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102.05 Remarks on Problem 100.I

Problem 100.I

Problem 100.I in the *Gazette* asks for solutions of the simultaneous equations

$$x_1 - \frac{1}{x_2} = x_2 - \frac{1}{x_3} = \ \dots \ = x_n - \frac{1}{x_1}, \tag{1}$$

where the x_j are non-zero real numbers and $n \geqslant 2$. The solution (see [1, 2]) is that either (i) $x_1 = \ \dots \ = x_n$, or (ii) n is even, and $x_{k+1} = \dfrac{1}{x_k}$, $k = 1, \dots, n-1$. Here we rephrase the problem by asking for all *periodic* solutions $x_1, x_2, \dots$ of the infinite set of equations

$$x_k - \frac{1}{x_{k+1}} = x_{k+1} - \frac{1}{x_{k+2}}, \qquad k = 1, 2, \dots, \tag{2}$$

and it is now natural to assume that $x_1, \dots, x_n$ are distinct, and $x_{n+1} = x_1$. If $n = 1$ then the solution is $x_1 = x_2 = \dots$, which is (i). If $n = 2$ then $x_3 = x_1 \neq x_2$ and $(x_1 - x_2)(x_1x_2 - 1) = 0$ so we obtain the solution (ii). As shown in [1, 2], there are no other real solutions. However, if we allow the x_j to be complex numbers, and write z_j in place of x_j (as we shall from now on), we find that there are other solutions. For example, providing that z_1 is not 0 or i, we have a periodic solution $z_1, z_2, \dots$ with $n = 3$, namely

$$z_1, z_2, z_3, z_4, \dots = z_1, \frac{1}{z_1 - i}, \frac{iz_1 + 1}{z_1}, z_1, \dots .$$

Next, as the reader can check, if we take $z_1 = (1 + \sqrt{2})i$ and $z_2 = -i$, then we obtain a solution with $z_5 = z_1$. This last example holds the clue to further progress. Instead of trying to *solve* the equations (2), we start with any z_1 and z_2, and then use the equations to generate a sequence $z_1, z_2, \dots$. The solutions to our problem are then precisely those sequences that are periodic. From this perspective, given z_1 and z_2, we let $\mu = z_1 - \dfrac{1}{z_2}$, and then generate the sequence z_k by the rule $z_{k+1} = \dfrac{1}{z_k - \mu}$. Thus we are now seeking periodic outcomes of an iterative process. Apart from raising the problem of complex solutions, this approach raises an interesting question, namely to explain why the constraint that the x_j are real, but otherwise arbitrary, should imply that $n \leqslant 2$.

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This problem is best solved in terms of Möbius transformations. A Möbius transformation is a map of the form $f(z) = \dfrac{az + b}{cz + d}$, where $ad - bc \neq 0$, and z is a complex variable. In our problem we have $z_{k+1} = f(z_k)$, where μ is the common value of the expressions in (2), and

$$f(z) = \frac{1}{z - \mu}, \qquad [f(z)]^{-1} = \frac{1}{f(z)} = z - \mu, \qquad f^{-1}(z) - \frac{1}{z} = \mu.$$

The equations (2) generate the sequence $z_1, f(z_1), f(f(z_1)), \ldots$, so the problem now is to find, for each n, the fixed points of the nth iterate f^n of f.

Now comes a more subtle point. As we have already solved the problem when $n \leqslant 2$, we shall now assume that $n \geqslant 3$. In this case $f^n(z_q) = z_{n+q} = z_q$ for all $q = 1, \ldots, n$, so that f^n has (at least) n *distinct* fixed points $z_1, \ldots, z_n$, where $n \geqslant 3$. Now the equation for the fixed points of a general Möbius map g is a quadratic equation, and the only way that this equation can have more than two roots is if $g(z) = z$ for all z; thus the problem is now one of *finding when the n th iterate f^n of f is the identity map*. For those familiar with the extensive theory of Möbius maps (which far exceeds that given in most textbooks on complex analysis) this is a well-rehearsed matter with a well-known solution. The account that follows explains this process and thereby solves our more general problem.

The normal form of f

We are studying the Möbius map $f(z) = \dfrac{1}{z - \mu}$. The fixed points of f are the solutions of $z(z - \mu) = 1$ and, providing that $\mu^2 + 4 \neq 0$ (which we shall assume), these fixed points, say

$$u = \frac{\mu + \sqrt{\mu^2 + 4}}{2}, \qquad v = \frac{\mu - \sqrt{\mu^2 + 4}}{2},$$

are distinct and satisfy

$$u^2 = u\mu + 1, \qquad uv - 1. \tag{3}$$

Next, we let

$$h(z) = \frac{z - u}{z - v}.$$

Then $hfh^{-1}(0) = hf(u) = h(u) = 0$; that is, the Möbius map hfh^{-1} fixes 0. Now the theory of Möbius maps allows us to work with ∞ as well, and, similarly, we find that $hfh^{-1}(\infty) = hf(v) = h(v) = \infty$; thus hfh^{-1} also fixes ∞. As hfh^{-1} is a Möbius map that fixes 0 and ∞, it must be of the form $hfh^{-1}(z) = \lambda z$, say, and therefore $hf^nh^{-1}(z) = (hfh^{-1})^n(z) = \lambda^n z$. Since hf^nh^{-1} is the identity map I if, and only if, f^n is, we see that $f^n = 1$ if, and only if, $\lambda^n = 1$, or

$$\lambda = \exp\left(\frac{2\pi iq}{n}\right) \tag{4}$$

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for some integer q. Further, as we have assumed that n is the smallest period of our solution, we see that $f^k \neq I$ for $k = 1, \ldots, n-1$, and this implies that $\lambda^k \neq 1$ when $k = 1, \ldots, n-1$. This means that λ is a primitive nth root of unity, and $\gcd\{q, n\} = 1$.

We must now find the relationship between μ and λ. As $hf(z) = \lambda h(z)$ for all z, we have

$$\frac{1 - u(z - \mu)}{1 - v(z - \mu)} = \lambda\left(\frac{z - u}{z - v}\right)$$

for all z, and if we let $z \to \infty$ in this we see from (3) that

$$\lambda = \frac{u}{v} = -u^2. \tag{5}$$

Thus, from (4), $u = -i\varepsilon \exp\dfrac{\pi i q}{n}$, for some ε which is either 1 or -1. Then, as $u^2 = u\mu + 1$, we find that

$$\mu = \frac{u^2 - 1}{u} = u - \frac{1}{u} = -2i\varepsilon \cos\left(\frac{\pi i q}{n}\right). \tag{6}$$

In conclusion, if we assume that a real solution to the original problem does exist, then μ is real; thus, from (6), $\mu = 0$. This means that $\dfrac{q}{n} = 2$ and as q and n are coprime, this shows that $q = 1$ and $n = 2$. Thus there is no *real* solution with $n \geqslant 3$. Finally, it is easy to see that, providing μ satisfies (6), there are complex solutions of period n. In fact, this whole discussion is about the transformation (by a Möbius map) of an analysis of the iterates of the map $z \to e^{\pi i r} z$, where r is a rational number. In order that all of the iterates of a given starting point lie on some circle through the fixed points of this map, the map can have order at most two.

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