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The Variational Iteration Method for an Inverse Problem of Finding a Source Parameter

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Abstract. An inverse problem of determining unknown source parameter in a parabolic equation is considered. The variational iteration method (VIM) is presented to solve inverse problems. The solution gives good approximations by VIM. A numerical example shows that the VIM works effectively for an inverse problem.

AMS subject classifications: 35K05, 35K99, 47J06, 49N45

Key words: Inverse problem, variational iteration method, parabolic equation.

1 Introduction

In this paper, we consider numerical procedures for the following parabolic equation with a time-dependent coefficient

$$u_t = \Delta u + \sum_{j=1}^{n} q_j(t) u_{x_j} + p(t)u(x,t) + f(x,t), \qquad x \in \Omega \subset R^n, \quad 0 < t \leq T, \tag{1.1a}$$

$$u(x,0) = \varphi(x), \qquad x \in \Omega, \tag{1.1b}$$

$$u(x,t) = g(x,t), \qquad (x,t) \in \partial\Omega \times [0,T], \tag{1.1c}$$

subject to the additional condition

$$\int_\Omega u(x,t)dx = a(t), \quad t \in [0,T], \tag{1.2}$$

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where $q_j(t)$, $j=1,\cdots,n$, $f(x,t)$, $\varphi(x)$, $g(x,t)$ and $a(t)$ are given continuous functions and Ω is a bounded domain with continuous boundary. We need to obtain a pair $\{u(x,t),p(t)\}$ from (1.1a)-(1.2).

If we let $u(x,t)$ represent the temperature, this problem can model a heat transfer process with an unknown source parameter $p(t)$. It can be regarded as a source identification problem. The parameter $p(t)$ needs to be determined from the additional condition (1.2).

The condition (1.2) is called the thermal energy measurement. It means that we should measure the thermal energy in every moment to determine the source parameter $p(t)$. The presence of an integral term in a boundary condition can greatly complicate the application of standard numerical techniques such as finite differences, finite elements, spectral methods, etc. It is therefore important to search effective means to deal with it. When one deals with (1.1a)-(1.2) by finite difference schemes, a very natural ideas is to utilize the numerical integration as in [1]. The boundary element method was presented in [2] to discretize the heat equation (1.1a). By means of the fundamental solution and the Green's formula, they converted the original differential equation to boundary integral equation.

The existence, uniqueness and continuous dependence of the solutions to this problem is discussed in [3,4].

The variational iteration method(VIM), which was proposed by He [5–8], has been proved by many authors to be a powerful mathematical tool to deal various kinds of linear and nonlinear problems [9–12]. In recent years, it was also used to deal with inverse problems. In [13], using VIM, the authors discussed a one-dimensional parabolic equation with an unknown parameter $p(t)$ and a constant coefficient q of the first order term. In [14,15], the authors solved high dimensional semi-linear inverse parabolic equation on an unbounded domain using VIM. In [16], the authors identified an unknown coefficient of the first order term of the reaction-diffusion equation using He's VIM. In this paper, we will use He's VIM to solve the inverse problem (1.1a)-(1.2) with variable coefficients $q_j(t)$ $(j=1,\cdots,n)$ and unknown coefficient $p(t)$ in a high dimensional bounded domain. In the end, we test the effect of VIM when the thermal energy measurement $a(t)$ has noise $\epsilon(t)$. We show the stability of this method.

The convergence of VIM was discussed in [17].

2 The variational iteration method

We simplify (1.1a) from the following transformation

$$v(x,t)=u(x,t)r(t),\quad r(t)=\exp\Big(-\int_0^t p(s)ds\Big). \tag{2.1}$$

Then

$$u(x,t)=\frac{v(x,t)}{r(t)},\quad p(t)=-\frac{r'(t)}{r(t)}. \tag{2.2}$$

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So the Eqs. (1.1a)-(1.1c) become

$$v_t = \Delta v + r(t)f(x,t) + \sum_{j=1}^{n} q_j(t) v_{x_j} \quad \text{in} \quad \Omega \times (0,T], \tag{2.3a}$$

$$v(x,0) = \varphi(x) \quad \text{in} \quad \Omega, \tag{2.3b}$$

$$v(x,t) = r(t)g(x,t) \quad \text{on} \quad \partial\Omega \times (0,T], \tag{2.3c}$$

subject to the additional condition

$$r(t) = \frac{\int_\Omega v(x,t)dx}{a(t)}, \quad t \in [0,T]. \tag{2.4}$$

The transformations (2.1) and (2.2) translate an unknown pair $\{u(x,t),p(t)\}$ of (1.1a)-(1.2) to an unknown pair $\{v(x,t),r(t)\}$ of (2.3a)-(2.4), but (2.4) gives simple relationship of $v(x,t)$ and $r(t)$, which creates a good condition for us to construct an effective sequence of iterations.

The VIM changes the partial differential equation to correction variational functional in the following form

$$v_{k+1}(x,t) = v_k + \int_0^t \lambda \Big[\frac{\partial v_k(x,s)}{\partial s} - \Delta \tilde{v}_k(x,s) - \tilde{r}_k(s) f(x,s) - \sum_{j=1}^{n} q_j(s) (\tilde{v}_k)_{x_j}(x,s) \Big] ds, \tag{2.5}$$

where $\tilde{v}_k$ and $\tilde{r}_k(t) = \frac{\int_\Omega v_k(x,t)dx}{a(t)}$ are considered as restricted variation which means $\delta\tilde{v}_k = 0$ and $\delta\tilde{r}_k = 0$ (see [7]). Here, λ is the general Lagrange multiplier, its optimal value is found by using variational theory.

Definition 2.1. A variable quantity F is a functional dependent on a function $w(x)$ if to each function $w(x)$ of a certain class of functions $w(x)$ corresponds with a value F. The variation of a functional $F[w(x)]$ is defined in the following form:

$$\delta F[w(x)] = \frac{\partial}{\partial \alpha} F[w(x) + \alpha \delta w] \Big|_{\alpha=0}. \tag{2.6}$$

As a well-known result, we have [18]:

Theorem 2.1. *If a functional $F[w(x)]$ which has a variation achieves a maximum or a minimum at $w = w^*(x)$, where $w(x)$ is an interior point of the domain of definition of the functional, then at $w = w^*(x)$,*

$$\delta F = 0. \tag{2.7}$$

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To find the optimal value of λ, we make correction functional stationary in the following form

$$\begin{aligned}\delta v_{k+1}(x,t) =& \delta v_k(x,t)\\ &+\delta\int_0^t \lambda\Big[\frac{\partial v_k(x,s)}{\partial s}-\Delta\tilde{v}_k(x,s)-\tilde{r}_k(s)f(x,s)-\sum_{j=1}^{n}q_j(s)(\tilde{v}_k)_{x_j}(x,s)\Big]ds\\ =&0,\end{aligned}$$

which results in

$$\delta v_{k+1}(x,t)=\delta v_k(x,t)+\int_0^t \lambda\frac{\partial\delta v_k(x,s)}{\partial s}ds=0. \tag{2.8}$$

Therefore, the stationary conditions and consequently the optimal value of λ are obtained in the following form:

$$1+\lambda(s)|_{s=t}=0,\quad \lambda'(s)=0, \tag{2.9}$$

which yields

$$\lambda(s)=-1. \tag{2.10}$$

Therefore, the iteration formulae can be expressed as follows

$$v_{k+1}(x,t)=v_k(x,t)-\int_0^t\Big[\frac{\partial v_k(x,s)}{\partial s}-\Delta v_k(x,s)-r_k(s)f(x,s)-\sum_{j=1}^{n}q_j(s)(v_k)_{x_j}(x,s)\Big]ds, \tag{2.11}$$

where

$$r_k(t)=\frac{\int_\Omega v_k(x,t)dx}{a(t)}. \tag{2.12}$$

Then we have the approximate solution of the Eqs. (1.1a)-(1.2):

$$u_k(x,t)=\frac{v_k(x,t)}{r_k(t)},\quad p_k(t)=-\frac{r_k'(t)}{r_k(t)}. \tag{2.13}$$

3 Numerical experiment

In this section, we apply the VIM presented in Section 2 to inverse problem (2.3a)-(2.4). Numerical results show that the VIM is very effective.

Consider Eqs. (1.1a)-(1.2) with $T=1$, $n=2$ and

$$\begin{aligned}&f(x,y,t)=[(2(\pi^2+t)\sin(\pi x)-\pi t\cos(\pi x))\sin(\pi y)+\pi t\sin(\pi x)\cos(\pi y)]e^t,\\ &q_1(t)=t,\quad q_2(t)=-t,\quad \varphi(x,y)=\sin(\pi x)\sin(\pi y),\quad g(x,y,t)\equiv 0,\quad a(t)=\frac{4e^t}{\pi^2},\end{aligned}$$

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for which the exact solution is

$$u(x,y,t)=e^t\sin(\pi x)\sin(\pi y),\quad p(t)=1-2t,$$

and then

$$r(t)=e^{t^2-t},\quad v(x,y,t)=e^{t^2}\sin(\pi x)\sin(\pi y).$$

If we take $v_0(x,y,t)=\varphi(x,y)$, using the variational iteration method, we have

$$v_1(x,y,t)=(1+t^2)\sin(\pi x)\sin(\pi y), \qquad r_1(t)=e^{-t}(1+t^2), \tag{3.1a}$$

$$v_2(x,y,t)=\frac{1}{2}(2+2t^2+t^4)\sin(\pi x)\sin(\pi y), \qquad r_2(t)=\frac{1}{2}e^{-t}(2+2t^2+t^4), \tag{3.1b}$$

$$v_3(x,y,t)=\frac{1}{6}(6+6t^2+3t^4+t^6)\sin(\pi x)\sin(\pi y), \quad r_3(t)=\frac{1}{6}e^{-t}(6+6t^2+3t^4+t^6), \tag{3.1c}$$

and so on. Considering (2.13), we can get the approximate solution of (1.1a)-(1.2)

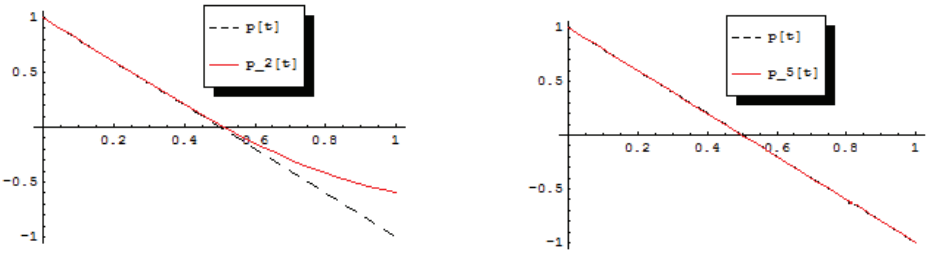
$$u_1(x,y,t)=e^t\sin(\pi x)\sin(\pi y), \qquad p_1(t)=1-2t+\frac{2t^3}{1+t^2}, \tag{3.2a}$$

$$u_2(x,y,t)=e^t\sin(\pi x)\sin(\pi y), \qquad p_2(t)=1-2t+\frac{2t^5}{2+2t^2+t^4}, \tag{3.2b}$$

$$u_3(x,y,t)=e^t\sin(\pi x)\sin(\pi y), \qquad p_3(t)=1-2t+\frac{2t^7}{6+6t^2+3t^4+t^6}, \tag{3.2c}$$

and so on. Actually, we can see the $u_k(x,y,t)$, $k=1,2,\cdots$, is the exact solution and $p_k(t)\to(1-2t)=p(t)$, as $k\to\infty$. We show it in Fig. 1 below.

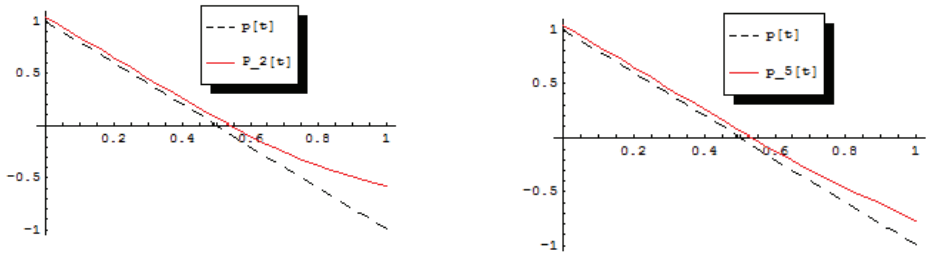
In the end, we test the effect of VIM when the thermal energy measurement $a(t)$ has noise $\epsilon(t)$. Taking $\epsilon(t)=10^{-3}e^t$, we can get the perturbation solution from VIM. We show the effect from Fig. 2 below.



(a) $k=2$ (b) $k=5$

Figure 1: The comparison of the $p_k(t)$ of the VIM and exact solution $p(t)$, $k=2$ and $k=5$.

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(a) $k=2$ (b) $k=5$

Figure 2: The errors between the exact solution $p(t)$ and the approximate solution $p_k^{\epsilon}(t)$ with the measurement noise $\epsilon(t)=10^{-3}e^t$.

4 Conclusions

In this paper, the VIM is presented for treating inverse problem of finding a source parameter. The major advantage of the method is that it can get a very effective approximate solution by only a few steps. Results of numerical examples demonstrate that the present method is quite effective to deal with inverse problems.

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