Algebraic analysis of stability and bifurcation for nonlinear flight dynamics

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ABSTRACT

This note presents an application of algebraic methods to derive exact conditions for certain nonlinear flight dynamical systems to exhibit stability and bifurcation. The roll-coupling flight model is taken as an example to show the feasibility of algebraic analysis. Some of the previous stability and bifurcation results obtained using numerical analysis for this model are confirmed.

NOMENCLATURE

α, β	angles of attack and sideslip, respectively
$\delta_a,\delta_e,\delta_r,\delta_T$	aileron, elevator, rudder, and thrust throttle deflections, respectively
Δ_1, Δ_2,...	Hurwitz determinants
I_x,I_y,I_z	roll, pitch, and yaw moments of inertia, respectively
J	Jacobian matrix
l,m,n	roll, pitch, and yaw moments, respectively
l_c	roll control parameter
p,q,r	body-axis roll, pitch, and yaw rates, respectively
φ, θ	body-axis roll and pitch Euler angles, respectively
prem	pseudo-remainder
$\mathbb{R}$	field of real numbers
$\boldsymbol{T}$	triangular set
δ	control vector
V	aircraft velocity
$\mathbf{x}$	state vector

1.0 INTRODUCTION

The approach used for algebraic analysis in this note is general and applicable to flight dynamics modeled as autonomous systems of ordinary differential equations of the form

$$d\mathbf{x}^T/dt = F(\delta, \mathbf{x}), \qquad \ldots (1)$$

where $\mathbf{x} = (\alpha,\beta,p,q,r,V,\theta,\varphi) \in \mathbb{R}^8$ is the state vector, $\delta = (\delta_e,\delta_a,\delta_r,\delta_T) \in \mathbb{R}^4$ is the control vector, and F is a vector of rational functions in δ_e, $\delta_a,\delta_r,\delta_T,\alpha,\beta,p,q,r,V$ and trigonometric functions of θ and φ with coefficients in $\mathbb{R}$, the field of real numbers (see, e.g., Refs 5, 6 and 11). Each state variable is a function of t and the real control parameters are independent of the derivation variable t. In general, the solutions of System (1) cannot be found and represented analytically. It is therefore necessary to analyse the qualitative behaviours such as stability, bifurcation, and chaos of the system. Most of this analysis has been carried out experimentally in the literature of flight dynamics by means of simulation and visualisation via numerical computation. In particular, the continuation method and phase portraits have been widely used for the analysis. Although numerical computation is practical and efficient, it has some drawbacks: the produced results are not certified, exact stability and bifurcation conditions on the parameters cannot be established, and observing the change of dynamical behaviours (bifurcation) with the variation of several parameters may be difficult. The algebraic approach based on symbolic computation can overcome some of these drawbacks. It provides an alternative way for (rigorous) qualitative analysis of nonlinear flight dynamics with parameters.

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In this note, we take the roll-coupling flight model as an example to show the feasibility of algebraic analysis of stability and bifurcation. Some of the previous stability and bifurcation results obtained using numerical analysis for this model are confirmed. The approach used here originates from Refs 10 and 15 and has been successfully applied to the analysis of stability and bifurcation for biological systems, where no trigonometric function is involved. The occurrence of trigonometric functions in the flight dynamical equations can be handled by introducing standard quadratic constraints between such functions, so the equations may be reduced to those of rational functions.

2.0 SIMPLIFIED ROLL-COUPLING MODEL

The classical roll-coupling problem has been studied in a number of papers[4,5,6,11,12,18] and the general dynamical system is described by five differential equations. To illustrate the application of algebraic methods for stability analysis, we first recall the simplified model[4] of the roll-coupling problem described by three differential equations of the form

$$\begin{aligned}\dot{p} &= a_1\beta + a_2 p - a_3 r + l_c,\\ \dot{r} &= (b_1 - b_2 p^2)\beta + b_3 r + b_4 p,\\ \dot{\beta} &= -r\,\mathrm{Cos}\,\alpha_0 + p\,\mathrm{Sin}\,\alpha_0 + c_1\beta,\end{aligned} \qquad \ldots (2)$$

where p, r are angular rates, β is the sideslip angle, and l_c is a control parameter. We take values for the other parameters as in Ref. 4:

$$a_1 = -14,\ a_2 = -7/5,\ a_3 = 0,$$

$$b_1 = 2,\ b_2 = 3/5,\ b_3 = -1/5,\ b_4 = 0,$$

$$\mathrm{Cos}\,\alpha_0 = 199/200,\ \mathrm{Sin}\,\alpha_0 = 499/5{,}000\ c_1 = 0$$

with $\alpha_0 \approx 0{\cdot}1$. Substituting these parametric values into the expressions on the right-hand side of System (2) and multiplying the resulting rational expressions by their constant denominators, we obtain the following three polynomials:

$$\begin{aligned}P_1 &= -70\beta - 7p + 5\,l_c,\\ P_2 &= 10\beta - 3\beta p^2 - r,\\ P_3 &= -4{,}975r + 499p.\end{aligned} \qquad \ldots (3)$$

These polynomials may be easily transformed into

$$\begin{aligned}T_1 &= 20{,}895p^3 - 14{,}925l_c p^2 - 76{,}636p + 49{,}750l_c,\\ T_2 &= 4{,}975r - 499p,\\ T_3 &= 70\beta + 7p - 5l_c\end{aligned} \qquad \ldots (4)$$

(note that $T_3 = -P_1$, $T_2 = -P_3$, and T_1 is obtained from P_2 by substituting the solution of $P_1 = 0$ for β and the solution of $P_3 = 0$ for r), such that Systems (3) and (4) have the same set of common zeros. Observe that System (4) is of triangular form in the sense that the variables r and β do not appear in T_1 and the variable β does not appear in T_2, so the common zeros of System (4) for (p,r,β) (with l_c as a parameter) may be found by determining first the zeros of T_1 for p, then the zeros of T_2 for r, and finally the zeros of T_3 for β. There are general algorithms for triangularising arbitrary sets of multivariate polynomials (see Section 4 and Refs 13 and 16).

Therefore, the problem of determining the number of equilibria of System (2) in terms of l_c is reduced to determining for what values of the parameter l_c the polynomial T_1 has 0, 1, 2, or 3 real zeros for p. This is a problem of real solution classification, which will be discussed in Section 4. There are available software packages, e.g., DISCOVERER (http://www.is.pku.edu.cn/~xbc/software.html) and RS/DV (http://www-salsa.lip6.fr/~rouillie/Software.html), which can be used to deal with the problem. By using any of these packages, it may be easily verified that T_1 has three real roots for any real value of the control parameter l_c, so System (2) always has three equilibria.

To determine the conditions on l_c for the stability of these equilibria, we compute the characteristic polynomial C of the Jacobian matrix of System (2):

$$C = \lambda^3 + 8/5\,\lambda^2 - 597/1{,}000\,p^2\lambda + 2{,}292/625\,\lambda + c, \text{ with}$$

$$c = -12{,}537/5{,}000\,p^2 + 597/500\,l_c p + 19{,}159/6{,}250\ ,$$

where r and β have been substituted by using $T_2 = 0$ and $T_3 = 0$. The *Hurwitz determinants* of C (see Section 4 and Refs 9,10,15) are

$$\Delta_1 = 8/5,\ \Delta_2 = 7{,}761/5{,}000\,p^2 - 597/500\,l_c p + 17{,}513/6{,}250,\ \Delta_3 = \Delta_2 c.$$

According to the Routh–Hurwitz criterion, for any equilibrium $(\bar{p},\bar{r},\bar{\beta})$ of System (2), if both Δ_2 and c are positive at $(p,r,\beta) = (\bar{p},\bar{r},\bar{\beta})$, then $(\bar{p},\bar{r},\bar{\beta})$ is stable. Thus by using DISCOVERER the following results can be established:

1. if $-l_c' < l_c < l_c'$, then System (2) has one and only one stable equilibrium;
2. if $l_c < -l_c'$ or $l_c > l_c'$, then System (2) has no stable equilibrium;
3. if $l_c = \pm\, l_c'$, then Hopf bifurcation occurs at one of the three equilibria of System (2),

where $l_c' = \sqrt{83687479645 - 1342915\,\sqrt{1510760769}}\,/\,49750 \approx 3{\cdot}56694049$ is the smaller of the two positive real roots of

$$6156717968750 l_c^4 - 416345211233875 l_c^2 + 4300559809007168.$$

Our results above confirm some of the results given by Goman and Khramtsovsky in Ref. 4, where $l_c' \approx 3{\cdot}55$ was obtained by using numerical computation.

3.0 REDUCED ROLL-COUPLING MODEL

Now we consider the reduced model of the roll-coupling problem as used in Refs 5,11,12, and 18, for which the dynamical equations have the the following form

$$\begin{aligned}\dot{\alpha} &= q - p\beta + z,\\ \dot{\beta} &= p\alpha - r + y,\\ \dot{p} &= -i_1 qr + l,\\ \dot{q} &= i_2 rp + m,\\ \dot{r} &= -i_3 pq + n.\end{aligned} \qquad \ldots (5)$$

In this reduced model, the angle α of attack and the sideslip angle β are both assumed to be small, so that Sin α, Cos α, Tan β are approximated by α, 1, β respectively. We take values for the parameters in System (5) as in Ref. 11 (aircraft B) and Ref. 12, so that

$$i_1 = \tfrac{16270}{23049},\quad i_2 = \tfrac{161311}{168090},\quad i_3 = \tfrac{145041}{184360},$$
$$z = -\tfrac{2831052123}{943750000}\,\alpha,\quad y = -\tfrac{2635807149}{47187500000}\,\beta,$$

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$$l = -\frac{11046667761459}{100039062500}\beta - \frac{66936485253}{3078125000}p + \frac{1216669055481}{800312500000}r - \frac{1636543372068}{5001953125}\delta_a,$$

$$m = -\frac{5918447522581281}{437734375000000}\alpha - \frac{7279010171450541}{218867187500000}\delta_e - \frac{317753335891439091}{175093750000000000}q - \frac{6857985666721707}{17509375000000000}\dot{\alpha},$$

$$n = \frac{12162492787869}{3273437500000}\beta - \frac{170741729763}{654687500000}r - \frac{111582502641}{65468750000}\delta_r,$$

where δ_a, δ_e, and δ_r are the control parameters for the aileron, elevator, and rudder deflections respectively. The floating-point numbers have been converted to rational numbers and all the subsequent computations are performed exactly.

Sinha and Ananthkrishnan[12] studied the pseudo-steady states of System (5) and their stability under the constraint $\beta = 0$ (i.e., zero-sideslip). Let the elevator deflection δ_e be fixed at $349/10^4$ rad, which is approximately 2°, as in Ref. 12. From the set consisting of β and the 5 polynomials on the right-hand side of System (5), one can compute a triangular set $\boldsymbol{T}$ (with respect to $\delta_r \prec p \prec q \prec r \prec \alpha \prec \beta$, see Refs 13 and 14), in which the first polynomial is

$$\begin{aligned} T = {} & 7669627842206598236673309520324707031250000000000000000000000000000000000000000\delta_r^5 \\ & + 7160304768053739671474707053458577411264035325037059686463820312500000000000000000\delta_r^3 \\ & - 1274745172771344216865839551136265105149861302614986229517872500000000000000000000\delta_a\delta_r^2 \\ & - 2023088392233844231226497401988854893220875634638857738909894200000000000000000000\delta_a^2\delta_r \\ & + 1774289636812458979087704327582443014457294905930756348080096118581726494806535654 1\delta_r \\ & + 2500765222960581697779853660492073902609889821506525256124051990310699700062120000 0\delta_a. \end{aligned}$$

Thus $T = 0$ gives the algebraic condition on δ_a and δ_r for System (5) to have real pseudo-steady states with $\beta = 0$. The curve $T = 0$ is plotted in Fig. 1(a), with zooming around (0;0) in Fig. 1(b). The rudder deflection takes value $\delta_r \approx 0{\cdot}51$ rad at the aileron deflection $\delta_a = -1/5$ rad as shown in Fig. 1(b). This confirms the numerical result given in Ref. 12.

Let R be the *pseudo-remainder* of the characteristic polynomial of the Jacobian matrix of System (5) with respect to $\boldsymbol{T}$ and $\Delta_1,...,\Delta_5$ be the Hurwitz determinants of R (see Section 4). $\Delta_1,...,\Delta_4$ and the constant term $a_5 = \Delta_5/\Delta_4$ of R are polynomials of total degrees 10, 20, 30, 40, 10 in δ_a, δ_r and consisting of 19, 67, 143, 247, 21 terms, respectively, with large integer coefficients. It can be verified† by using DISCOVERER that $\Delta_1,...,\Delta_4$ are all positive along the curve $T = 0$ for $-1/5 < \delta_a < 0$. However, the sign of a_5 changes from negative to positive at $\delta_a = \bar{\delta}_a \approx -0{\cdot}142372$. So $\bar{\delta}_a$ is a critical value at which the stability of zero-sideslip roll manoeuvres changes. This confirms the numerical result in Ref. 12 that the zero-sideslip roll manoeuvres become dynamically unstable beyond the value $\delta_a = \bar{\delta}_a$ of aileron deflection. The value $\bar{\delta}_a$ is one of the four real roots of a univariate polynomial of degree eight with integer coefficients of over 275 digits. This real root may be isolated as

$$\bar{\delta}_a \in \left[-\frac{1222966033}{8589934592}, -\frac{2445932065}{17179869184}\right].$$

4.0 ALGEBRAIC METHODS FOR STABILITY AND BIFURCATION ANALYSIS

Rename the state variables in the flight dynamical System (1) so that the state vector can be written as $(x_1,...,x_m) \in \mathbb{R}^m$. If there are trigonometric functions of θ and φ such as Sin θ, Cos θ, and Tan θ in System (1), then introduce new variables, e.g., $x_{m+1} = \text{Sin } \theta$, $x_{m+2} = \text{Cos } \theta$, add the quadratic constraint $x_{m+1}^2 + x_{m+2}^2 - 1 = 0$, replace Tan θ in (1) by x_{m+1}/x_{m+2}, and so on similarly. Let all the state and introduced variables be $x = (x_1,...,x_n)$. Then the components of F in (1) become rational functions P_i/Q_i in δ and x. Write $P_i = 0$, $Q_i \neq 0$ for all i and include additional constraints in x and δ if there are any. In this way, one can form a system of polynomial equations

$$P_1(\delta,x) = 0,...,P_n(\delta,x) = 0 \qquad \ldots (6)$$

and inequalities

$$Q_1(\delta,x) \bowtie 0,...,Q_s(\delta,x) \bowtie 0, \qquad \ldots (7)$$

where $\bowtie$ can take any of $\neq, >, \geq, <, \leq$. The equilibria of System (1) may be read off from the common real solutions of System (6-7) depending on the parameters δ. Thus the general problem of determining the number of equilibria of System (1) is reduced to the following algebraic problem.

Problem A. For any given integer $k \geq 0$, determine the conditions on δ for System (6-7) to have exactly k distinct real solutions for x.

To derive conditions on δ for k equilibria to be stable, we use the standard method of Lyapunov by considering the Jacobian matrix $J(\delta,x)$ of (1). An equilibrium $\bar{x}$ of (1) is asymptotically stable if all the eigenvalues of $J(\delta,\bar{x})$ have negative real parts, and unstable if $J(\delta,\bar{x})$ has at least one eigenvalue with positive real part. The well-known Routh–Hurwitz criterion states that all the eigenvalues of $J(\delta,\bar{x})$ have negative real parts if and only if the leading principal minors $\Delta_1,...,\Delta_m$ of the Hurwitz matrix (formed with the coefficients of the characteristic polynomial A of $J(\delta,\bar{x})$ and 0 as entries, see, e.g., Refs 9,10 and 15), called the Hurwitz determinants of A, are all positive at $x = \bar{x}$.

Therefore, deriving the conditions on δ for (1) to have k stable equilibria is reduced to establishing the conditions on δ for Δ_i to be

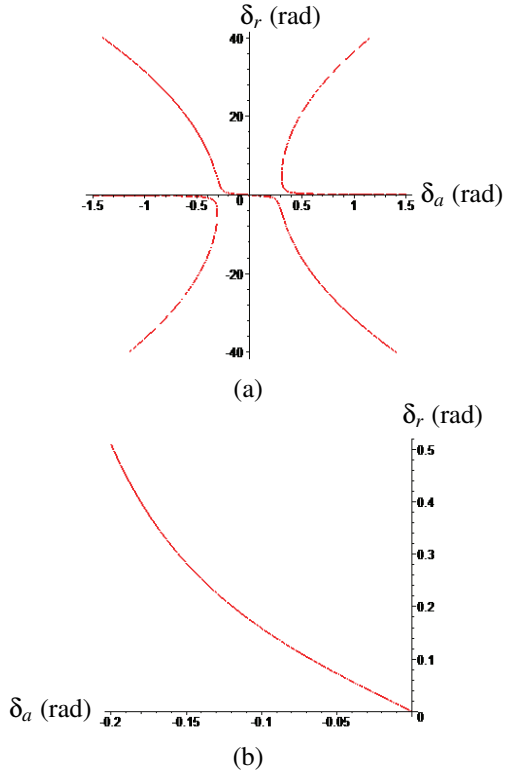


Figure 1. The algebraic curve $T = 0$.

† The verification for the positiveness of Δ_3 is the most difficult. It took more than eight hours in Maple 9 running on a laptop with 2·2GHz CPU and 1GB memory.

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0, positive, or negative at k equilibria. In general the real value of an equilibrium $\bar{x}$ cannot be exactly computed, so one cannot simply evaluate the values of Δ_i at $x = \bar{x}$. Therefore, we arrive at the following algebraic problem for deriving stability conditions, which was formulated first in Ref. 15.

Problem B. For any given integer $k \geq 0$, determine the conditions on δ for Δ_i to be 0, positive, or negative at k isolated real solutions of System (6-7).

To solve problems A and B, we employ algebraic methods to triangularise sets of multivariate polynomials (as a generalisation of Gaussian elimination for linear systems) and to classify real solutions of triangularised systems of polynomial equations with inequalities. Without entering into technical details, let us try to explain the functionality and some underlying ideas of the methods. The interested reader may consult Refs 1,13,16,17 and references therein.

The triangularisation of System (3) to (4) has been done by simple substitution of rational solutions of linear equations. Such substitution is not always possible because polynomial equations of higher degree are not necessarily solvable in term of rational functions. An operation that generalises the substitution is to *pseudo-divide* a polynomial G by another polynomial F, say $F = Ix_i^d + F_0$ with $d = \deg(F,x_i)$, with respect to the variable x_i. This division yields a *pseudo-remainder* R, denoted by $\text{prem}(G,F,x_i)$, and a *pseudo-quotient* Q such that

$$IG = QF + R, \deg(R,x_i) < d,$$

for some integer $s \geq 0$. Then, under the conditions $F = 0$ and $I \neq 0$, the polynomial equation $G = 0$, usually of degree greater than or equal to d, may be replaced by $R = 0$ of degree smaller than d. Repeating such replacement allows one to eliminate variables from some polynomials in question. On the basis of variable elimination, algorithms have been designed to triangularise any given polynomial set, e.g., $\boldsymbol{P} = \{P_1,...,P_n\}$ with P_i as in (6), and to decompose it into finitely many triangular sets of the form

$$\boldsymbol{T} = [T_1(\delta, x_1), T_2(\delta, x_1, x_2),...,T_n(\delta, x_1, x_2,...,x_n)]$$

(see the books by Wu[16] and the author[13]). The process of triangular decomposition is rather tedious and software tools (such as the Epsilon package described in Ref. 14) are needed for performing the involved symbolic computations. With respect to $\boldsymbol{T}$ one can form the pseudo-remainder of G, $\text{prem}(G,\boldsymbol{T})$, in the following way: $\text{prem}(G,\boldsymbol{T}) = \text{prem}(\cdots \text{prem}(\text{prem}(G,T_n,x_n),T_{n-1},x_{n-1}),...,T_1,x_1)$. Triangular sets may also be computed by means of Gröbner bases[1,13], an alternative algebraic method that has been well developed and implemented in popular computer algebra systems such as Maple (http://www.maplesoft.com/) and Mathematica (http://www.wolfram.com/).

To derive conditions on δ for System (6–7) to have k real solutions or for Δ_i to be 0, positive, or negative at k real solutions of System (6–7), one can use the recently developed method of Yang and Xia[17] or Lazard and Rouillier[8] by first decomposing $\boldsymbol{P}$ into triangular sets $\boldsymbol{T}_i$ and then computing an algebraic variety V_i from each $\boldsymbol{T}_i$ and Q_j, Δ_i; V_i decomposes the real space $\mathbb{R}^4$ of parameters δ into cells, such that the number of distinct real solutions of System (6–7) is constant and the signs of the inequality polynomials Q_j, Δ_i are invariant in each cell not meeting V_i. The cells may be constructed by using the partial cylindrical algebraic decomposition algorithm of Collins and Hong[2] and the number of distinct real solutions of (6–7) and the signs of Q_j, Δ_i in each cell may be determined by computing them at a sample (rational) point in the cell. In this way, the number of real solutions of (6–7) in terms of the parameters δ can be completely identified. The exact conditions on δ are determined from the polynomials that define V_i.

Finally, we note that bifurcation occurs when some of the eigenvalues of the Jacobian matrix $J = J(\delta,\bar{x})$ of (1) at an equilibrium $\bar{x}$ have zero real parts, but none of them has a positive real part. When J has a zero eigenvalue, the determinant of J is 0 and this is the case of *fold bifurcation*[7]. A more sophisticated situation is *Hopf bifurcation* when J has a pair of purely imaginary eigenvalues and the other eigenvalues of J all have negative real parts. In this case, limit cycles may bifurcate from $\bar{x}$ of the system with small perturbation. The condition for Hopf bifurcation may be given in terms of the Hurwitz determinants and the constant term of the characteristic polynomial of J according to a simple criterion established by El Kahoui and Weber[3]. Therefore, the problem of deriving exact conditions on δ for System (1) to exhibit bifurcation may also be reduced to the algebraic problems A and B.

5.0 CONCLUSION

We have shown that exact stability and bifurcation conditions on the parameters for the roll-coupling flight model may be derived by using algebraic methods with symbolic computation. The derived conditions have confirmed some of the stability and bifurcation results obtained previously using numerical analysis. The algebraic approach is general and applicable to other nonlinear flight dynamical systems and thus can serve as an alternative tool for stability and bifurcation analysis (in particular for certifying numerical results). However, this approach involves heavy computation with large symbolic expressions, which limit its application to flight dynamics with only a few parameters and not with tabular data.

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