J. K-Theory **14** (2015), 642–700
doi:10.1017/is014008019jkt278
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Tannakization in derived algebraic geometry

by

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Abstract

In this paper we begin studying tannakian constructions in ∞-categories and combine them with the theory of motivic categories developed by Hanamura, Levine, and Voevodsky. This paper is the first in a series of papers. For the purposes above, we first construct a derived affine group scheme and its representation category from a symmetric monoidal ∞-category, which we shall call the tannakization of a symmetric monoidal ∞-category. It can be viewed as an ∞-categorical generalization of work of Joyal-Street and Nori. Next we apply it to the stable ∞-category of mixed motives equipped with the realization functor of a mixed Weil cohomology. We construct a derived motivic Galois group which represents the automorphism group of the realization functor, and whose representation category satisfies an appropriate universal property. As a consequence, we construct an underived motivic Galois group of mixed motives, which is a pro-algebraic group and has nice properties. Also, we present basic properties of derived affine group schemes in the Appendix.

Key Words: tannakian construction; symmetric monoidal ∞-category; motives

Mathematics Subject Classification 2010: 18D10, 18G55, 55U30.

1. Introduction

In this paper we begin to study tannakian constructions in ∞-categories and combine them with the theory of motivic categories developed by Hanamura, Levine, and Voevodsky. This paper is the first in a series of papers (cf. [26], [27]). We develop tannakian constructions in higher category theory, which is well-suited to the theory of mixed motives, and we then apply it to a mixed motivic category. In this Introduction, we begin by explaining the role of categorical results in a logical order, but it seems more appropriate to read first Section 1.2 where we give illustrations of motivation.

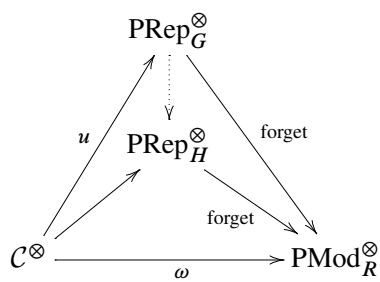
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1.1.

On the one hand, the theory of Galois and Tannakian categories and their dualities are beautiful duality theories in their own right [50], [19], [14], but on the other hand one of important aspects is the role as a powerful machine, which enables us to obtain invariants from abstract categories. For example, the étale fundamental groups of schemes and Picard-Vessiot Galois groups were constructed by means of these theories. Joyal-Street [30] and Nori constructed the machinery which approximates symmetric monoidal categories and graphs with (neutral) tannakian abelian categories (the braided case was also treated in [30]). This machinery is powerful: Joyal-Street applied it to quantum groups, and Nori used it to construct the Nori's category of motives (see e.g. [2]). We here informally call this approximation the tannakization of categories.

Tannakization of ∞-categories. We first construct tannakization in the setting of higher categories, i.e., ∞-categories. In this Introduction, by an ∞-category we informally mean a (weak) higher category, in which all n-morphisms are weakly invertible for $n > 1$ (cf. [7]). (There are several theories which provide "models" of such categories. We use the well-developed theory of quasi-categories from the next Section.) Let $\mathcal{C}^{\otimes}$ be a symmetric monoidal small ∞-category. For a commutative ring spectrum R, we let $\mathrm{Mod}_R^{\otimes}$ be the symmetric monoidal ∞-category of R-module spectra. Let $\mathrm{PMod}_R^{\otimes}$ be the symmetric monoidal full subcategory of $\mathrm{Mod}_R^{\otimes}$ spanned by dualizable R-module spectra (cf. Section 2). Let CAlg_R be the ∞-category of commutative R-ring spectra. Let $\omega : \mathcal{C}^{\otimes} \to \mathrm{PMod}_R^{\otimes}$ be a symmetric monoidal functor. Then our result can be roughly stated as follows (see Theorem 4.15):

Theorem 1.1 *There exists a derived affine group scheme G over R (explained below) which represents the automorphism group of ω. Furthermore, there is a symmetric monoidal functor $u : \mathcal{C}^{\otimes} \to \mathrm{PRep}_G^{\otimes}$ which makes the outer triangle in*



commute in the ∞-category of symmetric monoidal ∞-categories (here $\mathrm{PRep}_G^{\otimes}$ is the symmetric monoidal ∞-category of dualizable R-module spectra equipped with G-actions) such that these possess the following universality: for any inner triangle

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consisting of solid arrows in the above diagram where H is a derived affine group scheme, there exists a unique (in an appropriate sense) morphism $f : H \to G$ of derived affine group schemes which induces $\mathrm{PRep}_G^{\otimes} \to \mathrm{PRep}_H^{\otimes}$ *(indicated by the dotted arrow) filling the above diagram.*

For simplicity, we usually refer to the pair $(G, u : \mathcal{C}^{\otimes} \to \mathrm{PRep}_G^{\otimes})$ as the tannakization. By Theorem 1.1 we can obtain "Tannaka-Galois type invariants" in the quite general setting. A derived affine group scheme is an analogue of affine group schemes in derived algebraic geometry. This notion plays an important role in this paper. In Appendix, we presents a basic theory of derived affine group schemes. This notion is inherently of homotopical nature. For example, the functor ω can possess higher automorphisms. The derived affine group scheme G captures all these higher data. Beside, derived affine group schemes are quite tractable geometric objects. It would be worth remarking that the tannakization procedure is applicable also to situations in which $\mathcal{C}^{\otimes}$ seems "non-tannakian". Typical examples are $\mathcal{C}^{\otimes} = \mathrm{PMod}_A^{\otimes}$ with A arbitrary. Even in the case, our tannakization gives us meaningful invariants. Adams spectral sequence is an example of such a phenomenon. In a separate paper [26], we prove that our tannakization includes bar construction of an augmented commutative ring spectrum and its equivariant versions as a special case. Therefore our tannakization can be also viewed as a generalization of bar constructions and equivariant bar constructions.

Our motivation comes from various important and interesting examples which lie in the realm of ∞-categories. For example, the triangulated category of mixed motives, due to Hanamura, Levine and Voevodsky, is of great interest in the view of a tannakian theory for higher categories. The category of mixed motives has a natural formulation of symmetric monoidal stable ∞-category. The stable ∞-category is equipped with realization functors of mixed Weil cohomology theories. One of important examples of stable ∞-categories which recently appeared might be a symmetric monoidal stable ∞-category of noncommutative motives by Blumberg-Gepner-Tabuada [11], that is the natural and universal domain for localizing (or additive) invariants such as algebraic K-theory, topological Hochschild homology and topological cyclic homology.

1.2. Mixed motives.

Our principal interests lie in the stable ∞-category of mixed motives and variants. In order to invite reader's attention to the motivation of our "higher approach", let us briefly recall the original approach to Galois groups of motives (cf. [3]). The formulation of a symmetric monoidal abelian category $\mathcal{MM}^{\otimes}$ of mixed motives was conjectured by Beilinson and Deligne as the "extension" of Grothendieck pure

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motives. The conjectural abelian category has been expected to be a tannakian category with respect to realization functors arising from Weil cohomology theories. Namely, there is an affine group scheme G which represents the automorphism group of the realization functor

$$\mathcal{MM}^{\otimes} \longrightarrow \mathrm{Vect}^{\otimes}$$

to the category of vector spaces, and $\mathcal{MM}^{\otimes}$ is equivalent to the category of finite dimensional representation of G. The automorphism group of the (Singular, de Rham) realization functor conjecturally describes periods via comparison of singular cohomology and de Rham cohomology. On the other hand, in the work of Hanamura, Levine and Voevodsky in nineties, the triangulated category of mixed motives was constructed. One also has its ∞-categorical enhancement DM (described in Section 5 of this paper). The conjectural tannakian category of mixed motives should appear as the heart of a certain (so-called motivic) t-structure on DM. However, the existence of a motivic t-structure is related (equivalent) to three conjectures: the Bloch-Beilinson-Murre filtration conjecture, the Beilinson-Soulé vanishing conjecture, and the Grothendieck standard conjectures, which are still largely inaccessible.

Roughly speaking, the idea here may be simply described as follows: If one has a motivic t-structure and the heart $\mathcal{MM}$, then there is a natural exact functor $\mathcal{D}^{\otimes}(\mathcal{MM}) \to \mathsf{DM}^{\otimes}$ of stable ∞-categories from the symmetric monoidal derived ∞-category $\mathcal{D}^{\otimes}(\mathcal{MM})$. Although an existence of a motivic t-structure does not imply that $\mathcal{D}^{\otimes}(\mathcal{MM}) \to \mathsf{DM}^{\otimes}$ is an equivalence, from an intuitive viewpoint of rational homotopy theory the functor $\mathcal{D}^{\otimes}(\mathcal{MM}) \to \mathsf{DM}^{\otimes}$ should be "2-connected" (under the assumption of its existence). Therefore, the derived affine group scheme arising from $\mathsf{DM}^{\otimes}$, i.e. its tannakization, may have the "underlying" affine group scheme which corresponds to the Tannaka dual of $\mathcal{MM}^{\otimes}$.

In Section 5, following our perspective we show how one can naturally obtain group objects having fundamental properties: Theorem 5.14, Theorem 5.15, Theorem 5.17, Proposition 5.19. We shall call them derived and underived motivic Galois groups for mixed motives. Let $\mathbf{K}$ be a field of characteristic zero and let $H\mathbf{K}$ denote the Eilenberg-MacLane spectrum. Let $\mathsf{DM}^{\otimes} := \mathsf{DM}^{\otimes}(k)$ be the $H\mathbf{K}$-linear symmetric monoidal stable ∞-category of mixed motives over a perfect field k. Let $\mathsf{DM}_{\vee}^{\otimes}$ be the symmetric monoidal full subcategory spanned by dualizable objects in $\mathsf{DM}^{\otimes}$. In $\mathsf{DM}^{\otimes}$, dualizable objects coincide with compact objects. The homotopy category of $\mathsf{DM}_{\vee}$ can be identified with the $\mathbf{K}$-linear triangulated category of geometric motives $DM_{gm}(k)$ constructed by Voevodsky (see e.g. [40], [53]). Let E be a mixed Weil cohomology theory with coefficients $\mathbf{K}$ in the sense of [13]. For example, l-adic étale cohomology, singular cohomology, de Rham cohomology and

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rigid cohomology give mixed Weil theories. Then we can construct the homological realization functor

$$\mathsf{R}_E : \mathsf{DM}_\vee^\otimes \longrightarrow \mathrm{PMod}_{H\mathbf{K}}^\otimes,$$

that is a symmetric monoidal exact functor. Note that the homotopy category of $\mathrm{PMod}_{H\mathbf{K}}^\otimes$ can be regarded as the triangulated category of bounded complexes of $\mathbf{K}$-vector spaces with finite dimensional cohomology groups. Applying Theorem 1.1 to the realization functor of a mixed Weil cohomology theory we obtain (see Definition 5.13, Theorem 5.14):

Theorem 1.2 *We obtain a derived affine group scheme* $\mathsf{MG}_E = \mathrm{Spec}\, B_E$ *over* $H\mathbf{K}$*, which represents the realization functor* $\mathsf{R}_E : \mathsf{DM}_\vee^\otimes \to \mathrm{PMod}_{H\mathbf{K}}^\otimes$ *and enjoys the universality described in Theorem 1.1. Here* B_E *is a commutative differential graded* $\mathbf{K}$*-algebra.*

One can also generalizes it to symmetric monoidal stable subcategories of $\mathsf{DM}_\vee^\otimes$ and obtains derived affine group schemes. We shall refer to $\mathsf{MG}_E = \mathrm{Spec}\, B_E$ as the derived motivic Galois group (with respect to E). We also construct the comparison torsor of Singular-de Rham realizations; see Theorem 5.15.

Our construction has the advantage that the geometric nature of MG_E allows us to construct the underived motivic Galois group MG_E which is a usual pro-algebraic group over $\mathbf{K}$. This MG_E is the "underlying" group scheme of MG_E that we mentioned above. It has the following nice and expected property (see Theorem 5.17):

Theorem 1.3 *Let* K *be a* $\mathbf{K}$*-field, that is, a commutative* $\mathbf{K}$*-algebra which is a field. Let* $\overline{\mathrm{Aut}(\mathsf{R}_E)}(K)$ *be the group of equivalence classes of automorphisms of* R_E*. Then there is a natural isomorphism of groups*

$$MG_E(K) \simeq \overline{\mathrm{Aut}(\mathsf{R}_E)}(K)$$

where $MG_E(K)$ *denotes the group of* K*-valued points. These isomorphisms are functorial among* $\mathbf{K}$*-fields in the obvious way.*

This result means that MG_E has coarse representability and plays a role analogous to those of coarse moduli spaces of moduli stacks in classical algebraic geometry. It is remarkable from both practical and theoretical aspects (cf. [26], [27]).

Though we work in the situation which is rather different from the original abelian setting explained above, our MG_E and MG_E are nicely consistent with the traditional line of a conjectural motivic t-structure on $DM_{gm}(k)$. We prove that if a motivic t-structure exists, then its heart (that constitutes a tannakian category) is equivalent to the tannakian category of finite dimensional representations of MG_E;

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see Proposition 5.19. Hence the passage from MG_E to MG_E can be thought of as a sort of the geometric counterpart of taking the heart (even though we do not know whether or not $\mathsf{DM}^{\otimes}$ admits a suitable t-structure). Our principle of correspondences is described as follows:

Category	Group
$\mathsf{DM}^{\otimes}$	MG_E
A conjectural heart	MG_E

Based on this paper, we pursue detailed study in subsequent papers. In [26] we compare our construction with the conventional motivic Galois group of mixed Tate motives [26], and in [27] we prove a conjectural motivic Levi decomposition of motivic Galois groups of mixed abelian motives. Here we would like to invite the reader's attention to the results obtained in the second [26] (to formulate the results in [27] we need some preparations): MG_E and MG_E can be thought of as a natural generalization of conventional motivic Galois group MTG for mixed Tate motives constructed in [8], [33], [34]. Let $\mathsf{DTM}_{\vee} \subset \mathsf{DM}_{\vee}$ be the stable ∞-category of mixed Tate motives, that is, the stable idempotent complete subcategory generated by Tate objects $\{\mathbf{K}(n)\}_{n\in\mathbb{Z}}$ (see [26, Section 6] for more details). The full subcategory $\mathsf{DTM}_{\vee}$ forms a symmetric monoidal ∞-category $\mathsf{DTM}^{\otimes}$, whose symmetric monoidal structure is induced by that of $\mathsf{DM}^{\otimes}$. In [26], we prove comparison results which can be informally summarized as follows:

Theorem 1.4 ([26]) *(i) Let* MTG *be a derived affine group scheme over* $H\mathbf{K}$ *obtained as the tannakization of the composite* $\mathsf{R}_T : \mathsf{DTM}_{\vee}^{\otimes} \hookrightarrow \mathsf{DM}^{\otimes} \overset{\mathsf{R}_E}{\to} \mathrm{PMod}_{H\mathbf{K}}^{\otimes}$ *(we omit the subscript* E*). Then* MTG *is equivalent to a derived affine group scheme obtained from the* $\mathbb{G}_m$*-equivariant bar construction of a commutative differential graded* $\mathbf{K}$*-algebra* $\overline{Q}$ *equipped with* $\mathbb{G}_m$*-action. That is to say, it is the Čech nerve of a morphism of derived stacks* $\mathrm{Spec}\, H\mathbf{K} \to [\mathrm{Spec}\,\overline{Q}/\mathbb{G}_m]$ *(cf. Appendix Example A.6 and [26]). The complex* $\overline{Q}$ *is described in term of Bloch's cycle complexes.*

(ii) Suppose that Beilinson-Soulé vanishing conjecture holds for k *(e.g.,* k *is a number field). Let* MTG *be the Tannaka dual of the tannakian category of mixed Tate motives; the heart of motivic* t*-structure on* $\mathsf{DTM}_{\vee}$ *(constructed under the vanishing conjecture, see [34], [33], [26, Section 7]). Then the affine group scheme* MTG *is the excellent coarse moduli space (cf. Appendix A.4 or [26, 7.3]) of* MTG.

(iii) Let $\mathsf{Art}^{\otimes}$ *be the symmetric monoidal stable idempotent complete full subcategory generated by motives of smooth zero-dimensional varieties, i.e., Artin*

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motives. Then the tannakization of $\mathsf{Art}^{\otimes}$ *equipped with a realization functor is the absolute Galois group* $\mathrm{Gal}(\bar{k}/k)$.

This result links the works on mixed Tate motives and the classical Galois theory to our results. To our knowledge, this is the first construction which can generalizes the works on mixed Tate motives. In a sense, the aspect of tannakization as a generalization of bar constructions allows one to construct a motivic Galois group of all mixed motives.

In summary, our new perspective leads to noteworthy features of MG_E and MG_E:

- MG_E has the representability of automorphisms of the realization functor from $\mathsf{DM}_{\vee}^{\otimes}$, and the universality. The pro-algebraic group MG_E has the "coarse" representability of the equivalence classes of automorphisms of the realization functor.

- The category of finite dimensional representations of MG_E is the equivalent to the abelian category of mixed motives obtained as the heart of a conjectural motivic t-structure on $\mathsf{DM}_{\vee}^{\otimes}$. Note that Nori's motivic Galois group (cf. [2]) does not possess this property.

- MG_E is a natural generalization of the conventional motivic Galois group for mixed Tate motives. Contrary to this, it is not clear whether or not Nori's motivic Galois group is a generalization of the group for mixed Tate motives. Furthermore, as shown in [27], one can study the structure of motivic Galois group; a conjectural Levi decompositions of motivic Galios groups.

We would like to emphasize that the theory of higher category theory (∞-categories) provide a natural and nice framework for our purposes. For a commutative ring spectrum A, the homotopy category of $\mathrm{PMod}_A^{\otimes}$ (or $\mathrm{Mod}_A^{\otimes}$) forms a triangulated category equipped with a symmetric monoidal structure. However, if we work with triangulated categories (to prove Theorem 1.1 in particular, representability), we encounter several technical problems including the problem concerning the absence of descent of morphisms in the homotopy category of PMod_A. This issue is crucial also in our previous work on the derived analogue of Tannaka duality for schemes and stacks [18] inspired by theory of thick subcategories. It turns out that ∞-categories give us an appropriate theory.

We should refer the reader to the recent preprints building on tannakian philosophy in higher categories; in [38, VIII] Lurie develops a version of Tannaka duality for stable ∞-category equipped with t-structures (we combine it with our

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results to prove Proposition 5.19), and in [55] Wallbridge develops another version, which were done independent of us (under the supervision of Toën [51]).

Organization. This paper is roughly organized as follows. In Section 2, we fix notation and convention. In Section 3 we give preliminaries which we need Section 4. Section 4 is devoted to the proof of Theorem 1.1. Section 5 contains our construction of derived and underived motivic Galois groups of mixed motives, and we prove Theorem 5.14, 5.17. Our study on motivic Galois groups will be continued in subsequent papers [26], [27]. In Section 6, we present some other examples without going into detail.

In Appendix we present basic definitions and results concerning derived group schemes. The notion of derived (affine) group schemes is important in our work. We hereby decide to develop the basic theory of derived affine group schemes in Appendix for reader's convenience.

• The author thanks the participants of the seminars on DAG-HAG at Kyoto University and Nagoya University, and SGAD, for helpful conversations related to the subject of this paper. Also, he thanks the workshop at Tohoku University in March 2012 where he presented the main contents of this paper. He thanks J. Morava and M. Hanamura for encouraging interest. He would like to express his gratitude to the referee for careful reading and various comments. The author is partly supported by Grant-in-aid for Scientific Research 23840003, Japan Society for the Promotion of Science.

2. Notation and Convention

We fix notation and convention.

∞-categories. Throughout this paper, we use the theory of quasi-categories. A quasi-category is a simplicial set which satisfies the weak Kan condition of Boardman-Vogt: A quasi-category S is a simplicial set such that for any $0 < i < n$ and any diagram

$$\begin{array}{ccc} \Lambda^n_i & \longrightarrow & S \\ \downarrow & \nearrow & \\ \Delta^n & & \end{array}$$

of solid arrows, there exists a dotted arrow filling the diagram. Here Λ^n_i is the i-th horn and Δ^n is the standard n-simplex. The theory of quasi-categories from higher categorical viewpoint has been extensively developed by Joyal and Lurie. Following [36] we shall refer to quasi-categories as ∞-categories. Our main references are [36] and [37] (see also [29], [38]). We often refer to a map $S \to T$ of ∞-categories

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as a functor. We call a vertex in an ∞-category S (resp. an edge) an object (resp. a morphism). By $s \in S$ we often mean that s is an object of S. For the rapid introduction to ∞-categories, we refer to [36, Chapter 1], [18, Section 2]. It should be emphasized that there are several alternative theories such as Segal categories, complete Segal spaces, simplicial categories, relative categories,... etc. For the quick survey on various approaches to $(\infty,1)$-categories and their relations, we refer the reader to [7].

- Δ: the category of linearly ordered finite sets (consisting of $[0],[1],\ldots,[n] = \{0,\ldots,n\},\ldots$).
- Δ^n: the standard n-simplex.
- N: the simplicial nerve functor (cf. [36, 1.1.5]).
- $\mathcal{C}^{op}$: the opposite ∞-category of an ∞-category $\mathcal{C}$.
- Let $\mathcal{C}$ be an ∞-category and suppose that we are given an object c. Then $\mathcal{C}_{c/}$ and $\mathcal{C}_{/c}$ denote the undercategory and overcategory respectively (cf. [36, 1.2.9]).
- Cat_∞: the ∞-category of small ∞-categories in a fixed Grothendieck universe (cf. [36, 3.0.0.1]). We employ the ZFC-axiom together with the universe axiom of Grothendieck. We have a sequence of universes $(\mathbb{N} \in)\mathbb{U} \in \mathbb{V} \in \mathbb{W} \in \ldots$. If x belongs to $\mathbb{V}$ (resp. $\mathbb{W}$), we call x large (resp. super-large). We will not use notation $\mathbb{U}$, $\mathbb{V}$, and $\mathbb{W}$, but we shall use the wording such as "small sets", "large ∞-categories", etc.
- $\widehat{\mathrm{Cat}}_\infty$: the super-large ∞-category of large ∞-categories.
- $\mathcal{S}$: ∞-category of small spaces (cf. [36, 1.2.16]).
- $\mathrm{h}(\mathcal{C})$: homotopy category of an ∞-category (cf. [36, 1.2.3.1]).
- $\mathrm{Fun}(A,B)$: the function complex for simplicial sets A and B.
- $\mathrm{Fun}_C(A,B)$: the simplicial subset of $\mathrm{Fun}(A,B)$ classifying maps which are compatible with given projections $A \to C$ and $B \to C$.
- $\mathrm{Map}(A,B)$: the largest Kan subcomplex of $\mathrm{Fun}(A,B)$ when A and B are ∞-categories.
- $\mathrm{Map}_C(A,B)$: the simplicial subset of $\mathrm{Map}(A,B)$ classifying maps which are compatible with given projections $A \to C$ and $B \to C$.

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- $\mathrm{Map}_{\mathcal{C}}(C, C')$: the mapping space from an object $C \in \mathcal{C}$ to $C' \in \mathcal{C}$ where $\mathcal{C}$ is an ∞-category. We usually view it as an object in $\mathcal{S}$ (cf. [36, 1.2.2]).

- $\mathrm{Ind}(\mathcal{C})$; ∞-category of Ind-objects in an ∞-category $\mathcal{C}$ (cf. [36, 5.3.5.1], [37, 6.3.1.13]).

Symmetric monoidal ∞-categories and spectra. We employ the theory of symmetric monoidal ∞-categories developed in [37]. We refer to [37] for its generalities. Let Fin_* be the category of marked finite sets (our notation is slightly different from [37]). Namely, objects are marked finite sets and a morphism from $\langle n \rangle_* := \{1 < \cdots < n\} \sqcup \{*\} \to \langle m \rangle_* := \{1 < \cdots < m\} \sqcup \{*\}$ is a (not necessarily order-preserving) map of finite sets which preserves the distinguished points $*$. Let $\alpha^{i,n} : \langle n \rangle_* \to \langle 1 \rangle_*$ be a map such that $\alpha^{i,n}(i) = 1$ and $\alpha^{i,n}(j) = *$ if $i \neq j \in \langle n \rangle_*$. A symmetric monoidal category is a coCartesian fibration $p : \mathcal{M}^{\otimes} \to \mathrm{N}(\mathrm{Fin}_*)$ (cf. [36, 2.4]) such that for any $n \geq 0$, $\alpha^{1,n} \ldots \alpha^{n,n}$ induce an equivalence $\mathcal{M}^{\otimes}_n \to (\mathcal{M}^{\otimes}_1)^{\times n}$ where $\mathcal{M}^{\otimes}_n$ and $\mathcal{M}^{\otimes}_1$ are fibers of p over $\langle n \rangle_*$ and $\langle 1 \rangle_*$ respectively. A symmetric monoidal functor is a map $\mathcal{M}^{\otimes} \to \mathcal{M}'^{\otimes}$ of coCartesian fibrations over $\mathrm{N}(\mathrm{Fin}_*)$, which carries coCartesian edges to coCartesian edges. Let $\mathrm{Cat}_{\infty}^{\Delta,\mathrm{sMon}}$ be the simplicial category of symmetric monoidal ∞-categories in which morphisms are symmetric monoidal functors. Hom simplicial sets are given by those defined in [36, 3.1.4.4]. Let $\mathrm{Cat}_{\infty}^{\mathrm{sMon}}$ be the simplicial nerve of $\mathrm{Cat}_{\infty}^{\Delta,\mathrm{sMon}}$ (see [37, 2,1.4.13]).

There are several approaches to a "good" theory of commutative ring spectra. Among these, we employ the theory of spectra and commutative ring spectra developed in [37].

We list some of notation.

- $\mathbb{S}$: the sphere spectrum.

- Mod_A: ∞-category of A-module spectra for a commutative ring spectrum A.

- PMod_A: the full subcategory of Mod_A spanned by compact objects (in Mod_A, an object is compact if and only if it is dualizable, see [6]) . We refer to objects in PMod_A as perfect A-module (spectra).

- Let $\mathcal{M}^{\otimes} \to \mathcal{O}^{\otimes}$ be a fibration of ∞-operads. We denote by $\mathrm{Alg}_{/\mathcal{O}^{\otimes}}(\mathcal{M}^{\otimes})$ the ∞-category of algebra objects (cf. [37, 2.1.3.1]). We often write $\mathrm{Alg}(\mathcal{M}^{\otimes})$ or $\mathrm{Alg}(\mathcal{M})$ for $\mathrm{Alg}_{/\mathcal{O}^{\otimes}}(\mathcal{M}^{\otimes})$. Suppose that $\mathcal{P}^{\otimes} \to \mathcal{O}^{\otimes}$ is a map of ∞-operads. $\mathrm{Alg}_{\mathcal{P}^{\otimes}/\mathcal{O}^{\otimes}}(\mathcal{M}^{\otimes})$: ∞-category of $\mathcal{P}$-algebra objects.

- $\mathrm{CAlg}(\mathcal{M}^{\otimes})$: ∞-category of commutative algebra objects in a symmetric monoidal ∞-category $\mathcal{M}^{\otimes} \to \mathrm{N}(\mathrm{Fin}_*)$.

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- CAlg_R: ∞-category of commutative algebra objects in the symmetric monoidal ∞-category $\mathrm{Mod}_R^{\otimes}$ where R is a commutative ring spectrum. When $R = \mathbb{S}$, we set $\mathrm{CAlg} = \mathrm{CAlg}_{\mathbb{S}}$. The ∞-category CAlg_R is equivalent to the undercategory $\mathrm{CAlg}_{R/}$ as an ∞-category.

- $\mathrm{Mod}_A^{\otimes}(\mathcal{M}^{\otimes}) \to \mathrm{N}(\mathrm{Fin}_*)$: symmetric monoidal ∞-category of A-module objects, where $\mathcal{M}^{\otimes}$ is a symmetric monoidal ∞-category such that (1) the underlying ∞-category admits a colimit for any simplicial diagram, and (2) its tensor product functor $\mathcal{M} \times \mathcal{M} \to \mathcal{M}$ preserves colimits of simplicial diagrams separately in each variable. Here A belongs to $\mathrm{CAlg}(\mathcal{M}^{\otimes})$ (cf. [37, 3.3.3, 4.4.2]).

Let $\mathcal{C}^{\otimes}$ be the symmetric monoidal ∞-category. We usually denote, dropping the subscript $\otimes$, by $\mathcal{C}$ its underlying ∞-category. We say that an object X in $\mathcal{C}$ is dualizable if there exist an object $X^{\vee}$ and two morphisms $e : X \otimes X^{\vee} \to \mathbf{1}$ and $c : \mathbf{1} \to X \otimes X^{\vee}$ with $\mathbf{1}$ a unit such that the composition

$$X \overset{\mathrm{Id}_X \otimes c}{\longrightarrow} X \otimes X^{\vee} \otimes X \overset{e \otimes \mathrm{Id}_X}{\longrightarrow} X$$

is equivalent to the identity, and

$$X^{\vee} \overset{c \otimes \mathrm{Id}_{X^{\vee}}}{\longrightarrow} X^{\vee} \otimes X \otimes X^{\vee} \overset{\mathrm{Id}_{X^{\vee}} \otimes e}{\longrightarrow} X^{\vee}$$

is equivalent to the identity. Dualizable object may also be referred to as strongly dualizable objects elsewhere. For example, for $R \in \mathrm{CAlg}$, compact and dualizable objects coincide in the symmetric monoidal ∞-category $\mathrm{Mod}_R^{\otimes}$ (cf. [6]). The symmetric monoidal structure of $\mathcal{C}$ induces that of the homotopy category $\mathrm{h}(\mathcal{C})$. If we consider X to be an object also in $\mathrm{h}(\mathcal{C})$, then X is dualizable in $\mathcal{C}$ if and only if X is dualizable in $\mathrm{h}(\mathcal{C})$.

3. Basic definitions and geometric systems

In this Section, we prepare some notions which we need in the next Section.

The ∞-category Cat_{∞} of small ∞-categories has the symmetric monoidal structure determined by the Cartesian product $\mathcal{C} \times \mathcal{D}$. We denote by $\mathrm{CAlg}(\mathrm{Cat}_{\infty})$ the ∞-category of commutative algebra (monoid) objects in the symmetric monoidal ∞-category Cat_{∞}. A symmetric monoidal ∞-category can be identified with a commutative algebra (monoid) object in Cat_{∞}; there is a natural categorical equivalence $\mathrm{Cat}_{\infty}^{\mathrm{sMon}} \simeq \mathrm{CAlg}(\mathrm{Cat}_{\infty})$. If $\mathcal{A}^{\otimes}, \mathcal{B}^{\otimes} \in \mathrm{CAlg}(\mathrm{Cat}_{\infty})$, we write $\mathrm{Map}^{\otimes}(\mathcal{A}^{\otimes}, \mathcal{B}^{\otimes})$ for $\mathrm{Map}_{\mathrm{CAlg}(\mathrm{Cat}_{\infty})}(\mathcal{A}^{\otimes}, \mathcal{B}^{\otimes})$.

Geometric R-system. We introduce the notion of geometric R-systems.

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Definition 3.1 Let $\mathcal{T}^{\otimes} : \mathrm{CAlg}_R \to \mathrm{Cat}_\infty^{\mathrm{sMon}} \simeq \mathrm{CAlg}(\mathrm{Cat}_\infty)$ be a functor satisfying the following properties:

(A1) Let $\mathcal{T} : \mathrm{CAlg}_R \to \mathrm{CAlg}(\mathrm{Cat}_\infty) \to \mathrm{Cat}_\infty$ be the composition with the forgetful functor. For any A, $\mathcal{T}(A)$ is stable and $\mathcal{T}(A) \to \mathcal{T}(B)$ is exact for any $A \to B$.

(A2) For any $T, T' \in \mathcal{T}(R)$, the Isom-functor $\mathrm{Isom}(T,T') : \mathrm{CAlg}_R \to \mathcal{S}$, which will be defined below, is representable by a derived affine scheme over R. For any $T, T' \in \mathcal{T}(R)$, Hom-functor $\mathrm{Hom}(T,T') : \mathrm{CAlg}_R \to \mathcal{S}$, which will be defined below, is representable by derived affine schemes over R.

If (A1) and (A2) hold, we refer to $\mathcal{T}^{\otimes}$ as a *geometric R-system*.

We here define $\mathrm{Hom}(T,T') : \mathrm{CAlg}_R \to \mathcal{S}$ as follows. Let $\theta_{\Delta^1}, \theta_{\partial\Delta^1}, \theta_{\phi} : \mathrm{Cat}_\infty \to \mathcal{S}$ be the functors corresponding to Δ^1, $\partial\Delta^1$ and the empty category ϕ respectively via the Yoneda embedding $\mathrm{Cat}_\infty^{op} \subset \mathrm{Fun}(\mathrm{Cat}_\infty, \mathcal{S})$. The inclusion $\partial\Delta^1 \hookrightarrow \Delta^1$ induces $\theta_{\Delta^1} \to \theta_{\partial\Delta^1}$. Note that θ_ϕ is equivalent to the constant functor whose value is the contractible space. The functor $\theta_{\partial\Delta^1}$ is equivalent to the 2-fold product of the functor $\mathrm{Cat}_\infty \to \mathcal{S}$ which carries an ∞-category $\mathcal{A}$ to the largest Kan complex $\mathcal{A}^{\simeq}$ (this functor can be constructed as the functor corepresentable by Δ^0). Therefore, if we let $\mathcal{F} \to \mathrm{CAlg}_R$ be a left fibration corresponding to $\mathrm{CAlg}_R \xrightarrow{\mathcal{T}} \mathrm{Cat}_\infty \xrightarrow{\theta_{\Delta^0}} \mathcal{S}$, then giving $\theta_\phi \circ \mathcal{T} \to \theta_{\partial\Delta^1} \circ \mathcal{T}$ amounts to giving two sections of $\mathcal{F} \to \mathrm{CAlg}_R$. In order to construct $\theta_\phi \circ \mathcal{T} \to \theta_{\partial\Delta^1} \circ \mathcal{T}$ from T and T', we give (ordered) two sections $\mathrm{CAlg}_R \to \mathcal{F}$. By [36, 3.3.3.4], a section corresponds to an object in the limit $\lim_A \mathcal{T}(A)$ of $\mathcal{T} : \mathrm{CAlg}_R \to \mathrm{Cat}_\infty$. Hence the images of T and T' in $\lim \mathcal{T}(A)$ give rise to $\theta_\phi \to \theta_{\partial\Delta^1}$. We define $\mathrm{Hom}(T,T')$ to be the fiber product $\theta_\phi \circ \mathcal{T} \times_{\theta_{\partial\Delta^1} \circ \mathcal{T}} \theta_{\Delta^1} \circ \mathcal{T}$ in $\mathrm{Fun}(\mathrm{CAlg}_R, \mathcal{S})$. For any $A \in \mathrm{CAlg}_R$, $\mathrm{Hom}(T,T')(A)$ is equivalent to (homotopy) fiber product

$$\{(T \otimes_R A, T' \otimes_R A)\} \times_{\mathrm{Map}(\partial\Delta^1, \mathcal{T}(A))} \mathrm{Map}(\Delta^1, \mathcal{T}(A))$$

in $\mathcal{S}$, where $T \otimes_R A$ and $T' \otimes_R A$ denote the images of T and T' in $\mathcal{T}(A)$ respectively. It is the mapping space from $T \otimes_R A$ to $T' \otimes_R A$. We let $\mathrm{Isom}(T,T')$ be the functor $\mathrm{CAlg}_R \to \mathcal{S}$ obtained by restricting objects in $\mathrm{Hom}(T,T')(A)$ to equivalences for each A (one can do this procedure by using corresponding left fibration). If $T = T'$, we write $\mathrm{End}(T)$ for $\mathrm{Hom}(T,T)$. We define $\mathrm{Aut}(T)$ to be $\mathrm{Isom}(T,T)$.

The followings are examples of geometric R-systems. In the next Section we will prove that these examples are geometric R-systems.

The following examples will be shown to be geometric R-systems (cf. Lemma 4.7, Corollary 4.10):

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Example 3.2 Let $\Theta : \mathrm{CAlg}_R \to \mathrm{CAlg}(\mathrm{Cat}_\infty)$ be the functor which carries A to $\mathrm{PMod}_A^\otimes$ and carries $A \to B$ to the base change functor $\mathrm{PMod}_A^\otimes \to \mathrm{PMod}_B^\otimes$. We can obtain this functor Θ as follows. By virtue of [37, 6.3.5.18], we have $\mathrm{CAlg}_R \to \mathrm{CAlg}(\widehat{\mathrm{Cat}}_\infty)_{\mathrm{Mod}_R^\otimes /}$ which carries A to $\mathrm{Mod}_A^\otimes$. Composing with the forgetful functor $\mathrm{CAlg}(\widehat{\mathrm{Cat}}_\infty)_{\mathrm{Mod}_R^\otimes /} \to \mathrm{CAlg}(\widehat{\mathrm{Cat}}_\infty)$ and restricting $\mathrm{Mod}_A^\otimes$ to $\mathrm{PMod}_A^\otimes$ we have $\Theta : \mathrm{CAlg}_R \to \mathrm{CAlg}(\mathrm{Cat}_\infty)$. More precisely, $\Theta : \mathrm{CAlg}_R \to \mathrm{CAlg}(\mathrm{Cat}_\infty)$ is given by the composition

$$\mathrm{CAlg}_R \to \mathrm{CAlg}(\widehat{\mathrm{Cat}}_\infty)_{\mathrm{Mod}_R^\otimes /} \to \mathrm{CAlg}(\widehat{\mathrm{Cat}}_\infty) \to \mathrm{CAlg}(\widehat{\mathrm{Cat}}_\infty)$$

where the right functor takes a symmetric monodal ∞-category to its full subcategory spanned by dualizable objects. This is a geometric R-system.

Example 3.3 Let S be a (small) Kan complex. Let $f : \mathrm{Cat}_\infty \to \mathrm{Cat}_\infty$ be the colimit-preserving functor which is determined by $(-) \times S$. Namely, f carries $\mathcal{C}$ to $\mathcal{C} \times S$. Its right adjoint functor $g : \mathrm{Cat}_\infty \to \mathrm{Cat}_\infty$ carries $\mathcal{C}$ to $\mathrm{Fun}(S,\mathcal{C})$. (To obtain this adjoint, consider the adjunction $(-) \times S : \mathrm{Set}_\Delta \rightleftarrows \mathrm{Set}_\Delta : \mathrm{Fun}(S,-)$, where Set_Δ denotes the category of simplicial sets. If both Set_Δ are endowed with Joyal model structure [36, 2.2.5.1], then this adjunction is a Quillen adjunction by [36, 2.2.5.4]. It gives rise to the required adjunction) Then $\mathrm{Fun}(\mathrm{N}(\mathrm{Fin}_*),\mathcal{C}) \to \mathrm{Fun}(\mathrm{N}(\mathrm{Fin}_*),\mathcal{C})$ induced by composition with g preserves commutative monoid (algebra) objects. Thus it gives rise to $g^S : \mathrm{CAlg}(\mathrm{Cat}_\infty) \to \mathrm{CAlg}(\mathrm{Cat}_\infty)$. Roughly speaking, g^S sends a symmetric monoidal ∞-category $\mathcal{C}^\otimes$ to $\mathrm{Fun}(S,\mathcal{C})$ endowed with the symmetric monoidal structure $\mathrm{Fun}(S,\mathcal{C}) \times \mathrm{Fun}(S,\mathcal{C}) \to \mathrm{Fun}(S,\mathcal{C})$ given by symmetric monoidal structure $\mathcal{C} \times \mathcal{C} \to \mathcal{C}$. We informally regard an object in $\mathrm{Fun}(S,\mathcal{C})$ as something like a fiber bundle of objects in $\mathcal{C}$ over the geometric realization $|S|$. Let us consider the composite

$$\Theta^S : \mathrm{CAlg}_R \xrightarrow{\Theta} \mathrm{CAlg}(\mathrm{Cat}_\infty) \xrightarrow{g^S} \mathrm{CAlg}(\mathrm{Cat}_\infty).$$

This is a geometric R-system.

Automorphism group functor and Isom-functor. Let $\mathcal{C}^\otimes$ be a symmetric monoidal small ∞-category. Let $\mathcal{T}^\otimes$ be a geometric R-system. Let $\omega : \mathcal{C}^\otimes \to \mathcal{T}^\otimes(R)$ be a symmetric monoidal functor. We write $\mathcal{C}$ for its underlying ∞-category. Let $\theta_{\mathcal{C}^\otimes} : \mathrm{CAlg}(\mathrm{Cat}_\infty) \to \mathcal{S}$ be the functor corresponding to $\mathcal{C}^\otimes$ via the Yoneda embedding $\mathrm{CAlg}(\mathrm{Cat}_\infty)^{op} \subset \mathrm{Fun}(\mathrm{CAlg}(\mathrm{Cat}_\infty),\mathcal{S})$. Then the composite

$$\xi : \mathrm{CAlg}_R \xrightarrow{\mathcal{T}^\otimes} \mathrm{CAlg}(\mathrm{Cat}_\infty) \xrightarrow{\theta_{\mathcal{C}^\otimes}} \mathcal{S}$$

carries A to the space equivalent to $\mathrm{Map}^\otimes(\mathcal{C}^\otimes,\mathcal{T}^\otimes(A))$. We can extend ξ to $\xi_* : \mathrm{CAlg}_R \to \mathcal{S}_*$ by using the symmetric monoidal functor ω. Here $\mathcal{S}_*$ denotes

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the ∞-category of pointed spaces, that is, $\mathcal{S}_{\Delta^0/}$. To explain this, let $\mathcal{M} \to \mathrm{CAlg}_R$ be a left fibration corresponding to ξ. An extension of ξ to ξ_* amounts to giving a section $\mathrm{CAlg}_R \to \mathcal{M}$ of the left fibration $\mathcal{M} \to \mathrm{CAlg}_R$. According to [36, 3.3.3.4] a section corresponds to an object in the ∞-category $\mathcal{L}$ which is the limit of the diagram of spaces (or ∞-categories) given by ξ; $A \mapsto \mathrm{Map}^{\otimes}(\mathcal{C}^{\otimes}, \mathcal{T}^{\otimes}(A))$. Thus if $\lim \mathcal{T}^{\otimes}(A)$ denotes the limit of $\mathcal{T}^{\otimes} : \mathrm{CAlg}_R \to \mathrm{CAlg}(\mathrm{Cat}_\infty)$, then $\mathcal{L}$ is equivalent to $\mathrm{Map}^{\otimes}(\mathcal{C}^{\otimes}, \lim \mathcal{T}^{\otimes}(A))$ as ∞-categories (or equivalently spaces). The natural functor $\mathcal{T}^{\otimes}(R) \to \lim \mathcal{T}^{\otimes}(A)$ induces $p : \mathrm{Map}^{\otimes}(\mathcal{C}^{\otimes}, \mathcal{T}^{\otimes}(R)) \to \mathrm{Map}^{\otimes}(\mathcal{C}^{\otimes}, \lim \mathcal{T}^{\otimes}(A)) \simeq \mathcal{L}$. The image $p(\omega)$ in $\mathcal{L}$ gives rise to a section $s : \mathrm{CAlg}_R \to \mathcal{M}$. Consequently, we have $\xi_* : \mathrm{CAlg}_R \to \mathcal{S}_*$ which extends ξ. We define $\mathrm{Aut}(\omega)$ to be the composite

$$\mathrm{CAlg}_R \xrightarrow{\xi_*} \mathcal{S}_* \xrightarrow{\Omega_*} \mathrm{Grp}(\mathcal{S}),$$

where the second functor is the based loop functor, and $\mathrm{Grp}(\mathcal{S})$ denotes the ∞-category of group objects in $\mathcal{S}$. We refer to $\mathrm{Aut}(\omega)$ as the automorphism group functor of $\omega : \mathcal{C}^{\otimes} \to \mathcal{T}^{\otimes}(R)$. For any $A \in \mathrm{CAlg}_R$, $\mathrm{Aut}(\omega)(A)$ is equivalent (as an object in $\mathcal{S}$) to the mapping space from the symmetric monoidal functor $\mathcal{C}^{\otimes} \to \mathcal{T}^{\otimes}(R) \to \mathcal{T}^{\otimes}(A)$ to itself in $\mathrm{Map}^{\otimes}(\mathcal{C}^{\otimes}, \mathcal{T}^{\otimes}(A))$. We often abuse notation and write $\mathrm{Aut}(\omega)$ also for the composition $\mathrm{CAlg}_R \to \mathrm{Grp}(\mathcal{S}) \to \mathcal{S}$ with the forgetful functor. Let $\omega' : \mathcal{C}^{\otimes} \to \mathcal{T}(R)^{\otimes}$ be another symmetric monodal functor. Let $c : \mathrm{CAlg}_R \to \mathcal{S}$ be the constant functor whose value is the contractible space. Recall that the section s corresponds to a natural transformation $c \to \xi$ which we denote by t_ω. Similarly, we obtain from ω' a natural transoformation $t_{\omega'} : c \to \xi$. Then we have the fibre product $c \times_\xi c$ of $c \xrightarrow{t_\omega} \xi \xleftarrow{t_{\omega'}} c : \mathrm{CAlg}_R \to \mathcal{S}$. We refer to this fibre product as the Isom functor from ω to ω' and denote it by $\mathrm{Isom}(\omega, \omega')$.

Let $\Omega : \mathcal{C} \to \mathcal{T}(R)$ be the underlying functor of ω. Let $\theta_{\mathcal{C}} : \mathrm{Cat}_\infty \to \mathcal{S}$ be the functor corresponding to $\mathcal{C}$ via the Yoneda embedding $\mathrm{Cat}_\infty^{op} \subset \mathrm{Fun}(\mathrm{Cat}_\infty, \mathcal{S})$. Consider the composite

$$\eta : \mathrm{CAlg}_R \xrightarrow{\mathcal{T}} \mathrm{Cat}_\infty \xrightarrow{\theta_{\mathcal{C}}} \mathcal{S}.$$

As in the above case, we can extend η to $\eta_* : \mathrm{CAlg}_R \to \mathcal{S}_*$ by $\Omega : \mathcal{C} \to \mathcal{T}(R)$. We define $\mathrm{Aut}(\Omega)$ to be the composite

$$\mathrm{CAlg}_R \xrightarrow{\eta_*} \mathcal{S}_* \xrightarrow{\Omega_*} \mathrm{Grp}(\mathcal{S}).$$

We refer to $\mathrm{Aut}(\Omega)$ as the automorphism group functor of $\Omega : \mathcal{C} \to \mathcal{T}(R)$. We often abuse notation and write $\mathrm{Aut}(\Omega)$ also for the composite $\mathrm{CAlg}_R \to \mathrm{Grp}(\mathcal{S}) \to \mathcal{S}$. If Ω' is the underlying functor of ω', we also have the Isom-functor $\mathrm{Isom}(\Omega, \Omega') : \mathrm{CAlg}_R \to \mathcal{S}$ from Ω to Ω' in a similar way.

https://doi.org/10.1017/is014008019jkt278 Published online by Cambridge University Press

4. Tannakization

The goal of this Section is to prove Theorem 4.11 and Theorem 4.15.

We first need to prove Lemmata concerning the structure of the ∞-category Cat_∞.

Lemma 4.1 *Let $\mathcal{C}$ and $\mathcal{D}$ be ∞-categories. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor. Then F is a categorical equivalence if and only if the composition induces equivalences*

$$f : \mathrm{Map}(\Delta^0, \mathcal{C}) \to \mathrm{Map}(\Delta^0, \mathcal{D}) \quad \text{and} \quad g : \mathrm{Map}(\Delta^1, \mathcal{C}) \to \mathrm{Map}(\Delta^1, \mathcal{D})$$

in $\mathcal{S}$.

Proof: The part of "only if" is clear. We will prove the "if" part. Let $\mathcal{C}^{\simeq}$ and $\mathcal{D}^{\simeq}$ be the largest Kan complexes in $\mathcal{C}$ and $\mathcal{D}$ respectively. The equivalence of f implies that the induced map $F^{\simeq} : \mathcal{C}^{\simeq} \to \mathcal{D}^{\simeq}$ is a homotopy equivalence (or equivalently, categorical equivalence). It follows that F is essentially surjective. Hence it suffices to show that F is fully faithful. Let C and C' be objects in $\mathcal{C}$. There exists a natural equivalence

$$\mathrm{Map}_{\mathcal{C}}(C, C') \simeq \mathrm{Map}(\Delta^1, \mathcal{C}) \times_{\mathrm{Map}(\partial\Delta^1, \mathcal{C})} \{(C, C')\}$$

in $\mathcal{S}$, where $\{(C, C')\} = \Delta^0 \to \mathrm{Map}(\partial\Delta^1, \mathcal{C})$ corresponds to C and C'. The induced map $\mathrm{Map}_{\mathcal{C}}(C, C') \to \mathrm{Map}_{\mathcal{D}}(F(C), F(C'))$ can be identified with

$$\mathrm{Map}(\Delta^1, \mathcal{C}) \times_{\mathrm{Map}(\partial\Delta^1, \mathcal{C})} \{(C, C')\} \to \mathrm{Map}(\Delta^1, \mathcal{D}) \times_{\mathrm{Map}(\partial\Delta^1, \mathcal{D})} \{(F(C), F(C'))\},$$

which is an equivalence in $\mathcal{S}$ by our assumption. □

We will construct the full subcategory $\langle \Delta^0, \Delta^1 \rangle$ of Cat_∞ by the following inductive steps. We first note that Cat_∞ is a presentable ∞-category since it is (equivalent to) the simplicial nerve of the simplicial category consisting of fibrant objects in the combinatorial model category of small marked simplicial sets, defined in [36, 3.1.3.7]. Choose a regular cardinal κ such that Cat_∞ is κ-accessible (cf. [36, 5.4.2]) and both Δ^0 and Δ^1 are κ-compact. Let $[\Delta^0, \Delta^1]_0$ be the full subcategory of Cat_∞, spanned by Δ^0 and Δ^1. We define a transfinite sequence

$$[\Delta^0, \Delta^1]_0 \to [\Delta^0, \Delta^1]_1 \to \cdots$$

of full subcategories indexed by ordinals smaller than κ. Supposing that $[\Delta^0, \Delta^1]_\alpha$ has been defined, we define $[\Delta^0, \Delta^1]_{\alpha+1}$ to be the full subcategory of Cat_∞ spanned by retracts of colimits of κ-small diagrams taking values in $[\Delta^0, \Delta^1]_\alpha$. Here colimits are taken in Cat_∞. If λ is a limit ordinal, $[\Delta^0, \Delta^1]_\lambda$ is defined to be $\bigcup_{\alpha<\lambda} [\Delta^0, \Delta^1]_\alpha$. We set $\langle \Delta^0, \Delta^1 \rangle = \bigcup_{\alpha<\kappa} [\Delta^0, \Delta^1]_\alpha$.

Lemma 4.2 *The full subcategory $\langle \Delta^0, \Delta^1 \rangle$ has κ-small colimits which are compatible with those in Cat_∞. Moreover, it is idempotent complete.*

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Proof: Let $f : I \to \langle \Delta^0, \Delta^1 \rangle$ be a functor where I is a κ-small simplicial set. We will show that the colimit of f in Cat_∞ belongs to $\langle \Delta^0, \Delta^1 \rangle$. Since I is κ-small, we have an ordinal τ smaller than κ, such that f factors through $[\Delta^0, \Delta^1]_\tau \subset \langle \Delta^0, \Delta^1 \rangle$. Then by our construction, the colimit in Cat_∞ belongs to $[\Delta^0, \Delta^1]_{\tau+1}$. Since Cat_∞ is idempotent complete and $\langle \Delta^0, \Delta^1 \rangle$ is closed under retracts, $\langle \Delta^0, \Delta^1 \rangle$ is idempotent complete. □

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Lemma 4.3 *The full subcategory $\langle \Delta^0, \Delta^1 \rangle$ is the smallest full subcategory having the properties:*

- *it includes Δ^0 and Δ^1,*
- *it has κ-small colimits, and the inclusion $\langle \Delta^0, \Delta^1 \rangle \to \mathrm{Cat}_\infty$ preserves κ-small colimits,*
- *it is idempotent complete.*

Moreover, the full subcategory $\langle \Delta^0, \Delta^1 \rangle$ is small.

Proof: By Lemma 4.2, it will suffice to prove that for each α, $[\Delta^0, \Delta^1]_\alpha$ is contained in the smallest full subcategory. We proceed by transfinite induction. The case of $\alpha = 0$ is obvious. Suppose that $[\Delta^0, \Delta^1]_\beta$ is contained in the smallest full subcategory where $\beta < \alpha$. Then by our construction in both successor and limit cases, $[\Delta^0, \Delta^1]_\alpha$ is so. To see the second claim, note that the (small) full subcategory consisting of κ-compact objects in Cat_∞ is idempotent complete and admits κ-small colimits which are compatible with those in Cat_∞. Thus the first claim implies that it contains $\langle \Delta^0, \Delta^1 \rangle$. It follows that $\langle \Delta^0, \Delta^1 \rangle$ is small. □

Let $\mathrm{Ind}_\kappa(\langle \Delta^0, \Delta^1 \rangle)$ be the full subcategory of $\mathrm{Fun}(\langle \Delta^0, \Delta^1 \rangle^{op}, \mathcal{S})$ spanned by colimits of κ-filtered diagrams taking values in $\langle \Delta^0, \Delta^1 \rangle \subset \mathrm{Fun}(\langle \Delta^0, \Delta^1 \rangle^{op}, \mathcal{S})$ (see [36, 5.3.5]). According to Lemma 4.2 and [36, 5.5.1.1], $\mathrm{Ind}_\kappa(\langle \Delta^0, \Delta^1 \rangle)$ is a presentable ∞-category.

Corollary 4.4 *The full subcategory $\langle \Delta^0, \Delta^1 \rangle$ coincides with the full subcategory $\mathcal{E}$ consisting of κ-compact objects in* $\mathrm{Ind}_\kappa(\langle \Delta^0, \Delta^1 \rangle)$.

Proof: Since $\langle \Delta^0, \Delta^1 \rangle$ is idempotent complete by Lemma 4.2, our assertion follows from [36, 5.4.2.4] which says that the natural inclusion $\langle \Delta^0, \Delta^1 \rangle \to \mathcal{E}$ is idempotent completion. □

Proposition 4.5 *Let $\theta : \mathrm{Ind}_\kappa(\langle \Delta^0, \Delta^1 \rangle) \to \mathrm{Cat}_\infty$ be a left Kan extension of $\langle \Delta^0, \Delta^1 \rangle \to \mathrm{Cat}_\infty$ that preserves κ-filtered colimits (cf. [36, 5.3.5.10]). Then θ is a categorical equivalence.*

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Remark 4.6 As a result of Proposition 4.5, Cat_∞ is "generated" by Δ^0 and Δ^1, that is, Cat_∞ is the smallest full subcategory of Cat_∞ which contains Δ^0 and Δ^1 and is closed under small colimits.

Proof: Note that according to Lemma 4.2 $\langle \Delta^0, \Delta^1 \rangle \to \mathrm{Cat}_\infty$ preserves κ-small colimits, and by Corollary 4.4 $\langle \Delta^0, \Delta^1 \rangle$ coincides with the full subcategory of κ-compact objects in $\mathrm{Ind}_\kappa(\langle \Delta^0, \Delta^1 \rangle)$. Therefore by [36, 5.5.1.9] θ preserves small colimits. Note that every object in $\langle \Delta^0, \Delta^1 \rangle$ is κ-compact in Cat_∞ (see the proof of Lemma 4.3). Therefore invoking [36, 5.3.5.11 (1)] we deduce that θ is fully faithful. By adjoint functor theorem [36, 5.5.2.9 (1)] to θ, there exists its right adjoint $\xi : \mathrm{Cat}_\infty \to \mathrm{Ind}_\kappa(\langle \Delta^0, \Delta^1 \rangle)$. Let $\mathcal{C}$ be a (small) ∞-category. To prove our assertion, it suffices to show that the counit map $\theta \circ \xi(\mathcal{C}) \to \mathcal{C}$ is a categorical equivalence. Now it can be checked by Lemma 4.1. □

Now we show that the example presented in Example 3.2 is a geometric R-system.

Lemma 4.7 *Let $M, N \in \mathrm{PMod}_R$. The functor $\mathrm{Hom}(M,N) : \mathrm{CAlg}_R \to \mathcal{S}$ is representable by a derived affine scheme over R. Moreover, $\mathrm{Isom}(M,N)$ is representable by a derived affine scheme over R. Namely, the example in Example 3.2 is a geometric R-system.*

Proof: Note that there exist natural equivalences

$$\begin{aligned}\mathrm{Map}_{\mathrm{Mod}_A}(M \otimes_R A, N \otimes_R A) &\simeq \mathrm{Map}_{\mathrm{Mod}_R}(M, N \otimes_R A) \\ &\simeq \mathrm{Map}_{\mathrm{CAlg}_R}(\mathrm{Sym}^*(M \otimes_R N^\vee), A)\end{aligned}$$

in $\mathcal{S}$, where $N^\vee$ is the dual object of N in PMod_R, and $\mathrm{Sym}^*(M \otimes_R N^\vee)$ is a free commutative R-ring spectrum determined by $M \otimes_R N^\vee$. Consequently, we conclude that $\mathrm{Spec}\,\mathrm{Sym}^*(M \otimes_R N^\vee)$ represents the functor $\mathrm{Hom}(M,N)$.

Next consider $\mathrm{Isom}(M,N) : \mathrm{CAlg}_R \to \mathcal{S}$. Put $\mathrm{Spec}\,C = \mathrm{Hom}(M,N)$. There is a universal morphism $v : M \otimes_R C \to N \otimes_R C$ on $\mathrm{Hom}(M,N)$ (such that for any $a : \mathrm{Spec}\,A \to \mathrm{Hom}(M,N) = \mathrm{Spec}\,C$ corresponding to $f : M \otimes_R A \to N \otimes_R A$, the pullback $v \otimes_C A$ is equivalent to f). Let K be the cokernel (cone) of v. According to [52, II, 1.2.10.1], there exist C_K in CAlg_R and a morphism $C \to C_K$ such that composition induces a fully faithful functor

$$\mathrm{Map}_{\mathrm{CAlg}_R}(C_K, A) \to \mathrm{Map}_{\mathrm{CAlg}_R}(C, A)$$

whose essential image consists of those morphisms $C \to A$ such that the base change $v \otimes_C A : M \otimes_R A \to N \otimes_R A$ is an equivalence. The functor $\mathrm{Isom}(M,N)$ is represented by $\mathrm{Spec}\,C_K$. □

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Let $\Omega, \Omega' : \mathcal{C} \to \mathcal{T}(R)$ be two functors. For any $A \in \mathrm{CAlg}_R$, we let Ω_A (resp. Ω'_A) to be the composite $\mathcal{C} \to \mathcal{T}(R) \to \mathcal{T}(A)$ where the second functor is induced by $R \to A$ and the first functor is Ω (resp. Ω'). Consider the functor $\mathrm{Isom}(\Omega, \Omega') : \mathrm{CAlg}_R \to \mathcal{S}$ given by $A \mapsto \mathrm{Map}_{\mathrm{Map}(\mathcal{C}, \mathrm{PMod}_A)}(\Omega_A, \Omega'_A)$ (see the previous Section).

Lemma 4.8 *Suppose that $\mathcal{C}$ is equivalent to either Δ^0 or Δ^1. Then $\mathrm{Isom}(\Omega, \Omega')$ is representable by a derived affine scheme over R.*

Proof: We treat the case of $\mathrm{Aut}(\Omega)$, i.e., $\Omega = \Omega'$. We first treat the case of Δ^0. Let $M = \Omega(\{0\}) \in \mathcal{T}(R)$, where $\{0\}$ denotes the object in Δ^0. In this case, $\mathrm{Aut}(\Omega)$ is representable by a derived affine scheme $\mathrm{Aut}(M)$ over R since $\mathcal{T}^{\otimes}$ is a geometric R-system.

Next we consider the case of $\mathcal{C} = \Delta^1$. Let $M := \Omega(\{0\}) \in \mathcal{T}(R)$ and $N := \Omega(\{1\}) \in \mathcal{T}(R)$, where $\{0\}$ and $\{1\}$ denote objects in Δ^1. Then $\mathrm{Aut}(\Omega)$ is representable by the fiber product of derived affine schemes

$$\mathrm{Aut}(M) \times_{\mathrm{Hom}(M,N)} \mathrm{Aut}(N)$$

where we regard $\mathrm{Hom}(M, N)$ as a derived affine scheme by (A2) of the definition of geometric R-systems. Here $\mathrm{Aut}(M) \to \mathrm{Hom}(M, N)$ and $\mathrm{Aut}(N) \to \mathrm{Hom}(M, N)$ are defined by the compsoition with $M \to N$.

The case of $\mathrm{Isom}(\Omega, \Omega')$ is similar. We should replace $\mathrm{Aut}(M)$, $\mathrm{Aut}(N)$ and $\mathrm{Hom}(M, N)$ (that appear in the above argument) by $\mathrm{Isom}(M, M')$, $\mathrm{Isom}(N, N')$ and $\mathrm{Hom}(M, N')$.

This completes the proof. □

Using Proposition 4.5 we first treat the case where we do not take account into symmetric monoidal structures.

Proposition 4.9 *Let $\mathcal{C}$ be a small ∞-category. Then $\mathrm{Isom}(\Omega, \Omega')$ is representable by a derived affine scheme over R.*

Proof: Let $\mathcal{R}$ be the full subcategory of Cat_∞ that consists of those ∞-catgories $\mathcal{C}$ such that for any pair of functors $\Omega, \Omega' : \mathcal{C} \rightrightarrows \mathcal{T}(R)$, $\mathrm{Isom}(\Omega, \Omega')$ is representable by a derived affine scheme. Then CAlg_R is closed under small colimits, and thus $\mathcal{R}$ is closed under small colimits in Cat_∞. By Lemma 4.8, Δ^0 and Δ^1 belong to $\mathcal{R}$. According to Proposition 4.5, $\mathcal{R}$ coincides with Cat_∞. □

Corollary 4.10 *The functor $\Theta^S : \mathrm{CAlg}_R \to \mathrm{CAlg}(\mathrm{Cat}_\infty)$ in Example 3.3 is a geometric R-system.*

Proof: Replace $\mathcal{C}$ in the proof of Proposition 4.9 by the Kan complex S. Then the proof together with Lemma 4.7 implies (A1) and the first half of (A2). The proof of the latter half of (A2) is similar. □

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Theorem 4.11 *Let $\mathcal{T}^{\otimes} : \mathrm{CAlg}_R \to \mathrm{CAlg}(\mathrm{Cat}_\infty)$ be a geometric R-system. Let $\mathcal{C}^{\otimes}$ be a symmetric monoidal small ∞-category and let $\omega, \omega' : \mathcal{C}^{\otimes} \to \mathcal{T}^{\otimes}(R)$ be two symmetric monoidal functors. Then* $\mathrm{Aut}(\omega)$ *is representable by a derived affine group scheme over R. Moreover,* $\mathrm{Isom}(\omega, \omega')$ *is representable by a derived affine scheme over R.*

Proof: For ease of notation, we let $\Gamma = \mathrm{N}(\mathrm{Fin}_*)$. Note first that a symmetric monoidal ∞-category can be regarded as a commutative monoid object in Cat_∞ (see [37, 2.4.2]). Let $\mathcal{C}^{\otimes}$ and $\mathcal{T}^{\otimes}(A)$ be symmetric monoidal ∞-categories. Hence we regard them as commutative monoid objects $p : \Gamma \to \mathrm{Cat}_\infty$ and $q_A : \Gamma \to \mathrm{Cat}_\infty$ respectively. We remark that $p(\langle n \rangle_*) \simeq \mathcal{C}^{\times n}$ and $q_A(\langle n \rangle_*) \simeq \mathcal{T}(A)^{\times n}$. We let $r_A : \Gamma \times \Delta^1 \to \mathrm{Cat}_\infty$ be the map corresponding to the composite $\omega_A : \mathcal{C}^{\otimes} \to \mathcal{T}^{\otimes}(R) \to \mathcal{T}^{\otimes}(A)$. Then by using [36, 4.2.1.8] twice $\mathrm{Aut}(\omega)(A)$ can be identified with the Kan complex

$$\mathrm{Fun}(\Gamma \times \Delta^1 \times \Delta^1, \mathrm{Cat}_\infty) \times_{\mathrm{Fun}(\Gamma \times \partial(\Delta^1 \times \Delta^1), \mathrm{Cat}_\infty)} \{(c_p \sqcup c_{q_A}) \cup (r_A \sqcup r_A)\},$$

where $\{(c_p \sqcup c_{q_A}) \cup (r_A \sqcup r_A)\}$ denotes the union $(c_p \sqcup c_{q_A}) \cup (r_A \sqcup r_A) : \Gamma \times \partial(\Delta^1 \times \Delta^1) \to \mathrm{Cat}_\infty$ such that $c_p : \Gamma \times \Delta^1 \times \{0\} \overset{\mathrm{pr}_1}{\to} \Gamma \overset{p}{\to} \mathrm{Cat}_\infty$, $c_{q_A} : \Gamma \times \Delta^1 \times \{1\} \overset{\mathrm{pr}_1}{\to} \Gamma \overset{q_A}{\to} \mathrm{Cat}_\infty$, and $r_A \sqcup r_A : \Gamma \times \partial\Delta^1 \times \Delta^1 \to \mathrm{Cat}_\infty$. Thus $\mathrm{Aut}(\omega)$ is given by

$$A \mapsto \mathrm{Map}(\Gamma \times \Delta^1 \times \Delta^1, \mathrm{Cat}_\infty) \times_{\mathrm{Map}(\Gamma \times \partial(\Delta^1 \times \Delta^1), \mathrm{Cat}_\infty)} \{(c_p \sqcup c_{q_A}) \cup (r_A \sqcup r_A)\},$$

where the right hand side is the (homotopy) fiber product. Let I be a small ∞-category. Let $f : I \to \Gamma$ be a functor. Consider the composite

$$\mathrm{CAlg}_R \overset{\mathcal{T}^{\otimes}}{\longrightarrow} \mathrm{Fun}(\Gamma, \mathrm{Cat}_\infty) \to \mathrm{Fun}(I, \mathrm{Cat}_\infty) \to \mathcal{S}$$

where the first functor is $\mathcal{T}^{\otimes} : \mathrm{CAlg}_R \to \mathrm{CAlg}(\mathrm{Cat}_\infty) \subset \mathrm{Fun}(\Gamma, \mathrm{Cat}_\infty)$, and the second functor is induced by the composition with f, and the third functor is representable by $p \circ f$. By $f^* r_R := r_R \circ (f \times \mathrm{Id}_{\Delta^1}) : I \times \Delta^1 \to \mathrm{Cat}_\infty$, we can extends the above composite to $\mathrm{CAlg}_R \to \mathcal{S}_*$ as in the previous Section. Composing with $\mathcal{S}_* \overset{\Omega_*}{\to} \mathrm{Grp}(\mathcal{S}) \to \mathcal{S}$ we have $\mathrm{CAlg}_R \to \mathcal{S}$, which we shall denote by $\mathrm{Aut}(\omega)_f$. This functor sends A to the (homotopy) fiber product

$$\mathrm{Map}(I \times \Delta^1 \times \Delta^1, \mathrm{Cat}_\infty) \times_{\mathrm{Map}(I \times \partial(\Delta^1 \times \Delta^1), \mathrm{Cat}_\infty)} \{(f^* c_p \sqcup f^* c_{q_A}) \cup (f^* r_A \sqcup f^* r_A)\}.$$

We claim that if I is either Δ^0 or Δ^1 then $\mathrm{Aut}(\omega)_f$ is representable by a derived affine scheme over R. The case of $I \simeq \Delta^0$ is reduced to Proposition 4.9; suppose that the image $f(\Delta^0)$ corresponds to $\langle n \rangle_*$. Recall that $q_A(\langle n \rangle_*)$ is equivalent to

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the n-fold product $\mathcal{T}(A)^{\times n}$ as ∞-categories. In this case, $\mathrm{Aut}(\omega)_f$ is given by $\mathrm{Aut}(\omega_n) : \mathrm{CAlg}_R \to \mathcal{S}$,

$$A \mapsto \prod_{1 \leq i \leq n} \mathrm{Aut}(\mathrm{pr}_i \circ \omega_{n,A})$$

where $\omega_{n,A}$ is the functor $p(\langle n \rangle_*) \to q_A(\langle n \rangle_*)$ induced by ω, and $\mathrm{pr}_i : \mathcal{T}(A)^{\times n} \to \mathcal{T}(A)$ is the i-th projection. Hence thanks to Proposition 4.9, this functor $\mathrm{Aut}(\mathrm{pr}_i \circ \omega_{n,A})$ is representable by a derived affine scheme over R. It follows that $\mathrm{Aut}(\omega_n)$ is representable by a derived affine scheme over R. When $f : I \simeq \Delta^1$ and $I \to \Gamma$ corresponds to $\langle m \rangle_* \to \langle n \rangle_*$, $\mathrm{Aut}(\omega)_f$ is representable by

$$\mathrm{Aut}(\omega_m) \times_{\mathrm{Aut}(\omega_{m,n})} \mathrm{Aut}(\omega_n)$$

where $\omega_{m,n}$ is the functor $p(\langle m \rangle_*) \to q(\langle n \rangle_*)$ induced by ω and $\langle m \rangle_* \to \langle n \rangle_*$. Thus this case is again reduced to Proposition 4.9. Next we observe that if L is the full subcategory of $(\mathrm{Cat}_\infty)_{/\Gamma}$ spanned by those $f : I \to \Gamma$ such that $\mathrm{Aut}(\omega)_f$ is representable by a derived affine scheme over R, then L is closed under small colimits. Note that the cartesian product $\times : \mathrm{Cat}_\infty \times \mathrm{Cat}_\infty \to \mathrm{Cat}_\infty$ preserves small colimits separately in each variable. Thus it follows that L is closed under small colimits from the definition of $\mathrm{Aut}(\omega)_f$ and the fact that Aff_R admits small limits, (see also Corollary A.13, [36, 1.2.13.8]). Since $\Delta^0 \to \Gamma, \Delta^1 \to \Gamma \in L$ and L is closed under small colimits, by Remark 4.6 the identity map $\Gamma \to \Gamma$ belongs to R. Consequently, Theorem 4.11 follows since $\mathrm{Aut}(\omega) : \mathrm{CAlg}_R \to \mathrm{Grp}(\mathcal{S}) \to \mathcal{S}$ is representable by a derived affine scheme over R.

The case of $\mathrm{Isom}(\omega, \omega')$ is similar to that of $\mathrm{Aut}(\omega)$. □

Proposition 4.12 *Let $\omega : \mathcal{C}^\otimes \to \mathrm{PMod}_R^\otimes$ be a symmetric monoidal functor where $\mathcal{C}^\otimes$ is a symmetric monoidal small ∞-category. (Here the geometric R-system is given in Example 3.2.) Then the functor* $\mathrm{Aut}(\omega)$ *is representable by a derived affine group scheme G over R.*

Proof: It follows from Theorem 4.11 and Lemma 4.7. □

We remark that in the above Proposition it is not assumed that every object in $\mathcal{C}^\otimes$ is dualizable.

Let $\mathcal{C}^\otimes$ be a symmetric monoidal small ∞-category and let $\omega : \mathcal{C}^\otimes \to \mathrm{PMod}_R^\otimes$ be a symmetric monoidal functor. Let G be a derived affine group scheme over R. Let $\mathrm{PRep}_G^\otimes$ be the symmetric monoidal stable ∞-category of perfect representations of G (see A.6). Since $\mathrm{PRep}_G^\otimes$ coincides with $\mathrm{PMod}_{\mathrm{B}G}^\otimes$, thus we use $\mathrm{PRep}_G^\otimes$ and $\mathrm{PMod}_{\mathrm{B}G}^\otimes$ interchangeably. Suppose that ω is extended to a symmetric monoidal functor $\mathcal{C}^\otimes \to \mathrm{PRep}_G^\otimes$. Namely, the composite $\mathcal{C}^\otimes \to \mathrm{PRep}_G^\otimes \to \mathrm{PMod}_R^\otimes$ with the forgetful functor is equivalent to ω. Next our goal is Proposition 4.14 which relates such

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extensions with actions on ω. Let $\mathrm{N}(\Delta)^{op} \to \mathrm{Aff}_R \subset \mathrm{Fun}(\mathrm{CAlg}_R, \widehat{\mathcal{S}})$ be a functor corresponding to G and let $\mathsf{B}G$ be its colimit. Let $(\mathrm{Aff}_R)_{/\mathsf{B}G}$ be the full subcategory of $\mathrm{Fun}(\mathrm{CAlg}_R, \widehat{\mathcal{S}})_{/\mathsf{B}G}$ spanned by objects $X \to \mathsf{B}G$ such that X are affine schemes, that is, objects which belong to the essential image of Yoneda embedding $\mathrm{Aff}_R \hookrightarrow \mathrm{Fun}(\mathrm{CAlg}_R, \widehat{\mathcal{S}})$. There is the natural projection $(\mathrm{Aff}_R)_{/\mathsf{B}G} \to \mathrm{Aff}_R$, that is a right fibration (cf. [36, 2.0.0.3]). Let $\pi : \mathrm{Spec}\, R \to \mathsf{B}G$ be the natural projection. This determines a map between right fibrations $\mathrm{Aff}_R = (\mathrm{Aff}_R)_{/\mathrm{Spec}\, R} \to (\mathrm{Aff}_R)_{/\mathsf{B}G}$ over Aff_R. Let $(\mathrm{Aff}_R)_{/\mathsf{B}G} \to \mathcal{S}^{op}$ be a functor which assigns $\mathrm{Map}^{\otimes}(\mathcal{C}^{\otimes}, \mathrm{PMod}_A^{\otimes})$ to $\mathrm{Spec}\, A$ in $(\mathrm{Aff}_R)_{/\mathsf{B}G}$. Here $\mathrm{Map}^{\otimes}(-,-)$ indicates the mapping space in $\mathrm{CAlg}(\mathrm{Cat}_\infty)$. More precisely, let

$$c : (\mathrm{Aff}_R)_{/\mathsf{B}G} \to \mathrm{Aff}_R \xrightarrow{\theta} \mathrm{CAlg}(\mathrm{Cat}_\infty)^{op} \to \mathcal{S}^{op}$$

be the composition where the first functor is the natural projection, and the third is the image of $\mathcal{C}^{\otimes}$ by Yoneda embedding $(\mathrm{CAlg}(\mathrm{Cat}_\infty))^{op} \to \mathrm{Fun}(\mathrm{CAlg}(\mathrm{Cat}_\infty), \mathcal{S})$. Let $\theta : \mathrm{Aff}_R \to \mathrm{CAlg}(\mathrm{Cat}_\infty)^{op}$ be the functor induced by Θ, which carries $\mathrm{Spec}\, A$ to $\mathrm{PMod}_A^{\otimes}$. By the unstraightening functor [36, 3.2] together with [36, 4.2.4.4] the composition $(\mathrm{Aff}_R)_{/\mathsf{B}G} \to \mathcal{S}^{op}$ gives rise to a right fibration $p : \mathcal{M} \to (\mathrm{Aff}_R)_{/\mathsf{B}G}$. The mapping space $\mathrm{Map}^{\otimes}(\mathcal{C}^{\otimes}, \mathrm{PRep}_G^{\otimes})$ is homotopy equivalent to the limit of spaces

$$\lim_{\mathrm{Spec}\, A \to \mathsf{B}G} \mathrm{Map}^{\otimes}(\mathcal{C}^{\otimes}, \theta(\mathrm{Spec}\, A))$$

where $\mathrm{Spec}\, A \to \mathsf{B}G$ run over $(\mathrm{Aff}_R)_{/\mathsf{B}G}$ and $\mathrm{PMod}_{\mathsf{B}G}^{\otimes} \simeq \lim_{\mathrm{Spec}\, A \to \mathsf{B}G} \theta(\mathrm{Spec}\, A)$ (see A.6 for $\mathrm{PMod}_{\mathsf{B}G}^{\otimes}$).

Lemma 4.13 *If we denote by* $\mathrm{Map}_{(\mathrm{Aff}_R)_{/\mathsf{B}G}}((\mathrm{Aff}_R)_{/\mathsf{B}G}, \mathcal{M})$ *the simplicial set of the sections of* $p : \mathcal{M} \to (\mathrm{Aff}_R)_{/\mathsf{B}G}$ *(i.e., the set of* n*-simplexes of* $\mathrm{Map}_{(\mathrm{Aff}_R)_{/\mathsf{B}G}}((\mathrm{Aff}_R)_{/\mathsf{B}G}, \mathcal{M})$ *is the set of* $(\mathrm{Aff}_R)_{/\mathsf{B}G} \times \Delta^n \to \mathcal{M}$ *over* $(\mathrm{Aff}_R)_{/\mathsf{B}G}$*), then there is a categorical equivalence*

$$\mathrm{Map}^{\otimes}(\mathcal{C}^{\otimes}, \mathrm{PMod}_{\mathsf{B}G}^{\otimes}) \simeq \mathrm{Fun}_{(\mathrm{Aff}_R)_{/\mathsf{B}G}}((\mathrm{Aff}_R)_{/\mathsf{B}G}, \mathcal{M}).$$

Proof: It follows from [36, 3.3.3.2]. □

Proposition 4.14 *Let* $\mathcal{C}^{\otimes}$ *be a symmetric monoidal small* ∞*-category. Then there is a natural equivalence*

$$\mathrm{Map}_{\mathrm{CAlg}(\mathrm{Cat}_\infty)_{/\mathrm{PMod}_R^{\otimes}}}(\mathcal{C}^{\otimes}, \mathrm{PRep}_G^{\otimes}) \simeq \mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}_R, \mathrm{Grp}(\mathcal{S}))}(G, \mathrm{Aut}(\omega))$$

in $\mathcal{S}$*. This equivalence is functorial in the following sense: Let* $L : \mathrm{GAff}_R \to \mathcal{S}^{op}$ *be the functor which assigns* G *to* $\mathrm{Map}_{\mathrm{CAlg}(\mathrm{Cat}_\infty)_{/\mathrm{PMod}_R^{\otimes}}}(\mathcal{C}^{\otimes}, \mathrm{PRep}_G^{\otimes})$*. Let* M :

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$\mathrm{GAff}_R \to \mathcal{S}^{op}$ *be the functor which assigns* G *to* $\mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}_R,\mathrm{Grp}(\mathcal{S}))}(G,\mathrm{Aut}(F))$. *(See the proof below for the formulations of* L *and* M*.) Then there exists a natural equivalence from* L *to* M.

Proof: In order to make our proof readable we first show the first assertion without defining L and M. The mapping space $\mathrm{Map}_{\mathrm{CAlg}(\mathrm{Cat}_\infty)_{/\mathrm{PMod}_R^\otimes}}(\mathcal{C}^\otimes,\mathrm{PMod}_{\mathsf{B}G}^\otimes)$ is the homotopy limit (i.e. the limit in $\mathcal{S}$)

$$\mathrm{Map}^\otimes(\mathcal{C}^\otimes,\mathrm{PMod}_{\mathsf{B}G}^\otimes) \times_{\mathrm{Map}^\otimes(\mathcal{C}^\otimes,\mathrm{PMod}_R^\otimes)} \{\omega\}$$

where $\{\omega\} = \Delta^0 \to \mathrm{Map}^\otimes(\mathcal{C}^\otimes,\mathrm{PMod}_R^\otimes)$ is determined by ω. We adopt the notation in the proof of Proposition 4.12; $(\mathrm{Aff}_R)_{/\mathsf{B}G}$ is the fiber product of simplicial sets $\mathrm{Fun}(\mathrm{CAlg}_R,\mathcal{S})_{/\mathsf{B}G} \times_{\mathrm{Fun}(\mathrm{CAlg}_R,\mathcal{S})} \mathrm{Aff}_R$, and $(\mathrm{Aff}_R)_{/\mathsf{B}G} \to \mathrm{Aff}_R$ is the (forgetful) right fibration. Note that the ∞-category obtained from the model category of (large) marked simplicial sets over Aff_R equipped with the Cartesian model structure [36, 3.1.3.7] by inverting weak equivalences is equivalent to $\mathrm{Fun}(\mathrm{CAlg}_R,\widehat{\mathrm{Cat}}_\infty)$. Therefore, the morphism $\mathrm{Spec}\,R \to \mathsf{B}G$ in $\mathrm{Fun}(\mathrm{CAlg}_R,\mathcal{S})$ gives rise to a map of Cartesian fibrations over Aff_R:

$$\begin{array}{ccc} \mathrm{Aff}_R & \longrightarrow & (\mathrm{Aff}_R)_{/\mathsf{B}G} \\ & {\scriptstyle \mathrm{id}} \searrow \quad \swarrow & \\ & \mathrm{Aff}_R. & \end{array}$$

(Keep in mind that the right fibration $(\mathrm{Aff}_R)_{/\mathsf{B}G} \to \mathrm{Aff}_R$ is a fibrant object in the model category of marked simplicial sets over Aff_R.) The fiber product of Kan complexes

$$P = \mathrm{Map}_{(\mathrm{Aff}_R)_{/\mathsf{B}G}}((\mathrm{Aff}_R)_{/\mathsf{B}G},\mathcal{M}) \times_{\mathrm{Map}_{(\mathrm{Aff}_R)_{/\mathsf{B}G}}(\mathrm{Aff}_R,\mathcal{M})} \{\omega\}$$

is a homotopy limit since the section $\mathrm{Aff}_R \to (\mathrm{Aff}_R)_{/\mathsf{B}G}$ is a monomorphism (that is, a cofibration in the Cartesian simplicial model category of marked simplicial sets $(\widehat{\mathrm{Set}}_\Delta^+)_{/(\mathrm{Aff}_R)_{/\mathsf{B}G}}$). Therefore, the induced map is a Kan fibration. Here $\Delta^0 = \{\omega\} \to \mathrm{Map}_{(\mathrm{Aff}_R)_{/\mathsf{B}G}}(\mathrm{Aff}_R,\mathcal{M})$ is determined by ω. Let $\mathcal{N} := \mathcal{M} \times_{(\mathrm{Aff}_R)_{/\mathsf{B}G}} \mathrm{Aff}_R$ where $\mathrm{Aff}_R \to (\mathrm{Aff}_R)_{/\mathsf{B}G}$ is determined by the natural map $\mathrm{Spec}\,R \to \mathsf{B}G$. Observe that there is a natural Cartesian equivalence $\mathcal{N} \times_{\mathrm{Aff}_R} (\mathrm{Aff}_R)_{/\mathsf{B}G} \simeq \mathcal{M}$ over $(\mathrm{Aff}_R)_{/\mathsf{B}G}$. Indeed, the right fibration $\mathcal{M} \to (\mathrm{Aff}_R)_{/\mathsf{B}G}$ corresponds to $h : (\mathrm{Aff}_R)_{/\mathsf{B}G} \to \mathcal{S}^{op}$ informally given by $\mathrm{Spec}\,A \mapsto \mathrm{Map}^\otimes(\mathcal{C}^\otimes,\mathrm{PMod}_A^\otimes)$. In particular, h factors through the forgetful functor $(\mathrm{Aff}_R)_{/\mathsf{B}G} \to \mathrm{Aff}_R$. Thus the composition

$$(\mathrm{Aff}_R)_{/\mathsf{B}G} \to \mathrm{Aff}_R \to (\mathrm{Aff}_R)_{/\mathsf{B}G} \to \mathcal{S}^{op}$$

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with the section $\mathrm{Aff}_R \to (\mathrm{Aff}_R)_{/\mathsf{B}G}$ and the forgetful functor $(\mathrm{Aff}_R)_{/\mathsf{B}G} \to \mathrm{Aff}_R$ is equivalent to h. We then have homotopy equivalences

$$\mathrm{Map}_{(\mathrm{Aff}_R)_{/\mathsf{B}G}}((\mathrm{Aff}_R)_{/\mathsf{B}G}, \mathcal{M}) \simeq \mathrm{Map}_{\mathrm{Aff}_R}((\mathrm{Aff}_R)_{/\mathsf{B}G}, \mathcal{N})$$

and

$$\mathrm{Map}_{(\mathrm{Aff}_R)_{/\mathsf{B}G}}(\mathrm{Aff}_R, \mathcal{M}) \simeq \mathrm{Map}_{\mathrm{Aff}_R}(\mathrm{Aff}_R, \mathcal{N}).$$

Thus P is homotopy equivalent to the fiber product

$$Q = \mathrm{Map}_{\mathrm{Aff}_R}((\mathrm{Aff}_R)_{/\mathsf{B}G}, \mathcal{N}) \times_{\mathrm{Map}_{\mathrm{Aff}_R}(\mathrm{Aff}_R, \mathcal{N})} \{\omega\}$$

which is also a homotopy limit, where $\Delta^0 = \{\omega\} \to \mathrm{Map}_{\mathrm{Aff}_R}(\mathrm{Aff}_R, \mathcal{N})$ is determined by the section $\mathrm{Aff}_R \to \mathcal{N}$ corresponding to $\omega : \mathcal{C}^{\otimes} \to \mathrm{PMod}_R^{\otimes}$. We let $\alpha_{\mathsf{B}G} : \mathrm{CAlg}_R \to \mathcal{S}$ be a map correspoindig to the right fibration $(\mathrm{Aff}_R)_{/\mathsf{B}G} \to \mathrm{Aff}_R$ via the straightening functor. Let $\alpha_* : \mathrm{CAlg}_R \to \mathcal{S}$ be a map correspoindig to the right fibration $\mathrm{Aff}_R \to \mathrm{Aff}_R$ of the identity map. There is the natural transformation $\alpha_* \to \alpha_{\mathsf{B}G}$ determined by $\mathrm{Aff}_R \to (\mathrm{Aff}_R)_{/\mathsf{B}G}$, which we consider to be a functor $\mathrm{CAlg}_R \to \mathcal{S}_{*,\geq 1}$. Here $\mathcal{S}_{*,\geq 1}$ denotes the full subcategory of $\mathcal{S}_*$ spanned by pointed connected spaces. Let $\alpha_{\mathcal{N}} : \mathrm{CAlg}_R \to \mathcal{S}_*$ be a functor corresponding to the right fibration $\mathcal{N} \to \mathrm{Aff}_R$ equipped with the section $\mathrm{Aff}_R \to \mathcal{N}$. Observe that $\mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}, \mathcal{S}_*)}(\alpha_{\mathsf{B}G}, \alpha_{\mathcal{N}})$ is homotopy equivalent to Q. By composition with $\Omega_* : \mathcal{S}_* \to \mathrm{Grp}(\mathcal{S})$ we have $G : \mathrm{CAlg}_R \xrightarrow{\mathsf{B}G} \mathcal{S}_{*,\geq 1} \simeq \mathrm{Grp}(\mathcal{S})$ (that is, the composition is the original derived group scheme G). Let $\alpha'_{\mathcal{N}}$ be an object in $\mathrm{Fun}(\mathrm{CAlg}_R, \mathcal{S}_{*,\geq 1})$ such that $\alpha'_{\mathcal{N}}(A)$ is the pointed connected component determined by $\alpha_{\mathcal{N}}(A)$. Then we obtain

$$\begin{aligned} Q &\simeq \mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}_R, \mathcal{S}_*)}(\alpha_{\mathsf{B}G}, \alpha_{\mathcal{N}}) \\ &\simeq \mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}_R, \mathcal{S}_*)}(\alpha_{\mathsf{B}G}, \alpha'_{\mathcal{N}}) \\ &\simeq \mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}_R, \mathrm{Grp}(\mathcal{S}))}(G, \mathrm{Aut}(\omega)). \end{aligned}$$

Next to see (and formulate) the latter assertion, we will define L and M. We first define L. Since a derived affine group scheme is a group object in the Cartesian symmetric monoidal ∞-category of Aff_R, thus GAff_R is naturally embedded into $\mathrm{Fun}(\mathrm{N}(\Delta)^{op}, \mathrm{Fun}(\mathrm{CAlg}_R, \mathcal{S}))$ as a full subcategory. Let $\mathrm{Fun}(\mathrm{N}(\Delta)^{op}, \mathrm{Fun}(\mathrm{CAlg}_R, \mathcal{S})) \to \mathrm{Fun}(\mathrm{CAlg}_R, \mathcal{S})$ be the functor taking each simplicial object $\mathrm{N}(\Delta)^{op} \to \mathrm{Fun}(\mathrm{CAlg}_R, \widehat{\mathcal{S}})$ to its colimit. Let $\rho : \mathrm{GAff}_R \to \mathrm{Fun}(\mathrm{CAlg}_R, \mathcal{S})$ be the composition. Note that G maps to $\mathsf{B}G$. By the straightening and unstraightening functors [36, 3.2] together with [36, 4.2.4.4], we have the categorical equivalence $\mathrm{Fun}(\mathrm{CAlg}_R, \widehat{\mathrm{Cat}}_\infty) \simeq \mathrm{N}((\widehat{\mathrm{Set}}^+_\Delta)_{/\mathrm{Aff}_R})^{cf})$ where $(\widehat{\mathrm{Set}}^+_\Delta)_{/\mathrm{Aff}_R}$

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is the category of (not necessarily small) marked simplicial sets, which is endowed with the Cartesian model structure in [36, 3.1.3.7] and $(-)^{cf}$ indicates full simplicial subcategory of cofibrant-fibrant objects. In particular, there is the fully faithful functor $\mathrm{Fun}(\mathrm{CAlg}_R, \widehat{\mathcal{S}}) \to \mathrm{N}((\widehat{\mathrm{Set}}^+_\Delta)_{/\mathrm{Aff}_R})^{cf})$ which carries $\mathsf{B}G$ to $(\mathrm{Aff}_R)_{/\mathsf{B}G} \to \mathrm{Aff}_R$. Composing all these functors we have the composition

$$\mathrm{GAff}_R \xrightarrow{\rho} \mathrm{Fun}(\mathrm{CAlg}_R, \mathcal{S}) \to \mathrm{N}((\widehat{\mathrm{Set}}^+_\Delta)_{/\mathrm{Aff}_R})^{cf}).$$

Since $\mathrm{GAff}_R \simeq (\mathrm{GAff}_R)_{\mathrm{Spec}\, R/}$, the composition is extended to $u : \mathrm{GAff}_R \to \mathrm{N}((\widehat{\mathrm{Set}}^+_\Delta)_{/\mathrm{Aff}_R})^{cf})_{\mathrm{Aff}_R/}$. Through Yoneda embedding

$$\mathrm{N}((\widehat{\mathrm{Set}}^+_\Delta)_{/\mathrm{Aff}_R})^{cf})_{\mathrm{Aff}_R/} \to \mathrm{Fun}((\mathrm{N}((\widehat{\mathrm{Set}}^+_\Delta)_{/\mathrm{Aff}_R})^{cf})_{\mathrm{Aff}_R/})^{op}, \widehat{\mathcal{S}})$$

we define $I : (\mathrm{N}((\widehat{\mathrm{Set}}^+_\Delta)_{/\mathrm{Aff}_R})^{cf})_{\mathrm{Aff}_R/})^{op} \to \widehat{\mathcal{S}}$ to be the functor corresponding to $\mathcal{N} \to \mathrm{Aff}_R$ equipped with the section ω. Composing I^{op} with $\mathrm{GAff}_R \to \mathrm{N}((\widehat{\mathrm{Set}}^+_\Delta)_{/\mathrm{Aff}_R})^{cf})_{\mathrm{Aff}_R/}$ we define L to be $\mathrm{GAff}_R \to \widehat{\mathcal{S}}^{op}$. To define M, consider the functor $\mathrm{Fun}(\mathrm{CAlg}_R, \mathrm{Grp}(\mathcal{S})) \to \widehat{\mathcal{S}}^{op}$ determined by $\mathrm{Aut}(\omega)$ via Yoneda embedding. Then we define M to be the composition

$$\mathrm{GAff}_R \hookrightarrow \mathrm{Fun}(\mathrm{CAlg}_R, \mathrm{Grp}(\mathcal{S})) \to \widehat{\mathcal{S}}^{op}.$$

To obtain $L \simeq M$, note that the unstraightening functor induces a fully faithful functor

$$\mathrm{Fun}(\mathrm{CAlg}_R, \widehat{\mathcal{S}}_*) \hookrightarrow \mathrm{N}((\widehat{\mathrm{Set}}^+_\Delta)_{/\mathrm{Aff}_R})^{cf})_{\mathrm{Aff}_R/}.$$

Let $\mathsf{N} : \mathrm{CAlg}_R \to \mathcal{S}_*$ be a functor corresponding to $\mathcal{N} \to \mathrm{Aff}_R$ equipped with the section ω, that is, N corresponds to $\alpha_* \to \alpha_{\mathcal{N}}$. Let $\mathrm{Fun}(\mathrm{CAlg}_R, \mathcal{S}_*) \to \widehat{\mathcal{S}}^{op}$ be the functor determined by N via Yoneda embedding. The functor L is equivalent to

$$\mathrm{GAff}_R \xrightarrow{u} \mathrm{Fun}(\mathrm{CAlg}_R, \widehat{\mathcal{S}}_*) \subset \mathrm{N}((\widehat{\mathrm{Set}}^+_\Delta)_{/\mathrm{Aff}_R})^{cf})_{\mathrm{Aff}_R/} \to \widehat{\mathcal{S}}^{op}.$$

Since the essential image of GAff_R in $\mathrm{Fun}(\mathrm{CAlg}_R, \mathcal{S}_*)$ is contained in $\mathrm{Fun}(\mathrm{CAlg}_R, \mathcal{S}_{*,\geq 1})$, for our purpose we may and will replace $\alpha_{\mathcal{N}}$ by $\alpha'_{\mathcal{N}}$ (in the construction of N) and assume that N belongs to $\mathrm{Fun}(\mathrm{CAlg}_R, \mathcal{S}_{*,\geq 1})$. Then we see that L is equivalent to

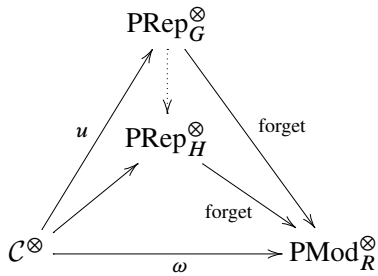
$$\mathrm{GAff}_R \to \mathrm{Fun}(\mathrm{CAlg}_R, \mathcal{S}_{*,\geq 1}) \simeq \mathrm{Fun}(\mathrm{CAlg}_R, \mathrm{Grp}(\mathcal{S})) \to \widehat{\mathcal{S}}^{op}$$

where the first functor is induced by u and the third functor is determined by $\mathrm{Aut}(\omega)$ via Yoneda embedding. Now the last composition is equivalent to M. □

Now we are ready to prove the following:

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Theorem 4.15 *Let $\mathcal{C}^{\otimes}$ be a symmetric monoidal small ∞-category. Let* $\mathrm{PMod}_R^{\otimes}$ *be the symmetric monoidal (small) ∞-category of perfect (dualizable) R-module spectra where R is a commutative ring spectrum. Let $\omega : \mathcal{C}^{\otimes} \to \mathrm{PMod}_R^{\otimes}$ be a symmetric monoidal functor. There are a derived affine group scheme G over R that represents the automorphism group functor* $\mathrm{Aut}(\omega)$*, and a symmetric monoidal functor $u : \mathcal{C}^{\otimes} \to \mathrm{PRep}_G^{\otimes}$ which makes the outer triangle in*



commute in $\mathrm{CAlg}(\mathrm{Cat}_\infty)$ *such that these possess the following universality: for any inner triangle consisting of solid arrows in the above diagram where H is a derived affine group scheme over R, there exists a morphism $f : H \to G$ of derived affine group schemes which induces $\mathrm{PRep}_G^{\otimes} \to \mathrm{PRep}_H^{\otimes}$ (indicated by the dotted arrow) filling the above diagram. Such f is unique up to a contractible space of choices.*

Proof: Take a derived affine group scheme G over R which represents $\mathrm{Aut}(\omega)$ by Proposition 4.12. By Proposition 4.14, we have a symmetric monoidal functor $\mathcal{C}^{\otimes} \to \mathrm{PRep}_G^{\otimes}$ that corresponds to the identity $G \simeq \mathrm{Aut}(\omega) \to \mathrm{Aut}(\omega)$. Then Proposition 4.14 implies our claim. □

We usually refer to $(G, \mathcal{C}^{\otimes} \xrightarrow{u} \mathrm{PMod}_G^{\otimes})$ (or simply G) in Theorem 4.15 as the tannakization of $\omega : \mathcal{C}^{\otimes} \to \mathrm{PMod}_R^{\otimes}$.

The following properties are easy but useful.

Proposition 4.16 *Let $\{\mathcal{C}_i^{\otimes}\}_{i \in I}$ be a (small) collection of symmetric monoidal ful subcategories of $\mathcal{C}^{\otimes}$. Assume that for any finite subset $J \subset I$, there is some $i \in I$ such that $\bigcup_{j \in J} \mathcal{C}_i \subset \mathcal{C}_i$. Suppose further that $\bigcup_{i \in I} \mathcal{C}_i = \mathcal{C}$. Let $\omega_i : \mathcal{C}_i^{\otimes} \hookrightarrow \mathcal{C}^{\otimes} \xrightarrow{\omega}$* $\mathrm{PMod}_R^{\otimes}$ *be the composite and let G_i be the tannakization of the composite. Then if G denotes the tannakization of ω, then $G \simeq \lim_{i \in I} G_i$.*

Proof: The collection $\{\mathcal{C}_i^{\otimes}\}_{i \in I}$ constitutes a filtered partially ordered set ordered by inclusions. As a consequence, according to [37, 3.2.3.2], the condition $\bigcup_{i \in I} \mathcal{C}_i = \mathcal{C}$ implies that $\mathcal{C}^{\otimes}$ is a colimit of $\{\mathcal{C}_i^{\otimes}\}_{i \in I}$ in $\mathrm{CAlg}(\mathrm{Cat}_\infty)$. It implies our claim (by noting the limit of derived affine schemes commutes with the limit as functors $\mathrm{CAlg}_R \to \mathcal{S}$). □

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Proposition 4.17 *We adopt the notion of the previous Proposition. Let $R \to R'$ be a morphism in* CAlg. *Then the tannakization of the composite $\mathcal{C}^{\otimes} \xrightarrow{\omega} \mathrm{PMod}_R^{\otimes} \xrightarrow{\otimes_R R'} \mathrm{PMod}_{R'}^{\otimes}$ is $G \times_{\mathrm{Spec}\, R} \mathrm{Spec}\, R'$.*

5. Motivic Galois groups

In this Section we will construct derived motivic Galois groups of mixed motives, underived motivic Galois groups, and variants and establish a number of fundamental properties (cf. Theorem 5.14, 5.15, 5.17, Proposition 5.19). The term "derived" in derived motivic Galois group stems from the "highly structured" category: stable ∞-category of mixed motives (see Remark 5.18). We first need to construct the realization functor of a mixed Weil cohomology theory in the ∞-categorical setting.

5.1. ∞-category of mixed motives

We construct the ∞-category of mixed motives. We first construct a stable ∞-category of motivic spectra. There are several approaches to construct it. Let S be a scheme separated and of finite type over a perfect field. Let Sm_S be the category of smooth scheme separated and of finite type over S. One can perform the construction of Morel and Voevodsky ([41], [54]) in the setting of ∞-categories. On the other hand, there are several model categories of motivic spectra (e.g., [28], [25], [17], [13]). Then the passage from model categories to ∞-categories allows us to have an ∞-category of motivic spectra. There are also DG-categorical formulations (for "DM"), see e.g. [5] [9] and we can pass to ∞-categories from them. In this paper we will adopt second approach. Especially, we use the model category of symmetric Tate spectra described in [13, 1.4.3], where Cisinski and Déglise introduced the theory of the mixed Weil theory which gives us the very powerful method for constructing realization functors.

Symmetric Tate spectra. We shall refer ourselves to [12] and [13] for the model category of symmetric Tate spectra. We here recall the minimal definitions for symmetric Tate spectra. Let R be an (ordinary) commutative ring and $\mathrm{Sh}(\mathrm{Sm}_S, R)$ the abelian category of Nisnevich sheaves of R-modules. Let $\mathrm{Comp}(\mathrm{Sh}(\mathrm{Sm}_S, R))$ be the category of complexes of objects in $\mathrm{Sh}(\mathrm{Sm}_S, R)$. This is a symmetric monoidal category. For the symmetric monoidal structure of complexes of objects in a symmetric monoidal abelian category, see e.g. [12, 3.1]. For any $X \in \mathrm{Sm}_S$, we write $R(X)$ for the Nisnevich sheaf associated to the presheaf given by $Y \mapsto \oplus_{f \in \mathrm{Hom}_{\mathrm{Sm}_S}(Y,X)} R \cdot f$ where $\oplus_{f \in \mathrm{Hom}_{\mathrm{Sm}_S}(Y,X)} R \cdot f$ is the free R-module generated by the set $\mathrm{Hom}_{\mathrm{Sm}_S}(Y, X)$. It gives rise to a functor $\mathrm{Sm}_S \to \mathrm{Comp}(\mathrm{Sh}(\mathrm{Sm}_S, R))$. Let $R(1)[1] \in \mathrm{Comp}(\mathrm{Sh}(\mathrm{Sm}_S, R))$ be the cokernel of the split monomorphism $R(S) \to$

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$R(\mathbb{G}_m)$ determined by the unit $S \to \mathbb{G}_m = \operatorname{Spec} S[t,t^{-1}]$ of the torus. A symmetric Tate sequence is a sequence $\{E_n\}_{n\in\mathbb{N}}$ where E_n is an object of $\mathrm{Comp}(\mathrm{Sh}(\mathrm{Sm}_S,R))$ which is equipped with an action by the symmteric group $\mathfrak{S}_n$ for each $n \in \mathbb{N}$. A morphism $\{E_n\}_{n\in\mathbb{N}} \to \{F_n\}_{n\in\mathbb{N}}$ is a collection of $\mathfrak{S}_n$-equivariant maps $E_n \to F_n$. Let $\mathrm{S}_{\mathrm{Tate}}(R)$ be the category of symmetric Tate sequences. Let $\mathfrak{S}'$ be the category of finite sets whose morphisms are bijections. Then the category of functors from $\mathfrak{S}'$ to $\mathrm{Comp}(\mathrm{Sh}(\mathrm{Sm}_S,R))$ is naturally equivalent to the category of symmetric Tate sequences (To $F : \mathfrak{S}' \to \mathrm{Comp}(\mathrm{Sh}(\mathrm{Sm}_S,R))$ we associate $\{E_n = F(\bar{n})\}_{n\in\mathbb{N}}$ if $\bar{n}$ is $\{1,\ldots,n\}$). For $E,F : \mathfrak{S}' \to \mathrm{Comp}(\mathrm{Sh}(\mathrm{Sm}_S,R))$, the tensor product is defined to be $\mathfrak{S}' \to \mathrm{Comp}(\mathrm{Sh}(\mathrm{Sm}_S,R))$ given by $N \mapsto \bigoplus_{N=P\sqcup Q} E(P)\otimes F(Q)$. It yields a symmetric monoidal structure on the category of symmetric Tate sequences. Let $\mathrm{Sym}(R(1))$ denote a symmetric Tate sequence $\{R(1)^{\otimes n}\}_{n\in\mathbb{N}}$ such that $\mathfrak{S}_n$ acts on $R(1)^{\otimes n}$ by permutation. The canonical isomorphism $R(1)^{\otimes n}\otimes R(1)^{\otimes m} \to R(1)^{\otimes n+m}$ is $\mathfrak{S}_n\times\mathfrak{S}_m$-equivariant when $\mathfrak{S}_n\times\mathfrak{S}_m$ acts on $R(1)^{\otimes n+m}$ through the natural inclusion $\mathfrak{S}_n\times\mathfrak{S}_m \to \mathfrak{S}_{n+m}$. Unwinding the definition of tensor product of symmetric Tate sequences we have a morphism

$$\mathrm{Sym}(R(1))\otimes\mathrm{Sym}(R(1)) \to \mathrm{Sym}(R(1))$$

which makes $\mathrm{Sym}(R(1))$ a commutative algebra object in $\mathrm{S}_{\mathrm{Tate}}$. Let $\mathrm{Sp}_{\mathrm{Tate}}(R)$ be the category of modules in $\mathrm{S}_{\mathrm{Tate}}(R)$ over the commutative algebra object $\mathrm{Sym}(R(1))$. We call an object in $\mathrm{Sp}_{\mathrm{Tate}}(R)$ a symmetric Tate spectrum. In [13, 1.4.2], the classes of stable $\mathbb{A}^1$-equivalences, stable $\mathbb{A}^1$-fibrations are defined (these are important, but we will not recall them here since we need preliminaries). In [13, 1.4.3] (see also [12]), the model category structure of $\mathrm{Sp}_{\mathrm{Tate}}(R)$ is constructed:

Proposition 5.1 *The category* $\mathrm{Sp}_{\mathrm{Tate}}(R)$ *is a stable proper cellular symmetric monoidal model category with stable* $\mathbb{A}^1$*-equivalences as weak equivalences, and stable* $\mathbb{A}^1$*-fibrations as fibrations.*

Remark 5.2 A pointed model category is stable if the suspention functor induces an equivalence of the homotopy category (cf. [24]).

Lemma 5.3 *The category* $\mathrm{Sp}_{\mathrm{Tate}}(R)$ *is presentable. In particular, it is a combinatorial model category.*

Proof: We first remark that our notion of presentable categories is equivalent to locally presentable categories in [1]. Observe that $\mathrm{S}_{\mathrm{Tate}}(R)$ is presentable. Since $\mathrm{Comp}(\mathrm{Sh}(\mathrm{Sm}_S,R))$ is presentable and $\mathrm{S}_{\mathrm{Tate}}(R)$ can be identified with the functor category from $\mathfrak{S}'$ to $\mathrm{S}_{\mathrm{Tate}}(R)$, thus by [36, 5.5.3.6] we see that $\mathrm{S}_{\mathrm{Tate}}(R)$ is presentable. Then according to [37, 3.4.4.2] the category $\mathrm{Sp}_{\mathrm{Tate}}(R)$ of modules over $\mathrm{Sym}(R(1))$ is presentable. □

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Let $\mathrm{Comp}(R)$ be the category of chain complexes of R-modules. There is a combinatorial symmetric monoidal model structure of $\mathrm{Comp}(R)$ whose weak equivalences are quasi-isomorphisms and whose fibrations are degreewise surjective maps. The complex R (concentrated in degree zero) is a cofibrant unit. This model structure is called the projective model structure ([24]). There is a symmetric monoidal functor $\mathrm{Comp}(R) \to \mathrm{Comp}(\mathrm{Sh}(\mathrm{Sm}_S, R))$ which carries a complex N to the constant functor with value N. For any $A \in \mathrm{Comp}(R) \to \mathrm{Comp}(\mathrm{Sh}(\mathrm{Sm}_S, R))$, we have the symmetric Tate spectrum $\{R(1)^{\otimes n} \otimes A\}_{n \in \mathbb{N}}$ such that $\mathfrak{S}_n$ acts on $R(1)^{\otimes n} \otimes A$ by permutation on $R(1)^{\otimes n}$. This determines the infinite suspention functor

$$\Sigma^\infty : \mathrm{Comp}(\mathrm{Sh}(\mathrm{Sm}_S, R)) \to \mathrm{Sp}_{\mathrm{Tate}}(R)$$

which is symmetric monoidal (see [13, 1.4.2.1]). According to [13, 1.2.5, 1.4.2], the composition

$$\mathrm{Comp}(R) \to \mathrm{Comp}(\mathrm{Sh}(\mathrm{Sm}_S, R)) \to \mathrm{Sp}_{\mathrm{Tate}}(R)$$

is a (symmetric monoidal) left Quillen functor. By composition, we also have

$$L : \mathrm{Sm}_S \to \mathrm{Comp}(\mathrm{Sh}(\mathrm{Sm}_S, R)) \overset{\Sigma^\infty}{\to} \mathrm{Sp}_{\mathrm{Tate}}(R).$$

Localizations. Now we recall an elegant localization method which transform model categories into ∞-categories (cf. [37, 1.3.4.1, 1.3.1.15, 4.1.3.4]). Let $(\mathcal{C}, W)$ be a pair of an ∞-category $\mathcal{C}$ and a collection W of edges in $\mathcal{C}$ which contains all degenerate edges. We say that a map $f : \mathcal{C} \to \mathcal{D}$ exhibits $\mathcal{D}$ as the ∞-category obtained from $\mathcal{C}$ by inverting the edges in W when for any ∞-category $\mathcal{E}$, the functor f induces a fully faithful functor $\mathrm{Fun}(\mathcal{D}, \mathcal{E}) \to \mathrm{Fun}(\mathcal{C}, \mathcal{E})$ whose essential image consists of functors which sends edges in W to equivalences in $\mathcal{E}$. The fibrant replacement $(\mathcal{C}, W) \to \mathcal{D}$ of the model category Set^+_Δ of marked simplicial sets (see [36, 3.1]) exhibits $\mathcal{D}$ as the ∞-category obtained from $\mathcal{C}$ by inverting the edges in W. For a model category $\mathbb{M}$, let $\mathbb{M}^c$ be the full subcategory consisiting of cofibrant objects and W the collection of edges in $\mathrm{N}(\mathbb{M}^c)$ which correspond to weak equivalences in $\mathbb{M}^c$. Then we denote by $\mathrm{N}(\mathbb{M}^c)_\infty$ the ∞-category obtained from $\mathrm{N}(\mathbb{M}^c)$ by inverting edges in W. When $\mathbb{M}$ is a combinatorial model category, $\mathrm{N}(\mathbb{M}^c)_\infty$ is a presentable ∞-category. A left Quillen equivalence $\mathbb{M} \to \mathbb{N}$ induces a categorical equivalence $\mathrm{N}(\mathbb{M}^c)_\infty \to \mathrm{N}(\mathbb{N}^c)_\infty$. A homotopy (co)limit diagram in $\mathbb{M}$ corresponds to a (co)limit diagram (see [37, 1.3.4.23, .1.3.4.24]). In virtue of [37, 4.1.3.4], if $\mathbb{M}$ is a symmetric monoidal model category, the localization $\mathrm{N}(\mathbb{M}^c) \to \mathrm{N}(\mathbb{M}^c)_\infty$ is promoted to a symmetric monoidal functor $\mathrm{N}(\mathbb{M}^c)^\otimes \to \mathrm{N}(\mathbb{M}^c)^\otimes_\infty$ whose underlying functor can be identified with $\mathrm{N}(\mathbb{M}^c) \to \mathrm{N}(\mathbb{M}^c)_\infty$. The tensor product $\mathrm{N}(\mathbb{M}^c)_\infty \times \mathrm{N}(\mathbb{M}^c)_\infty \to \mathrm{N}(\mathbb{M}^c)_\infty$ preserves small colimits separately

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in each variable since for any $M \in \mathbb{M}^c$, $(-) \otimes M : \mathbb{M} \to \mathbb{M}$ and $M \otimes (-) : \mathbb{M} \to \mathbb{M}$ are left Quillen functors.

Next we apply this localization to the symmetric monoidal left Quillen functor $\mathrm{Comp}(R) \to \mathrm{Sp}_{\mathrm{Tate}}(R)$. Then we have a symmetric monoidal functor of symmetric monoidal presentable ∞-categories

$$\mathrm{N}(\mathrm{Comp}(R)^c)^{\otimes}_{\infty} \longrightarrow \mathrm{N}(\mathrm{Sp}_{\mathrm{Tate}}(R)^c)^{\otimes}_{\infty}$$

which preserves small colimits. We set $\mathsf{D}^{\otimes}(R) = \mathrm{N}(\mathrm{Comp}(R)^c)^{\otimes}$ and $\mathsf{Sp}^{\otimes}_{\mathrm{Tate}}(R) = \mathrm{N}(\mathrm{Sp}_{\mathrm{Tate}}(R)^c)^{\otimes}_{\infty}$. When we consider the underlying ∞-category, we drop the superscript $\otimes$. The following Proposition implies that the ∞-categories $\mathsf{D}(R)$ and $\mathsf{Sp}_{\mathrm{Tate}}(R)$ are stable.

Proposition 5.4 *Let $\mathbb{M}$ be a combinatorial stable model category. Then the ∞-category $\mathrm{N}(\mathbb{M}^c)_{\infty}$ is stable and presentable.*

Proof: The presentability is due to [37, 1.3.4.22].

Let $\mathcal{C} = \mathrm{N}(\mathbb{M}^c)_{\infty}$. We first observe that $\mathcal{C}$ is pointed, that is, there is an object which is both initial and final. By [37, 1.3.4.24], initial objects and final objects $\mathcal{C}$ correspond to homotopy initial objects and a homotopy final objects in $\mathbb{M}$ respectively. Consequently, $\mathcal{C}$ has a zero object since $\mathbb{M}$ is pointed. Since $\mathcal{C}$ is presentable, it has small colimits and limits. Therefore by [38, I, 10.12], it remains to prove that the suspension functor Σ induces a categorical equivalence $\mathcal{C} \to \mathcal{C}$. Note that by our assumption and [37, 1.3.4.24] the suspention functor induces an equivalence of the homotopy category

$$\Sigma : \mathrm{h}(\mathcal{C}) \longrightarrow \mathrm{h}(\mathcal{C}).$$

In particular, $\Sigma : \mathcal{C} \to \mathcal{C}$ is essentially surjective. We claim that $\Sigma : \mathcal{C} \to \mathcal{C}$ is fully faithful. It will suffices to show that the suspention functor induces a homotopy equivalence $\mathrm{Map}_{\mathcal{C}}(C, D) \to \mathrm{Map}_{\mathcal{C}}(\Sigma(C), \Sigma(D))$ for any two objects $C, D \in \mathcal{C}$. Note that $\mathrm{Map}_{\mathcal{C}}(C, D)$ is pointed by the zero map and $\mathrm{Map}_{\mathcal{C}}(C, D)$ is a group object in $\mathrm{h}(\mathcal{S})$. We may assume that a base point is the zero map. The natural map $\mathrm{Map}_{\mathcal{C}}(\Sigma(C), D) \to \Omega \mathrm{Map}_{\mathcal{C}}(C, D)$ is a homotopy equivalence. It follows that the n-th homotopy group $\pi_n(\mathrm{Map}_{\mathcal{C}}(C, D))$ can be identified with $\pi_0(\mathrm{Map}_{\mathcal{C}}(\Sigma^n(C), D))$. We conclude that the map $\pi_n(\mathrm{Map}_{\mathcal{C}}(C, D)) \to \pi_n(\mathrm{Map}_{\mathcal{C}}(\Sigma(C), \Sigma(D)))$ can be identified with the bijective map

$$\pi_0(\mathrm{Map}_{\mathcal{C}}(\Sigma^n(C), D)) \to \pi_0(\mathrm{Map}_{\mathcal{C}}(\Sigma^{n+1}(C), \Sigma(D))),$$

as desired. □

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Let $\mathbf{K}$ be a field of characteristic zero. Let $\mathbf{HK}$ be the motivic Eilenberg-MacLane spectrum which is a commutative algebra object in $\mathrm{Sp}_{\mathrm{Tate}}(\mathbf{K})$ (see e.g. [45]).

When $\mathbf{R}$ is a commutative algebra object in $\mathrm{Sp}_{\mathrm{Tate}}(\mathbf{K})$ we denote by $\mathrm{Sp}_{\mathrm{Tate}}(\mathbf{R})$ the category of module objects in $\mathrm{Sp}_{\mathrm{Tate}}(\mathbf{K})$ over $\mathbf{R}$ (see [47, Section 4]).

According to [13, 1.5.2] built on [47, 4.1], there is a combinatorial symmetric monoidal model category structure on $\mathrm{Sp}_{\mathrm{Tate}}(\mathbf{R})$ such that a morphism is a weak equivalence (resp. fibration) in $\mathrm{Sp}_{\mathrm{Tate}}(\mathbf{R})$ if the underlying morphism in $\mathrm{Sp}_{\mathrm{Tate}}(\mathbf{K})$ is a weak equivalence (resp. fibration). The base change functor $\mathrm{Sp}_{\mathrm{Tate}}(\mathbf{K}) \to \mathrm{Sp}_{\mathrm{Tate}}(\mathbf{HK})$ is a symmetric monoidal left Quillen functor. By inverting by weak equivalences we have a symmetric monoidal functor of symmetric monoidal ∞-categories

$$\mathsf{Sp}^{\otimes}_{\mathrm{Tate}}(\mathbf{K}) \to \mathsf{Sp}^{\otimes}_{\mathrm{Tate}}(\mathbf{HK}) := \mathrm{N}(\mathrm{Sp}^{\otimes}_{\mathrm{Tate}}(\mathbf{HK})^{c})^{\otimes}_{\infty}$$

which preserves small colimits. We remark that $\mathrm{Sp}_{\mathrm{Tate}}(\mathbf{HK})$ is a stable model category and thus $\mathsf{Sp}^{\otimes}_{\mathrm{Tate}}(\mathbf{HK})$ is stable by Proposition 5.4.

Remark 5.5 There is no reason to assume that $\mathbf{K}$ is a field of characteristic zero in the above discussion. We can replace $\mathbf{K}$ by an arbitrary commutative ring R. But in what follows we use the notion of mixed Weil theory which works over $\mathbf{K}$.

Remark 5.6 Let S be the Zariski spectrum of a perfect field k. Let R be an ordinary commutative ring. Let Cor_R be the Suslin-Voevodsky's R-linear category of finite correspondences. Here by an R-linear category, we mean a category enriched over the symmetric monoidal category of R-modules. An R-linear functor means an (obvious) enriched functor. See [32] for the overview of enriched categories. An object in Cor_R is a smooth scheme over S, that is, an object in Sm_S. The hom R-module $\mathrm{Hom}_{\mathrm{Cor}_R}(X,Y)$ is a free R-module generated by the set of reduced irreducible closed subscheme $W \in X \times_k Y$ such that the natural morphism $W \to X$ is finite and its image is an irreducible component of X. The composition

$$\mathrm{Hom}_{\mathrm{Cor}_R}(X,Y) \otimes_R \mathrm{Hom}_{\mathrm{Cor}_R}(Y,Z) \to \mathrm{Hom}_{\mathrm{Cor}_R}(X,Z), \quad W \otimes W' \mapsto W' \circ W,$$

where W and W' are actual reduced irreducible subschemes, is determined by $W' \circ W =$ the push-forward by the projection $X \times_k Y \times_k Z \to X \times_k Z$ of the intersection product $(W \times_k Z) \cap (X \times_k W')$. By the formula $X \otimes Y = X \times_S Y$ Cor_R is a symmetric monoidal category. There is a natural map $\mathrm{Sm}_S \to \mathrm{Cor}_R$ which sends a smooth scheme X to X and sends morphisms $X \to Y$ to their graphs in $X \times_k Y$. A Nisnevich sheaf of (R-modules) with transfers is a contravariant R-linear functor on Cor_R into the category of R-modules, which is a Nisnevich sheaf on the restriction to Sm_S. Let $\mathrm{Sh}(\mathrm{Cor}_R)$ be the abelian category of Nisnevich sheaves with transfers. As the construction of the model category $\mathrm{Sp}_{\mathrm{Tate}}(R)$, in [12, 7.15] the symmetric

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monoidal model category of $\mathrm{DM}(S)$ is constructed (we here employ the notation $\mathrm{DM}(S)$ in [12, 7.15]): we start with the category $\mathrm{Comp}(\mathrm{Sh}(\mathrm{Cor}_R))$ and take the localization of it by $\mathbb{A}^1$-homotopy equivalence and stabilize the Tate sphere (this is only the rough strategy, for the detail we refer the reader to [12]). Suppose $R = \mathbf{K}$. There is a left Quillen adjoint symmetric monoidal functor $\mathrm{Sp}_{\mathrm{Tate}}(\mathbf{HK}) \to \mathrm{DM}(S)$, which induces a Quillen equivalence (proved by using alteration [45, Theorem 68], [13, 2.7.9.1]). It gives rise to an equivalence of symmetric monoidal stable ∞-categories

$$\mathsf{Sp}_{\mathrm{Tate}}(\mathbf{HK}) \to \mathsf{DM}(k) := \mathrm{N}(\mathrm{DM}(S)^c)_\infty.$$

Thanks to [13, 2.7.10] compact objects and dualizable objects coincide in $\mathsf{Sp}_{\mathrm{Tate}}(\mathbf{HK})$. (We say that an object is dualizable if it have a strong dual in the sense in loc. cite.) The full subcategory $\mathsf{Sp}_{\mathrm{Tate}}(\mathbf{HK})_{\mathrm{cpt}}$ of the homotopy category of $\mathsf{Sp}_{\mathrm{Tate}}(\mathbf{HK}) \simeq \mathsf{DM}(k)$ spanned by compact objects is equivalent to Voevodsky's category $DM_{gm}(k)$ of geometric motives with coefficients in $\mathbf{K}$. The triangulated category $DM_{gm}(k)$ is anti-equivalent to Hanamura's category [20] and Levine's category [35] (with rational coefficients).

We summarize the properties of $\mathsf{Sp}_{\mathrm{Tate}}(\mathbf{HK}) \simeq \mathsf{DM}(k)$ as follows:

Proposition 5.7 *The ∞-category $\mathsf{Sp}_{\mathrm{Tate}}(\mathbf{HK}) \simeq \mathsf{DM}(k)$ is stable and presentable. Moreover, it is compactly generated (cf. [36, 5.5.7.1]). Both compact objects and dualizable obejcts coincide.*

Proof: See Proposition 5.4 and Remark 5.6. □

Mixed Weil cohomologies. Suppose that the base scheme S is a perfect field k. Let E be a mixed Weil theory in the sense of Cisinski-Déglise [13, Section 2.1]. A mixed Weil theory is a presheaf E on Sm_S (or the category of affine smooth k-schemes) of commutative differential graded $\mathbf{K}$-algebras which satisfies $\mathbb{A}^1$-homotopy invariance, the descent property and axioms on dimension, stability, Künneth formula (see for the detail [13, 2.1.2]). For example, in loc. cite., it is shown that algebraic and analytic de Rham cohomologies, rigid cohomology, and l-adic étale cohomology are mixed Weil theories. To a mixed Weil theory E we associate a commutative algbera object $\mathbf{E}$ in $\mathrm{Sp}_{\mathrm{Tate}}(\mathbf{K})$, that is, a commutative ring spectrum (see [13, 2.1.5]). Let $\mathbf{HK} \otimes_{\mathbf{K}} \mathbf{E}$ be the (derived) tensor product which is a commutative algebra object in $\mathrm{Sp}_{\mathrm{Tate}}(\mathbf{K})$ (see [13, 2.7.8] and its proof). By [13, 2.7.6], the natural homomorphism $\mathbf{E} \to \mathbf{HK} \otimes_{\mathbf{K}} \mathbf{E}$ (induced by the structure homomorphism $\mathbf{K} \to \mathbf{HK}$) is an isomorphism in the homotopy category of commutative algebra objects. The homomorphism $\mathbf{E} \to \mathbf{HK} \otimes_{\mathbf{K}} \mathbf{E}$ determines a symmetric monoidal functor $\mathrm{Sp}^{\otimes}_{\mathrm{Tate}}(\mathbf{E}) \to \mathrm{Sp}^{\otimes}_{\mathrm{Tate}}(\mathbf{HK} \otimes_{\mathbf{K}} \mathbf{E})$ which is left Quillen. The induced symmetric monoidal functor $\rho : \mathsf{Sp}^{\otimes}_{\mathrm{Tate}}(\mathbf{E}) \to \mathsf{Sp}^{\otimes}_{\mathrm{Tate}}(\mathbf{HK} \otimes_{\mathbf{K}} \mathbf{E})$ is an equivalence (since the underlying functor is a categorical equivalence). Similarly,

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there is a symmetric monoidal functor $\mathsf{Sp}^{\otimes}_{\mathrm{Tate}}(\mathbf{HK}) \to \mathsf{Sp}^{\otimes}_{\mathrm{Tate}}(\mathbf{HK} \otimes_{\mathbf{K}} \mathbf{E})$ determined by the natural homomorphism $\mathbf{HK} \to \mathbf{HK} \otimes_{\mathbf{K}} \mathbf{E}$. Composing these functors we obtain

$$\mathsf{D}^{\otimes}(\mathbf{K}) \to \mathsf{Sp}^{\otimes}_{\mathrm{Tate}}(\mathbf{K}) \to \mathsf{Sp}^{\otimes}_{\mathrm{Tate}}(\mathbf{HK}) \simeq \mathsf{DM}^{\otimes}(k) \to \mathsf{Sp}^{\otimes}_{\mathrm{Tate}}(\mathbf{HK} \otimes_{\mathbf{K}} \mathbf{E}) \overset{\rho^{-1}}{\to} \mathsf{Sp}^{\otimes}_{\mathrm{Tate}}(\mathbf{E})$$

where ρ^{-1} is a homotopy inverse of ρ.

Lemma 5.8 *Let $\phi : \mathcal{C} \to \mathcal{D}$ be an exact functor of stable ∞-categories. Let $\mathrm{h}(\mathcal{C})$ and $\mathrm{h}(\mathcal{D})$ be the homotopy categories of $\mathcal{C}$ and $\mathcal{D}$ respectively. Suppose that $\mathrm{h}(\phi) : \mathrm{h}(\mathcal{C}) \to \mathrm{h}(\mathcal{D})$ is a categorical equivalence of ordinary categories. Then ϕ is a categorical equivalence.*

Proof: It is clear that ϕ is essentially surjective. It suffices to show that for $M, N \in \mathcal{C}$, ϕ induces an equivalence

$$\mathrm{Map}_{\mathcal{C}}(M,N) \to \mathrm{Map}_{\mathcal{D}}(\phi(M),\phi(N))$$

in $\mathcal{S}$. We are reduced to proving that the composition

$$\begin{aligned}\pi_0(\mathrm{Map}_{\mathcal{C}}(\Sigma^n M,N))) &\simeq \pi_n\,(\mathrm{Map}_{\mathcal{C}}(M,N)) \\ &\to \pi_n(\mathrm{Map}_{\mathcal{D}}(\phi(M),\phi(N))) \simeq \pi_0(\mathrm{Map}_{\mathcal{C}}(\Sigma^n \phi(M),\phi(N)))\end{aligned}$$

is a bijective where $\pi_n(-)$ denotes the n-th homotopy group and Σ is the suspention functor that is compatible with ϕ. Now our assertion follows from our assumption. □

Lemma 5.9 *The composition $\mathsf{D}^{\otimes}(\mathbf{K}) \to \mathsf{Sp}^{\otimes}_{\mathrm{Tate}}(\mathbf{E})$ is an equivalence of symmetric monoidal ∞-categories.*

Proof: It is enough to show that the underlying functor is a categorical equivalence. By Lemma 5.8 it suffices to prove that the induced functor of homotopy categories $\mathrm{h}(\mathsf{D}(\mathbf{K})) \to \mathrm{h}(\mathsf{Sp}_{\mathrm{Tate}}(\mathbf{E}))$ is an equivalence. The right adjoint of this functor is described as $D_{\mathbf{A}^1}(k,\mathbf{E}) = \mathrm{h}(\mathsf{Sp}_{\mathrm{Tate}}(\mathbf{E})) \to D(\mathbf{K}) = \mathrm{h}(\mathsf{D}(\mathbf{K}))$ given by $M \mapsto \mathbf{R}\mathrm{Hom}_{\mathbf{E}}(\mathbf{E}, M)$ where we use the notation $D_{\mathbf{A}^1}(k,\mathbf{E})$, $D(\mathbf{K})$ and $\mathbf{R}\mathrm{Hom}_{\mathbf{E}}(\mathbf{E},M)$ in [13] (namely, the right adjoint is given by the "Hom complex" $\mathbf{R}\mathrm{Hom}_{\mathbf{E}}(\mathbf{E},M)$ in $\mathrm{h}(\mathsf{Sp}_{\mathrm{Tate}}(\mathbf{E}))$). This right adjoint is an equivalence by [13, 2.7.11] and thus $\mathrm{h}(\mathsf{D}(\mathbf{K})) \to \mathrm{h}(\mathsf{Sp}_{\mathrm{Tate}}(\mathbf{E}))$ is so. □

Let $H\mathbf{K}$ be the (not motivic) Eilenberg-MacLane commutative ring spectrum of $\mathbf{K}$ in Sp.

Proposition 5.10 *There is an equivalence $\mathrm{Mod}^{\otimes}_{H\mathbf{K}} \to \mathsf{D}^{\otimes}(\mathbf{K})$ of symmetric monoidal ∞-categories.*

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Proof: This immediately follows from [37, 8.1.2.13]. □

Remark 5.11 There is no need to assume that **K** is a field. The proof is valid for any commutative ring.

Definition 5.12 By Proposition 5.10 and Lemma 5.9, we obtain a symmetric monoidal functor

$$\mathsf{R}_E : \mathsf{Sp}^{\otimes}_{\mathrm{Tate}}(\mathbf{HK}) \simeq \mathsf{DM}^{\otimes}(k) \to \mathsf{Sp}^{\otimes}_{\mathrm{Tate}}(\mathbf{HK} \otimes_{\mathbf{K}} \mathbf{E}) \xrightarrow{\rho^{-1}} \mathsf{Sp}^{\otimes}_{\mathrm{Tate}}(\mathbf{E}) \simeq \mathrm{Mod}^{\otimes}_{H\mathbf{K}}.$$

We refer to is as the realization functor assosiated to E.

5.2. The construction of motivic Galois groups and the comparison torsor

For a mixed Weil thoery E, we have

$$\mathsf{Sp}^{\otimes}_{\mathrm{Tate}}(\mathbf{HK}) \simeq \mathsf{DM}^{\otimes}(k) \xrightarrow{\mathsf{R}_E} \mathrm{Mod}^{\otimes}_{H\mathbf{K}}.$$

For example, suppose that $S = \mathrm{Spec}k$ is the Zariski spectrum of a field of characteristic zero and $\mathbf{K} = k$. Let E be the mixed Weil theory of algebraic de Rham cohomology and $L(X)$ the image of smooth scheme $X \in \mathrm{Sm}_S$ in $\mathsf{Sp}^{\otimes}_{\mathrm{Tate}}(\mathbf{HK})$. Then R_E carries $L(X)$ to the dual of the complex computing the de Rham cohomology of X.

By Remark 5.6, in $\mathrm{Mod}^{\otimes}_{H\mathbf{K}}$ and $\mathsf{Sp}^{\otimes}_{\mathrm{Tate}}(\mathbf{HK})$, compact objects and dualizable objects coincide respectively. This diagram induces the diagram of full subcategories of dualiziable objects whose underlying ∞-categories are small stable idempotent complete ∞-categoires:

$$\mathsf{Sp}^{\otimes}_{\mathrm{Tate}}(\mathbf{HK})_{\vee} \simeq \mathsf{DM}^{\otimes}_{\vee}(k) \xrightarrow{\mathsf{R}_E} \mathrm{PMod}^{\otimes}_{H\mathbf{K}}$$

where R_E is the restriction of the realization functor (we abuse notation).

Definition 5.13 We apply Theorem 4.15 to the realization functor R_E and obtain a derived affine group scheme MG_E over $H\mathbf{K}$ which we shall call the derived motivic Galois group associated to the mixed Weil theory E. There is a diagram of symmetic monoidal stable idempotent complete ∞-categories

$$\begin{array}{ccc} \mathsf{DM}^{\otimes}_{\vee}(k) & \longrightarrow & \mathrm{PRep}^{\otimes}_{\mathsf{MG}_E} \\ & {\scriptstyle \mathsf{R}_E} \searrow \quad \swarrow & \\ & \mathrm{PMod}^{\otimes}_{H\mathbf{K}} & \end{array}$$

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where $\mathrm{PRep}^{\otimes}_{\mathsf{MG}_E}$ is the symmetric monoidal stable idempotent complete ∞-category of perfect representations of MG_E (see Appendix A.6) and $\mathrm{PMod}^{\otimes}_{\mathsf{MG}_E} \to \mathrm{PMod}^{\otimes}_{H\mathbf{K}}$ is the forgetful functor. When E is clear, we often write MG for MG_E. If we let $\mathsf{MG}_E = \mathrm{Spec}\, B_E$, then we can choose B_E to be a commutative differential graded $\mathbf{K}$-algebra B_E by virtue of the well-known categorical equivalence between the ∞-category of commutative $H\mathbf{K}$-ring spectra and that of commutative differential graded $\mathbf{K}$-algebras (cf. e.g. [37, 8.1.4.11]).

Applying Theorem 4.15 to this situation we have

Theorem 5.14 (**Representability and Universality**) *The derived affine group scheme* $\mathsf{MG}_E = \mathrm{Spec}\, B_E$ *represents the automorphism group functor* $\mathrm{Aut}(\mathsf{R}_E)$ *and the universality described in Theorem 4.15.*

Apart from the fundamental and intrinsic interest, the representability and universality are useful for the technical purpose. For instance, the representablity (and the coarse representability) plays an important role in Section 5.4 (also in subsequent papers). The universality can be used to obtain a natural homomorphism $G_{\mathrm{Hodge}} \to \mathsf{MG}_E$ by constructing Hodge realization functor $\mathsf{DM}^{\otimes}_{\triangledown}(k) \to \mathcal{D}^{\otimes}(MHS)$ (though we do not give such a construction in this paper). Here G_{Hodge} is the Tannaka dual group of the category of mixed Hodge structures and E is the Betti cohomology theory. Since $\mathbf{K}$ is a field of characteristic zero, to work with MG_E, we may employ complicial algebraic geometry [52, II, 2.3]. But when one wants to apply our tannakization to the integral Betti realization and obtain motivic Galois group over $H\mathbb{Z}$, we need the brave new derived algebraic geometry [52, II, 2.4], [38].

Next we suppose that the base field k is embedded into the complex field $\mathbb{C}$; $\sigma : k \to \mathbb{C}$. Let R_{Be} and R_{dR} be the realization functors associated to Betti (singular) cohomology and algebraic de Rham cohomology respectively. Let $\mathrm{Isom}(\mathsf{R}_{\mathrm{Be}} \otimes_{H\mathbb{Q}} Hk, \mathsf{R}_{\mathrm{dR}}) : \mathrm{CAlg}_{Hk} \to \mathcal{S}$ be the Isom-functor (see Section 3). Here $\mathsf{R}_{\mathrm{Be}} \otimes_{H\mathbb{Q}} Hk : \mathrm{CAlg}_{Hk} \to \mathcal{S}$ is the composite $\mathsf{DM}^{\otimes}_{\triangledown}(k) \overset{\mathsf{R}_{\mathrm{Be}}}{\to} \mathrm{PMod}^{\otimes}_{H\mathbb{Q}} \overset{\otimes_{H\mathbb{Q}} Hk}{\to} \mathrm{PMod}^{\otimes}_{Hk}$. Thanks to Theorem 4.11 we deduce

Theorem 5.15 (**Algebraicity of Comparison torsor**) *There is a derived affine scheme* $\mathsf{Isom}_{\mathrm{Be,dR}} = \mathrm{Spec}\, B_{\mathrm{Be,dR}}$ *that represents the Isom-functor* $\mathrm{Isom}(\mathsf{R}_{\mathrm{Be}} \otimes_{H\mathbb{Q}} Hk, \mathsf{R}_{\mathrm{dR}})$.

The derived affine scheme $\mathsf{Isom}_{\mathrm{Be,dR}}$ comes up with the (left) action of $\mathsf{MG}_{\mathrm{dR}}$ and the (right) action of $\mathsf{MG}_{\mathrm{Be}} \otimes_{H\mathbb{Q}} Hk$. It is a torsor under each action (in an appropriate sense). This torsor is important. We sketch the fundamental interests on $\mathsf{Isom}_{\mathrm{Be,dR}}$. The comparison $\mathsf{R}_{\mathrm{Be}} \otimes_{H\mathbb{Q}} H\mathbb{C} \simeq \mathsf{R}_{\mathrm{dR}} \otimes_{Hk} H\mathbb{C}$ induced by the de Rham

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theorem gives rise to the classifying morphism

$$\omega : \operatorname{Spec} H\mathbb{C} \to \mathsf{Isom}_{\mathrm{Be,dR}} = \operatorname{Spec} B_{\mathrm{Be,dR}}.$$

The "image" of $\omega^* : B_{\mathrm{Be,dR}} \to H\mathbb{C}$ (defined in a suitable way; the image of $H_*(B_{\mathrm{Be,dR}}) \to \mathbb{C}$) is given by periods of mixed motives. We will discuss this issue in detail in a subsequent work.

5.3. Variants.

We can also construct a derived affine group scheme from any symmetric monoidal (full) subcategory in $\mathsf{DM}_{\vee}^{\otimes}(k)$. Let $\mathsf{S}^{\otimes} \subset \mathsf{DM}_{\vee}^{\otimes}(k)$ be a symmetric monoidal full subcategory. In virtue of Theorem 4.15 the composite

$$\mathsf{S}^{\otimes} \hookrightarrow \mathsf{DM}_{\vee}^{\otimes}(k) \xrightarrow{\mathrm{R}_E} \mathrm{PMod}_{H\mathbf{K}}^{\otimes}$$

yields a derived affine group scheme $\mathsf{MG}_E(\mathsf{S}^{\otimes})$ over $H\mathbf{K}$. Full subcategories of mixed Tate motives, Artin motives and so on have been very important examples. As mentioned in Introduction, we will investigate the tannakizations of these full subcategories in a separate paper [26].

Let X be a smooth scheme over k. Let m be an integer. Let $\mathsf{DM}_{\vee}^{\otimes}(k)_{X(m)}$ denotes the smallest symmetric monoidal idempotent complete stable subcategory which contains $L(X)(m)$. The underlying stable ∞-category is the smallest stable subcategory which contains $(L(X)(m))^{\otimes n}$ $(n \geq 0)$ and is closed under retracts. In this case, we write $\mathsf{MG}_E(X(m))$ for $\mathsf{MG}_E(\mathsf{DM}_{\vee}^{\otimes}(k)_{X(m)})$.

Proposition 5.16 *There exists a natural equivalence of derived affine group schemes*

$$\mathsf{MG}_E \xrightarrow{\sim} \lim_{(X,m)} \mathsf{MG}_E(X(m))$$

where the right hand side is the (small) limit of derived affine group schemes, and pairs (X,m) run over smooth projective schemes X and integers $m \in \mathbb{Z}$. The index category of a limit of the right hand side is the poset of full subcategories $\mathsf{DM}_{\vee}^{\otimes}(k)_{X(m)}$.

Proof: It is enough to show that the colimit

$$\operatorname{colim}_{(X,m)} \mathsf{DM}_{\vee}^{\otimes}(k)_{X(m)}$$

in $\mathrm{CAlg}(\mathrm{Cat}_{\infty})$ is equivalent to $\mathsf{DM}_{\vee}^{\otimes}(k)$. It will suffice to prove that the colimit of the diagram of $\{\mathsf{DM}_{\vee}(k)_{X(m)}\}_{(X,m)}$ in Cat_{∞} is equivalent to the ∞-category $\mathsf{DM}_{\vee}^{\otimes}(k)$. We are reduced to showing that for any $M \in \mathsf{DM}_{\vee}^{\otimes}(k)$ there exists

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$L(X)(m)$ such that M belongs to $\mathsf{DM}^{\otimes}_{\vee}(k)_{X(m)}$. Note that by Proposition 5.7 there exists a finite collection of objects $\{L(X_1)(m_1),\dots,L(X_r)(m_r)\}$ such that M lies in the smallest stable subcategory which contains all $L(X_i)(m_i)$ and is closed under retracts. Therefore we easily see that there exists $L(X)(m)$ such that $M \in \mathsf{DM}^{\otimes}_{\vee}(k)_{X(m)}$. □

5.4. Coarse moduli spaces and underived motivic Galois groups.

We will construct a usual affine group scheme MG_E over $\mathbf{K}$ (i.e., pro-algebraic group) from MG_E, which is more tractable and has an expected nice property (cf. Theorem 5.17). The group scheme MG_E appears as the *coarse moduli space* of MG_E. That is, MG_E is an analogue of coarse moduli spaces of moduli stacks that play important role in the classical algebraic geometry (cf. [31]). This MG_E is also well-suited to the traditional conjectural line of motivic t-structure (cf. Proposition 5.19).

For this purpose, let $\mathsf{dga}_{\mathbf{K}}$ be the category of commutative differential graded $\mathbf{K}$-algebras. By virtue of [23, 2.2.1] or [37, 8.1.4.10], there is a combinatorial model category structure on $\mathsf{dga}_{\mathbf{K}}$, in which weak equivalences are quasi-isomorphisms (of underlying complexes), and fibrations are those maps which induce levelwise surjective maps. Let $\mathsf{dga}^{\geq 0}_{\mathbf{K}}$ be the full subcategory of $\mathsf{dga}_{\mathbf{K}}$ spanned by those objects such that $A^i = 0$ for any $i < 0$ (here we use the cohomological indexing). Note that $\mathsf{dga}^{\geq 0}_{\mathbf{K}}$ consists of not connective but "coconnective" objects. According to [23, 2.2.1], there is a combinatorial model category structure on $\mathsf{dga}^{\geq 0}_{\mathbf{K}}$, in which weak equivalences are quasi-isomorphisms, and fibrations are those maps which induce levelwise surjective maps (here one can choose a Sullivan algebra as a cofibrant replacement, see e.g. [22]). It gives rise to a Quillen adjunction

$$\tau : \mathsf{dga}_{\mathbf{K}} \rightleftarrows \mathsf{dga}^{\geq 0}_{\mathbf{K}} : \iota$$

where the right adjoint $\iota : \mathsf{dga}^{\geq 0}_{\mathbf{K}} \to \mathsf{dga}_{\mathbf{K}}$ is the inclusion. The left adjoint sends A to its quotient by differential graded ideal generated by elements $a \in A^i$ for $i < 0$. By the localization, we have the induced adjunction

$$\tau : \mathrm{CAlg}_{H\mathbf{K}} \simeq \mathrm{N}(\mathsf{dga}^{c}_{\mathbf{K}})_{\infty} \rightleftarrows \mathrm{N}((\mathsf{dga}^{\geq 0}_{\mathbf{K}})^{c})_{\infty} : \iota$$

(cf. [37, 1.3.4.26] and adjoint functor theorem [37, 5.5.2.9]), where the right adjoint is fully faithful (we abuse notation and continue to use notation τ and ι). For the first equivalence, see [37, 8.1.4.11]. Hence by adjunction, $\mathrm{Spec}\,\tau B_E$ represents $\mathrm{N}((\mathsf{dga}^{\geq 0}_{\mathbf{K}})^{c})_{\infty} \hookrightarrow \mathrm{CAlg}_{H\mathbf{K}} \xrightarrow{\mathrm{Aut}(\mathsf{R}_E)} \mathrm{Grp}(\mathcal{S})$ (recall $\mathsf{MG}_E = \mathrm{Spec}\,B_E$). Next let $\mathsf{dga}^{0}_{\mathbf{K}}$ be the full subcategory of $\mathsf{dga}_{\mathbf{K}}$ spanned by objects such that

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$A^i = 0$ for $i \neq 0$, that is, the category of commutative $\mathbf{K}$-algebras. Then there is a natural adjunction $\mathrm{N}(\mathsf{dga}_{\mathbf{K}}^0) \rightleftarrows \mathrm{N}((\mathsf{dga}_{\mathbf{K}}^{\geq 0})^c)_\infty$, where the right adjoint carries A to the (homologically) connective cover of A, i.e. $H^0(A)$, and the left adjoint is the natural inclusion. Note that since $\mathbf{K}$ is a field there is a natural isomorphism $H^0(A \otimes B) \simeq H^0(A) \otimes H^0(B)$ for $A, B \in \mathsf{dga}_{\mathbf{K}}^{\geq 0}$. Therefore, the affine scheme $\mathrm{Spec}\, H^0(\tau B_E)$ inherits a group structure from $\mathrm{Spec}\, \tau B_E$. The natural map $H^0(\tau B_E) \to \tau B_E$ induces a morphism $\mathrm{Spec}\, \tau B_E \to \mathrm{Spec}\, H^0(\tau B_E)$ as functors $\mathrm{N}((\mathsf{dga}_{\mathbf{K}}^{\geq 0})^c)_\infty \to \mathrm{Grp}(\mathcal{S})$. It is universal among morphims to usual affine group schemes. We henceforce abuse notation and write MG_E and $\mathrm{Spec}\, \tau B_E$ also for the restrictions of $\mathrm{CAlg}_{H\mathbf{K}} \simeq \mathrm{N}(\mathsf{dga}_{\mathbf{K}}^c)_\infty \to \mathrm{Grp}(\mathcal{S})$ to $\mathrm{N}((\mathsf{dga}_{\mathbf{K}}^{\geq 0})^c)_\infty$ and $\mathrm{N}(\mathsf{dga}_{\mathbf{K}}^0)_\infty$. Note that if we regard MG_E and $\mathrm{Spec}\, \tau B_E$ as functors $\mathrm{N}((\mathsf{dga}_{\mathbf{K}}^{\geq 0})^c)_\infty \to \mathrm{Grp}(\mathcal{S})$, then these coincide. As functors $\mathrm{N}(\mathsf{dga}_{\mathbf{K}}^0) \to \mathrm{Grp}(\mathcal{S})$, $\mathrm{Spec}\, \tau B_E \to \mathrm{Spec}\, H^0(\tau B_E)$ can be viewed as $\mathsf{MG}_E|_{\mathrm{N}(\mathsf{dga}_{\mathbf{K}}^0)} \to \mathrm{Spec}\, H^0(\tau B_E)$. Thus $\mathrm{Spec}\, H^0(\tau B_E)$ is the coarse moduli space of MG_E (see Section A.4). We shall refer to the affine group scheme (i.e. pro-algebraic group)

$$MG_E = \mathrm{Spec}\, H^0(\tau B_E)$$

as the underived motivic Galois group (or simply the motivic Galois group). This construction also works for variants of MG_E. The affine group scheme MG_E has the following important property:

Theorem 5.17 (**Coarse representability of motivic Galois groups**) *Let K be a $\mathbf{K}$-field, that is, a commutative $\mathbf{K}$-algebra which is a field. Let $\overline{\mathrm{Aut}(\mathsf{R}_E)}(K)$ be the group of equivalence classes of automorphisms of R_E, that is, $\pi_0(\mathrm{Aut}(\mathsf{R}_E)(HK))$. Then there is a natural isomorphism of groups*

$$MG_E(K) \simeq \overline{\mathrm{Aut}(\mathsf{R}_E)}(K)$$

where $MG_E(K)$ denotes the group of K-valued points. These isomorphisms are functorial among $\mathbf{K}$-fields in the obvious way.

Proof: We first suppose that $K = \mathbf{K}$. Let $B'_E = \tau B_E$. There is a natural morphism $\mathrm{Spec}\, B'_E \to \mathrm{Spec}\, H^0(B'_E)$ corresponding to $H^0(B'_E) \to B'_E$. Here we regard $\mathrm{Spec}(-)$ as a functor $\mathrm{N}((\mathsf{dga}_{\mathbf{K}}^{\geq 0})^c)_\infty \to \mathrm{Grp}(\mathcal{S})$. Let $u : \mathrm{Spec}\, \mathbf{K} \to \mathrm{Spec}\, H^0(B'_E)$ be the unit morphism and let $\mathrm{Spec}\, C = \mathrm{Spec}\, \mathbf{K} \times_{\mathrm{Spec}\, H^0(B'_E)} \mathrm{Spec}\, B'_E$ be the associated (homotopy) fiber in $\mathrm{N}((\mathsf{dga}_{\mathbf{K}}^{\geq 0})^c)_\infty^{op}$. Observe that $H^n(B'_E)$ is a free $H^0(B'_E)$-module for any $n \geq 0$. To see this, note that the commutative Hopf graded algebra $H^*(B'_E)$ induces a coaction $H^n(B'_E) \to H^n(B'_E) \otimes_{\mathbf{K}} H^0(B'_E)$ of the commutative Hopf algebra $H^0(B'_E)$. This action commutes with the coaction of $H^0(B'_E)$ on itself. Consequently, the quasi-coherent module $H^n(B'_E)$ on $\mathrm{Spec}\, H^0(B'_E)$ descends to $\mathrm{Spec}\, \mathbf{K} \simeq \mathrm{Spec}\, H^0(B'_E)/\mathrm{Spec}\, H^0(B'_E)$. It follows that $H^n(B'_E)$ has the

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form $L \otimes_{\mathbf{K}} H^0(B'_E)$ where L is a vector space. The freeness implies that $H^n(C) \simeq \mathbf{K} \otimes_{H^0(B'_E)} H^n(B'_E)$. In particular, $H^0(C) \simeq \mathbf{K}$. Hence by [38, VIII, 4.4.8], for any usual commutative $\mathbf{K}$-algebra R, $\mathrm{Map}_{\mathrm{N}((\mathrm{dga}_{\mathbf{K}}^{\geq 0})^c)_\infty^{op}}(\mathrm{Spec}\, R, \mathrm{Spec}\, C)$ is connected. Therefore we see that the full subcategory of $\mathrm{Map}_{\mathrm{N}((\mathrm{dga}_{\mathbf{K}}^{\geq 0})^c)_\infty^{op}}(\mathrm{Spec}\, R, \mathrm{Spec}\, B'_E)$ spanned by morphisms $\mathrm{Spec}\, R \to \mathrm{Spec}\, B'_E$ lying over (the connected component of) u, is connected. If we replace u by another $\mathbf{K}$-valued point v of MG_E via a translation by group action, the same conclusion holds. Therefore, we have a natural isomorphism $\overline{\mathrm{Aut}(\mathsf{R}_E)}(\mathbf{K}) \simeq MG_E(\mathbf{K})$. For a general $\mathbf{K}$-field K, if we replace B'_E by the base change $B'_E \otimes_{\mathbf{K}} K$, then the same argument works. □

We will give several remarks.

Remark 5.18 The tannakian view of motives is originated from Grothendieck's idea. For the original ideas of motivic Galois groups and motivations, we refer the reader to [3] (e.g., fascinating topics are periods of mixed motives via $\mathsf{DM}^{\otimes}(k)$, polylogs and so on, though we concentrate on the foundational issues in this paper). The guiding principle behind our work is that the stable ∞-category of mixed motives (or so-called geometric motives) should naturally constitute a "tannakian category" in the setting of ∞-categories. It is considered to be a version of the original idea, which is generalized to the realm of higher category theory. As mentioned in the Introduction, a conjectural abelian (furthermore tannakian) category of mixed motives is defined as the heart of $\mathsf{DM}_{\vee}^{\otimes}(k)$ endowed with a conjectural motivic t-structure. Here a motivic t-structure is a non-degenerate t-structure on the homotopy category of $\mathsf{DM}_{\vee}(k)$, such that $\otimes : \mathsf{DM}_{\vee}(k) \times \mathsf{DM}_{\vee}(k) \to \mathsf{DM}_{\vee}(k)$ and the realization functor are t-exact. The existence of a motivic t-structure is a hard problem, and recently Beilinson [7] shows that the existence of a motivic t-structure implies Grothendieck's standard conjectures (cf. [3, Chapitre 5]) in characteristic zero. Conversely, Hanamura [21] proves that a "generalized standard conjectures" (i.e., Grothendieck's standard conjectures, Beilinson-Soulé vanishing conjecture, and Bloch-Beilinson-Murre filtration conjecture) imply the existence of a motivic t-structure. (It is worth remarking that a construction of a motivic Galois group for numerical pure motives also needs the standard conjectures, see [3].) These conjectures in full generality are largely inaccessible by now. The idea of us is to start with the ∞-category $\mathsf{DM}^{\otimes}(k)$ endowed with the realization functor into ∞-category of complexes, partly motivated by "derived tannakian philosophy". The reader might raise an objection to our construction of the motivic Galois group as a derived affine group scheme. (But we can extract a usual group scheme from it as above.) We do not think that this is the drawback. Rather, the derived affine group scheme $\mathsf{MG}_E = \mathrm{Spec}\, B_E$ should capture the interesting new data of "highly structured" category $\mathsf{DM}(k)$ of mixed motives

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which may not arise from a conjectural abelian category of mixed motives. Suppose that a motivic t-structure exists and let $\mathcal{MM}$ be its heart. Let $D^b(\mathcal{MM})$ be the bounded derived category (if exists) and let $D^b(\mathcal{MM}) \to DM_{gm}(k)$ be the natural functor. The problem whether or not $D^b(\mathcal{MM}) \to DM_{gm}(k)$ is an equivalence is mysterious. Thus, at least a priori, we can think that $\mathsf{DM}_{\vee}^{\otimes}(k)$ has richer information than $\mathcal{MM}$. We morally think of the part of higher and lower homotopy data of MG_E as the data of $\mathsf{DM}(k)$ which can not be determined by the abelian category $\mathcal{MM}$. From an intuitive point of view, homologically positive degree part of B_E should vanish under the exsitence of a motivic t-structure, and homologically negative degrees of B_E (or $\mathrm{Spec}\,\tau B_E$) should encode higher automorphisms of the realization functor (which do not arise from data of the heart).

In the case of mixed Tate motives, Beilinson-Soulé vanishing conjecture implies the existence of a motivic t-structure on the triangulated subcategory of mixed Tate motives, by the work of Kriz-May [33], Levine [34]. In [8] and [33], the bar construction of a "motivic dg-algebra" is used, and it yields a derived affine group scheme. Recently, using bar constructions Spitzweck has constructed the derived affine group scheme such that its representation category is equivalent to the (∞-)category of (integral) mixed Tate motives, see [49]. This construction can be viewed as Beilinson-Soulé vanishing conjecture-free and $K(\pi,1)$-property-free approach. In [26], as mentioned before, we study the tannakizaiton of ∞-category of mixed Tate motives, which is related to the so-called motivic Galois group for mixed Tate motives.

The pro-algebraic group MG_E constructed in the unconditional fashion is related to a conjectural abelian category of mixed motives. The following Proposition connects our approach with the traditional line (the representability plays a major role in the proof). Apart from our machinery, we also make use of Lurie's theory of locally dimensional ∞-categories (taking acccount of t-structures) developed in [38, VIII, Section 5].

Proposition 5.19 (**Comparison with a conjectural tannakian category**) *Suppose that the homotopy category of* $\mathsf{DM}_{\vee}(k)$ *admits a motivic* t*-structure* $(\mathsf{DM}_{\vee,\geq 0}(k), \mathsf{DM}_{\vee,\leq 0}(k))$ *in the sense of the previous Remark (here* $\mathsf{DM}_{\vee,\geq 0}(k)$ *and* $\mathsf{DM}_{\vee,\leq 0}(k)$ *denote the full subcategory of* $\mathsf{DM}_{\vee}(k)$ *determined by the* t*-structure on the homotopy category). Then its heart is equivalent to the symmetric monoidal abelian category of finite dimensional* $\mathbf{K}$*-linear representations of* MG_E.

Proof: We let $(\mathrm{Ind}(\mathsf{DM}_{\vee,\geq 0}(k)), \mathrm{Ind}(\mathsf{DM}_{\vee,\leq 0}(k)))$ be the induced t-structure on $\mathrm{Ind}(\mathsf{DM}_{\vee}(k)) = \mathsf{DM}(k)$ (cf. [38, VIII, 5.4.1]) and let $\overline{\mathsf{DM}}^{\otimes}(k)$ be its left completion of $\mathsf{DM}^{\otimes}(k)$ with respect to this t-structure (cf. [37, 1.2.1.17], [38, VIII, 4.6.17]). Under the existence of motivic t-structure, according to [26, Remark 7.12] we

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may assume further that $\overline{\mathsf{DM}}^{\otimes}(k)$ is locally dimensional in the sense of Lurie [38, VIII]. Let $\mathrm{CAlg}^{\mathrm{dis}}_{H\mathbf{K}}$ be the full subcategory of $\mathrm{CAlg}_{H\mathbf{K}}$ spanned by objects A such that $\pi_i(A) = 0$ for $i \neq 0$ (this category can be viewed as the nerve of category of commutative $\mathbf{K}$-algebras). We let $\overline{\mathrm{Aut}}(\overline{\mathsf{R}}_E) : \mathrm{CAlg}^{\mathrm{dis}}_{H\mathbf{K}} \to \mathrm{Grp}(\mathcal{S}^{\mathrm{dis}})$ be the functor given informally by the formula $A \mapsto \pi_0(\mathrm{Aut}(\overline{\mathsf{R}}_E)(A))$ where $\overline{\mathsf{R}}_E : \overline{\mathsf{DM}}^{\otimes}(k) \to \mathrm{Mod}^{\otimes}_{H\mathbf{K}}$ is the symmetric monoidal functor induced by R_E (see the homotopy equivalence displayed below), and $\mathrm{Aut}(\overline{\mathsf{R}}_E)$ is the automorphism group functor $\mathrm{CAlg}^{\mathrm{dis}}_{H\mathbf{K}} \to \mathrm{Grp}(\mathcal{S})$ defined in the obvious way. Here $\mathcal{S}^{\mathrm{dis}} \subset \mathcal{S}$ denotes the full subcategory spanned by discrete small spaces, i.e., small sets. Then as a result of the theory of locally dimensional ∞-categories (more precisely, [38, 5.2.12, 5.6.1, 5.6.19 and their proof]) we deduce that $\overline{\mathrm{Aut}}(\overline{\mathsf{R}}_E)$ is represented by the Tannaka dual group G of the neutral tannakian category consisting of dualizable objects of the heart of $\overline{\mathsf{DM}}^{\otimes}(k)$. The neutral tannakian category consisting of dualizable objects of the heart of $\overline{\mathsf{DM}}^{\otimes}(k)$ (or equivalently $\mathsf{DM}^{\otimes}$) is the heart of $\mathsf{DM}^{\otimes}_{\vee}(k)$. It suffices to show that $G \simeq MG_E$. To see this, for $A \in \mathrm{CAlg}^{\mathrm{dis}}_{H\mathbf{K}}$ we set $\mathrm{Mod}_{A,\geq 0} = \{X \in \mathrm{Mod}_A |\ \pi_i(X) = 0 \text{ for } i < 0\}$ and $\mathrm{Mod}_{A,\leq 0} = \{X \in \mathrm{Mod}_A |\ \pi_i(X) = 0 \text{ for } i > 0\}$. Combining the universality of left completions [38, VIII, 4.6.17] and the fact $(\mathrm{Mod}_{A,\geq 0}, \mathrm{Mod}_{A,\leq 0})$ is an accessible, left and right complete t-structure (see [37, 1.2.1, 1.3.3, 1.4.4] for these notions), we have homotopy equivalences

$$\begin{aligned}\mathrm{Map}^{\mathrm{L,\,rex}}_{\mathrm{CAlg}(\widehat{\mathrm{Cat}}_\infty)}(\overline{\mathsf{DM}}^{\otimes}(k), \mathrm{Mod}^{\otimes}_A) &\simeq \mathrm{Map}^{\mathrm{L,\,rex}}_{\mathrm{CAlg}(\widehat{\mathrm{Cat}}_\infty)}(\mathsf{DM}^{\otimes}(k), \mathrm{Mod}^{\otimes}_A) \\ &\simeq \mathrm{Map}^{\mathrm{rex}}_{\mathrm{CAlg}(\widehat{\mathrm{Cat}}_\infty)}(\mathsf{DM}^{\otimes}_{\vee}(k), \mathrm{Mod}^{\otimes}_A)\end{aligned}$$

where the notation $\mathrm{Map}^{\mathrm{rex}}$ (resp. $\mathrm{Map}^{\mathrm{L,\,rex}}$) indicates the full subcategory spanned by right t-exact functors (resp. colimit-preserving right t-exact functors) (cf. [37, 1.3.3]), and the second equivalence is given by left Kan extensions. Moreover, the realization functor $\mathsf{R}_E : \mathrm{Ind}(\mathsf{DM}_{\vee}(k))^{\otimes} \simeq \mathsf{DM}^{\otimes}(k) \to \mathrm{Mod}^{\otimes}_{H\mathbf{K}}$ is right t-exact, that is, $\mathrm{Ind}(\mathsf{DM}_{\vee,\geq 0})$ maps to $\mathrm{Mod}_{H\mathbf{K},\geq 0}$. The functor $\overline{\mathsf{R}}_E$ is induced by R_E through the above equivalence. Hence we have $\mathrm{Aut}(\mathsf{R}_E) \simeq \mathrm{Aut}(\overline{\mathsf{R}}_E)$ as (restricted) functors $\mathrm{CAlg}^{\mathrm{dis}}_{H\mathbf{K}} \to \mathrm{Grp}(\mathcal{S})$, and thus $\mathrm{Spec}\tau B_E$ represents $\mathrm{Aut}(\overline{\mathsf{R}}_E) : \mathrm{CAlg}^{\mathrm{dis}}_{H\mathbf{K}} \to \mathrm{Grp}(\mathcal{S})$. (Notice that we here use Theorem 5.14.) Then we have a natural morphism $\mathrm{Spec}\tau B_E \to G \simeq \overline{\mathrm{Aut}}(\overline{\mathsf{R}}_E)$, that is regarded as a morphism of functors $\mathrm{CAlg}^{\mathrm{dis}}_{H\mathbf{K}} \to \mathrm{Grp}(\mathcal{S})$. It induces an equivalence of spaces

$$\mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{dis}}_{H\mathbf{K}}, \mathrm{Grp}(\mathcal{S}))}(G, F) \to \mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{dis}}_{H\mathbf{K}}, \mathrm{Grp}(\mathcal{S}))}(\mathrm{Spec}\tau B_E, F)$$

for any $F : \mathrm{CAlg}^{\mathrm{dis}}_{H\mathbf{K}} \to \mathrm{Grp}(\mathcal{S}^{\mathrm{dis}}) \subset \mathrm{Grp}(\mathcal{S})$. In particular, $\mathrm{Spec}\tau B_E \to G$ is universal among morphisms to usual affine group schemes. On the other hand, $\mathrm{Spec}\tau B_E \to MG_E$ is also universal among morphisms to usual affine group

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schemes. Therefore, we conclude that $MG_E \simeq G$. Hence MG_E can be identified with the Tannaka dual group of the heart of a conjectural motivic t-structure on the homotopy category of $\mathsf{DM}^{\otimes}_{\vee}(k)$. □

Remark 5.20 There has been Nori's abelian category of mixed motives (see [2]) and its motivic Galois group $\mathsf{MG}_{\mathrm{Nori}}$. It is natural to consider that the relationship between our MG_E and MG_E and $\mathsf{MG}_{\mathrm{Nori}}$ (here E should be the Betti cohomology theory). Our MG_E and MG_E are directly related with $\mathsf{DM}^{\otimes}(k)$ and its realization functor, and this question depends on a relation between $\mathsf{DM}^{\otimes}(k)$ and (∞-categorical setup of) the derived category of Nori's abelian category, which seems out of reach at the present time.

Remark 5.21 There is the natural functor $\mathsf{DM}^{\otimes}_{\vee}(k) \longrightarrow \mathrm{PRep}^{\otimes}_{\mathsf{MG}_E}$. It seems reasonable to conjecture or expect that this functor is an equivalence. This conjecture is a refinement of the conjecture [3, 22.1.4.1 (ii)] which says that the realization functor is conservative, that is, $\mathsf{R}_E(M) = 0$ implies that $M = 0$. Unfortunately, we have no idea to attack this problem. Of course, this functor is universal among functors into the ∞-categories of complexes of the representations of affine groups over $\mathbf{K}$. Namely, let $f : \mathsf{DM}^{\otimes}_{\vee}(k) \to \mathrm{PRep}^{\otimes}_{G}$ be a functor which commutes with functors to $\mathrm{PMod}^{\otimes}_{H\mathbf{K}}$ where G is a usual affine group scheme over $\mathbf{K}$ (considered as the derived affine group scheme). Then there exists a homomorphism $G \to \mathsf{MG}_E$ which induces $\mathrm{PMod}^{\otimes}_{\mathsf{MG}_E} \to \mathrm{PMod}^{\otimes}_{G}$ such that the composition $\mathsf{DM}^{\otimes}_{\vee}(k) \to \mathrm{PRep}^{\otimes}_{\mathsf{MG}_E} \to \mathrm{PRep}^{\otimes}_{G}$ is equivalent to f. An example of such G we should keep in mind is the Tannaka dual group of the abelian category of finite dimensional continuous l-adic representations of the absolute Galois group (when $\mathbf{K} = \mathbb{Q}_l$ and E is the mixed Weil theory of l-adic étale cohomology). Another important example is the Tannaka dual group of the abelian category of mixed Hodge structures.

Remark 5.22 In view of $\mathbb{A}^1$-homotopy theory and non-commutative motives, we are also interested in versions of other cohomology theories. We conclude this Section by a K-theoretic analogue. Let $\mathbf{MSS}(S)$ be the symmetric monoidal stable model category of symmetric $\mathbb{P}^1$-spectra (see e.g. [42, A. 6,4]). Let $p : \mathrm{Spec}\mathbb{C} \to S$ be a morphism to the base scheme $S = \mathrm{Spec}k$ where k is a field of characteristic zero. According to [42, A.7.2] there is a *stable topological realization*

$$R_{\mathbb{C}} : \mathrm{N}(\mathbf{MSS}(S))^{\otimes}_{\infty} \overset{p^*}{\to} \mathrm{N}(\mathbf{MSS}(\mathrm{Spec}\mathbb{C}))^{\otimes}_{\infty} \overset{\mathrm{Sing}_{\mathbb{C}}}{\to} \mathrm{Sp}^{\otimes}$$

that is a colimit-preserving symmetric monoidal functor. Here $\mathrm{Sp}^{\otimes}$ denotes the symmetric monoidal ∞-category of spectra. It also gives rise to

$$\mathrm{Mod}_{BGL}(\mathrm{N}(\mathbf{MSS}(S))^{\otimes}_{\infty}) \to \mathrm{Mod}^{\otimes}_{R_{\mathbb{C}}(BGL)}$$

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where BGL is the $\mathbb{P}^1$-spectrum of algebraic K-theory (see [42, 1.2.5]) and $R_{\mathbb{C}}(BGL)$ represents the complex K-theory. Let $R^{\vee}_{\mathbb{C}}$ be the composition

$$\mathrm{PMod}_{BGL}(\mathrm{N}(\mathbf{MSS}(S))^{\otimes}_{\infty}) \to \mathrm{Mod}_{BGL}(\mathrm{N}(\mathbf{MSS}(S))^{\otimes}_{\infty}) \to \mathrm{Mod}^{\otimes}_{R_{\mathbb{C}}(BGL)}$$

where $\mathrm{PMod}_{BGL}(\mathrm{N}(\mathbf{MSS}(S))^{\otimes}_{\infty})$ is the full subcategory of dualizable objects in $\mathrm{Mod}_{BGL}(\mathrm{N}(\mathbf{MSS}(S))^{\otimes}_{\infty})$. Then $\mathrm{Aut}(R^{\vee}_{\mathbb{C}})$ is represented by a derived affine group scheme over the spectrum representing complex K-theory.

6. Other examples

In this Section, we will present some other examples for the applications of tannakizations. To avoid letting this Section get too long, we only mention examples which one can define quickly.

6.1. *Perfect complexes of derived stacks*

Let R be a commutative ring spectrum. Let $\mathscr{X}$ be a derived stack over R (for this notion, we refer to [52], [38], or [26]). Let $\mathrm{Perf}^{\otimes}(\mathscr{X})$ be the symmetric monoidal stable ∞-category of perfect complexes on $\mathscr{X}$. Here we define $\mathrm{Perf}^{\otimes}(\mathscr{X})$ to be the limit $\lim_{\mathrm{Spec}\, A \to \mathscr{X}} \mathrm{PMod}^{\otimes}_{A}$ in $\mathrm{CAlg}(\mathrm{Cat}_{\infty})$ where $\mathrm{Spec}\, A \to \mathscr{X}$ run over smooth morphisms with $A \in \mathrm{CAlg}_{R}$. Let $p : \mathrm{Spec}\, R \to \mathscr{X}$ be a morphism of derived stacks over R. We have the pullback functor

$$p^{*} : \mathrm{Perf}^{\otimes}(\mathscr{X}) \to \mathrm{PMod}^{\otimes}_{R} \simeq \mathrm{Perf}^{\otimes}(\mathrm{Spec}\, R)$$

which is an R-linear symmetric monoidal exact functor. It gives rise to its tannakziation; a derived affine group scheme over R. We can think this as a generalization of bar constructions of commutative ring spectra. In [26] we study this issue in detail.

6.2. *Topological spaces*

Let R be a connective commutative ring spectrum. Let $\mathrm{CAlg}^{\mathrm{con}}_{R}$ be the full subcategory of CAlg_{R} spanned by connective spectra A, that is, $\pi_i(A) = 0$ for $i < 0$. Let S be a topological space which we regard as an object in $\mathcal{S}$. Let $p : \Delta^0 \to S$ denote a point. We can view S as a constant functor belonging to $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{con}}_{R}, \mathcal{S})$. Let $\mathrm{Perf}^{\otimes}(S)$ be the limit $\lim_{\mathrm{Spec}\, R \to S} \mathrm{PMod}^{\otimes}_{R}$ where $\mathrm{Spec}\, R \to S$ run over $\mathrm{Mod}_{\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{con}}_{R}, \mathcal{S})}(\mathrm{Spec}\, R, S)$. We may think of $\mathrm{Perf}^{\otimes}(S)$ as the symmetric

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monoidal ∞-category of perfect complexes on S with R-coefficients. The symmetric monoidal ∞-category $\mathrm{Perf}^{\otimes}(S)$ is a small stable idempotent complete ∞-category. Then the prescribed point $p : \Delta^0 \to S$ induces

$$\mathrm{Perf}^{\otimes}(S) \longrightarrow \mathrm{Perf}^{\otimes}(\Delta^0)$$

where $\mathrm{Perf}^{\otimes}(\Delta^0) \simeq \mathrm{Perf}^{\otimes}(\mathrm{Spec}\, R) \simeq \mathrm{PMod}_R^{\otimes}$. We then apply the tannakization functor to this diagram. We denote by $G(S, p)$ the associated derived affine group scheme over R. When $R = H\mathbb{Q}$, it would be interesting to compare the rational homotopy theory and $G(S, p)$ over $H\mathbb{Q}$. We speculate on the relation to the de Rham homotopy theory. For simplicity, S is simply connected of finite type. Let $A_{PL}(S)$ be the polynomial de Rham algebra of S over $\mathbb{Q}$ (see e.g. [10]), which is a commutative differential graded $\mathbb{Q}$-algbera. Since the coefficient is $\mathbb{Q}$, we may regard $A_{PL}(S)$ as a coconnective commutative ring spectrum over $H\mathbb{Q}$ (that is, $\pi_i(A_{PL}(S)) = 0$ for $i > 0$). Let $\mathrm{Spec}(A_{PL}(S))$ be the functor $\mathrm{CAlg}_{H\mathbb{Q}} \to \mathcal{S}$ corepresentable by $A_{PL}(S)$. There is a natural base point $\rho : \mathrm{Spec}(H\mathbb{Q}) \to \mathrm{Spec}(A_{PL}(S))$ induced by $\Delta^0 \to S$. The associated Čech nerve of ρ determines a simplicial diagram $\mathrm{N}(\Delta)^{op} \to \mathrm{Aff}_{H\mathbb{Q}}$ which is a derived affine group scheme $G_{PL}(S)$. That is to say, the Čech nerve comes from the bar construction. Then the relationship with de Rham homotopy theory should be described by an equivalence $G(S, p) \simeq G_{PL}(S)$ of derived group schemes over $H\mathbb{Q}$ ($G_{PL}(S)$ is obtained by the tannakization of the forgetful functor $\mathrm{PMod}^{\otimes}_{A_{PL}(S)} \to \mathrm{PMod}^{\otimes}_{H\mathbb{Q}}$).

A. Derived group schemes

A.1. Derived schemes

Before proceeding to derived (affine) group schemes, let us review derived schemes and fix our convention. Let R be a commutative ring spectrum. Recall that CAlg denotes the ∞-category of commutative ring spectra (commutaive algebra objects in Sp, i.e. E_∞-rings in [37]).

Let us recall the notion of étale and flat morphisms in CAlg. We say that a morphism $A \to B$ in CAlg is étale (resp. flat) if it has the following properties:

1. $\pi_0(A) \to \pi_0(B)$ is étale (resp. flat),
2. the isomorphism $\pi_0(B) \otimes_{\pi_0(A)} \pi_n(A) \simeq \pi_n(B)$ of abelian groups for any $n \in \mathbb{Z}$.

If an étale (resp. flat) morphism $A \to B$ induces a surjective morphism $\mathrm{Spec}\,\pi_0(B) \to \mathrm{Spec}\,\pi_0(A)$, we say that $A \to B$ is étale (resp. flat) surjective.

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Let $A \to B^\bullet$ be an coaugmented cosimplicial objects in CAlg_R. We say that $A \to B^\bullet$ is an étale hypercover if for any $n \geq 0$, the natural morphism $\mathrm{cosk}_{n-1}(B^\bullet)_n \to B^n$ is étale surjective, and $A \to B^0$ is étale surjective. Here we abute notation by writing $\mathrm{cosk}_{n-1}(B^\bullet)_n$ for the coskeleton when we consider $B^\bullet$ to be the simplicial object in $(\mathrm{CAlg}_R)^{op}$.

Let $\mathcal{S}$ denote the ∞-category of small spaces. We say that a functor (or presheaf) $P : \mathrm{CAlg}_R \to \mathcal{S}$ is a (hypercomplete étale) sheaf if the following two properties hold:

- if $\{A_\lambda\}$ is a finite family of objects in CAlg_R, then $P(\sqcap_\lambda A_\lambda) \simeq \sqcap_\lambda P(A_\lambda)$
- Let $A \to B^\bullet$ be an étale hypercover. Then we have $P(A) \simeq \lim(P(B^\bullet))$, where $\lim(P(B^\bullet))$ denotes a limit of the cosimplicial diagram.

Let $\mathrm{Sh}(\mathrm{CAlg}_R^{et})$ be the full subcategory of $\mathrm{Fun}(\mathrm{CAlg}_R, \widehat{\mathcal{S}})$ spanned by sheaves. ($\widehat{\mathcal{S}}$ is the ∞-category of spaces in an enlarged universe.) For any A in CAlg_R, we define $\mathrm{Spec}\, A$ to be a functor $\mathrm{CAlg}_R \to \mathcal{S}$ corepresentable by A. This functor is a sheaf. Namely, $\mathrm{Spec}\, A$ belongs to $\mathrm{Sh}(\mathrm{CAlg}_R^{et})$. We shall refer to $\mathrm{Spec}\, A$ as the derived affine scheme (over R) associated to A. Let $\mathrm{Aff}_R \subset \mathrm{Sh}(\mathrm{CAlg}_R^{et})$ be the full subcategory spanned by derived affine schemes over R. Yoneda's Lemma implies that $\mathrm{Aff}_R \simeq (\mathrm{CAlg}_R)^{op}$. If R is the sphere spectrum, then we usually write Aff for Aff_R.

A derived scheme is informally a "geometric object" which is "Zariski locally" isomorphic to a derived affine scheme. In [38], it has been developed the approach of ringed ∞-topoi to the definition of derived schemes and derived Deligne-Mumford stacks. We here take the definition of derived schemes which is similar to [52]. A derived scheme over R which has affine diagonal is a sheaf (that is, a contravariant functor which satisfies the descent condition as above) $X : \mathrm{Aff}_R^{op} \to \widehat{\mathcal{S}}$ which has the following properties (i) and (ii):

(i) For any two morphisms (natural transformations) $a : \mathrm{Spec}\, A \to X$ and $b : \mathrm{Spec}\, B \to X$ with derived affine schemes $\mathrm{Spec}\, A$ and $\mathrm{Spec}\, B$ over R, then the fiber product $\mathrm{Spec}\, A \times_X \mathrm{Spec}\, B$ is representable by a derived affine scheme $\mathrm{Spec}\, C$.

(ii) There exist the disjoint union of derived affine schemes $\sqcup_{\lambda \in I} \mathrm{Spec}\, A_\lambda$ and a morphism $p : \sqcup_{\lambda \in I} \mathrm{Spec}\, A_\lambda \to X$ such that for any $q : \mathrm{Spec}\, B \to X$ and any $\lambda \in I$, the base change $\sqcup_\lambda \mathrm{Spec}\, C_\lambda \to \mathrm{Spec}\, B$ is an étale morphism and it induces an open immersion $\mathrm{Spec}\, \pi_0(C_\lambda) \to \mathrm{Spec}\, \pi_0(B)$ for each $\lambda \in I$, and a surjective morphism $\sqcup_\lambda \mathrm{Spec}\, \pi_0(C_\lambda) \to \mathrm{Spec}\, \pi_0(B)$ of ordinary schemes, where $\mathrm{Spec}\, C_\lambda := \mathrm{Spec}\, A_\lambda \times_X \mathrm{Spec}\, B$.

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We denote by Sch_R the full subcategory spanned by derived schemes over R. (We assume that all derived schemes have affine diagonal.) We shall refer to [38], [43], [52] for the generalities on derived schemes and derived stacks.

Remark A.1 In this paper we work with the derived algebraic geometry over non-connective commutative ring spectra (this point is relevant to motivic applications).

Remark A.2 Let $\mathrm{CAlg}^{\mathrm{con}}$ be the ∞-category of connective commutative ring spectra. There is an adjoint pair

$$\mathrm{CAlg}^{\mathrm{con}} \rightleftarrows \mathrm{CAlg}$$

where the left adjoint $\mathrm{CAlg} \to \mathrm{CAlg}^{\mathrm{con}}$ carries A to its connective cover, and the right adjoint is the natural inclusion. If $\mathrm{CAlg}^{\mathrm{dis}}$ denotes the full subcategory of $\mathrm{CAlg}^{\mathrm{con}}$ spanned by Eilenberg-MacLane ring spectra of ordinary commutative rings, then there is also an adjoint pair

$$\mathrm{CAlg}^{\mathrm{con}} \rightleftarrows \mathrm{CAlg}^{\mathrm{dis}}$$

where the right adjoint $\mathrm{CAlg}^{\mathrm{dis}} \to \mathrm{CAlg}^{\mathrm{con}}$ is the natural inclusion, and the left adjoint carries A to $\pi_0(A)$. In particular, if A is a commutative ring spectrum, then $\pi_0(A)$ is an ordinary commutative ring.

A.2. Derived group schemes

A (ordinary) group scheme over a scheme S is a scheme G which is endowed with morphisms $S \to G$ and $G \times_S G \to G$ that satisfies the usual group axioms. If one employs the functorial point of view, then a group scheme is a group-valued functor on the category of commutative rings, which is representable by a scheme. The notion of derived group schemes is similar to that of group schemes. The point is that to define the notion of derived group schemes we will replace the ordinary category of commutative rings by CAlg. As the case of derived schemes, the notion of group-valued functors on CAlg is not useless. We should treat functors into group objects in $\mathcal{S}$. We first recall the notion of group objects in ∞-categorical settings (these are also commonly called group-like A_∞-spaces in operadic contexts). We refer to [51] [49] for accounts of this subject including related notions.

Definition A.3 Let $\mathcal{C}$ be an ∞-category which admits finite limits. A monoid object is a map $f : \mathrm{N}(\Delta)^{op} \to \mathcal{C}$ having the property: $f([0])$ is a final object, and for each $n \in \mathbb{N}$, inclusions $\{i-1, i\} \hookrightarrow [n]$ for $1 \leq i \leq n$ induce an equivalence

$$f([n]) \to f([1]) \times \ldots \times f([1])$$

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where the right hand side is the n-fold product. We denote by $\mathrm{Mon}(\mathcal{C})$ the full subcategory of $\mathrm{Fun}(\mathrm{N}(\Delta)^{op},\mathcal{C})$ spanned by monoid objects. The idea of this definition goes back to the notion of Segal's Γ-spaces (cf. [48]).

A groupoid object in $\mathcal{C}$ is a functor $f : \mathrm{N}(\Delta)^{op} \to \mathcal{C}$ with the following property: for every n and every partition $[n] = S \cup S'$ such that $S \cap S'$ has one element which we denote by s, the diagram

$$\begin{array}{ccc} f([n]) & \longrightarrow & f(S) \\ \downarrow & & \downarrow \\ f(S') & \longrightarrow & f(\{s\}) \end{array}$$

is a pullback diagram in $\mathcal{C}$ (see [36, 6.1.2]). We say that a groupoid object $f : \mathrm{N}(\Delta)^{op} \to \mathcal{C}$ is a group object if $f([0])$ is a final object in $\mathcal{C}$. We denote by $\mathrm{Grp}(\mathcal{C})$ the full subcategory of $\mathrm{Fun}(\mathrm{N}(\Delta)^{op},\mathcal{C})$ that is spanned by group objects in $\mathcal{C}$. Note that $\mathrm{Grp}(\mathcal{C})$ is a full subcategory of $\mathrm{Mon}(\mathcal{C})$.

Definition A.4 A derived group scheme over R is a functor

$$G : \mathrm{CAlg}_R \longrightarrow \mathrm{Grp}(\mathcal{S})$$

such that the composite $\mathrm{CAlg}_R \to \mathrm{Grp}(\mathcal{S}) \to \mathcal{S}$ is representable by a derived scheme X, where the second map $\mathrm{Grp}(\mathcal{S}) \to \mathcal{S}$ is induced by $\{[1]\} \subset \Delta$. If X is affine, then we shall call it an derived affine group scheme.

The ∞-category $\mathrm{Grp}(\mathcal{S})$ admits a simple description. Let $\mathcal{S}_*$ be the ∞-category of pointed spaces. Namely, $\mathcal{S}_*$ is the (homotopy) fiber of $\mathrm{Fun}(\Delta^1,\mathcal{S}) \to \mathrm{Fun}(\{0\},\mathcal{S}) \simeq \mathcal{S}$ over the contractible space $* \in \mathcal{S}$. Let $\mathcal{S}_{*,\geq 1}$ be the full subcategory of $\mathcal{S}_*$ spanned by pointed connected spaces. Then by [36, 7.2.2.11] we have a functor

$$\mathcal{S}_{*,\geq 1} \longrightarrow \mathrm{Fun}(\mathrm{N}(\Delta)^{op},\mathcal{S}_*)$$

which to any $* \to X \in \mathcal{S}_{*,\geq 1}$ associates the groupoid of the Čech nerve, and it induces an equivalence $\mathcal{S}_{*,\geq 1} \simeq \mathrm{Grp}(\mathcal{S}_*)$. Since an initial object in $\mathcal{S}_*$ is a final object, we easily see that there is a natural equivalence $\mathrm{Grp}(\mathcal{S}_*) \simeq \mathrm{Grp}(\mathcal{S})$ induced by the forgetful functor $\mathcal{S}_* \to \mathcal{S}$ (cf. [36, 7.2.2.5, 7.2.2.10]). By this identification $\mathcal{S}_{*,\geq 1} \simeq \mathrm{Grp}(\mathcal{S})$, the functor $\mathrm{Grp}(\mathcal{S}) \to \mathcal{S}$ induced by $[1] \in \Delta$ is equivalent to the composite

$$\mathcal{S}_{*,\geq 1} \xrightarrow{\Omega} \mathcal{S}_* \longrightarrow \mathcal{S}$$

where Ω is the loop space functor and the second map is the forgetful functor. Thus one can say that a derived group scheme is a functor $G : \mathrm{CAlg}_R \to \mathcal{S}_{*,\geq 1}$ such that the composite

$$\mathrm{CAlg}_R \xrightarrow{G} \mathcal{S}_{*,\geq 1} \xrightarrow{\Omega} \mathcal{S}_* \to \mathcal{S}$$

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is representable by a derived scheme.

Remark A.5 Note that giving a functor $G : \mathrm{CAlg}_R \to \mathrm{Fun}(\mathrm{N}(\Delta)^{op}, \mathcal{S})$ is equivalent to giving a functor $G' : \mathrm{N}(\Delta)^{op} \to \mathrm{Fun}(\mathrm{CAlg}_R, \mathcal{S})$. Using [36, 5.1.2.3] we see that the condition G factors through $\mathrm{Grp}(\mathcal{S})$ is equivalent to the condition that G' is a group object in $\mathrm{Fun}(\mathrm{CAlg}_R, \mathcal{S})$. Consequently, we have an equivalence

$$\mathrm{Fun}(\mathrm{CAlg}_R, \mathrm{Grp}(\mathcal{S})) \simeq \mathrm{Grp}(\mathrm{Fun}(\mathrm{CAlg}_R, \mathcal{S})).$$

An object $\mathrm{Grp}(\mathrm{Fun}(\mathrm{CAlg}_R, \mathcal{S}))$ is a derived group scheme if and only if the image by

$$\mathrm{Grp}(\mathrm{Fun}(\mathrm{CAlg}_R, \mathcal{S})) \to \mathrm{Fun}(\mathrm{CAlg}_R, \mathcal{S})$$

induced by $[1] \in \Delta$ is a derived scheme. Thus a derived group scheme over R is a group object of the ∞-category of derived schemes over R. The ∞-category of derived group schemes over R is equivalent to $\mathrm{Grp}(\mathrm{Sch}_R)$.

A.3. Commutative Hopf ring spectrum

We focus on the case of derived affine group schemes. An (usual) affine group schemes is regarded as the Zariski spectrum of a commutative Hopf-algebra. We will give a similar description in our setting. By Remark A.5, giving a derived affine group scheme is equivalent to giving a functor $G : \mathrm{N}(\Delta) \to \mathrm{CAlg}_R$ such that $G^{op} : \mathrm{N}(\Delta)^{op} \to \mathrm{CAlg}_R^{op} = \mathrm{Aff}_R$ is a group object in $\mathrm{CAlg}_R^{op} = \mathrm{Aff}_R$. We regard G as a functor $G : \mathrm{N}(\Delta)^{op} \to \mathrm{Aff}_R$, which is a group object. A monoid object $M : \mathrm{N}(\Delta)^{op} \to \mathrm{Aff}_R$ is a group object if and only if

$$\alpha^* \times \beta^* : M([2]) \to M([1]) \times M([1])$$

is an equivalence where $\alpha : \{0,2\} \hookrightarrow [2]$ and $\beta : \{0,1\} \hookrightarrow [2]$. We have the natural fully faithful functor

$$\mathrm{Grp}(\mathrm{Aff}_R) \to \mathrm{Fun}(\mathrm{N}(\Delta), \mathrm{CAlg}_R).$$

We refer to an object in $\mathrm{Fun}(\mathrm{N}(\Delta), \mathrm{CAlg}_R)$ which lies in the essential image of this functor as a *commutative Hopf ring spectrum* over R. We refer to the essential image, we denote by CHopf_R, as the ∞-category of commutative Hopf ring spectra over R. Note that there is a natural categorical equivalence $\mathrm{CHopf}_R^{op} \simeq \mathrm{Grp}(\mathrm{Aff}_R)$, which we refer to as the ∞-category of derived affine group schemes over R. Also, we set $\mathrm{GAff}_R := \mathrm{CHopf}_R^{op}$. Let $\mathrm{Fun}'(\mathrm{N}(\Delta)^{op}, \mathrm{Aff}_R) := \mathrm{Mon}(\mathrm{Aff}_R)$ be the full subcategory of $\mathrm{Fun}(\mathrm{N}(\Delta)^{op}, \mathrm{Aff}_R)$ spanned by monoid objects. We refer to an object in the essential image of $\mathrm{Fun}'(\mathrm{N}(\Delta)^{op}, (\mathrm{CAlg}_R)^{op}) \subset \mathrm{Fun}(\mathrm{N}(\Delta), \mathrm{CAlg}_R)$ as a commutative bi-ring spectra over R. We remark the standard fact: if M is a monoid object in $\mathcal{S}$, M is a group object in $\mathcal{S}$ if and only if a monoid $\pi_0(M)$ is a group.

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A.4. Derived group schemes, group schemes and examples

Let G be a derived group scheme over a commutative ring spectrum R. We will explain how to associate to G a (usual) group scheme $\bar{G}$ over $\pi_0(R)$. For simplicity, we here assume that G is affine, i.e., $G = \operatorname{Spec} A$. We impose some conditions on G. Let us suppose either of conditions:

(i) G is flat over R.

(ii) A and R are connective, that is, $\pi_i(A) = \pi_i(R) = 0$ for $i < 0$.

We first treat the case (i). In this case, according to [37, 8.2.2.13] there is an isomorphism $\pi_0(A \otimes_R A) \simeq \pi_0(A) \otimes_{\pi_0(R)} \pi_0(A)$ of commutative rings. Hence the group object $G : \mathrm{N}(\Delta)^{op} \to \mathrm{Aff}_R$ induces a group structure $\bar{G} : \mathrm{N}(\Delta)^{op} \to \mathrm{Aff}^0_{\pi_0(R)}$ of $\bar{G} = \operatorname{Spec}\pi_0(A)$, where $\mathrm{Aff}^0_{\pi_0(R)}$ denotes the ∞-category of ordinary affine schemes over $\pi_0(R)$.

Next we consider the case (ii). In this case, we also have an isomorphism $\pi_0(A \otimes_R A) \simeq \pi_0(A) \otimes_{\pi_0(R)} \pi_0(A)$ of commutative rings. Thus a similar argument shows that $\bar{G} := \operatorname{Spec}\pi_0(A)$ inherits a group structure. In addition, $\bar{G}$ is equivalent to the composite

$$G_0 : \mathrm{CAlg}^{\mathrm{dis}}_{H\pi_0(R)} \hookrightarrow \mathrm{CAlg}_R \xrightarrow{G} \mathrm{Grp}(\mathcal{S}) \xrightarrow{\pi_0} \mathrm{Grp}(\mathcal{S}^{\mathrm{dis}})$$

where $\mathrm{CAlg}^{\mathrm{dis}}_{H\pi_0(R)}$ is the nerve of the category of usual commutative $\pi_0(R)$-rings, the first functor is the natural functor, and $\mathcal{S}^{\mathrm{dis}}$ is the category of small sets. A group scheme H over $\pi_0(R)$ is said to be the excellent coarse moduli space of a derived group scheme G if H represents the above composite G_0. Assume further that $\pi_0(R)$ is a field. If the morphism $G \to H$ in $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{dis}}_{H\pi_0(R)}, \mathrm{Grp}(\mathcal{S}))$ is universal among morphisms into usual affine group schemes over $\pi_0(R)$, then we call H the coarse moduli space of G.

Conversely, we may regard a flat group scheme G over $\pi_0(R)$ as a derived group scheme that is flat over $H\pi_0(R)$. Here $H\pi_0(R)$ is the Eilenberg-MacLane spectrum, which is a discrete commutative ring spectrum. Set $G = \operatorname{Spec} B$. Then the usual tensor product $B \otimes_{\pi_0(R)} B$ of commutative rings coincides with the "derived" tensor product of HB and HB over $H\pi_0(R)$ in CAlg. Consequently, G can be viewed as a derived group scheme. The ∞-category of derived affine group schemes over $H\pi_0(R)$ contains the nerve of the category of affine group schemes which are flat over $\pi_0(R)$ as a full subcategory.

Finally, we give some examples of derived affine group schemes, which do not necessarily come from usual flat group schemes.

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Example A.6 Let $s : A \to R$ be an augmentation map in CAlg_R. Then we have a section $s^* : \mathrm{Spec}\, R \to \mathrm{Spec}\, A$. Let $G := \mathrm{Spec}\, R \times_{\mathrm{Spec}\, A} \mathrm{Spec}\, R$. The projection morphism $G \times_R G \simeq \mathrm{Spec}\, R \times_{\mathrm{Spec}\, A} \mathrm{Spec}\, R \times_{\mathrm{Spec}\, A} \mathrm{Spec}\, R \xrightarrow{p_{13}} \mathrm{Spec}\, R \times_{\mathrm{Spec}\, A} \mathrm{Spec}\, R \simeq G$ deterrmies a "multiplication". To make this idea precise consider the Čech nerve $\mathrm{N}(\Delta)^{op} \to \mathrm{Aff}_R$ associated to s^* (see [36, 6.1.2.11]). This Čech nerve is a derived affine group scheme over R whose underlying scheme is $\mathrm{Spec}\, R \times_{\mathrm{Spec}\, A} \mathrm{Spec}\, R$. In CAlg_R, this construction is known as a bar construction.

Example A.7 Let R be a commutative ring spectrum. Let $M \in \mathrm{PMod}_R$. Let $f : \mathrm{CAlg}_R \to \mathrm{Grp}(\mathcal{S})$ be a functor given by $A \mapsto \mathrm{Aut}(M \otimes_R A)$ (see Example 3.2). Then according to Lemma 4.7, f is representable by a derived affine group scheme over R. See also Example 3.3.

Example A.8 Let $\mathbb{S}[\mathbb{CP}^\infty] := \Sigma^\infty \mathbb{CP}^\infty_+$ be the unreduced suspention spectrum of the classifying space $\mathbb{CP}^\infty$. The commutative monoid structure in $\mathcal{S}$ (that is, E_∞-structure) of $\mathbb{CP}^\infty$ induces a commutative algebra structure on $\mathbb{S}[\mathbb{CP}^\infty]$. Namely, $\mathbb{S}[\mathbb{CP}^\infty] \in \mathrm{CAlg}$. The diagonal map $\mathbb{CP}^\infty \to \mathbb{CP}^\infty \times \mathbb{CP}^\infty$ makes $\mathbb{S}[\mathbb{CP}^\infty]$ a commutative Hopf ring spectrum and thus $\mathrm{Spec}\mathbb{S}[\mathbb{CP}^\infty]$ is a derived affine group scheme over $\mathbb{S}$ (see [44, 12.1]).

Example A.9 Let k be a number field. In [49] Spitzweck constructed the derived affine group scheme $G = \mathrm{Spec}\, B$ over $H\mathbb{Z}$ such that the ∞-category of $H\mathbb{Z}$-spectra with action of G (see below) is equivalent to the stable ∞-category of Voevodsky's category $\mathsf{DM}(k)$ of integer coefficients generated by Tate motives. (His results are much stronger, see [49].)

A.5. ∞-categories of commutative bi-ring spectra and commutative Hopf ring spectra

We will prove that ∞-categories of commutative Hopf ring spectra and commutative bi-ring spectra have good properties, that is, these are presentable ∞-categories. To this end, we first give a slighly modified description of commutative bi-ring spectra.

The ∞-category CAlg_R has the natural coCartesian symmetric monoidal structure (cf. [37]) which we will specify by a coCartesian fibration $\mathrm{CAlg}_R^\otimes \to \mathrm{N}(\mathrm{Fin}_*)$. Let $\mathrm{Ass}^\otimes \to \mathrm{N}(\mathrm{Fin}_*)$ denote the associative ∞-operad (see [37, 4.1.1.3] for the definition of associative ∞-operad $\mathrm{Ass}^\otimes$). The projection

$$p : \mathrm{CAlg}_R^{m\otimes} := \mathrm{CAlg}_R^\otimes \times_{\mathrm{N}(\mathrm{Fin}_*)} \mathrm{Ass}^\otimes \to \mathrm{Ass}^\otimes$$

is a monoidal ∞-category (cf. [37, 4.1.1.10]), that is, the underlying monoidal ∞-category of $\mathrm{CAlg}_R^\otimes$. Let us recall the construction of the opposite monoidal ∞-category. Let $\mathcal{M}^\otimes \to \mathrm{Ass}^\otimes$ be a monoidal ∞-category. Let $F_{\mathcal{M}^\otimes} : \mathrm{Ass}^\otimes \to \widehat{\mathrm{Cat}}_\infty$

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be a functor corresponding to $\mathcal{M}^{\otimes} \to \mathrm{Ass}^{\otimes}$ via the straightening functor (see [36, 3.2] for the straightening and unstraightening functors). Let $\mathrm{Op} : \widehat{\mathrm{Cat}}_{\infty} \to \widehat{\mathrm{Cat}}_{\infty}$ be the natural (auto)equivalence which carries S to the opposite category S^{op}. The composite $\mathrm{Op} \circ F_{\mathcal{M}^{\otimes}} : \mathrm{Ass}^{\otimes} \to \widehat{\mathrm{Cat}}_{\infty}$ defines a monoidal ∞-category $\mathcal{M}^{\otimes}_{op} \to \mathrm{Ass}^{\otimes}$ via the unstraightening functor. Let $\mathcal{M}$ be the underlying ∞-category of $\mathcal{M}^{\otimes}$. Roughly speaking, $\mathcal{M}^{\otimes}_{op} \to \mathrm{Ass}^{\otimes}$ is the ∞-category $\mathcal{M}^{op}$ endowed with the monoidal structure given by $\otimes^{op} : (\mathcal{M} \times \mathcal{M})^{op} \to \mathcal{M}^{op}$ where $\otimes$ indicates the monoidal operation of $\mathcal{M}$. If a monoidal ∞-category $\mathcal{N}^{\otimes} \to \mathrm{Ass}^{\otimes}$ is equivalent to $\mathcal{M}^{\otimes}_{op} \to \mathrm{Ass}^{\otimes}$, then we shall refer to $\mathcal{N}^{\otimes} \to \mathrm{Ass}^{\otimes}$ as the opposite monoidal ∞-category of $\mathcal{M}^{\otimes} \to \mathrm{Ass}^{\otimes}$. If we replace $\mathrm{Ass}^{\otimes}$ by $\mathrm{N}(\mathrm{Fin}_*)$, we obtain the opposite symmetric monoidal ∞-category of a symmetric monoidal ∞-category.

Let $q : (\mathrm{CAlg}_R)^{m\otimes}_{op} \to \mathrm{Ass}^{\otimes}$ denote the opposite monoidal ∞-category of p. We define $\mathrm{CoAlg}(\mathrm{CAlg}^{\otimes}_R)$ to be $\mathrm{Alg}_{/\mathrm{Ass}^{\otimes}}((\mathrm{CAlg}_R)^{m\otimes}_{op})^{op}$ where $\mathrm{Alg}_{/\mathrm{Ass}^{\otimes}}((\mathrm{CAlg}_R)^{m\otimes}_{op})$ is the ∞-category of algebra objects. We refer to $\mathrm{CoAlg}(\mathrm{CAlg}^{\otimes}_R)$ as the ∞-category of commutative bi-ring spectra over R (or commutative bi-ring R-module spectra).

Now we show that this definition is compatible with the above definition. The opposite symmetric monoidal ∞-category $(\mathrm{CAlg}_R)^{\otimes}_{op} \to \mathrm{N}(\mathrm{Fin}_*)$ of $p : \mathrm{CAlg}^{\otimes}_R \to \mathrm{N}(\mathrm{Fin}_*)$ is a Cartesian monoidal ∞-category (cf. [37, 2.4.0.1]). By [37, 2.4.1.9], there is a Cartesian structure [37, 2.4.1.1] $(\mathrm{CAlg}_R)^{\otimes}_{op} \to (\mathrm{CAlg}_R)^{op}$ and it induces the second categorical equivalence in

$$\begin{aligned}\mathrm{Alg}_{/\mathrm{Ass}^{\otimes}}((\mathrm{CAlg}_R)^{\otimes}_{op}) &\simeq \mathrm{Alg}_{/\mathrm{N}(\Delta)^{op}}((\mathrm{CAlg}_R)^{\otimes}_{op} \times_{\mathrm{Ass}^{\otimes}} \mathrm{N}(\Delta)^{op}) \\ &\simeq \mathrm{Fun}'(\mathrm{N}(\Delta)^{op}, (\mathrm{CAlg}_R)^{op}),\end{aligned}$$

where the first equivalence is induced by the map $\mathrm{Cut} : \mathrm{N}(\Delta)^{op} \to \mathrm{Ass}^{\otimes}$ defined in [37, 4.1.2.5] and [37, 4.1.2.15], and $\mathrm{Fun}'(\mathrm{N}(\Delta)^{op}, (\mathrm{CAlg}_R)^{op})$ is the full subcategory of monoid objects (see Appendix A.3). Remark that $f : \mathrm{N}(\Delta)^{op} \to (\mathrm{CAlg}_R)^{op}$ lies in $\mathrm{Fun}'(\mathrm{N}(\Delta)^{op}, (\mathrm{CAlg}_R)^{op})$ if and only if maps $\{i-1, i\} \hookrightarrow [n]$ with $1 \leq i \leq n$ induce an equivalence $\otimes_{1 \leq i \leq n} f([1]) \to f([n])$ for each n, and $f([0]) \simeq R$. Consequently, $\mathrm{CoAlg}(\mathrm{CAlg}^{\otimes}_R)$ is naturally equivalent to $\mathrm{Fun}'(\mathrm{N}(\Delta), \mathrm{CAlg}_R)$ where $\mathrm{Fun}'(\mathrm{N}(\Delta), \mathrm{CAlg}_R)$ again denotes the full subcategory of $\mathrm{Fun}(\mathrm{N}(\Delta), \mathrm{CAlg}_R)$ spanned by comonoid objects.

Let $a = \{0,2\} \hookrightarrow [2]$ and $b = \{0,1\} \hookrightarrow [2]$. Let $C : \mathrm{N}(\Delta) \to \mathrm{CAlg}_R$ be in $\mathrm{Fun}'(\mathrm{N}(\Delta), \mathrm{CAlg}_R) \simeq \mathrm{CoAlg}(\mathrm{CAlg}^{\otimes}_R)$. The object C is a commutative Hopf ring spectrum if and only if $C(a)$ and $C(b)$ determine $u : C([1]) \to C([1]) \otimes_R C([1])$ and $v : C([1]) \to C([1]) \otimes_R C([1])$ such that $u \otimes v : C([1]) \otimes_R C([1]) \to C([1]) \otimes_R C([1])$ is an equivalence in CAlg_R. The spectrum R is a unit of the symmetric monoidal ∞-category $\mathrm{CAlg}^{\otimes}_R$ and thus R is promoted to an object in $\mathrm{CoAlg}(\mathrm{CAlg}^{\otimes}_R)$. Clearly, R is a commutative Hopf ring spectrum. The ∞-category CHopf_R is contained in

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$\mathrm{CoAlg}(\mathrm{CAlg}_R^{\otimes})$ as a full subcategory. Yoneda lemma implies the natural functor $\mathrm{CHopf}_R^{op} \to \mathrm{Fun}(\mathrm{CAlg}_R, \mathrm{Grp}(\mathcal{S}))$ is fully faithful.

Lemma A.10 *The natural inclusion* $\mathrm{Fun}'(\mathrm{N}(\Delta), \mathrm{CAlg}_R) \to \mathrm{Fun}(\mathrm{N}(\Delta), \mathrm{CAlg}_R)$ *preserves small colimits. Moreover, the inclusion* $\mathrm{CHopf}_R \hookrightarrow \mathrm{CoAlg}(\mathrm{CAlg}_R^{\otimes}) \simeq \mathrm{Fun}'(\mathrm{N}(\Delta), \mathrm{CAlg}_R)$ *preserves small colimits.*

Proof: We will prove the first assertion. Let I be a small ∞-category and $I \to \mathrm{Fun}'(\mathrm{N}(\Delta), \mathrm{CAlg}_R)$ a functor. We will claim that a colimit of the composition $q : I \to \mathrm{Fun}'(\Delta, \mathrm{CAlg}_R) \to \mathrm{Fun}(\mathrm{N}(\Delta), \mathrm{CAlg}_R)$ satisfies the comonoid condition. For $\lambda \in I$, we set $A_\lambda = q(\lambda)([1])$. Note that $q([0]) \simeq R$ and $q(\lambda)([n])$ is equivalent to the n-fold tensor product $A_\lambda \otimes_R \ldots \otimes_R A_\lambda$. By [36, 5.1.2.3], the n-th term of the colimit of q is $\mathrm{colim}(A_\lambda \otimes_R \ldots \otimes_R A_\lambda)$ (indexed by I) in CAlg_R. It will suffice to prove that for each $n \in \mathbb{N}$, inclusions $\{i-1, i\} \hookrightarrow [n]$ for $1 \leq i \leq n$ induces an equivalence

$$\mathrm{colim}(A_\lambda) \otimes_R \ldots \otimes_R \mathrm{colim}(A_\lambda) \to \mathrm{colim}(A_\lambda \otimes_R \ldots \otimes_R A_\lambda).$$

According to [36, 4.4.2.7], we may assume that I is either a pushout diagram or a coproduct diagram. For simplicity, suppose that $n = 2$. (The general case is straightforward.) Note that the symmetric monoidal structure of CAlg_R is coCartesian. In the coproduct case, $(\otimes_\lambda A_\lambda) \otimes_R (\otimes_\lambda A_\lambda) \simeq \otimes_\lambda (A_\lambda \otimes_R A_\lambda)$. In the case of pushout, for a diagram $A \leftarrow C \to B$ in CAlg_R, we have an equivalence $(A \otimes_C B) \otimes_R (A \otimes_C B) \simeq (A \otimes_R A) \otimes_{C \otimes_R C} (B \otimes_R B)$. Hence our claim follows.

Next we will prove that the inclusion

$$\mathrm{CHopf}_R \hookrightarrow \mathrm{CoAlg}(\mathrm{CAlg}_R^{\otimes}) \simeq \mathrm{Fun}'(\mathrm{N}(\Delta), \mathrm{CAlg}_R)$$

preserves small colimits. Let I be a small ∞-category and $I \to \mathrm{CHopf}_R$ a functor. Let $q : I \to \mathrm{CHopf}_R \hookrightarrow \mathrm{Fun}'(\Delta, \mathrm{CAlg}_R)$ be the composition. We adopt the notation similar to the above paragraph. We claim that the colimit of q belongs to CHopf_R. By assumption, $a = \{0,2\} \hookrightarrow [2]$ and $b = \{0,1\} \hookrightarrow [2]$ and the colimits induce a diagram

$$\begin{array}{ccccc} \mathrm{colim} A_\lambda & \longrightarrow & \mathrm{colim}(A_\lambda \otimes_R A_\lambda) & \longleftarrow & \mathrm{colim} A_\lambda \\ & \searrow^{\mathrm{colim}(a_*)} & \downarrow & \swarrow_{\mathrm{colim}(b_*)} & \\ & & \mathrm{colim}(A_\lambda \otimes_R A_\lambda) & & \end{array}$$

where the the upper horizontal diagram is the colimit of the coproduct diagrams $A_\lambda \to A_\lambda \otimes_R A_\lambda \leftarrow A_\lambda$. The vertical arrow in the middle is an equivalence (by our assumption). Moreover, in the previous paragraph, we have shown that the upper horizontal diagram exhibits $\mathrm{colim}(A_\lambda \otimes_R A_\lambda)$ as the coproduct of $\mathrm{colim} A_\lambda$ and $\mathrm{colim} A_\lambda$. This implies that the colimit of q belongs to CHopf_R. □

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Proposition A.11 *The ∞-category* $\mathrm{CoAlg}(\mathrm{CAlg}_R^{\otimes})$ *is a presentable ∞-category.*

Proof: Let $\mathcal{C}$ be a subcategory of $\widehat{\mathrm{Cat}}_\infty$ such that:

- objects are ∞-categories $\mathcal{X}$ such that $\mathcal{X}^{op}$ is an accessible ∞-category,
- morphisms are functors $F : \mathcal{X} \to \mathcal{Y}$ such that $F^{op} : \mathcal{X}^{op} \to \mathcal{Y}^{op}$ are accessible functors.

Note that $\mathrm{Op} : \widehat{\mathrm{Cat}}_\infty \to \widehat{\mathrm{Cat}}_\infty$ which sends $\mathcal{X}$ to $\mathcal{X}^{op}$ is a categorical equivalence. Moreover by [36, 5.4.7.3] the limit of accessible ∞-categories in $\widehat{\mathrm{Cat}}_\infty$ exists and it is an accessible ∞-category. These observations together with [36, 5.4.4.3, 5.1.2.3] imply that $\mathcal{C} \subset \widehat{\mathrm{Cat}}_\infty$ satisfies the conditions (a), (b), (c) in [36, 5.4.7.11]. Since the monoidal structure on CAlg_R is compatible with small colimits, combined with [37, 3.2.3.4] we can apply [36, 5.4.7.14] to deduce that $\mathrm{CoAlg}(\mathrm{CAlg}_R^{\otimes})$ is accessible. Finally, $\mathrm{CoAlg}(\mathrm{CAlg}_R^{\otimes})$ admits small colimits since $\mathrm{Fun}(\mathrm{N}(\Delta),\mathrm{CAlg}_R)$ is presentable and $\mathrm{CoAlg}(\mathrm{CAlg}_R^{\otimes}) \subset \mathrm{Fun}(\mathrm{N}(\Delta),\mathrm{CAlg}_R)$ is stable under small colimits. □

Proposition A.12 *The ∞-category* CHopf_R *is a presentable ∞-category.*

Proof: Let $V \to \mathrm{N}(\Delta)$ denote the inclusion corresponding to

$$\begin{array}{ccc} & & a = \{0,2\} \\ & & \downarrow \\ b = \{0,1\} & \longrightarrow & c = [2] \end{array}$$

where two maps are inclusions. Namely, V has exactly three objects a,b,c, and non-degenerate maps are $a \to c$ and $b \to c$. The composition with $V \to \mathrm{N}(\Delta)$ determines a map $\mathrm{Fun}'(\mathrm{N}(\Delta),\mathrm{CAlg}_R) \to \mathrm{Fun}(V,\mathrm{CAlg}_R)$. For $p : V \to \mathrm{CAlg}_R$, p induces $p(a) \otimes p(b) \to p(c)$ since $p(a) \otimes p(b)$ is a coproduct of $p(a)$ and $p(b)$. By left Kan extension it yields $\mathrm{Fun}(V,\mathrm{CAlg}_R) \to \mathrm{Fun}(\Delta^1,\mathrm{CAlg}_R)$ which carries p to $p(a) \otimes p(b) \to p(c)$, and we have the composition $\sigma : \mathrm{Fun}'(\mathrm{N}(\Delta),\mathrm{CAlg}_R) \to \mathrm{Fun}(\Delta^1,\mathrm{CAlg}_R)$. By the definition of CHopf_R, we have a homotopy cartesian square

$$\begin{array}{ccc} \mathrm{CHopf}_R & \longrightarrow & \mathrm{Fun}'(\mathrm{N}(\Delta),\mathrm{CAlg}_R) \\ \downarrow & & \downarrow \sigma \\ \mathrm{Fun}_{\simeq}(\Delta^1,\mathrm{CAlg}_R) & \xrightarrow{\tau} & \mathrm{Fun}(\Delta^1,\mathrm{CAlg}_R) \end{array}$$

where $\mathrm{Fun}_{\simeq}(\Delta^1,\mathrm{CAlg}_R)$ is the full subcategory of $\mathrm{Fun}(\Delta^1,\mathrm{CAlg}_R)$ spanned by maps $\Delta^1 \to \mathrm{CAlg}_R$ which correspond to equivalences in CAlg_R, and τ is

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the inclusion. Since $\mathrm{Fun}_{\simeq}(\Delta^1,\mathrm{CAlg}_R) \simeq \mathrm{CAlg}_R$, τ preserves small colimits. According to [36, 5.1.2.3], we see that σ preserves small colimits (by noting that by Lemma A.10 $\mathrm{Fun}'(\mathrm{N}(\Delta),\mathrm{CAlg}_R) \to \mathrm{Fun}(\mathrm{N}(\Delta),\mathrm{CAlg}_R)$ preserves small colimits). Note that by Proposition A.11 and [36, 5.4.4.3] $\mathrm{Fun}'(\mathrm{N}(\Delta),\mathrm{CAlg}_R)$, $\mathrm{Fun}(\Delta^1,\mathrm{CAlg}_R)$ and $\mathrm{Fun}_{\simeq}(\Delta^1,\mathrm{CAlg}_R)$ are presentable ∞-categories (we remark that CAlg_R is presentable). Thus by virtue of [36, 5.5.3.13] we see that CHopf_R is also presentable. □

As a corollary of these results, we have

Corollary A.13 *Let* GAff_R *be the* ∞*-category of derived affine group schemes over* R*. Then* GAff_R *has small colimits and limits. The forgetful functor* $\mathrm{GAff}_R \to \mathrm{Aff}_R$ *preserves small limits.*

A.6. Representations of commutative bi-ring spectra and commutative Hopf ring spectra

We will construct a functor $\mathrm{CoAlg}(\mathrm{CAlg}) \to \widehat{\mathrm{Cat}}_\infty$ which carries $B \in \mathrm{CoAlg}(\mathrm{CAlg})$ to the stable presentable ∞-category Rep_B consisting of spectra endowed with coaction of B. Informally, Rep_B is the ∞-category of spectra N endowed with action of the derived monoid scheme $\mathrm{Spec}\, B$ which associates an *automorphism* $N \otimes V \xrightarrow{\sim} N \otimes V$ to each point $\mathrm{Spec}\, V \to \mathrm{Spec}\, B$ with $V \in \mathrm{CAlg}$. Thus when B does not lie in CHopf, roughly speaking, Mod_B (which we are going to define) does not coincide with the ∞-category of "comodules" of B. We believe that the notation Rep_B is little confusing.

Before we define the ∞-category Rep_B for $B \in \mathrm{CoAlg}(\mathrm{CAlg})$, we recall the ∞-category $\widehat{\mathrm{Cat}}_\infty^{\mathrm{L,st}}$ of stable presnetable ∞-categories and the functor $\mathrm{CAlg} \to \widehat{\mathrm{Cat}}_\infty^{\mathrm{L,st}}$ which to any $R \in \mathrm{CAlg}$ associates the ∞-category Mod_R of left R-module spectra.

Let $\widehat{\mathrm{Cat}}_\infty^{\mathrm{L,st}}$ be the ∞-category of presentable stable ∞-categories where morphisms are colimit-preserving functors. (This category is a subcategory of $\widehat{\mathrm{Cat}}_\infty$.) There is a natural symmetric monoidal structure on $\widehat{\mathrm{Cat}}_\infty^{\mathrm{L,st}}$ which commutes with small colimit separately in each variable (see [37, 6.3.2]). For $\mathcal{C},\mathcal{D} \in \widehat{\mathrm{Cat}}_\infty^{\mathrm{L,st}}$, the tensor product $\mathcal{C} \otimes \mathcal{D}$ has the following universality: There is a functor $\mathcal{C} \times \mathcal{D} \to \mathcal{C} \otimes \mathcal{D}$ which preserves small colimits separately in each variable, and if $\mathcal{E}$ belongs to $\widehat{\mathrm{Cat}}_\infty^{\mathrm{L,st}}$ and $\mathrm{Fun}_c(\mathcal{C} \times \mathcal{D},\mathcal{E})$ denotes the full subcategory of $\mathrm{Fun}(\mathcal{C} \times \mathcal{D},\mathcal{E})$ spanned by functors which preserve small colimits separately in each variable, then the composition induces a categorical equivalence

$$\mathrm{Fun}^{\mathrm{L}}(\mathcal{C} \otimes \mathcal{D},\mathcal{E}) \to \mathrm{Fun}_c(\mathcal{C} \times \mathcal{D},\mathcal{E}),$$

where $\mathrm{Fun}^{\mathrm{L}}(-,-)$ on the left side of the equivalence indicates the full subcategory of $\mathrm{Fun}(-,-)$ spanned by colimit-preserving functors. An object $\mathrm{CAlg}(\widehat{\mathrm{Cat}}_\infty^{\mathrm{L,st}})$ can

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be regarded as a symmetric monoidal stable presentable ∞-category whose tensor product preserves small colimits separately in each variable.

Let $\mathrm{LM}^{\otimes}$ be the ∞-operad of left modules (see for the definition [37, 4.2.1.7]). Consider the symmetric monoidal ∞-category $\mathrm{Sp}^{\otimes} \to \mathrm{N}(\mathrm{Fin}_*)$ of spectra. The natural fibration $\mathrm{LM}^{\otimes} \to \mathrm{Ass}^{\otimes}$ and its section $\mathrm{Ass}^{\otimes} \hookrightarrow \mathrm{LM}^{\otimes}$ of ∞-operads described in [37, 4.2.1.9, 4.2.1.10] determine a map

$$\phi : \mathrm{LMod}(\mathrm{Sp}) = \mathrm{Alg}_{\mathrm{LM}^{\otimes}/\mathrm{Ass}^{\otimes}}(\mathrm{Sp}^{\otimes}) \to \mathrm{Alg}_{\mathrm{Ass}^{\otimes}/\mathrm{N}(\mathrm{Fin}_*)}(\mathrm{Sp}^{\otimes}).$$

By [37, 6.3.3.15] ϕ is a coCartesian fibrarion (informally for $R \to R' \in \mathrm{Alg}_{\mathrm{Ass}^{\otimes}/\mathrm{N}(\mathrm{Fin}_*)}(\mathrm{Sp}^{\otimes})$ and $(R, M) \in \mathrm{LMod}(\mathrm{Sp})$, $M \to M \otimes_R R'$ is a coCartesian edge lying over it). Thus the straightening functor gives rise to $\mathrm{Alg}_{\mathrm{Ass}^{\otimes}/\mathrm{N}(\mathrm{Fin}_*)}(\mathrm{Sp}^{\otimes}) \to \widehat{\mathrm{Cat}}_{\infty}$ which factors through $\mathrm{Alg}_{\mathrm{Ass}^{\otimes}/\mathrm{N}(\mathrm{Fin}_*)}(\mathrm{Sp}^{\otimes}) \to \widehat{\mathrm{Cat}}_{\infty}^{\mathrm{L,st}}$. It is extended to a functor between the ∞-categories of commutative algebra objects

$$\mathrm{CAlg}(\mathrm{Alg}_{\mathrm{Ass}^{\otimes}/\mathrm{N}(\mathrm{Fin}_*)}(\mathrm{Sp}^{\otimes})) \to \mathrm{CAlg}(\widehat{\mathrm{Cat}}_{\infty}^{\mathrm{L,st}})$$

(cf. [37, 6.3.5.16]). As explained in the proof of [37, 6.3.5.18], the unique bifunctor $\mathrm{Ass}^{\otimes} \times \mathrm{Comm}^{\otimes} \to \mathrm{Comm}^{\otimes}$ of ∞-operads (here the ∞-operad $\mathrm{Comm}^{\otimes}$ is determined by the identity map $\mathrm{Comm}^{\otimes} := \mathrm{N}(\mathrm{Fin}_*) \to \mathrm{N}(\mathrm{Fin}_*)$) exhibits $\mathrm{Comm}^{\otimes}$ as a tensor product of $\mathrm{Ass}^{\otimes}$ and $\mathrm{Comm}^{\otimes}$. It follows a categorical equivalence

$$\mathrm{CAlg}(\mathrm{Sp}^{\otimes}) \to \mathrm{CAlg}(\mathrm{Alg}_{\mathrm{Ass}^{\otimes}/\mathrm{N}(\mathrm{Fin}_*)}(\mathrm{Sp}^{\otimes})).$$

Thus we have

$$\Theta : \mathrm{CAlg} \longrightarrow \mathrm{CAlg}(\widehat{\mathrm{Cat}}_{\infty}^{\mathrm{L,st}})$$

which carries A to $\mathrm{Mod}_R^{\otimes}$.

Next using Θ, for any $B \in \mathrm{CoAlg}(\mathrm{CAlg}_R^{\otimes})$ we will define an ∞-category Mod_B in a functorial fashion. Remember that the ∞-category $\mathrm{CoAlg}(\mathrm{CAlg}_R^{\otimes})$ is equivalent to the ∞-category $\mathrm{Fun}'(\mathrm{N}(\Delta), \mathrm{CAlg}_R)$ of comonoid objects. The functor Θ naturally induces $\Theta_R : \mathrm{CAlg}_R \simeq \mathrm{CAlg}_{R/} \to \mathrm{CAlg}(\widehat{\mathrm{Cat}}_{\infty}^{\mathrm{L,st}})_{\mathrm{Mod}_R^{\otimes}/}$ (the first equivalence follows from [37, 3.4.1.7]). Hence composing $\mathrm{CoAlg}(\mathrm{CAlg}_R^{\otimes}) \simeq \mathrm{Fun}'(\mathrm{N}(\Delta), \mathrm{CAlg}_R)$ with it we have

$$\mathrm{CoAlg}(\mathrm{CAlg}_R^{\otimes}) \to \mathrm{Fun}(\mathrm{N}(\Delta), \mathrm{CAlg}(\widehat{\mathrm{Cat}}_{\infty}^{\mathrm{L,st}})_{\mathrm{Mod}_R^{\otimes}/}).$$

Since $\mathrm{CAlg}(\widehat{\mathrm{Cat}}_{\infty}^{\mathrm{L,st}})_{\mathrm{Mod}_R^{\otimes}/}$ admits small limits (because it is presentable by [36, 5.5.3.11]), there is a right adjoint of

$$\mathrm{CAlg}(\widehat{\mathrm{Cat}}_{\infty}^{\mathrm{L,st}})_{\mathrm{Mod}_R^{\otimes}/} \to \mathrm{Fun}(\mathrm{N}(\Delta), \mathrm{CAlg}(\widehat{\mathrm{Cat}}_{\infty}^{\mathrm{L,st}})_{\mathrm{Mod}_R^{\otimes}/})$$

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induced by the obvious map $\mathrm{N}(\Delta) \to \Delta^0$. Namely, the right adjoint

$$\mathrm{Fun}(\mathrm{N}(\Delta), \mathrm{CAlg}(\widehat{\mathrm{Cat}}_\infty^{\mathrm{L,st}})_{\mathrm{Mod}_R^\otimes/}) \to \mathrm{CAlg}(\widehat{\mathrm{Cat}}_\infty^{\mathrm{L,st}})_{\mathrm{Mod}_R^\otimes/}$$

sends $\mathrm{N}(\Delta) \to \mathrm{CAlg}(\widehat{\mathrm{Cat}}_\infty^{\mathrm{L,st}})_{\mathrm{Mod}_R^\otimes/}$ to its limit. Combining all together we have

$$\mathrm{CoAlg}(\mathrm{CAlg}_R^\otimes) \to \mathrm{Fun}(\mathrm{N}(\Delta), \mathrm{CAlg}(\widehat{\mathrm{Cat}}_\infty^{\mathrm{L,st}})_{\mathrm{Mod}_R^\otimes/}) \to \mathrm{CAlg}(\widehat{\mathrm{Cat}}_\infty^{\mathrm{L,st}})_{\mathrm{Mod}_R^\otimes/}$$

and for $B \in \mathrm{CoAlg}(\mathrm{CAlg}_R^\otimes)$ we set its image $\mathrm{Mod}_R^\otimes \to \mathrm{Mod}_B^\otimes \in \mathrm{CAlg}(\widehat{\mathrm{Cat}}_\infty^{\mathrm{L,st}})_{\mathrm{Mod}_R^\otimes/}$ which we refer to as the R-linear symmetric monoidal ∞-category of representations of the commutative bi-ring spectrum B (here R-linear structure means a symmetric monoidal colimit-preserving functor $\mathrm{Mod}_R^\otimes \to \mathrm{Rep}_G^\otimes$). If $G = \mathrm{Spec}\, B$ is a derived affine group (monoid) scheme over R, then we often write Rep_G for Rep_B.

Proposition A.14 *The ∞-category Rep_G is a stable presentable ∞-category endowed with a symmetric monoidal structure which preserves small colimits separately in each variable.*

Let $\mathrm{Rep}_G^\otimes$ denote the symmetric monoidal ∞-category of representations of G. The unit $u : \mathrm{Spec}\, R \to G$ induces a symmetric monoidal functor $u^*\mathrm{Rep}_G^\otimes \to \mathrm{Mod}_R^\otimes$. Let $\mathrm{PRep}_G^\otimes$ be the symmetric monoidal full subcategory of $\mathrm{Rep}_G^\otimes$ spanned by dualizable objects. An object $M \in \mathrm{Rep}_G$ lies in PRep_G if and only if $u^*(M)$ lies in PMod_R. We refer to PRep_G as the ∞-category of perfect representations of G. We can easily deduce the following:

Proposition A.15 *The ∞-category PRep_G is a small stable idempotent complete ∞-category endowed with a symmetric monoidal structure which preserves finite colimits separately in each variable.*

Let $(\mathrm{CAlg}_R)^{op} \hookrightarrow \mathrm{Fun}(\mathrm{CAlg}_R, \widehat{\mathcal{S}})$ be Yoneda embedding, where $\widehat{\mathcal{S}}$ denotes the ∞-category of (not necessarily small) spaces, i.e. Kan complexes. We shall refer to objects in $\mathrm{Fun}(\mathrm{CAlg}_R, \widehat{\mathcal{S}})$ as presheaves on CAlg_R or simply functors. By left Kan extension of Θ_R, we have a colimit-preserving functor

$$\overline{\Theta}_R : \mathrm{Fun}(\mathrm{CAlg}_R, \widehat{\mathcal{S}}) \to \mathrm{CAlg}(\widehat{\mathrm{Cat}}_\infty)^{op}.$$

For $X \in \mathrm{Fun}(\mathrm{CAlg}_R, \widehat{\mathcal{S}})$, we write $\mathrm{Mod}_X^\otimes$ for $\overline{\Theta}_R(X)$. We denote by $\mathrm{PMod}_X^\otimes$ the full subcategory spanned by dualizable objects. Let G be a derived affine group scheme and let $\psi : \mathrm{N}(\Delta)^{op} \to \mathrm{Aff}_R$ be the corresponding simplicial object. Let $\mathrm{N}(\Delta)^{op} \xrightarrow{\psi} (\mathrm{CAlg}_R)^{op} \hookrightarrow \mathrm{Fun}(\mathrm{CAlg}_R, \widehat{\mathcal{S}})$ be the composition and let $\mathsf{B}G$ denote the colimit. Remember $\overline{\Theta}_R(\mathsf{B}G) = \mathrm{Mod}_{\mathsf{B}G}^\otimes \simeq \mathrm{Rep}_G^\otimes$ and $\mathrm{PMod}_{\mathsf{B}G}^\otimes \simeq \mathrm{PRep}_G^\otimes$.

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https://doi.org/10.1017/is014008019jkt278 Published online by Cambridge University Press

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Received: February 17, 2014

https://doi.org/10.1017/is014008019jkt278 Published online by Cambridge University Press