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Mild mixing of certain interval-exchange transformations

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Abstract. We prove that irreducible, linearly recurrent, type W interval-exchange transformations are always mild mixing. For every irreducible permutation, the set of linearly recurrent interval-exchange transformations has full Hausdorff dimension.

1. *Introduction*

Fix a permutation π of $\{1, \ldots, d\}$ and positive lengths $\lambda_1, \ldots, \lambda_d$ that sum to one. Put $\lambda_0 = 0$. Write

$$I_i = [\lambda_0 + \cdots + \lambda_{i-1}, \lambda_0 + \cdots + \lambda_i)$$

for each $1 \leq i \leq d$. The *interval-exchange transformation* on $[0, 1)$ determined by the data (λ, π) is the map $T : [0, 1) \to [0, 1)$ given by

$$Tx = x - \sum_{j<i} \lambda_j + \sum_{\pi j<\pi i} \lambda_j$$

for all x in I_i. Thus T rearranges the intervals $I_1, \ldots, I_d$ according to the permutation π. Every interval-exchange transformation preserves Lebesgue measure on $[0, 1)$. Katok [**Kat80**] showed that interval-exchange transformations are never mixing.

If for some $1 \leq k < d$ the set $\{1, \ldots, k\}$ is π invariant, then any interval-exchange transformation with permutation π is the concatenation of interval-exchange transformations on fewer intervals. It is therefore typical to assume that no such k exists. In this case, π is said to be *irreducible*. Veech [**Vee82**] and Masur [**Mas82**, Theorem 1] showed independently that for any irreducible permutation π and almost every choice of $\lambda_1, \ldots, \lambda_d$ in the simplex $\lambda_1 + \cdots + \lambda_d = 1$ the corresponding interval-exchange transformation is uniquely ergodic. Veech [**Vee84**, Theorem 1.4] proved that almost every interval-exchange transformation is rigid. Avila and Forni [**AF07**, Theorem A] proved that almost every interval-exchange transformation is weak mixing whenever π is not a rotation of $\{1, \ldots, d\}$.

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In this paper, we prove that for any permutation in an infinite class introduced by Veech [**Vee84**] and studied by Chaves and Nogueira [**CN01**] under the moniker of type W permutations, the set of λ for which the interval-exchange transformation determined by (λ, π) is mild mixing has full Hausdorff dimension.

THEOREM 1. *For every irreducible, type W permutation π on $\{1, \ldots, d\}$, the set of lengths $(\lambda_1, \ldots, \lambda_d)$ for which (λ, π) is mild mixing has full Hausdorff dimension.*

In fact, we will prove in §5 that whenever π is type W and λ is such that (λ, π) is linearly recurrent (see §3), then (λ, π) is mild mixing. It follows from work of Kleinbock and Weiss [**KW04**] that, for a fixed irreducible permutation π, the set of such λ in the simplex has full Hausdorff dimension. (See Theorem 3 below.) Chaika *et al* [**CCM13**] have extended Kleinbock and Weiss's result by showing that the set is winning for Schmidt's game, which implies full Hausdorff dimension.

For interval exchanges on three intervals, Theorem 1 is a consequence of work by Ferenczi *et al* [**FHZ05**, Theorem 4.1] who showed that, for such interval exchanges, linear recurrence implies minimal self-joinings and hence mild mixing. Boshernitzan and Nogueira [**BN04**, Theorem 5.3] showed that all interval-exchange transformations that are type W and linearly recurrent are weakly mixing. We remark that examples due to Himli [**Hmi06**] prove this is not the case in general: indeed, taking $m = 4$ in the first example of §3 therein gives a permutation that is not type W and an interval-exchange transformation that has a non-constant eigenfunction; one can verify that the interval-exchange transformation is linearly recurrent provided α is badly approximable.

For flows on surfaces Frączek and Lemańczyk [**FL06**] showed that special flows over irrational rotations with bounded partial quotients whose roof function is piecewise absolutely continuous and has non-zero sum of jumps is always mild mixing. Subsequently, Frączek, Lemańczyk and Lesigne [**FLL07**] gave a criterion for a piecewise constant roof function over an irrational rotation with bounded partial quotients to be mild mixing. Using this criterion, Frączek [**Fra09**] has shown that, when the genus is at least two, the set of Abelian differentials for which the vertical flow is mild mixing is dense in every stratum of moduli space. More recently, Kanigowski and Kułaga-Przymus [**KK15**] showed that roof functions over interval-exchange transformations having symmetric logarithmic singularities at some of the discontinuities of the interval-exchange transformation give rise to special flows that are mild mixing.

Sections 2 and 3 contain the facts we need about type W and linearly recurrent interval-exchange transformations, respectively. In §4, we discuss rigidity and mild mixing.

2. *Type W permutations*

Fix a permutation π on $\{1, \ldots, d\}$. Define a permutation σ of $\{0, \ldots, d\}$ by

$$\sigma(j) = \begin{cases} \pi^{-1}(1) - 1 & \text{if } j = 0, \\ d & \text{if } \pi(j) = d, \\ \pi^{-1}(\pi(j) + 1) - 1 & \text{otherwise,} \end{cases}$$

and write $\Sigma(\pi)$ for the set of orbits of σ. This auxiliary permutation was introduced by Veech [**Vee82**]. It describes which intervals are adjacent after an application of any interval-exchange transformation defined by the permutation π.

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We can represent $\Sigma(\pi)$ as a directed graph with $\{0, \omega_1, \ldots, \omega_{d-1}, 1\}$ as its set of vertices: writing $\omega_0 = 0$ and $\omega_d = 1$, there is an edge from ω_i to ω_j if and only if $\sigma(i) = j$. The edge with source 0 and the edge with target 1 correspond, respectively, to the first two cases in the definition of σ. Call this graph the *endpoint identification graph*. Say that π is *type W* if the vertices 0 and 1 are in distinct components of the graph. Some examples of type W permutations are given in [**CDK09**, §5].

Define

$$T_+(a) = \lim_{x \to a+} T(x), \quad T_-(a) = \lim_{x \to a-} T(x)$$

for all a in $[0, 1)$ and $(0, 1]$, respectively. The endpoint identification graph contains the edge $0 \to \omega_k$ if and only if $0 = T_+(\omega_k)$ and the edge $\omega_j \to 1$ if and only if $T_-(\omega_j) = 1$. All other edges $\omega_j \to \omega_k$ correspond to equalities $T_-(\omega_j) = T_+(\omega_k)$, where $\omega_j \neq 0$ and $\omega_k \neq 1$.

3. *Linear recurrence*

Fix an irreducible permutation π on $\{1, \ldots, d\}$. Given lengths $\lambda_1, \ldots, \lambda_d$, put $\beta_i = \lambda_1 + \cdots + \lambda_i$ for all $1 \leq i \leq d-1$. Let $D = \{\beta_1, \ldots, \beta_{d-1}\}$. One says that (λ, π) satisfies the *infinite distinct orbits condition* if $D \cap T^{-n} D = \varnothing$ for all n in $\mathbb{N}$. Keane [**Kea75**] showed that the infinite distinct orbits condition implies minimality.

Assuming the infinite distinct orbits condition, the set

$$\bigcup \{T^{-i} D : 0 \leq i \leq n\}$$

partitions $[0, 1)$ into sub-intervals of positive length. Write ϵ_n for the length of the shortest interval in this partition. It was observed in [**Bos88**] that if T satisfies the infinite distinct orbits condition and $J \subset [0, 1)$ is an interval of length at most ϵ_n, then there are times $p, q \geq 0$ with $p + q = n - 1$ such that the sets

$$T^{-p} J, T^{-p+1} J, \ldots, J, \ldots, T^{q-1} J, T^q J \tag{2}$$

are disjoint intervals. Moreover, $T^k J = T^{k-1} J + \alpha_k$ for all $-p < k \leq q$, which is to say that each interval is a translate of the previous one under T. We call (2) the *tower* defined by J. Call $T^{-p} J$ the *bottom floor* of the tower and $T^q J$ the *top floor* of the tower; see Figure 1 for a schematic diagram. Write τ for the union of the sets in (2). If J contains a discontinuity of T, then $q = 0$, and if J contains a discontinuity of T^{-1}, then $p = 0$. Note that disjointness of the intervals implies that $n\epsilon_n \leq 1$ so it must be that $\epsilon_n \to 0$ as $n \to \infty$.

An interval-exchange transformation is *linearly recurrent* if

$$\inf\{n\epsilon_n : n \in \mathbb{N}\} \geq c$$

for some positive constant c. Thus, for linearly recurrent interval-exchange transformations, the tower (2) determined by any interval J of length ϵ_n has measure at least c.

Fix an irreducible permutation π. We conclude this section with a proof of the following result.

THEOREM 3. *For every irreducible permutation π, the set of λ for which the interval-exchange transformation (λ, π) is linearly recurrent is strong winning.*

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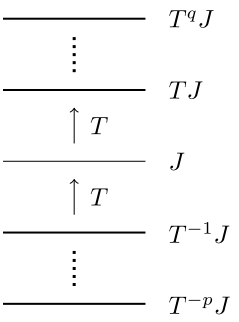


FIGURE 1. The tower determined by the interval J. Each interval is a translate of the one below under T.

An interval-exchange transformation T is *badly approximable* if

$$\inf\{n|q - T^n(p)| : n \in \mathbb{N}\} > 0$$

holds for all discontinuities p, q of T. Note that badly approximable implies the infinite distinct orbits condition. It follows from the proof of [**CCM13**, Theorem 1.4] that those λ for which (λ, π) is badly approximable are strong winning and, in particular, have full Hausdorff dimension. To prove Theorem 3, it therefore suffices to prove that every badly approximable interval-exchange transformation is linearly recurrent. We give a proof of this folklore result for completion.

PROPOSITION 4. *Fix an irreducible permutation π. If, for some λ, the interval-exchange transformation defined by (λ, π) is badly approximable, then it is linearly recurrent.*

Proof. Fix an interval-exchange transformation T that is badly approximable. Let c be the minimum of the quantities

$$\inf\{n|q - T^n(p)| : n \in \mathbb{N}\}$$

over all discontinuities p, q of T. By hypothesis, c is positive. Let η be the minimal spacing between discontinuities of T.

Suppose that T is not linearly recurrent. Then $n\epsilon_n \le c/2$ and $\epsilon_n < \eta$ for some $n \ge 2$. Fix $0 \le l \le m \le n$ and discontinuities p, q of T such that $\epsilon_n = |T^l(q) - T^m(p)|$. We claim that, if $1 \le l$, there is no discontinuity of T^{-1} between $T^m(p)$ and $T^l(q)$. Indeed, all discontinuities of T^{-1} are of the form $T^i(r)$ for some $i \in \{1, 2\}$ and some discontinuity r of T, so the existence of a discontinuity of T^{-1} between $T^m(p)$ and $T^l(q)$ contradicts $\epsilon_n = |T^l(q) - T^m(p)|$. Thus $|T^{l-1}(q) - T^{m-1}(p)| = \epsilon_n$. Iterating gives $|q - T^{m-l}(p)| = \epsilon_n$. Since $\epsilon_n < \eta$, we must have $0 < m - l$. But then

$$c \le \inf\{n|q - T^n(p)| : n \in \mathbb{N}\} \le (m - l)|q - T^{m-l}(p)| = (m - l)\epsilon_n \le n\epsilon_n \le c/2,$$

which is absurd. □

4. *Rigidity*

A measure-preserving transformation T on a probability space $(X, \mathscr{B}, \mu)$ is *rigid* if for every f in $\mathrm{L}^2(X, \mathscr{B}, \mu)$ there is a sequence $i \mapsto n_i$ such that $T^{n_i} f \to f$ in $\mathrm{L}^2(X, \mathscr{B}, \mu)$. We remark (see, for instance, [**Ber+14**, §2]) that rigidity is equivalent to the existence

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of a sequence $i \mapsto n_i$ in $\mathbb{N}$ with $n_i \to \infty$ such that for every f in $\mathrm{L}^2(X, \mathscr{B}, \mu)$ one has $T^{n_i} f \to f$ in $\mathrm{L}^2(X, \mathscr{B}, \mu)$.

A measure-preserving transformation T is *mildly mixing* if it has no non-trivial rigid factors. As described in [**Wal72**], this is equivalent to the absence of non-constant functions f in $\mathrm{L}^2(X, \mathscr{B}, \mu)$ such that $T^{n_i} f \to f$ in $\mathrm{L}^2(X, \mathscr{B}, \mu)$. Indeed, given a particular sequence $i \mapsto n_i$, the subspace

$$\{f \in \mathrm{L}^2(X, \mathscr{B}, \mu) : T^{n_i} f \to f\}$$

can be shown to be of the form $\mathrm{L}^2(X, \mathscr{C}, \mu)$ for some T invariant sub-σ-algebra $\mathscr{C}$, and the corresponding factor is non-trivial if the above subspace contains non-constant functions. Conversely, any rigid function on a factor lifts to a rigid function on the original system.

5. *Proof of main theorem*

In this section, we will prove Theorem 1. We begin with the following lemma.

LEMMA 5. *Let T be an ergodic, measure-preserving transformation on a probability space $(X, \mathscr{B}, \mu)$. If, given f in $\mathrm{L}^2(X, \mathscr{B}, \mu)$, one can find a constant $\rho > 0$ such that*

$$\mu(\{x \in X : |f(T^{i+1}x) - f(T^i x)| < \delta \text{ for all } -b \le i \le b\}) \ge \rho$$

for all $\delta > 0$ and all $b \in \mathbb{N}$, then f is constant.

Proof. For each $\delta > 0$, the sets

$$\{x \in X : |f(T^{i+1}x) - f(T^i x)| < \delta \text{ for all } -b \le i \le b\}$$

are decreasing as $b \to \infty$ so their intersection has measure at least ρ, by hypothesis. This intersection is T invariant so has full measure by ergodicity. In particular, $\{x \in X : |f(Tx) - f(x)| < \delta\}$ has full measure. Since $\delta > 0$ is arbitrary, f is T invariant almost surely and therefore constant by ergodicity. □

By Theorem 3, the following result implies Theorem 1.

THEOREM 6. *A linearly recurrent, type W interval-exchange transformation must be mildly mixing.*

Proof. Fix a type W interval-exchange transformation T on $[0, 1)$ that is linearly recurrent. Let

$$0 \to \zeta_1 \to \cdots \to \zeta_s \to 0$$

be the loop in the endpoint identification graph that contains 0. Put $c = \inf\{n\epsilon_n : n \in \mathbb{N}\}$. Assume that for some f in $\mathrm{L}^2(X, \mathscr{B}, \mu)$ there is a sequence $n_i \to \infty$ such that $T^{n_i} f \to f$ in $\mathrm{L}^2(X, \mathscr{B}, \mu)$. Write f_l for $f \circ T^l$. We show that, for every $\delta > 0$ and every $b \in \mathbb{N}$, the set

$$\{x \in [0, 1) : |f_i(Tx) - f_i(x)| < \delta \text{ for all } -b \le i \le b\}$$

has measure at least $c/10$. It will then follow from Lemma 5 and ergodicity of all linearly recurrent, type W interval-exchange transformations [**BN04**, Theorem 5.2] that f is constant. Thus the only rigid functions in $\mathrm{L}^2(X, \mathscr{B}, \mu)$ are the constant functions and T is mildly mixing.

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Fix $\delta > 0$ and $b \in \mathbb{N}$. Using Lusin's theorem, one can find a compact set $K \subset [0, 1)$ with $\mu(K) \geq 1 - c/300s$ on which each of $T^{-b}f, \ldots, f, \ldots, T^b f$ is uniformly continuous. Fix $\eta > 0$ so small that whenever $x, y \in K$ with $|x - y| < \eta$ one has $|f(T^i x) - f(T^i y)| < \delta/8s$ for all $-b \leq i \leq b$. Fix using Chebychev's inequality and the fact that $\epsilon_n \to 0$ a time $n \in \mathbb{N}$ such that:

(1) $\epsilon_n < \min\{c/200, \eta, \epsilon_1/4\}$; and

(2) the set

$$G = \left\{ x \in [0, 1) : |f_i(T^n x) - f_i(x)| < \frac{\delta}{8s} \text{ for all } -b \leq i \leq b \right\}$$

has measure at least $1 - c/300s$.

Put $H = G \cap K \cap T^{-n}K$. We have $\mu(H) \geq 1 - c/100s$.

For each $1 \leq i \leq s$, let I_i be the interval $[\zeta_i - \epsilon_n/2, \zeta_i + \epsilon_n/2)$. As described in §3, each I_i determines a tower

$$T^{-(n-1)}I_i, T^{-(n-2)}I_i, \ldots, T^{-1}I_i, I_i$$

of disjoint intervals with total measure at least c. Write τ_i for this tower. Put $I_0 = [0, \epsilon_n/2)$. Similarly, it is the roof of a tower τ_0 with total measure at least $c/2$. We claim that 90% of the points x in τ_0 have all the following properties.

(1) $x \in T^{-\ell}I_0$ with $\ell \neq n - 1$.

(2) $x \in H$ and $T^{-1}x \in H$.

(3) For every $1 \leq i \leq s$, one can find points y_i in $H \cap T^{-\ell}[\zeta_i - \epsilon_n/2, \zeta_i)$ and z_i in $H \cap T^{-\ell}[\zeta_i, \zeta_i + \epsilon_n/2)$.

This follows from the following arguments.

(1) 99% of the points in τ_0 are not in the bottom level since $100\epsilon_n < c/2$.

(2) $\mu(H \cap TH) \geq 1 - c/50s$ because $\mu(TH) = \mu(H)$. It follows from $\mu(\tau_0) \geq c/2$ that $\mu(\tau_0 \cap H \cap TH) \geq c/2 - c/50s \geq c/10$. Thus at least 20% of the points in τ_0 satisfy property 2.

(3) Suppose one can find r distinct levels $-l_1, \ldots, -l_r$ of τ_0 and, for each $-l_i$, a tower τ_{m_i} with $H \cap T^{-l_i}[\zeta_{m_i} - \epsilon_n/2, \zeta_{m_i}) = \varnothing$ or $H \cap T^{-l_i}[\zeta_{m_i}, \zeta_{m_i} + \epsilon_n/2) = \varnothing$. By the pigeonhole principle, at least r/s of these levels have the same defective tower, say, τ_m. But then $\mu(\tau_m \backslash H) \geq (r/s)(\epsilon_n/2) \geq (r/s)(c/2n)$. But $\mu(\tau_m \backslash H) \leq c/100s$ so $r/n \leq 2\%$. Thus 97% of the points in τ_0 enjoy property 3.

All told, at least 15% of the points in τ_0 satisfy all three properties.

Fix a point x in τ_0 such that all three properties hold. In particular, let $y_1, \ldots, y_s, z_1, \ldots, z_s$ and let ℓ be as in property 3. For every $1 \leq j \leq s - 1$ and every edge $\zeta_j \to \zeta_{j+1}$, we have the estimates

$$|f_i(y_j) - f_i(T^n y_j)| < \frac{\delta}{8s}, \quad |f_i(T^n z_{j+1}) - f_i(z_{j+1})| < \frac{\delta}{8s} \tag{7}$$

for all $-b \leq i \leq b$ since both y_j and z_{j+1} belong to G. Also

$$|z_{j+1} - y_{j+1}| < \eta$$

since $\{y_{j+1}, z_{j+1}\} \subset T^{-\ell}[\zeta_{j+1} - \epsilon_n/2, \zeta_{j+1} + \epsilon_n/2)$ and

$$|T^n y_j - T^n z_{j+1}| < \eta$$

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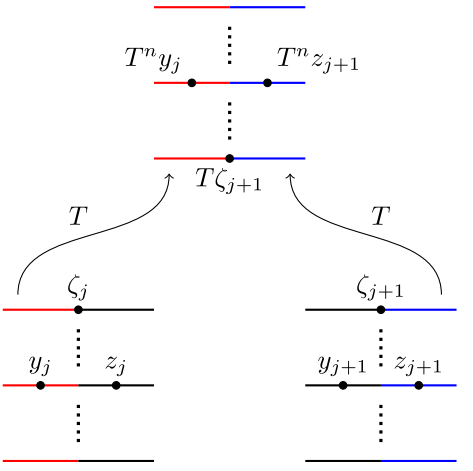


FIGURE 2. The towers and their relationships for an edge $\zeta_j \to \zeta_{j+1}$ in the endpoint identification graph with $0 \notin \{\zeta_j, \zeta_{j+1}\}$.

since $[T\zeta_{j+1} - \epsilon_n/2, T\zeta_{j+1} + \epsilon_n/2)$ contains $\{T^{\ell+1}y_j, T^{\ell+1}z_{j+1}\}$ and is the bottom floor of a tower of width ϵ_n. (See Figure 2 for a schematic diagram.) These two inequalities imply that

$$|f_i(z_{j+1}) - f_i(y_{j+1})| < \frac{\delta}{8s}, \quad |f_i(T^n y_j) - f_i(T^n z_{j+1})| < \frac{\delta}{8s}$$

for all $-b \le i \le b$ since $\{y_j, y_{j+1}, z_{j+1}\} \subset K \cap T^{-n}K$. Combining these with (7) gives

$$|f_i(y_j) - f_i(y_{j+1})| < \frac{\delta}{2s} \tag{8}$$

for all $-b \le i \le b$.

Considering next the edge $\zeta_s \to 0$, we have the estimates

$$|f_i(x) - f_i(T^n x)| < \frac{\delta}{8s}, \quad |f_i(T^n y_s) - f_i(y_s)| < \frac{\delta}{8s} \tag{9}$$

for all $-b \le i \le b$ since $\{x, y_s\} \in G$. Also

$$|T^n x - T^n y_s| < \eta$$

since $[T(0) - \epsilon_n/2, T(0) + \epsilon_n/2)$ contains $\{T^{\ell+1}x, T^{\ell+1}y_s\}$ and is the bottom floor of a tower of width ϵ_n. (See Figure 3 for a schematic diagram.) Thus

$$|f_i(T^n x) - f_i(T^n y_s)| \le \frac{\delta}{8s}$$

for all $-b \le i \le b$ since $\{x, y_s\} \subset T^{-n}K$. Combining with (9) gives

$$|f_i(x) - f_i(y_s)| \le \frac{\delta}{2s} \tag{10}$$

for all $-b \le i \le b$.

Using (8) for all $1 \le j \le s-1$ together with (10) yields

$$|f_i(x) - f_i(y_1)| < \frac{\delta}{2} \tag{11}$$

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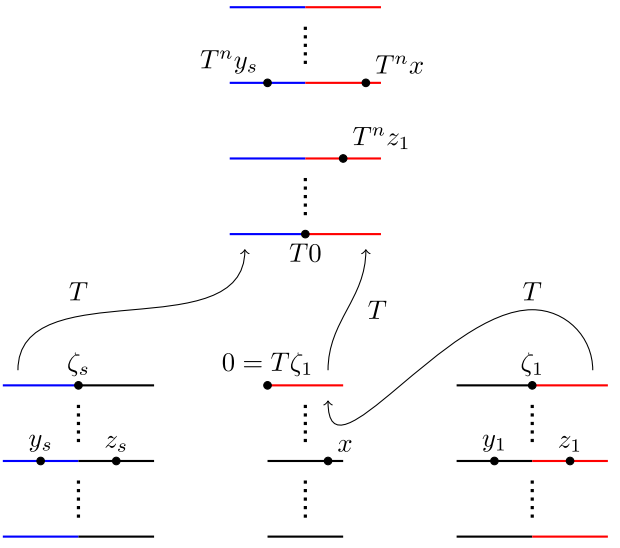


FIGURE 3. The towers and their relationships for the edges $0 \to \zeta_1$ and $\zeta_s \to 0$ in the endpoint identification graph. Note that $T^n z_1$ and $T^{n-1}x$ are in the same level since $T\zeta_1 = 0$.

for all $-b \le i \le b$. Now we use the edge $0 \to \zeta_1$ to pick up some invariance. First, note that

$$|f_i(z_1) - f_i(T^n z_1)| < \frac{\delta}{8s}, \quad |f_i(T^n(T^{-1}x)) - f_i(T^{-1}x)| < \frac{\delta}{8s} \tag{12}$$

for all $-b \le i \le b$ since $\{z_1, T^{-1}x\} \subset H$. Also

$$|y_1 - z_1| < \eta$$

because $\{y_1, z_1\} \subset T^{-\ell}[\zeta_1 - \epsilon_n/2, \zeta_1 + \epsilon_n/2)$ and

$$|T^n z_1 - T^n(T^{-1}x)| < \eta$$

because $T^n z_1$ and $T^n(T^{-1}x)$ are both in the interval $T^{n-\ell-1}[0, \epsilon_n/2)$. (See Figure 3 for a schematic diagram.) These two estimates imply that

$$|f_i(y_1) - f_i(z_1)| < \frac{\delta}{8s}, \quad |f_i(T^n z_1) - f_i(T^n(T^{-1}x))| < \frac{\delta}{8s}$$

for all $-b \le i \le b$ because $\{y_1, z_1, T^n(T^{-1}x), T^n z_1\} \subset K$. Together with (12) these imply that

$$|f_i(y_1) - f_i(T^{-1}x)| < \frac{\delta}{2s}$$

for all $-b \le i \le b$. Finally, combining with (11) gives

$$|f_i(x) - f_i(T^{-1}x)| < \delta$$

for all $-b \le i \le b$.

Since x was an arbitrary point in τ_0 satisfying properties 1, 2 and 3, the set

$$\{x \in [0, 1) : |f_i(x) - f_i(T^{-1}x)| < \delta \text{ for all } -b \le i \le b\}$$

has measure at least $15c/200$. Since $\delta > 0$ and $b \in \mathbb{N}$ were arbitrary, the function f is constant by Lemma 5. □

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