processes and thus a reduction to the small deviation results on fractional Brownian motion, cf. [**17**]. We illustrate this connection in the next sub-section.

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4.3.2. *Basic example.* Let $H \in (0, 1)$ be the fractionality parameter. Let W_H be a fractional Brownian motion and let $U_H(t) = \mathrm{e}^{-Ht/2} W_H(\mathrm{e}^t)$, $t \in \mathbb{R}$, be a fractional Ornstein–Uhlenbeck (OU) process. (There are several other ways to extend the classical OU-process to the fractional case. We refer to [13] for alternative definitions and further references.) In other words, it is a real centered Gaussian stationary process with covariance

$$K_H(t) = \tfrac{1}{2}(\mathrm{e}^{Ht} + \mathrm{e}^{-Ht} - |\mathrm{e}^{t/2} - \mathrm{e}^{-t/2}|^{2H}).$$

The asymptotic behavior of the corresponding spectral density $m_H(u)$ is well known, see e.g. [10, Proposition 1],

$$m_H(u) \sim \frac{\Gamma(2H+1)\sin(\pi H)}{2\pi} |u|^{-2H-1} =: M_H |u|^{-2H-1} \quad \text{as } u \to \infty. \tag{4.5}$$

This is essentially due to the behavior of the covariance at the origin,

$$K_H(t) = 1 - \frac{|t|^{2H}}{2} + O(|t|^{1+\min\{1,2H\}}) \quad \text{as } t \to 0.$$

It will be also useful for us to consider integrated versions of fractional Brownian motion and their stationary versions. Let us denote $V_h := W_h$ for $h \in (0, 1)$, and define processes V_h for all non-integer positive $h > 1$ inductively, by

$$V_{h+1}(t) := \int_0^t V_h(s)\, ds, \quad t \geqslant 0.$$

It is easy to see that the process V_h is h-self-similar. Therefore, $\{U_h(t) = \mathrm{e}^{-ht} V_h(\mathrm{e}^t), t \in \mathbb{R}\}$ is a stationary process with the covariance function

$$K_h(t) = \mathrm{e}^{-ht} \mathbb{E}\,(V_h(1) V_h(\mathrm{e}^t)), \quad t \in \mathbb{R},$$

for all positive non-integer values of the parameter h. We can also easily find the inductive formula for the spectral measures of U_h. Indeed, for any $h > 1$, $t \in \mathbb{R}$, we have

$$U_h'(t) = -h\mathrm{e}^{-ht} V_h(\mathrm{e}^t) + \mathrm{e}^{-ht}\mathrm{e}^t V_{h-1}(\mathrm{e}^t) = -hU_h(t) + U_{h-1}(t).$$

Rewrite this identity as

$$U_{h-1}(t) = U_h'(t) + hU_h(t)$$

and translate it in the language of spectral measures. Let μ_h denote the spectral measure of U_h. Recall that U_h has a spectral representation

$$U_h = \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}tu} Z_h(du),$$

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where Z_h is a centered random measure with orthogonal values on $\mathbb{R}$ such that $\mathbb{E}\,|Z_h(\cdot)|^2 = \mu_h(\cdot)$. Since

$$U_h'(t) = \int_R \mathrm{e}^{\mathrm{i}tu}\mathrm{i}u\, Z_h(du),$$

we obtain

$$U_{h-1}(t) = \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}tu}(\mathrm{i}u + h) Z_h(du).$$

By the uniqueness of the spectral representation, it follows that $(\mathrm{i}u+h)Z_h(du) = Z_{h-1}(du)$ and we finally obtain

$$\mu_h(du) = \frac{\mu_{h-1}(du)}{|\mathrm{i}u + h|^2} = \frac{\mu_{h-1}(du)}{u^2 + h^2}.$$

It follows from (4.5) that μ_h has a spectral density m_h satisfying

$$m_h(u) \sim M_H\, |u|^{-2h-1} \quad \text{as } u \to \infty.$$

(Here and elsewhere $H := \{h\}$ is the fractional part of h.)

Assuming condition (4.2) on the weight to hold and applying Theorem 4.2 with $r = 2h + 1$, $M = M_H$ we obtain as $\varepsilon \to 0$,

$$\begin{aligned} &\ln \mathbb{P}\left(\int_{\mathbb{R}} q(t)|U_h(t)|^2\, dt \leqslant \varepsilon^2\right) \\ &\quad \sim -\left(\frac{1}{(2h+1)\sin(\pi/(2h+1))}\int_{\mathbb{R}} q(t)^{1/(2h+1)}\, dt\right)^{(2h+1)/2h} \\ &\qquad \times \frac{h(\Gamma(2H+1)\sin(\pi H))^{1/2h}}{\varepsilon^{1/h}}. \end{aligned} \tag{4.6}$$

In view of the identity

$$\int_{\mathbb{R}} q(t)\,|U_h(t)|^2\, dt = \int_0^\infty \rho(t)\,|V_h(t)|^2\, dt$$

with the weight

$$\rho(t) := \frac{q(\ln t)}{t^{2h+1}}, \quad t > 0,$$

formula (4.6) immediately yields an equivalent result for the weighted L_2-norm of V_h. The small ball asymptotics for the weighted L_2-norm of V_h and U_h was obtained in [**20**, Theorems 3.1, 3.3 and 4.2] but only for the weights with bounded support.

One should also mention [**11**, **17**] where small deviations of more general weighted L_p-norms, $1 \leqslant p \leqslant +\infty$ were studied for fractional Brownian motions and for Riemann–Liouville processes.

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4.3.3. *Proper complex processes.* In all the previous examples the spectral density m satisfied (4.1) with $M_+ = M_-$. For complex-valued processes this condition may be violated. By repeating the proof of Theorem 2.4 and using asymptotics (4.3) we obtain the following analogue of Theorem 2.4 for complex-valued processes with continuous spectra.

THEOREM 4.3. *Let $\{X(t), t \in \mathbb{R}\}$ be a complex centered mean-square continuous stationary proper Gaussian process. Assume that it has a spectral density satisfying the asymptotic condition (4.1) with some $r > 1$. Let q be a summable weight on $\mathbb{R}$ satisfying (4.2).*

Then we have, as $\varepsilon \to 0$,

$$\ln \mathbb{P}\left(\int_0^{2\pi} q(t)|X(t)|^2\,dt \leqslant \varepsilon^2\right) \sim -\left(\frac{M_-^{1/r} + M_+^{1/r}}{2r\sin(\pi/r)} \int_{\mathbb{R}} q(t)^{1/r}\,dt\right)^{r/(r-1)} \times \frac{(r-1)(2\pi)^{1/(r-1)}}{\varepsilon^{2/(r-1)}}.$$

Apart from the weight integration domain, the constant in the limit is exactly the same as in Theorem 2.4.

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A. *Appendix.* Here we collect some statements from [2–5]. Recall that $\mathcal{F} : L_2(\mathbb{R}) \to L_2(\mathbb{R})$ is standard Fourier transform. For a compact operator T in $L_2(\mathbb{R})$ we denote by $s_n(T)$ its singular values and by

$$N(\lambda; T) := \#\{n : s_n(T) \geqslant \lambda\}$$

the distribution function of $s_n(T)$. Define

$$\Delta_\theta(T) = \lim_{\lambda \to 0_+} \lambda^\theta N(\lambda; T).$$

Denote by ℓ_δ and $\ell_{\delta,\infty}$ the spaces of sequences (x_j) (with $j \in \mathbb{N}$ or $j \in \mathbb{Z}$) such that, respectively,

$$\|(x_j)\|_\delta := \left(\sum_j |x_j|^\delta\right)^{1/\delta} < \infty; \qquad \|\!|(x_j)|\!\|_{\delta,\infty} := \sup_j (|j|^{1/\delta}|x_j|) < \infty.$$

PROPOSITION A.1 (A particular case of [3, Theorems 1(b) and 2]). *Let*

$$(\mathcal{A}u)(s) = b(s)\mathcal{F}^{-1}(a(\xi)\mathcal{F}(cu)(\xi)), \tag{A.1}$$

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where functions $b, c \in L_2(\mathbb{R})$ have compact supports while a has the form

$$a(\xi) = \zeta(\xi) M(\operatorname{sgn}(\xi)) |\xi|^{-r},$$

here $r > 1$, $M : \{-1, 1\} \to [0, +\infty)$ and ζ is a smooth cut-off function vanishing in a neighborhood of the origin. Then

$$\Delta_{1/r}(\mathcal{A}) = \frac{1}{2\pi} \int_{\mathbb{R}} \int_{\mathbb{R}\setminus\{0\}} \mathbf{1}_{\{|b(t)|\,|c(t)|\,|M(\operatorname{sgn}(\xi))|\,|\xi|^{-r} \geqslant 1\}}\, d\xi\, dt.$$

PROPOSITION A.2 (A particular case of [4, Corollary 4 and Lemma 3]). *Let operator $\mathcal{A}$ have the form (A.1).*

(1) *Suppose that weight functions b and c satisfy the assumptions of Proposition A.1 while $a(\xi) = \zeta(\xi)\psi(\xi)$, where ζ is a smooth cut-off function vanishing in a neighborhood of the origin and $\psi \in L_\infty(\mathbb{R})$, $\psi(\xi) = o(|\xi|^{-r})$ as $|\xi| \to \infty$, $r > 1$. Then $\Delta_{1/r}(\mathcal{A}) = 0$.*

(2) *Suppose that functions a, b and c satisfy the assumptions of Proposition A.1. Let $\operatorname{supp}(b) \subset I_1$, $\operatorname{supp}(c) \subset I_2$, where I_1 and I_2 are closed bounded segments with non-overlapping interiors. Then $\Delta_{1/r}(\mathcal{A}) = 0$.*

PROPOSITION A.3 (A particular case of [2, §5.7]). *Let*

$$(\widetilde{\mathcal{A}}u)(t) = b(t)\mathcal{F}^{-1}(a(\xi)u(\xi)),$$

where $a, b \in L_2(\mathbb{R})$. Define sequences $v(a)$ and $v(b)$ according to (4.4).

Let $v(a) \in \ell_{\delta,\infty}$, $v(b) \in \ell_\delta$ for some $\delta \in (0, 2)$. Then $(s_j(\widetilde{\mathcal{A}})) \in \ell_{\delta,\infty}$, and

$$\|\!|(s_j(\widetilde{\mathcal{A}}))|\!\|_{\delta,\infty} \leqslant \text{const} \cdot \|\!|v(a)|\!\|_{\delta,\infty} \cdot \|v(b)\|_\delta,$$

where constant depends on δ.

PROPOSITION A.4 (Corollary 5 in [5, §11.6]). *If $\Delta_\theta(T_1)$, $\Delta_\theta(T_2)$ are finite then*

$$|(\Delta_\theta(T_1))^{1/(\theta+1)} - (\Delta_\theta(T_2))^{1/(\theta+1)}| \leqslant (\Delta_\theta(T_1 - T_2))^{1/(\theta+1)}.$$

In particular, if $\Delta_\theta(T_1 - T_2) = 0$, then

$$\Delta_\theta(T_1) = \Delta_\theta(T_2).$$

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