NOTE ON NILPOTENT AND SOLVABLE ALGEBRAS

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In general, the class of a nilpotent linear algebra of dimension n is at most $n+1$, and the index, or derived length, of a solvable linear algebra of dimension n is at most n. In this note it is shown that, for a nilpotent linear algebra of dimension n satisfying $x^2 = 0$ for all x, the class is at most n; and bounds are obtained for the indices of solvable Lie algebras.

(1) *Nilpotent linear algebras.* Let $\mathbf{A}$ be a linear algebra, not necessarily associative, over a commutative field. A sequence $\{\mathbf{A}^i\}$ of subspaces of $\mathbf{A}$ is defined by $\mathbf{A}^1 = \mathbf{A}$, $\mathbf{A}^{i+1} = \mathbf{A}^i\mathbf{A}$ $(i \geqslant 1)$; thus $\mathbf{A}^{i+1}$ is the subspace of $\mathbf{A}^i$ spanned by the elements of $\mathbf{A}$ of the form xy, where $x \in \mathbf{A}^i$ and $y \in \mathbf{A}$. A second sequence $\{\mathbf{A}_i\}$ is defined by $\mathbf{A}_1 = \mathbf{A}$, $\mathbf{A}_{i+1} = \mathbf{A}\mathbf{A}_i$ $(i \geqslant 1)$. Following Albert ((2), p. 528) we say that $\mathbf{A}$ is nilpotent if $\mathbf{A}^r = 0 = \mathbf{A}_r$ for some r; the minimum value of r for which these conditions are satisfied will be called the *class* of $\mathbf{A}$.

If $\mathbf{A}^{j+1} = \mathbf{A}^j$, then $\mathbf{A}^k = \mathbf{A}^j$ $(k \geqslant j)$. Thus if $\mathbf{A}$ is nilpotent, and of finite dimension, any non-zero member of the sequence $\{\mathbf{A}^i\}$ is of lower dimension than the preceding member; a similar property is satisfied by the members of the sequence $\{\mathbf{A}_i\}$. It follows that the class of a nilpotent algebra of dimension n is at most $n+1$, and that, if $\mathbf{A}^n \neq 0$, $\mathbf{A}^i$ $(i = 1, \ldots, n)$ is of dimension $n+1-i$; similarly, if $\mathbf{A}_n \neq 0$, $\mathbf{A}_i$ $(i = 1, \ldots, n)$ is of dimension $n+1-i$.

Theorem 1. *If* $\mathbf{A}$ *is a nilpotent linear algebra of dimension* n (>1), *such that* $x^2 = 0$ *for all* $x \in \mathbf{A}$, *then the class of* $\mathbf{A}$ *is at most* n.

Proof. If $x^2 = 0$ for all x, then $(y+z)^2 = 0$ for all y, z and therefore $yz = -zy$, so that multiplication in $\mathbf{A}$ is anticommutative. If $\mathbf{A}$ is nilpotent, and $\mathbf{A}^n \neq 0$, then $\mathbf{A}^2$ is of dimension $n-1$. Let $e_1, e_2, \ldots, e_{n-1}$ be a basis for $\mathbf{A}^2$. Since $\mathbf{A}^2$ is a subspace of $\mathbf{A}$, an element e_n can be chosen so that $e_1, e_2, \ldots, e_{n-1}, e_n$ is a basis for $\mathbf{A}$. The elements of $\mathbf{A}^2$ consist of all linear combinations of e_ie_j $(i,j = 1, 2, \ldots, n)$; since $e_je_i = -e_ie_j$ and $e_i^2 = 0$, only those products e_ie_j for which $i<j$ need be considered. The elements of $\mathbf{A}^3$ consist of all linear combinations of e_ie_j $(i = 1, \ldots, n-1;\ j = 1, \ldots, n)$ and again only those products e_ie_j for which $i<j$ need be considered. But this implies that $\mathbf{A}^2 = \mathbf{A}^3$, and therefore, since $\mathbf{A}$ is nilpotent, $\mathbf{A}^2 = 0$. If $n > 1$, this is contrary to the hypothesis $\mathbf{A}^n \neq 0$. Since multiplication in $\mathbf{A}$ satisfies the anticommutative law, $\mathbf{A}^i = \mathbf{A}_i$ for all i. Hence $\mathbf{A}_n$ is also zero, and so the class of $\mathbf{A}$ is at most n.

An example of a nilpotent linear algebra of dimension n and class n, such that $x^2 = 0$ for all x, is given by $e_ie_n = -e_ne_i = e_{i-1}$ $(i = 2, \ldots, n-1)$, and $e_ie_j = 0$ otherwise. Thus the bound given by Theorem 1 for the class of algebras of this type is the best possible.

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2. *Solvable linear algebras.* Another sequence of subspaces of a linear algebra $\mathbf{A}$ consists of the successive derived algebras $\mathbf{A}^{(i)}$ defined by $\mathbf{A}^{(0)} = \mathbf{A}$, $\mathbf{A}^{(i+1)} = \mathbf{A}^{(i)}\mathbf{A}^{(i)}$ $(i \geqslant 0)$. The subspace $\mathbf{A}^{(i)}$ is called the *i-th derived algebra* of $\mathbf{A}$, and the sequence $\{\mathbf{A}^{(i)}\}$ is called the *derived sequence* of $\mathbf{A}$. Again following Albert ((2), p. 528) we say that $\mathbf{A}$ is *solvable of index*† r if $\mathbf{A}^{(r)} = 0$, but $\mathbf{A}^{(r-1)} \neq 0$. It is easily shown that a nilpotent algebra is solvable; but solvability does not imply nilpotency.

By an argument similar to that used for nilpotent algebras, it can be shown that the index of a solvable algebra of dimension n is at most n, and that if $\mathbf{A}^{(n-1)} \neq 0$, the subspace $\mathbf{A}^{(i)}$ $(i = 0, 1, \ldots, n-1)$ is of dimension $n-i$.

The remainder of this note is concerned with Lie algebras; a linear algebra $\mathbf{L}$ is a Lie algebra if $x^2 = 0$ for all $x \in \mathbf{L}$, and the Jacobi identity $x(yz) + y(zx) + z(xy) = 0$ is satisfied for all $x, y, z \in \mathbf{L}$.

THEOREM 2. *The index of a solvable Lie algebra of dimension* n (> 2) *over a field of characteristic zero, is not greater than* $[\log_2(n-2)] + 2$.

Proof. If $\mathbf{L}$ is a solvable Lie algebra of dimension n over a field of characteristic zero, its first derived algebra is nilpotent (Jacobson (3), p. 880). Since the dimension of $\mathbf{L}^{(1)}$ is at most $n-1$, it follows from Theorem 1 that its class is at most‡ $n-1$. The rth derived algebra of $\mathbf{L}^{(1)}$ is $\mathbf{L}^{(r+1)}$, which is spanned by products of 2^r elements of $\mathbf{L}^{(1)}$; by a well-known property of Lie algebras (see, for example, Magnus (4), p. 106) all such products are contained in the subspace $(\mathbf{L}^{(1)})^{2^r}$. Hence $\mathbf{L}^{(r+1)} = 0$ if $2^r > n-2$, and so, if $n > 2$, the index of $\mathbf{L}$ is at most $[\log_2(n-2)] + 2$.

It will be shown below that the first derived algebra of a solvable Lie algebra over a field of non-zero characteristic need not be nilpotent. Thus the proof of Theorem 2 given above applies only to fields of characteristic zero. We shall obtain a weaker bound for the index of a solvable Lie algebra over a field of non-zero characteristic by using the following theorem:

THEOREM 3. *Let* $\mathbf{L}$ *be a solvable Lie algebra of dimension* n *and index* 3, *over a field* $\mathbf{F}$. *Then, if the dimensions of* $\mathbf{L}^{(1)}$ *and* $\mathbf{L}^{(2)}$ *are* $n-1$ *and* $n-2$ *respectively, the characteristic of* $\mathbf{F}$ *is a factor of* $n-2$.

Proof §. Let $e_1, e_2, \ldots, e_n$ be a basis for $\mathbf{L}$ such that $e_1, e_2, \ldots, e_{n-1}$ is a basis for $\mathbf{L}^{(1)}$ and $e_1, e_2, \ldots, e_{n-2}$ is a basis for $\mathbf{L}^{(2)}$. Then, since $\mathbf{L}^{(3)} = 0$, $e_\alpha e_\beta = 0$; also, all products are independent of e_n, and $e_\alpha e_{n-1}$ is independent of e_{n-1}. In these circumstances the Jacobi condition is satisfied for all x, y and z if and only if the constants of structure c_{ij}^k satisfy

$$c_{\beta,n}^{n-1} c_{n-1,\alpha}^{\gamma} + c_{n,\alpha}^{n-1} c_{n-1,\beta}^{\gamma} = 0, \tag{1}$$

$$c_{n-1,n}^{n-1} c_{n-1,\alpha}^{\beta} + \sum_{\gamma=1}^{n-2} (c_{n,\alpha}^{\gamma} c_{\gamma,n-1}^{\beta} + c_{\alpha,n-1}^{\gamma} c_{\gamma,n}^{\beta}) = 0, \tag{2}$$

$$\sum_{\gamma=1}^{n-2} c_{\alpha,n-1}^{\gamma} c_{\gamma,n}^{n-1} = 0. \tag{3}$$

† The term 'derived length' is frequently used in place of 'index'.

‡ The class of a nilpotent Lie algebra can take the maximum value given by Theorem 1; in fact the algebra defined at the end of § 1 is a Lie algebra.

§ In the proof of this theorem, Greek suffixes take the values 1 to $n-2$ and Latin suffixes the values 1 to n.

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Let P and Q be the matrices whose (α, β)th elements are $c^{\beta}_{\alpha, n-1}$ and $c^{\beta}_{\alpha, n}$ respectively. Since the subspace $\mathbf{L}^{(2)}$ is of dimension $n-2$ and is spanned by elements of the form $\sum_{\gamma=1}^{n-2} c^{\gamma}_{\alpha, n-1} e_{\gamma}$, the matrix P is non-singular. Hence equations (3) are satisfied if and only if $c^{n-1}_{\gamma, n} = 0$; in these circumstances equations (1) are satisfied identically. Equations (2) can be written

$$kP = PQ - QP, \tag{4}$$

where $k = c^{n-1}_{n-1, n}$. Premultiplying both sides of (4) by P^{-1}, we get

$$kI = Q - P^{-1}QP,$$

and therefore, taking the trace of both sides,

$$(n-2)\, k = 0. \tag{5}$$

Since $\mathbf{L}^{(1)}$ contains the element e_{n-1}, and $c^{n-1}_{\beta, \gamma} = 0 = c^{n-1}_{\gamma, n}$, k cannot be zero. Therefore, from (5), the characteristic of $\mathbf{F}$ is a factor of $n-2$.

Any solution of (4) for k, P, Q, where k is non-zero and P is non-singular, provides an example of a Lie algebra satisfying the conditions of Theorem 3. For instance, if P and Q are square matrices of order $n-2$ defined over a field whose characteristic is a factor of $n-2$ by

$$P = \begin{bmatrix} 1 & 1 & 0 & \dots & \dots & 0 & 0 \\ 0 & 1 & 1 & \dots & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & \dots & 0 & 0 \\ \vdots & & & & & & \vdots \\ 0 & 0 & 0 & \dots & \dots & 1 & 1 \\ 0 & 0 & 0 & \dots & \dots & 0 & 1 \end{bmatrix}, \quad Q = \begin{bmatrix} 1 & 0 & 0 & \dots & \dots & 0 & 0 \\ 1 & 2 & 0 & \dots & \dots & 0 & 0 \\ 0 & 2 & 3 & \dots & \dots & 0 & 0 \\ 0 & 0 & 3 & \dots & \dots & 0 & 0 \\ \vdots & & & & & & \vdots \\ 0 & 0 & 0 & \dots & \dots & n-3 & 0 \\ 0 & 0 & 0 & \dots & \dots & n-3 & 0 \end{bmatrix},$$

then $PQ - QP = P$. The corresponding Lie algebra is given by

$$\begin{aligned} e_{\alpha} e_{n-1} &= -e_{n-1} e_{\alpha} &&= e_{\alpha} + e_{\alpha+1} && (\alpha = 1, \dots, n-3), \\ e_{n-2} e_{n-1} &= -e_{n-1} e_{n-2} &&= e_{n-2}, \\ e_{\alpha} e_{n} &= -e_{n} e_{\alpha} &&= \alpha e_{\alpha} + (\alpha - 1)\, e_{\alpha-1} && (\alpha = 2, \dots, n-2), \\ e_{1} e_{n} &= -e_{n} e_{1} &&= e_{1}, \\ e_{n-1} e_{n} &= -e_{n} e_{n-1} &&= e_{n-1}, \end{aligned}$$

with all other products of the form $e_i e_j$ zero.

This example not only proves the existence of algebras satisfying the conditions of Theorem 3, but also shows that the first derived algebra of a solvable Lie algebra over a field of non-zero characteristic need not be nilpotent. For, in the algebra $\mathbf{L}^{(1)}$, $e_{n-2} e_{n-1} = e_{n-2}$, and so $\mathbf{L}^{(1)}$ is not nilpotent.

3. *The derived sequence of a Lie algebra.* Let $\mathbf{A}^{(j)}$ and $\mathbf{A}^{(k)}$ be any two members of the derived sequence of a linear algebra $\mathbf{A}$, and suppose that $k > j$. If their dimensions are s and t respectively, the dimension of the difference algebra $\mathbf{A}^{(j)} - \mathbf{A}^{(k)}$ is $s - t$ (see, for example, Albert (1), p. 27). The derived algebra of $\mathbf{A}^{(j)} - \mathbf{A}^{(k)}$ is easily shown to be $\mathbf{A}^{(j+1)} - \mathbf{A}^{(k)}$. If $\mathbf{A}$ is a Lie algebra, all the members of the derived sequence and all the difference algebras formed from them are also Lie algebras.

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THEOREM 4. *Let* $\mathbf{L}^{(i)}$, $\mathbf{L}^{(i+1)}$, $\mathbf{L}^{(i+2)}$ *and* $\mathbf{L}^{(i+3)}$ *be four successive members of the derived sequence of a Lie algebra* $\mathbf{L}$ *over a field* $\mathbf{F}$ *of characteristic* p. *Then, if the difference algebras* $\mathbf{L}^{(i)} - \mathbf{L}^{(i+1)}$ *and* $\mathbf{L}^{(i+1)} - \mathbf{L}^{(i+2)}$ *are each of dimension* 1, *the dimension of* $\mathbf{L}^{(i+2)} - \mathbf{L}^{(i+3)}$ *is a multiple of* p. *In particular, if* $p = 0$, $\mathbf{L}^{(i+2)} = \mathbf{L}^{(i+3)}$.

Proof. If the Lie algebra $\mathbf{L}^{(i)} - \mathbf{L}^{(i+3)}$ is of dimension m, then the dimensions of its first and second derived algebras are $m-1$ and $m-2$ respectively; also its third derived algebra is zero. Therefore, by Theorem 3, the characteristic p of $\mathbf{F}$ is a factor of $m-2$, and so the dimension of $\mathbf{L}^{(i+2)} - \mathbf{L}^{(i+3)}$ is a multiple of p. In particular, if $p = 0$, $\mathbf{L}^{(i+2)} - \mathbf{L}^{(i+3)}$ is zero.

THEOREM 5. *The index of a solvable Lie algebra of dimension* n *over a field of characteristic* p *is not greater than* $n - [\frac{1}{3}n]$ *if* $p \neq 2$ *or* $n - [\frac{1}{4}(n+1)]$ *if* $p = 2$.

Proof. Let $\mathbf{L}$ be a solvable Lie algebra of dimension n over a field $\mathbf{F}$. Since $\mathbf{L}$ is solvable, two successive members of the derived sequence of $\mathbf{L}$ coincide if and only if they are both zero. Thus the dimension of a difference algebra of the form $\mathbf{L}^{(i)} - \mathbf{L}^{(i+1)}$, where $\mathbf{L}^{(i)} \neq 0$, is at least 1. But Theorem 4 states that, if two successive difference algebras of this form have dimension 1, the dimension of the next is a multiple of p. Therefore, if $p \neq 2$, the maximum possible value of the index of $\mathbf{L}$ corresponds to the case in which the dimensions of the successive difference algebras of the form $\mathbf{L}^{(i)} - \mathbf{L}^{(i+1)}$ are alternately 1 and 2, except for the last two non-zero algebras of this form, which can both be of dimension 1. Therefore the index of $\mathbf{L}$ cannot exceed $n - [\frac{1}{3}n]$. If $p = 2$, the maximum possible value of the index corresponds to the case in which two successive difference algebras of the form $\mathbf{L}^{(i)} - \mathbf{L}^{(i+1)}$ have dimension 1, whilst the next has dimension 2. It follows that, in this case, the index of $\mathbf{L}$ cannot exceed $n - [\frac{1}{4}(n+1)]$.

Theorem 5 is valid in the case $p = 0$, but does not provide such a good bound for the index of a solvable Lie algebra as that given by Theorem 2. It remains to be shown whether the bound given by Theorem 2 is the best possible in the case $p = 0$, and whether the bounds given by Theorem 5 are the best possible in the case $p \neq 0$.

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