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L^2 estimates for convolution operators with oscillating kernels

BY YIBIAO PAN

Department of Mathematics and Statistics, University of Pittsburgh, Pittsburgh, PA 15260, *U.S.A.*

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1. *Introduction*

Consider the operator

$$T_\lambda f(x) = \int_{\mathbb{R}^n} e^{i\lambda\Phi(x,y)} K(x-y)\,\phi(x,y) f(y)\,dy, \tag{1·1}$$

where $\Phi \in C^\infty$, $\phi \in C_0^\infty$, and K is smooth away from the origin and homogeneous of degree $-\mu$, for some $\mu \geqslant 0$. The following theorem was proved in [**4**].

THEOREM 1·1. *If* rank $(\partial^2\Phi/\partial x_j\,\partial y_k) \geqslant k$, $n-k < \mu < n$, *then*

$$\|T_\lambda\|_{L^2\to L^2} \leqslant C\lambda^{-(n-\mu)/2}. \tag{1·2}$$

Theorem 1·1 generalized an earlier result of Phong and Stein in [**6**] for operators with bilinear phase functions. However, the most interesting problem concerns the norm of the operators in the critical case, which occurs when $\mu = n-k$, $0 \leqslant k \leqslant n$. The problem is to determine when the optimal decay estimate

$$\|T_\lambda\|_{L^2\to L^2} \leqslant C\lambda^{-\frac{1}{2}k} \tag{1·3}$$

holds. There are two extreme cases which are relatively well understood, as follows.

(i) $k = n$, $\mu = 0$. In this case the Hessian of Φ is non-singular, and K is homogeneous of degree 0. It was shown in [**4**] that $\|T_\lambda\|_{L^2\to L^2}$ has the optimal decay $\lambda^{-n/2}$. If K is assumed to be smooth, the result is previously known as the generalized Hausdorff–Young inequality (see, e.g. [**1**]).

(ii) $k = 0$, $\mu = n$. In this case we have no assumption on the rank of the Hessian of Φ and we want to know if $\|T_\lambda\|_{L^2\to L^2}$ is uniformly bounded. If $\Phi \equiv 0$, T_λ is reduced to the usual singular integral operators and it is bounded on L^2 if and only if K satisfies

$$\int_{x\in S^{n-1}} K(x)\,d\sigma(x) = 0. \tag{1·4}$$

For operators with non-trivial phase functions, the result of Ricci and Stein in [**7**] implies that $\|T_\lambda\|$ is uniformly bounded if K satisfies (1·4) and Φ is a real polynomial (see [**6**, **7**]). On the basis of their work, the author proved that for general phase functions, $\|T_\lambda\|_{L^2\to L^2}$ are uniformly bounded if some entry of the Hessian of Φ does not vanish of infinite order (while the rank of the Hessian of Φ may be zero). It is also known that the uniform boundedness does not hold if this condition is removed (see [**3**, **4**]).

The generic case $0 < k < n$ appears to be much harder. So far the only known result concerns operators with bilinear phases, for which the complete solution was given in [**5**]. It can be stated as follows.

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THEOREM 1·2. *Let (Bx, y) be a real bilinear form, and $\Phi(x, y) = (Bx, y)$. Assume that* rank $(B) = k$, $0 < k < n$, *$K(x)$ is homogeneous of degree $-(n-k)$ and smooth away from the origin. Let $r(B) =$* range (B).

(i) *Suppose $r(B) = r(B^t)$, where B^t is the transpose of B. Then* (1·3) *holds if and only if $K|_{r(B)^\perp}$ has mean value zero over the unit sphere in $r(B)^\perp$.*

(ii) *If $r(B) \neq r(B^t)$, then* (1·3) *always holds.*

This suggests that when the phase function is 'sufficiently symmetric' for (1·3) to hold, K has to have some cancellation property.

For generic phase functions (not necessarily bilinear), the problem becomes more complicated. First of all, it is not clear how we should formulate the cancellation condition in Theorem 1·2 for general phase functions, since the Hessian matrix of Φ is no longer a constant matrix and the subspace $r(B)$ may vary from point to point. Secondly, as we shall see later, (1·3) may fail even if $K(x)$ has the best cancellation property (e.g. $K(x)$ is odd).

The purpose of this paper is to study this intriguing problem and present both positive and negative results. It is revealed that operators with real analytic phases and operators with merely smooth phases behave quite differently. We shall restrict our attention to $k = 1$, $\mu = n-1$, and we consider convolution type operators only.

2. *Statement of result and a few lemmas*

$$\text{Let} \quad T_\lambda f(x) = (e^{i\lambda\Phi(\cdot)}K(\cdot)\,\phi(\cdot)) * f(x) = \int_{\mathbb{R}^n} e^{i\lambda\Phi(x-y)}K(x-y)\,\phi(x-y)f(y)\,dy, \tag{2·1}$$

where $\Phi \in C^\infty$, $\phi \in C_0^\infty$. Let $K(x)$ be homogeneous of degree $-(n-1)$ and smooth away from the origin. Let $H_\Phi(x) = (\partial^2\Phi(x)/\partial x_j\,\partial x_k)$, and $B = H_\Phi(0)$.

THEOREM A. *Let K, T_λ be given as above. Assume that Φ is real analytic in* supp (ϕ), rank $(H_\Phi(x)) \geqslant 1$, *for $x \in$* supp (ϕ) *and* rank $(B) = 1$. *If $K|_{r(B)^\perp}$ has vanishing mean-value over the unit sphere of $r(B)^\perp$, then*

$$\|T_\lambda f\|_{L^2} \leqslant C\lambda^{-\frac{1}{2}}\|f\|_{L^2} \tag{2·2}$$

holds uniformly in λ.

Remark 1. Let $E = r(B)^\perp$. We have dim $(E) = n-1$. Let L be a linear transformation which maps $\mathbb{R}^{n-1}$ onto E, i.e. $L(\mathbb{R}^{n-1}) = E$. The condition that $K|_E$ has vanishing mean value over the unit sphere of E means that

$$\int_{z \in S^{n-2}} K(Lz)\,d\sigma(z) = 0. \tag{2·3}$$

It can be verified that the condition is independent of the choice of L.

Remark 2. The theorem becomes false if the assumption on the analyticity of Φ is dropped (see Section 4).

Since the operator T_λ is of convolution type, by taking Fourier transform we have

$$\widehat{T_\lambda f}(\xi) = m_\lambda(\xi)\hat{f}(\xi),$$

where

$$m_\lambda(\xi) = \int_{\mathbb{R}^n} e^{i(\xi\cdot x+\lambda\Phi(x))}K(x)\,\phi(x)\,dx. \tag{2·4}$$

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Using Plancherel's theorem, we see that, to prove (2·2), it suffices to prove that

$$\|m_\lambda\|_\infty \leqslant C\lambda^{-\frac{1}{2}} \tag{2·5}$$

uniformly in λ.

We now list and prove a few lemmas that are needed in the proof of Theorem A. First we have the following version of Van der Corput's lemma.

LEMMA 2·1. *Suppose that $\psi \in C_0^\infty(\mathbb{R}^m)$, ϕ is real-valued, and for some multi-index α with $|\alpha| > 0$,*

$$\left(\frac{\partial}{\partial x}\right)^\alpha \phi(x) \geqslant 1$$

throughout the support of ψ. Then

$$\left| \int_{\mathbb{R}^m} e^{i\lambda\phi(x)}\psi(x)\,dx \right| \leqslant c_k \lambda^{-\frac{1}{k}}(\|\psi\|_\infty + \|\nabla\psi\|_1)$$

with $k = |\alpha|$, and the constant c_k is independent of λ and ψ and remains bounded as long as the C^{k+1} norm of ϕ remains bounded.

A proof of this lemma can be found in [**8**].

LEMMA 2·2. *Assume that $P(x)$ is a real polynomial in $x, x \in \mathbb{R}^m$, and $k(x)$ satisfies*

$$|k(x)| \leqslant A|x|^{-m}, \quad |\nabla k(x)| \leqslant A|x|^{-m-1},$$

and $\|\hat{k}\|_\infty \leqslant B$. Then

$$\left| \text{p.v.} \int_{\mathbb{R}^m} e^{iP(x)}k(x)\,dx \right| \leqslant C, \tag{2·6}$$

with a constant C that depends only on A, B and the degree of P, and not on the coefficients of P.

This lemma is essentially theorem 5 in [**8**], except that in [**8**], $k(x)$ is assumed to be homogeneous of degree $-m$ and to satisfy

$$\int_{x \in S^{m-1}} k(x)\,d\sigma(x) = 0. \tag{2·7}$$

The conditions on $k(x)$ in Lemma 2·2 are weaker, but (2·6) remains true (see [**7**], theorem 1).

LEMMA 2·3. *Let $k(x)$ be given as in Lemma 2·2, $\phi \in C^\infty$ and $\psi \in C_0^\infty$. Assume that for some multi-index α, $|\alpha| \geqslant 2$, $\partial^\alpha\phi/\partial x^\alpha(0) = c_0 \neq 0$. Then there exists a constant C such that*

$$\left| \text{p.v.} \int_{\mathbb{R}^m} e^{i\xi\cdot x}\, e^{i\lambda\phi(x)}k(x)\,\psi(x)\,dx \right| \leqslant C \tag{2·8}$$

uniformly in $\xi \in \mathbb{R}^m$, and $\lambda \in \mathbb{R}^1$. The constant C may depend on A, B, c_0, and the $C^{m|\alpha|+1}$ norm of ϕ.

Proof of Lemma 2·3. Let $k = |\alpha|$, and $\|\phi\|_{C^{mk+1}} \leqslant A$. Let $\theta \in C^\infty$ be such that $0 \leqslant \theta \leqslant 1$, $\theta(x) = 0$ if $|x| \leqslant \frac{1}{2}$ and $\theta(x) = 1$ if $|x| \geqslant 1$. Write

$$\int_{\mathbb{R}^m} e^{i\xi\cdot x}\, e^{i\lambda\phi(x)}k(x)\,\psi(x)\,dx = I_1 + I_2 + I_3,$$

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where
$$I_1 = \int_{\mathbb{R}^m} e^{i\xi\cdot x} e^{i\lambda\phi(x)} k(x)\,\psi(x)\,(1-\theta)\,(\lambda^{1/(mk)}x)\,dx,$$

$$I_2 = \int_{\mathbb{R}^m} e^{i\xi\cdot x} e^{i\lambda\phi(x)} k(x)\,\psi(x)\,\theta(\lambda^{1/(mk)}x)\,(1-\theta)\,(2Ax/c_0)\,dx,$$

and
$$I_3 = \int_{\mathbb{R}^m} e^{i\xi\cdot x} e^{i\lambda\phi(x)} k(x)\,\psi(x)\,\theta(\lambda^{1/(mk)}x)\,\theta(2Ax/c_0)\,dx.$$

The estimate of I_3 is straightforward:

$$|I_3| \leqslant \int_{|x|\geqslant c_0/(4A)} |k(x)\,\psi(x)|\,dx \leqslant (4A/c_0)^m \|\psi\|_1.$$

For the estimate of I_1, we use Lemma 2·2. Let

$$P(x) = \sum_{|\beta|\leqslant mk-1} \frac{\phi^\beta(0)}{\beta!} x^\beta.$$

We have
$$|\phi(x)-P(x)| \leqslant A|x|^{mk},$$

and

$$\begin{aligned}
|I_1| &\leqslant \left| \int_{\mathbb{R}^m} e^{i\lambda P(x)} k(x)\,\psi(x)\,(1-\theta)\,(\lambda^{1/(mk)}x)\,dx \right| \\
&\quad + \int_{\mathbb{R}^m} |e^{i\lambda(\phi(x)-P(x))}-1|\,|k(x)\,\psi(x)\,(1-\theta)\,(\lambda^{1/(mk)}x)|\,dx \\
&\leqslant \left| \int_{\mathbb{R}^m} \hat{\psi}(w)\,dw \int_{\mathbb{R}^m} \widehat{(1-\theta)}(w')\,dw' \int_{\mathbb{R}^m} e^{i(\lambda P(x)+x\cdot w+\lambda^{1/(mk)}x\cdot w')} k(x)\,dx \right| \\
&\quad + A_1 \int_{|x|\leqslant\lambda^{-1/(mk)}} \lambda|x|^{mk-m}\,dx \leqslant C\|\hat{\psi}\|_1\,\|\widehat{(1-\theta)}\|_1 + C\lambda(\lambda^{-1/(mk)})^{mk} \leqslant C,
\end{aligned}$$

where we have applied Lemma 2·2 to the first term and used the observation that, as polynomials in x, the degrees of $(\lambda P(x)+x\cdot w+\lambda^{1/(mk)}x\cdot w')$ and $P(x)$ are equal. For the estimation of I_2, we use Lemma 2·1. The phase function in I_2 is $\xi\cdot x+\lambda\phi(x)$, not $\lambda\phi(x)$. But we have

$$\frac{\partial^\alpha}{\partial x^\alpha}(\xi\cdot x+\lambda\phi(x)) = \lambda\frac{\partial^\alpha\phi}{\partial x^\alpha},$$

because $|\alpha| > 1$. A quick check of the proof of Lemma 2·1 reveals that it remains true if $i\lambda\phi(x)$ is replaced by $i(\xi\cdot x+\lambda\phi(x))$. Let $\eta(x) = k(x)\,\psi(x)\,\theta(\lambda^{1/(mk)}x)\,(1-\theta)\,(2Ax/c_0)$. We have

$$\|\eta\|_\infty + \|\nabla\eta\|_1 \leqslant C\lambda^{m/(mk)} = \lambda^{1/k},$$

which implies that
$$|I_2| \leqslant C\lambda^{-1/k}(\|\eta\|_\infty + \|\nabla\eta\|_1) \leqslant C.$$

This concludes the proof of Lemma 2·3. ∎

We now understand how the constant in (2·8) depends on c_0, and so we can prove the following.

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LEMMA 2·4. *Let $k(x), \psi(x)$ be given as in Lemma 2·3 and let $I \subset \mathbb{R}^1$ be a closed interval. Assume that $\phi(x,t)$ is real-analytic in* $\operatorname{supp}(\psi) \times I$. *Then there exists a constant C such that*

$$\left| \text{p.v.} \int_{\mathbb{R}^m} e^{i\xi \cdot x} e^{i\lambda\phi(x,t)} k(x)\, \psi(x)\, dx \right| \leqslant C \tag{2·9}$$

uniformly in $\xi \in \mathbb{R}^m$, $\lambda \in \mathbb{R}^1$ and $t \in I$.

Proof of Lemma 2·4. Let

$$I_\phi(\xi, \lambda, t) = \text{p.v.} \int_{\mathbb{R}^m} e^{i\xi \cdot x} e^{i\lambda\phi(x,t)} k(x)\, \psi(x)\, dx.$$

We shall prove that for each $t_0 \in I$, there is a $\delta = \delta(t_0) > 0$ such that $I_\phi(\xi, \lambda, t)$ is uniformly bounded in $\xi \in \mathbb{R}^m$, $\lambda \in \mathbb{R}^1$ and $|t - t_0| < \delta$. Without loss of generality, we may assume that $\phi(0,t) = 0$, $\nabla_x \phi(0,t) = 0$, but $\phi(x,t)$ is not identically zero (otherwise there is nothing to prove).

First we assume that $\phi(x, t_0)$, as a function of x, is not identically zero. Then there is an $\alpha = \alpha(t_0), |\alpha| > 1$ such that $\partial^\alpha \phi / \partial x^\alpha (0, t_0) = c_0 \neq 0$. Choose $\delta = \delta(t_0)$ such that $|\partial^\alpha \phi / \partial x^\alpha (0,t)| > \frac{1}{2} c_0$ for $|t - t_0| < \delta$. By Lemma 2·3, there is a constant $C = C(t_0)$ such that

$$|I_\phi(\xi, \lambda, t)| \leqslant C \tag{2·10}$$

for all $\xi \in \mathbb{R}^m$, $\lambda \in \mathbb{R}^1$ and $t \in (t_0 - \delta, t_0 + \delta)$.

If $\phi(x, t_0)$ is identically zero for a given t_0, there exists a positive integer l and a real analytic function $\phi_1(x,t)$ such that

$$\phi(x,t) = (t - t_0)^l \phi_1(x,t)$$

for all t, and $\phi_1(x, t_0)$ is not identically zero. Therefore we can find $\delta = \delta(t_0)$ such that

$$|I_{\phi_1}(\xi, \lambda, t)| \leqslant C \tag{2·11}$$

uniformly in $\xi \in \mathbb{R}^m$, $\lambda \in \mathbb{R}^1$ and $|t - t_0| < \delta$. Since $I_\phi(\xi, \lambda, t) = I_{\phi_1}(\xi, \lambda(t - t_0)^l, t)$, (2·11) implies that $|I_\phi(\xi, \lambda, t)| \leqslant C$, uniformly in $\xi \in \mathbb{R}^m$, $\lambda \in \mathbb{R}^1$ and $|t - t_0| < \delta$.

Finally, by using a compactness argument, we see that (2·9) is true for all $\xi \in \mathbb{R}^m$, $\lambda \in \mathbb{R}^1$ and $t \in I$. ∎

LEMMA 2·5. *Suppose $x \in \mathbb{R}^1$, $\phi(x)$ is real and smooth, $\phi'(x_0) = 0, \phi''(x_0) \neq 0$. If $\psi \in C_0^\infty$ and the support of ψ is a sufficiently small neighbourhood of x_0, which is contained in $\{|x - x_0| \leqslant 1\}$, then*

$$\int_{\mathbb{R}^m} e^{i\lambda\phi(x)} \psi(x)\, dx = A_0 \lambda^{-\frac{1}{2}} + E(\lambda, \phi, \psi), \tag{2·12}$$

where $A_0 = (2\pi i)^{\frac{1}{2}} (\phi''(x_0))^{-\frac{1}{2}} e^{i\lambda\phi(x_0)}$, and $E(\lambda, \phi, \psi)$ satisfies

$$|E(\lambda, \phi, \psi)| \leqslant C\lambda^{-\frac{3}{2}} (1 + \|\psi'\|_\infty + \|\psi''\|_\infty + \lambda^{-\frac{1}{2}} \|\psi'''\|_\infty + \lambda^{-1} \|\psi^{(4)}\|_\infty). \tag{2·13}$$

It is well known that the integral in (2·12) has $A_0 \lambda^{-\frac{1}{2}}$ as the main term in its asymptotic expansion. Such results can be found in [**8**] or [**2**]. Because, in our application, ψ will be dependent on λ, it is very important for us to know precisely how the derivatives of ψ are involved in the error term. Since no such result seem to exist in the literature, we shall include a proof of Lemma 2·5 in an appendix.

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3. *Proof of Theorem A*

We now prove Theorem A. As we mentioned in Section 2, to prove the theorem it suffices to prove the uniform estimate

$$|m_\lambda(\xi)| \leqslant C\lambda^{-\frac{1}{2}}, \tag{3·1}$$

for the multiplier m_λ, which is given by (2·4). Without loss of generality, we may assume that $\Phi(0) = 0$, $\nabla_x \Phi(0) = 0$. Since $\operatorname{rank}(H_\Phi) \geqslant 1$, we may assume that the support of ϕ is contained in a small ball centred at the origin. (Away from the singularity the estimate (3.1) follows immediately from Lemma 2·5.)

We now make a simple reduction. Since the matrix $B = H_\Phi(0)$ is symmetric and $\operatorname{rank}(B) = 1$, we can find an $n \times n$ orthogonal matrix U such that

$$UBU^t = (b'_{ij})_{1 \leqslant i,j \leqslant n}, \tag{3·2}$$

where $b'_{11} \neq 0$, and $b'_{ij} = 0$ if $(i,j) \neq (1,1)$. Then

$$m_\lambda(\xi U) = \int_{\mathbb{R}^n} e^{i(\xi \cdot x + \lambda\Phi(xU))} K(xU)\, \phi(xU)\, dx.$$

Let $\Phi_1(x) = \Phi(xU)$, $K_1(x) = K(xU)$. Then $H_{\Phi_1}(0) = UBU^t$. The condition that $K|_{r(B)^\perp}$ has vanishing mean value is equivalent to

$$\int_{z \in S^{n-2}} K_1(0,z)\, d\sigma(z) = 0. \tag{3·3}$$

So without loss of generality we may assume that $B = H_\Phi(0)$ is of the form in (3·2) and K satisfies (3·3). Let $x' = (x_2, \ldots, x_n)$. We may also assume that $\phi(x) = \phi_1(x_1)\,\phi_2(x')$ for some $\phi_1 \in C_0^\infty(\mathbb{R}^1)$ and $\phi_2 \in C_0^\infty(\mathbb{R}^{n-1})$. Take $\chi \in C_0^\infty(\mathbb{R}^{n-1})$, such that $\chi \equiv 1$ on $\operatorname{supp}(\phi_2)$. Let $\beta \in C^\infty(\mathbb{R}^n)$, $\beta = 0$ if $|x| \leqslant \frac{1}{2}$, $\beta = 1$ if $|x| \geqslant 1$. We write

$$\begin{aligned} m_\lambda(\xi) &= \int_{\mathbb{R}^n} e^{i(\xi \cdot x + \lambda\Phi(x))} K(x)\, \phi(x)\, (1-\beta)(\lambda^{\frac{1}{2}} x)\, dx \\ &\quad + \int_{\mathbb{R}^n} e^{i(\xi \cdot x + \lambda\Phi(x))} K(x)\, \phi(x)\, \beta(\lambda^{\frac{1}{2}} x)\, dx \\ &= J_1 + J_2. \end{aligned} \tag{3·4}$$

The estimate for J_1 is straightforward:

$$|J_1| \leqslant C \int_{|x| \leqslant \lambda^{-\frac{1}{2}}} \frac{dx}{|x|^{n-1}} \leqslant C\lambda^{-\frac{1}{2}}. \tag{3·5}$$

Now turning to the estimation of $J_2 = J_2(\lambda, \xi)$, we write

$$J_2(\lambda, \xi) = \int_{\mathbb{R}^{n-1}} e^{i\xi' \cdot x'} J_0(\lambda, \sigma, x')\, \phi_2(x')\, dx', \tag{3·6}$$

where $\sigma = \lambda^{-1}\xi_1$ and

$$J_0(\lambda, \sigma, x') = \int_{\mathbb{R}^1} e^{i\lambda(\sigma x_1 + \Phi(x_1, x'))} K(x_1, x')\, \beta(\lambda^{\frac{1}{2}} x_1, \lambda^{\frac{1}{2}} x')\, \phi_1(x_1)\, dx_1. \tag{3·7}$$

Consider the function $F : \mathbb{R}^n \times \mathbb{R}^1 \to \mathbb{R}^n$, given by

$$F(x,y) = y + \frac{\partial \Phi}{\partial x_1}(x_1, x'). \tag{3·8}$$

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We have $F(0,0)=0$, and $\partial F/\partial x_1(0,0)=a_0\neq 0$. We can find $b>0$, and a real analytic function $\eta:\{|x'|<b,|y|<b\}\to\mathbb{R}^1$ such that $x_1=\eta(x',y)$ is the unique solution of $F(x_1,x',y)=0$. We may assume that $\operatorname{supp}(\phi)\subset\{|x|<b\}$ and that $|\nabla\Phi(x)|\leqslant\frac{1}{4}b$, $|\partial^2\Phi/\partial x_1^2(x)|\geqslant\frac{1}{2}a_0$ for $x\in\operatorname{supp}(\phi)$.

If $|\sigma|=|\lambda^{-1}\xi_1|\geqslant b$, we have

$$\left|\frac{\partial}{\partial x_1}(\sigma x_1+\Phi(x_1,x'))\right|\geqslant\tfrac{1}{2}b.$$

Using integration by parts, we get

$$|J_0(\lambda,\sigma,x')|\leqslant C\lambda^{-\frac{1}{2}},$$

which implies that $|J_2(\lambda,\xi)|\leqslant C\lambda^{-\frac{1}{2}}$. (In fact one can show that $|J_0|\leqslant C_N\lambda^{-N}$, for all $N>0$.)

From now on, we assume that $|\sigma|<b$, $x'\in\operatorname{supp}(\phi_2)$. Since $x_1=\eta(x',\sigma)$ is a non-degenerate critical point of $(\sigma x_1+\Phi(x_1,x'))$ (as a function of x_1), by Lemma 2·5, we have

$$J_0(\lambda,\sigma,x')=(2\pi i)^{\frac{1}{2}}\lambda^{-\frac{1}{2}}e^{i\lambda G(x',\sigma)}K_0(x',\sigma,\lambda)+E(x',\sigma,\lambda), \tag{3·9}$$

where
$$G(x',\sigma)=\sigma\eta(x',\sigma)+\Phi(\eta(x',\sigma),x'),$$
$$K_0(x',\sigma,\lambda)=K(\eta(x',\sigma),x')\,\psi(\eta(x',\sigma),x')\,\beta(\lambda^{\frac{1}{2}}\eta(x',\sigma),\lambda^{\frac{1}{2}}x')$$
with
$$\psi(\eta(x',\sigma),x')=\phi_1(\eta(x',\sigma))\,\chi(x')\,(\partial^2\Phi/\partial x_1^2\,(\eta(x',\sigma),x'))^{-\frac{1}{2}},$$

and $E(x',\sigma,\lambda)$ is the error term. By Lemma 2·5, we have

$$\left|\int_{\mathbb{R}^{n-1}}e^{i\xi'\cdot x'}E(x',\sigma,\lambda)\,\psi_2(x')\,dx'\right|$$
$$\leqslant C\lambda^{-\frac{3}{2}}\left(C\lambda+\sum_{s=0}^{2}\lambda^{-s/2}\int_{\mathbb{R}^{n-1}}(|x'|+\lambda^{-\frac{1}{2}})^{-(n+s+1)}\,dx'\right)\leqslant C\lambda^{-\frac{1}{2}}. \tag{3·10}$$

We shall need the following lemma. The proof of the lemma will be given at the end of this section.

LEMMA 3·1. *The function $K_0(x',\sigma,\lambda)$ satisfies*
(i) $|K_0(x',\sigma,\lambda)|\leqslant A|x'|^{-(n-1)}$,
(ii) $|\nabla_{x'}K_0(x',\sigma,\lambda)|\leqslant A|x'|^{-n}$,
(iii) $\|\widehat{K_0}\|_\infty\leqslant A$,
where $\widehat{K_0}$ is the Fourier transform of $K_0(x',\sigma,\lambda)$ in the x'-variable, and A is independent of σ and λ.

Now we continue the proof of Theorem A. Invoking Lemma 2·4 and Lemma 3·1, we have

$$\left|\text{p.v.}\int_{\mathbb{R}^{n-1}}e^{i\xi'\cdot x'}e^{i\lambda G(x',\sigma)}K_0(x',\sigma,\lambda)\,\psi_2(x')\,dx'\right|\leqslant C, \tag{3·11}$$

uniformly in $\xi'\in\mathbb{R}^{n-1}$, $\lambda\in\mathbb{R}^1$, and $|\sigma|\leqslant b$. Combining (3·4–3·7) and (3·9–3·11) we get

$$\|m_\lambda\|_\infty\leqslant C\lambda^{-\frac{1}{2}}.$$

The proof is complete. ∎

In order to prove Lemma 3·1, we need the following

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LEMMA 3·2. *Let $K(x)$ be homogeneous of degree $-(n-1)$, smooth away from the origin, and suppose that*

$$\int_{x'\in S^{n-2}} K(0,x')\,d\sigma(x') = 0.$$

Then

$$\left|\int_{\mathbb{R}^{n-1}} e^{i\xi'\cdot x'}K(h,x')\,dx'\right| \leqslant C,$$

uniformly in $\xi'\in\mathbb{R}^{n-1}$ and $h\in\mathbb{R}^1$.

This lemma is contained in [**5**] (p. 160, lemma 3·1). We now prove Lemma 3·1.

Proof of Lemma 3·1. Parts (i) and (ii) of the lemma follow directly from the assumptions on K and β. To prove (iii), we write

$$K(\eta(x',\sigma),x') = (K(\eta(x',\sigma),x')-K(\eta(0,\sigma),x'))+K(\eta(0,\sigma),x'). \tag{3·12}$$

By differentiating the equation

$$\sigma+\frac{\partial\Phi}{\partial x_1}(\eta(x',\sigma),x') = 0$$

we get

$$\frac{\partial^2\Phi}{\partial x_1^2}(\eta(x',\sigma),x')\frac{\partial\eta}{\partial x_j}(x',\sigma)+\frac{\partial^2\Phi}{\partial x_1\,\partial x_j}(\eta(x',\sigma),x') = 0 \quad \text{for } j = 2,\dots,n. \tag{3·13}$$

We have $|\partial^2\Phi/\partial x_1^2(\eta(x',\sigma),x')| \geqslant \frac{1}{2}a_0$ for $|x'|\leqslant b$ and $|\sigma|\leqslant b$. By (3·2) we also have $\partial^2\Phi/\partial x_1\,\partial x_j(0,0) = 0$ for $j = 2,\dots,n$. Therefore we have

$$\left|\frac{\partial\eta}{\partial x_j}(x',\sigma)\right| \leqslant C(|\eta(x',\sigma)|+|x'|) \tag{3·14}$$

for $j = 2,\dots,n$, and

$$\begin{aligned}|K(\eta(x',\sigma),x')-K(\eta(0,\sigma),x')| &\leqslant \sum_{s=2}^{n}|x_s|\sup_{|z'|\leqslant|x'|}\left|\frac{\partial}{\partial z_s}(K(\eta(z',\sigma),x'))\right| \\ &\leqslant \sum_{s=2}^{n}|x_s|\sup_{|z'|\leqslant|x'|}\left|\frac{\partial K}{\partial x_1}(\eta(z',\sigma),x')\right|\left|\frac{\partial\eta}{\partial z_s}(z',\sigma)\right| \\ &\leqslant C\sum_{s=2}^{n}|x_s|\sup_{|z'|\leqslant|x'|}\frac{|\eta(z',\sigma)|+|z'|}{(|\eta(z',\sigma)|+|x'|)^n} \leqslant C|x'|^{-n+2}\end{aligned} \tag{3·15}$$

uniformly in σ. Therefore we have

$$\begin{aligned}&\left|\int_{\mathbb{R}^{n-1}} e^{i\xi'\cdot x'}(K(\eta(x',\sigma),x')-K(\eta(0,\sigma),x'))\,\psi(\eta(x',\sigma),x')\,\beta(\lambda^{\frac{1}{2}}\eta(x',\sigma),\lambda^{\frac{1}{2}}x')\,dx'\right| \\ &\qquad\leqslant \int_{|x'|\leqslant C}|x'|^{-n+2}\,dx' \leqslant C.\end{aligned} \tag{3·16}$$

On the other hand, by (3·3) and Lemma 3·2 we have

$$\left|\int_{\mathbb{R}^{n-1}} e^{ix'\cdot\xi'}K(h,x')\,dx'\right| \leqslant C \tag{3·17}$$

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uniformly in ξ' and h. We also have

$$\int_{\mathbb{R}^{n-1}} e^{ix'\cdot\xi'} K(\eta(0,\sigma),x')\,\beta(\lambda^{\frac{1}{2}}\eta(x',\sigma),\lambda^{\frac{1}{2}}x')\,dx'$$

$$= \int_{\mathbb{R}^{n-1}} e^{i(\lambda^{-\frac{1}{2}}\xi')x'} K(\lambda^{\frac{1}{2}}\eta(0,\sigma),x')\,\beta(\lambda^{\frac{1}{2}}\eta(\lambda^{-\frac{1}{2}}x',\sigma),x')\,dx' \quad (3{\cdot}18)$$

$$= \int_{\mathbb{R}^{n-1}} e^{i(\lambda^{-\frac{1}{2}}\xi')x'} K(\lambda^{\frac{1}{2}}\eta(0,\sigma),x')\,dx' - \int_{\mathbb{R}^{n-1}} e^{i(\lambda^{-\frac{1}{2}}\xi')x'} K(\lambda^{\frac{1}{2}}\eta(0,\sigma),x')\,\gamma_{\lambda,\sigma}(x')\,dx', \quad (3{\cdot}19)$$

where $\gamma_{\lambda,\sigma}(x') = (1-\beta)(\lambda^{\frac{1}{2}}\eta(\lambda^{-\frac{1}{2}}x',\sigma),x')$. Clearly we have $\operatorname{supp}(\gamma_{\lambda,\sigma}) \subset \{|x'| \leqslant 1\}$ and

$$\left\| \frac{\partial^\alpha \gamma_{\lambda,\sigma}}{\partial x'^\alpha} \right\|_\infty \leqslant C \quad \text{for all } \alpha \in \mathbb{N}^{n-1}.$$

This implies that $\|\widehat{\gamma_{\lambda,\sigma}}\|_1 \leqslant C$. Hence by (3·17),

$$\left| \int_{\mathbb{R}^{n-1}} e^{i(\lambda^{-\frac{1}{2}}\xi')x'} K(\lambda^{\frac{1}{2}}\eta(0,\sigma),x')\,\gamma_{\lambda,\sigma}(x')\,dx' \right|$$

$$\leqslant \|\widehat{\gamma_{\lambda,\sigma}}\|_1 \sup_{v\in\mathbb{R}^{n-1}} \left| \int_{\mathbb{R}^{n-1}} e^{iv\cdot x'} K(\lambda^{\frac{1}{2}}\eta(0,\sigma),x')\,dx' \right| \leqslant C.$$

Since the first term in (3·19) is already known to be uniformly bounded (by setting $h = \lambda^{\frac{1}{2}}\eta(0,\sigma)$ in (3·17)), we get

$$\left| \int_{\mathbb{R}^{n-1}} e^{i\xi'\cdot x'} K(\eta(0,\sigma),x')\,\beta(\lambda^{\frac{1}{2}}\eta(x',\sigma),\lambda^{\frac{1}{2}}x')\,dx' \right| \leqslant C. \quad (3{\cdot}20)$$

Observe that the L^1 norm of the Fourier transform of $\psi(\eta(x',\sigma),x')$ (as a function in x') is uniformly bounded. Using similar argument and (3·20), we get

$$\left| \int_{\mathbb{R}^{n-1}} e^{i\xi'\cdot x'} K(\eta(0,\sigma),x')\,\psi(\eta(x',\sigma),x')\,\beta(\lambda^{\frac{1}{2}}\eta(x',\sigma),\lambda^{\frac{1}{2}}x')\,dx' \right| \leqslant C. \quad (3{\cdot}21)$$

Combining (3·12), (3·16) and (3·21), we get

$$\|\widehat{K_0}\|_\infty \leqslant C.$$

The proof of Lemma 3·1 is complete. ∎

4. *An operator with smooth phase function*

In this section we prove that Theorem A does not remain if the assumption on the analyticity of the phase function is removed.

The example that we shall present here was briefly mentioned in [**4**]. A more detailed discussion will be given below.

Let $\{a_n\}$ be a sequence of positive numbers such that $a_1 = 1$ and such that $a_{n+1} \leqslant a_n/(4(n+1))$. Let $g(t)$ be an odd C^∞ function on $[-1,1]$ such that $g(t) = 0$ for $t \in [a_n, na_n]$, and $g(t) > 0$ for $t \in ((n+1)a_{n+1}, a_n)$. Set

$$\phi(t) = \int_0^t \int_0^\tau g(s)\,ds\,d\tau.$$

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The function ϕ was first constructed by Nagel and Wainger in [3] in their investigation of Hilbert transforms along curves. We observe that the function ϕ is convex, and there are non-zero real numbers σ_n and ζ_n such that $\phi(t) = \sigma_n t + \zeta_n$ for $t \in [a_n, na_n]$. Although ϕ is in C^∞, it is not real analytic. In fact, $\phi^{(k)}(0) = 0$ for all k.

Let $n = 2$, $k = 1$, $\mu = n - k = 1$. Let $x = (x_1, x_2)$, $\Phi(x) = x_1^2 + \phi(x_2)$. Take a function $r \in C_0^\infty(\mathbb{R}^1)$, which satisfies $r(t) = 1$ if $|t| \leqslant \frac{1}{2}$, $r(t) = 0$ if $|t| \geqslant 1$. Let $\psi(x) = r(x_1)\, r(x_2)$. We have rank $(H_\Phi(x)) \geqslant 1$ and rank $(H_\Phi(0)) = 1$. Let $K(x) = x_2/(x_1^2 + x_2^2)$. Clearly K is smooth away from the origin and homogeneous of degree -1. Since K is odd, it satisfies (2·3). Let

$$T_\lambda f(x) = \int_{\mathbb{R}^2} e^{i\lambda\Phi(x-y)} K(x-y)\, \psi(x-y) f(y)\, dy. \tag{4·1}$$

We have the following result:

THEOREM B. *Let Φ, K and T_λ be given as above. Then, for any $A > 0$, there exists a $\lambda \in \mathbb{R}^1$ such that $\|T_\lambda\|_{L^2 \to L^2} \geqslant A\lambda^{-\frac{1}{2}}$.*

To prove Theorem B, we need a few simple lemmas.

LEMMA 4·1. *Let*

$$I(\lambda, a) = \int_{\mathbb{R}^1} e^{i\lambda t^2} \frac{r(t)}{t^2 + a^2}\, dt.$$

Then there exist $c_0 \neq 0$ and $N_1, N_2 > 0$ such that, for $|a| > N_1 \lambda^{-\frac{1}{2}}$, $\lambda > N_2$,

$$I(\lambda, a) = c_0\, a^{-2}\lambda^{-\frac{1}{2}} + I_1(\lambda, a), \tag{4·2}$$

where $I_1(\lambda, a)$ satisfies

$$|I_1(\lambda, a)| \leqslant \tfrac{1}{2}|c_0|\, a^{-2}\lambda^{-\frac{1}{2}}. \tag{4·3}$$

Proof of Lemma 4·1. Write

$$I(\lambda, a) = \frac{1}{a^2}\int_{\mathbb{R}^1} e^{i\lambda t^2} r(t)\, dt - \frac{1}{a^2}\int_{\mathbb{R}^1} e^{i\lambda t^2} \frac{t^2 r(t)}{t^2 + a^2}\, dt. \tag{4·4}$$

Using integration by parts, we have

$$\left| \int_{\mathbb{R}^1} e^{i\lambda t^2} \frac{t^2 r(t)}{t^2 + a^2}\, dt \right| \leqslant \left| \frac{1}{2i\lambda} \int_{\mathbb{R}^1} e^{i\lambda t^2} \left(\frac{tr(t)}{t^2 + a^2} \right)' dt \right| \leqslant C\lambda^{-1} \int_{\mathbb{R}^1} \frac{dt}{t^2 + a^2} \leqslant c_2 (\lambda a)^{-1}.$$

For the first term in (4·4) we use

$$\int_{\mathbb{R}^1} e^{i\lambda t^2} r(t)\, dt = c_0\, \lambda^{-\frac{1}{2}} + R(\lambda), \tag{4·5}$$

and $|R(\lambda)| \leqslant c_1 \lambda^{-\frac{3}{2}} \leqslant (|c_0|/4)\lambda^{-\frac{1}{2}}$, if $\lambda > N_2 = 4c_1/|c_0|$. By (4·4) and (4·5), we get

$$I(\lambda, a) = c_0\, a^{-2}\lambda^{-\frac{1}{2}} + I_1(\lambda, a),$$

where

$$I_1(\lambda, a) = a^{-2} R(\lambda) - a^{-2} \int_{\mathbb{R}^1} e^{i\lambda t^2} \frac{t^2 r(t)}{t^2 + a^2}\, dt.$$

If we let $N_1 = 4c_2/|c_0|$, $|a| > N_1 \lambda^{-\frac{1}{2}}$, $\lambda > N_2$, then

$$I_1(\lambda, a)| \leqslant \tfrac{1}{4}|c_0|\, |a^{-2}\lambda^{-\frac{1}{2}}| + c_2 |a^{-3}\lambda^{-1}| \leqslant \tfrac{1}{2}|c_0|\, a^{-2}\lambda^{-\frac{1}{2}}. \quad \blacksquare$$

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LEMMA 4·2. *Let $I(\lambda, a)$ be given as in Lemma 4·1. Then we have*

(i) $|I(\lambda, a)| \leqslant C|a|^{-2}\lambda^{-\frac{1}{2}}$,

(ii) $\left|\frac{d}{da}I(\lambda, a)\right| \leqslant C|a|^{-3}\lambda^{-\frac{1}{2}}$,

for $\lambda > 0$, $a \in \mathbb{R}^1$. The constant C is independent of a and λ.

Proof of Lemma 4·2. To prove (i), we use Van der Corput's lemma. We have

$$|I(\lambda, a)| \leqslant C\lambda^{-\frac{1}{2}}\left(\left\|\frac{r(t)}{t^2+a^2}\right\|_\infty + \left\|\frac{d}{dt}\left(\frac{r(t)}{t^2+a^2}\right)\right\|_1\right) \leqslant C|a|^{-2}\lambda^{-\frac{1}{2}}.$$

(ii) can be proved similarly.

LEMMA 4·3. *If*

$$F(\eta, \lambda, h) = \int_{-h}^{h} e^{i\eta \cdot a} aI(\lambda, a)\, da,$$

then there is a constant $C > 0$ such that

$$|F(\eta, \lambda, h)| \leqslant C\lambda^{-\frac{1}{2}} \tag{4·6}$$

uniformly in η, λ and h.

Proof of Lemma 4·3. First we assume that $|h| \geqslant |\eta|^{-1}$. We write

$$F(\eta, \lambda, h) = \int_{|a| \leqslant |\eta|^{-1}} (e^{i\eta a} - 1)\, aI(\lambda, a)\, da + \int_{|\eta|^{-1} \leqslant |a| \leqslant |h|} e^{i\eta a} aI(\lambda, a)\, da, \tag{4·7}$$

where we used the fact that $aI(\lambda, a)$ is an odd function of a. Using Lemma 4·2 for the first term and integration by parts for the second term, we get

$$|F(\eta, \lambda, h)| \leqslant C\lambda^{-\frac{1}{2}}.$$

If $|h| < |\eta|^{-1}$, the second term disappears, and the first integral is taken over $\{|a| \leqslant |h|\}$. The estimate is still valid. ∎

Having established these lemmas, we are now ready to prove Theorem B. The proof follows the ideas of Nagel and Wainger in [3].

Proof of Theorem B. Let $(\xi, \eta) \in \mathbb{R}^2$, and m_λ be the multiplier of T_λ, i.e.

$$\widehat{T_\lambda f} = m_\lambda \hat{f}.$$

We have
$$m_\lambda(\xi, \eta) = \int_{\mathbb{R}^2} e^{i(\xi x_1 + \eta x_2)}\, e^{i\lambda(x_1^2 + \phi(x_2))} \frac{x_2}{x_1^2 + x_2^2} r(x_1)\, r(x_2)\, dx_1\, dx_2. \tag{4·8}$$

Setting $\xi = 0$, we have

$$\begin{aligned} m_\lambda(0, \eta) &= \int_{-1}^{1} e^{i(\eta x_2 + \lambda\phi(x_2))} r(x_2)\, x_2 I(\lambda, x_2)\, dx_2 \\ &= \int_{-1}^{1} e^{i(\eta x_2 + \lambda\phi(x_2))} x_2 I(\lambda, x_2)\, dx_2 + \int_{-1}^{1} e^{i(\eta x_2 + \lambda\phi(x_2))} (r(x_2) - 1)\, x_2 I(\lambda, x_2)\, dx_2 \\ &= J_1(\eta, \lambda) + J_2(\eta, \lambda). \end{aligned} \tag{4·9}$$

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Using part (i) of Lemma 4·2, we have

$$|J_2(\eta,\lambda)| \leqslant \|r'\|_\infty \int_{-1}^{1} x_2^2 |I(\lambda,x_2)|\,dx_2 \leqslant C\lambda^{-\frac{1}{2}}, \tag{4·10}$$

while the first term can be written as

$$J_1(\eta,\lambda) = 2i\int_0^1 \sin(\eta a+\lambda\phi(a))\,aI(\lambda,a)\,da. \tag{4·11}$$

Now we fix n, let $\eta = -\lambda\sigma_n$ and break the integral in (4·11) into six parts:

$$J_1(-\lambda\sigma_n,\lambda) = \int_0^{\lambda^{-\frac{1}{2}}} + \int_{\lambda^{-\frac{1}{2}}}^{\frac{1}{2}a_n} + \int_{\frac{1}{2}a_n}^{a_n} + \int_{a_n}^{na_n} + \int_{na_n}^{2na_n} + \int_{2na_n}^{1}. \tag{4·12}$$

We shall prove that there is a number λ_n such that

$$|m_{\lambda_n}(0,-\lambda_n\sigma_n)| \geqslant \tfrac{1}{4}|c_0|(\log n)\lambda_n^{-\frac{1}{2}}. \tag{4·13}$$

Let $M_n' = (\phi'(a_n)-\phi'(\frac{1}{2}a_n))^{-1}$, $M_n' = (\phi'(2na_n)-\phi'(a_n))^{-1}$. Both M_n' and M_n'' are finite. Let $M_n = \max\{N_1^2 a_n^{-2}, N_2, M_n'^2, M_n''^2\}$ and $\lambda > M_n$. Invoking Lemma 4·1, we have the following estimate for the fourth term in (4·12):

$$\left|\int_{a_n}^{na_n} \sin(-\lambda\sigma_n a+\lambda\phi(a))\,aI(\lambda,a)\,da\right| = |c_0\lambda^{-\frac{1}{2}}\sin(\lambda\zeta_n)|\log n$$

$$+\left|\sin(\lambda\zeta_n)\int_{a_n}^{na_n} aI_1(\lambda,a)\,da\right| \geqslant \tfrac{1}{2}|c_0|\lambda^{-\frac{1}{2}}|\sin(\lambda\zeta_n)|\log n. \tag{4·14}$$

Using part (i) of Lemma 4·2, Lemma 4·3 and the fact that $\phi(a) \leqslant Ca^2$, we have the following estimate for the first term in (4·12):

$$\left|\int_0^{\lambda^{-\frac{1}{2}}} \sin(-\lambda\sigma_n a+\lambda\phi(a))\,aI(\lambda,a)\,da\right|$$

$$\leqslant \left|\int_0^{\lambda^{-\frac{1}{2}}} \sin(-\lambda\sigma_n a)\,aI(\lambda,a)\,da\right| + \lambda\int_0^{\lambda^{-\frac{1}{2}}} |\phi(a)||aI(\lambda,a)|\,da \leqslant C\lambda^{-\frac{1}{2}}. \tag{4·15}$$

By part (i) of Lemma 4·2, the third and fifth terms in (4·12) are bounded by

$$C\lambda^{-\frac{1}{2}}\left(\int_{\frac{1}{2}a_n}^{a_n}\frac{da}{a} + \int_{na_n}^{2na_n}\frac{da}{a}\right) \leqslant C\lambda^{-\frac{1}{2}}. \tag{4·16}$$

Now consider the second term in (4·12). If $a\in(\lambda^{-\frac{1}{2}},\frac{1}{2}a_n)$, we have

$$\sigma_n-\phi'(a) = \phi'(a_n)-\phi'(a) \geqslant \phi'(a_n)-\phi'(a_n/2) \geqslant \lambda^{-\frac{1}{2}}.$$

We also note that $\sigma_n-\phi'(a)$ is monotonic. By Van der Corput's lemma, we have for $t\in(\lambda^{-\frac{1}{2}},\frac{1}{2}a_n)$

$$\left|\int_{\lambda^{-\frac{1}{2}}}^{t} e^{i(-\lambda\sigma_n a+\lambda\phi(a))}\,da\right| \leqslant \lambda^{-\frac{1}{2}}. \tag{4·17}$$

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Using Lemma 4·2, (4·17) and integration by parts, we have

$$\left|\int_{\lambda^{-\frac{1}{2}}}^{\frac{1}{2}a_n} e^{i(-\lambda\sigma_n a+\lambda\phi(a))}\, aI(\lambda,a)\,da\right| = \left|\int_{\lambda^{-\frac{1}{2}}}^{\frac{1}{2}a_n} \frac{d}{dt}\left(\int_{\lambda^{-\frac{1}{2}}}^{t} e^{i(-\lambda\sigma_n a+\lambda\phi(a))}\,da\right) tI(\lambda,t)\,dt\right| \leqslant C\lambda^{-\frac{1}{2}}, \tag{4·18}$$

which implies that the second term in (4·12) is bounded by $C\lambda^{-\frac{1}{2}}$. Similar argument gives the same bound for the sixth term in (4·12) (observe that

$$\phi'(na_n) = \phi'(a_n) = \sigma_n).$$

To summarize, we now have, for $\lambda > M_n$,

$$|m_\lambda(0, -\lambda\sigma_n)| \geqslant \tfrac{1}{2}|c_0|\,\lambda^{-\frac{1}{2}}|\sin(\lambda\zeta_n)|\log n - C\lambda^{-\frac{1}{2}}, \tag{4·19}$$

where the constant C is independent of n. We choose a number λ_n such that $\lambda_n > M_n$ and $\sin(\lambda_n\zeta_n) = 1$. This implies that

$$|m_{\lambda_n}(0, -\lambda_n\sigma_n)| \geqslant \tfrac{1}{4}|c_0|\,(\log n)\,\lambda_n^{-\frac{1}{2}}. \tag{4·20}$$

Since $m_{\lambda_n}(\cdot)$ is continuous, (4·20) implies that

$$\|m_{\lambda_n}\|_\infty \geqslant \tfrac{1}{4}|c_0|\,(\log n)\,\lambda_n^{-\frac{1}{2}}. \tag{4·21}$$

Theorem B follows from (4·21) and Plancherel's theorem. ∎

5. *Further remarks*

A natural problem is to prove the analogue of Theorem A for all k with $1 \leqslant k < n$. It is not hard to see that all the argument presented in this paper works for $1 \leqslant k \leqslant n$, except the proof of Lemma 2·4. To obtain estimate (1·3) for $2 \leqslant k \leqslant n$ under conditions that are similar to those in Theorem A, one needs to prove the uniform boundedness of the integral $I_\phi(\xi,\lambda,t)$ (defined in the proof of Lemma 2·4) for $\xi\in\mathbb{R}^m$, $\lambda\in\mathbb{R}^1$, and t in a compact set in $\mathbb{R}^k$. To be more specific, let $k(x)$, $\psi(x)$ be given as in Lemma 2·2, Γ be a compact set in $\mathbb{R}^k$ ($k \geqslant 2$), and $\phi(x,t)$ be real analytic in $\operatorname{supp}(\psi)\times\Gamma$. Let

$$I_\phi(\lambda,t) = \text{p.v.}\int_{\mathbb{R}^m} e^{i\lambda\phi(x,t)}k(x)\,\psi(x)\,dx. \tag{5·1}$$

Problem. Prove that there exists a constant $C > 0$ such that

$$|I_\phi(\lambda,t)| \leqslant C \tag{5·2}$$

uniformly in $\lambda\in\mathbb{R}^1$ and $t\in\Gamma$.

Such an estimate would also be interesting in its own right, because it would provide a true generalization of a theorem of Stein (Lemma 2·2). To see this, we let $d(m,N)$ be the dimension of the linear space of polynomials in m variables with degree at most N. Given such a polynomial $P(x) = \sum_{|\alpha|\leqslant N} a_\alpha x^\alpha$, we write

$$P(x) = \lambda \sum_{|\alpha|\leqslant N} (\lambda^{-1}a_\alpha)\,x^\alpha,$$

where $\lambda = (\sum a_\alpha^2)^{\frac{1}{2}}$. Let $t = (t_\alpha)_{|\alpha|\leqslant N}\in\mathbb{R}^{d(m,N)}$. By choosing

$$\phi(x,t) = \sum_{|\alpha|\leqslant N} t_\alpha x^\alpha,$$

and $\Gamma = S^{d(m,N)-1}$, we see that (5·2) implies (2·6).

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6. *Appendix*

Here we shall prove Lemma 2·5. Without loss of generality, we may assume that $x_0 = 0$, $\phi(0) = \phi'(0) = 0$. By making a change of variable, we can further assume that $\phi(x) = x^2$. Fix a function $\alpha(x) \in C_0^\infty(\mathbb{R}^1)$ which satisfies $\alpha(x) = 1$ if $|x| \leqslant 1$. Since $\operatorname{supp}(\psi) \subset \{|x| \leqslant 1\}$, we have

$$\int_{\mathbb{R}^1} e^{i\lambda x^2}\psi(x)\,dx = \int_{\mathbb{R}^1} e^{i\lambda x^2}(\psi(x)-\psi(0))\,\alpha(x)\,dx + \psi(0)\int_{\mathbb{R}^1} e^{i\lambda x^2}\alpha(x)\,dx. \tag{6·1}$$

Using standard results (e.g. proposition 6 on p. 319 [**8**]), we have

$$\left|\int_{\mathbb{R}^1} e^{i\lambda x^2}\alpha(x)\,dx - (\pi i)^{\frac{1}{2}}\lambda^{-\frac{1}{2}}\right| \leqslant C\lambda^{-\frac{3}{2}}, \tag{6·2}$$

where C is independent of ψ and λ. To deal with the first term, we write

$$\psi(x)-\psi(0) = x\int_0^1 \psi'(tx)\,dt = x\psi_1(x),$$

where $\psi_1(x) = \int_0^1 \psi'(tx)\,dt$. Using integration by parts, we have

$$\begin{aligned}\int_{\mathbb{R}^1} e^{i\lambda x^2}(\psi(x)-\psi(0))\,\alpha(x)\,dx &= \int_{\mathbb{R}^1} e^{i\lambda x^2}x\psi_1(x)\,\alpha(x)\,dx\\ &= -\frac{1}{2i\lambda}\int_{\mathbb{R}^1} e^{i\lambda x^2}(\psi_1(x)\,\alpha(x))'\,dx\\ &= -\frac{1}{2i\lambda}\left(\int_{\mathbb{R}^1} e^{i\lambda x^2}(\psi_1(x)\,\alpha(x))'\,\alpha(\lambda^{\frac{1}{2}}x)\,dx\right)\\ &\quad -\frac{1}{2i\lambda}\left(\int_{\mathbb{R}^1} e^{i\lambda x^2}(\psi_1(x)\,\alpha(x))'\,(1-\alpha)(\lambda^{\frac{1}{2}}x)\,dx\right).\end{aligned} \tag{6·3}$$

The first term is trivially bounded by

$$\lambda^{-1-\frac{1}{2}}(\|\psi_1'\|_\infty + \|\psi_1\|_\infty).$$

For the second term, we use integration by parts again:

$$\begin{aligned}\left|\frac{1}{2i\lambda}\int_{\mathbb{R}^1} e^{i\lambda x^2}(\psi_1(x)\,\alpha(x))'\,(1-\alpha)(\lambda^{\frac{1}{2}}x)\,dx\right| &\leqslant C\lambda^{-3}\\ &\quad\times\int_{\mathbb{R}^1}\left|\frac{d}{dx}\left(\frac{1}{x}\frac{d}{dx}\left(\frac{1}{x}(\psi_1(x)\,\alpha(x))'\,(1-\alpha)(\lambda^{\frac{1}{2}}x)\right)\right)\right|dx\\ &\leqslant \lambda^{-\frac{3}{2}}\|\psi_1'\|_\infty + \lambda^{-2}\|\psi_1''\|_\infty + \lambda^{-\frac{5}{2}}\|\psi_1'''\|_\infty.\end{aligned} \tag{6·4}$$

Observe that $\|\psi_1^{(k)}\|_\infty \leqslant \|\psi^{(k+1)}\|_\infty$. Therefore we have

$$\left|\int_{\mathbb{R}^1} e^{i\lambda x^2}(\psi(x)-\psi(0))\,\alpha(x)\,dx\right| \leqslant \lambda^{-\frac{3}{2}}(\|\psi'\|_\infty + \|\psi''\|_\infty + \lambda^{-\frac{1}{2}}\|\psi'''\|_\infty + \lambda^{-1}\|\psi^{(4)}\|_\infty). \tag{6·5}$$

Combining (6·1), (6·2) and (6·5), Lemma 2·5 is proved. ∎

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