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ERROR ESTIMATES FOR STOKES PROBLEM WITH TRESCA FRICTION CONDITIONS

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Abstract. In this paper, we present and study a mixed variational method in order to approximate, with the finite element method, a Stokes problem with Tresca friction boundary conditions. These non-linear boundary conditions arise in the modeling of mold filling process by polymer melt, which can slip on a solid wall. The mixed formulation is based on a dualization of the non-differentiable term which define the slip conditions. Existence and uniqueness of both continuous and discrete solutions of these problems is guaranteed by means of continuous and discrete inf-sup conditions that are proved. Velocity and pressure are approximated by *P1 bubble-P1* finite element and piecewise linear elements are used to discretize the Lagrange multiplier associated to the shear stress on the friction boundary. Optimal *a priori* error estimates are derived using classical tools of finite element analysis and two uncoupled discrete inf-sup conditions for the pressure and the Lagrange multiplier associated to the fluid shear stress.

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1. Introduction

No-slip hypothesis at fluid-wall interface leads to good agreement with experimental observations for Newtonian fluids which is no longer true for non-Newtonian fluid [27]. For example, in the flow of certain high molecular weight linear polymers through circular dies, the exit flow rate has been found to be a discontinuous function of pressure drop over a certain range of shear rates [22]. This observation is consistent with the hypothesis that the velocity at the wall does not vanish. Several studies have been made and showed not only that slip takes place when a threshold is reached but also it's the origin of many defects and instabilities in the polymer injection process [32]. The Tresca boundary conditions used in this paper could model this phenomenon, since it states, in one hand, that when the magnitude of the fluid shear stress, on some part of the fluid boundary, reaches a given threshold value, then the tangential fluid velocity may be different to zero, *i.e.* fluid may slip on this part of the boundary. In the other hand, when the fluid shear stress magnitude is strictly below the threshold

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value, then the tangential velocity is zero. These boundary conditions are completed with a non penetration condition (*i.e.* normal velocity is equal to zero). Let us mention that a slip boundary condition could be used also in modeling the blood flow through a diseased artery (see [31] and the reference therein).

The first attempt to integrate this kind of boundary conditions in a numerical simulation of a fluid flow is due to Doltsinis *et al.* [12] and Fortin and Côté [14]. Since that, many papers were published simulating various flows with such boundary conditions, see [29] and references therein. In [18], numerical results were obtained using the augmented Lagrangian method and a block relaxation algorithm.

The mathematical analysis of this class of non linear boundary conditions, that leads to variational inequality problems, has been introduced by Fujita in [15], where he investigated some hydrodynamics problems with leak and slip boundary conditions involving subdifferential property. In [16], Fujita studied the variational inequality of the second kind formulation (where the velocity and pressure are the only unknowns) of the Stokes problem with slip boundary conditions which leads to a non-differentiable problem. He established the existence and uniqueness of a weak solution to the resulting problem. Saito in [30,31] studied the regularity of Fujita's weak solution. Recently, based on the penalty method, finite element approximation of the non-differentiable problem introduced by Fujita has been proposed in [25] and several error estimates with strong regularity assumption on the velocity field are obtained.

The aim of this work is to contribute to the numerical analysis of Stokes problems with Tresca (slip) boundary conditions in the two dimensional case. Our first purpose is to introduce another Lagrange multiplier related to the tangential component of the fluid stress on the slip zone, that will transform the non-differentiable problem into a smooth one, and to prove that this new formulation has a unique solution. This method has been previously used in the study of contact problems with friction in elasticity (see for example [23]). In the present work, we adapt it in order to take into account the presence of two Lagrange multipliers: the fluid pressure, associated to the incompressibility condition and defined in the whole domain, and the tangential component of the fluid stress, defined only on a part of the fluid domain boundary. Our second goal is to propose a mixed finite element approximation of the continuous three-field mixed problem and to carry out the convergence analysis in order to obtain optimal *a priori* error estimates.

The paper is organized as follows. In Section 2, the main notations and the functional framework are presented. In Section 3, we introduce the equations modeling the Stokes problem with Tresca boundary conditions. Then, in Section 4, we establish and study a continuous mixed variational formulation. In this section a coupled continuous inf-sup condition, involving the fluid pressure and the new Lagrange multiplier associated to the Tresca boundary conditions, is proved. Using this result an existence and uniqueness result is established in the continuous case. The following section is devoted to study a mixed finite element approximation and a coupled discrete inf-sup condition is proved, which allows us to state an existence and uniqueness result for the discrete formulation. In Section 6, *a priori* error estimates are derived. We show an optimal convergence order of $h^{3/4}$ with $\mathbf{H}^2(\Omega)$ regularity assumption on the velocity.

2. Preliminary and notations

We need to set some notations and recall some functional tools necessary for our analysis. Let $\Omega \subset \mathbb{R}^2$ be an open bounded set with Lipschitz boundary $\partial\Omega$.

In what follows, the Euclidean norm of a point $\mathbf{x} \in \mathbb{R}^2$ is denoted by $|\mathbf{x}|$. The Lebesgue space $L^2(\Omega)$ is endowed with the norm:

$$||p||_0 = \left(\int_\Omega |p(\mathbf{x})|^2 \, \mathrm{d}\Omega \right)^{\frac{1}{2}}, \qquad \forall p \in L^2(\Omega),$$

corresponding to the scalar product:

$$(p, q) = \int_\Omega p(\mathbf{x})\, q(\mathbf{x}) \mathrm{d}\Omega,$$

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while $L^2_0(\Omega)$ is the closed subspace of $L^2(\Omega)$ defined by:

$$L^2_0(\Omega) = \left\{ p \in L^2(\Omega) \text{ such that } \int_\Omega p(\mathbf{x})\, \mathrm{d}\Omega = 0 \right\}.$$

We denote by $L^\infty(\Omega)$ the space of bounded functions on Ω which will be endowed with the norm

$$||\psi||_{L^\infty(\Omega)} = \operatorname*{ess\,sup}_{\mathbf{x}\in\Omega} |\psi(\mathbf{x})|, \qquad \forall \psi \in L^\infty(\Omega).$$

We make constant use of the standard Sobolev space $H^m(\Omega)$, $m \geq 1$, provided with the norm:

$$||\psi||_m = \left(\sum_{0\leq|\alpha|\leq m} ||\partial^\alpha \psi||^2_0 \right)^{1/2},$$

where α is a multi-index.

Fractional Sobolev spaces $H^\varrho(\Omega)$, $\varrho \in \mathbb{R}_+ \backslash \mathbb{N}$ are defined by

$$H^\varrho(\Omega) = \{ \varphi \in H^m(\Omega) \text{ such that } ||\varphi||_{\varrho,\Omega} < +\infty \},$$

with

$$||\varphi||_{\varrho,\Omega} = \left(||\varphi||^2_m + \sum_{|\alpha|=m} \int_\Omega \int_\Omega \frac{(\partial^\alpha \varphi(\mathbf{x}) - \partial^\alpha \varphi(\mathbf{y}))^2}{|\mathbf{x}-\mathbf{y}|^{2+2\theta}} \, \mathrm{d}\Omega_{\mathbf{x}} \, \mathrm{d}\Omega_{\mathbf{y}} \right)^{\frac{1}{2}}$$

with m being the integer part of ϱ and θ its decimal parts.

The closure in $H^\varrho(\Omega)$ of $\mathcal{D}(\Omega)$, the space of infinitely differentiable functions with compact support in Ω, is denoted by $H^\varrho_0(\Omega)$.

On any measurable part $\Gamma \subseteq \partial\Omega$ we introduce the space $H^{\frac{1}{2}}(\Gamma)$ as follows

$$H^{\frac{1}{2}}(\Gamma) = \left\{ \varphi \in L^2(\Gamma) \text{ such that } ||\varphi||_{\frac{1}{2},\Gamma} < +\infty \right\},$$

where

$$||\psi||_{\frac{1}{2},\Gamma} = \left(||\psi||^2_{0,\Gamma} + \int_\Gamma \int_\Gamma \frac{(\psi(\mathbf{x}) - \psi(\mathbf{y}))^2}{|\mathbf{x}-\mathbf{y}|^2} \mathrm{d}\Gamma_{\mathbf{x}} \mathrm{d}\Gamma_{\mathbf{y}} \right)^{\frac{1}{2}} \quad \text{and} \quad ||\psi||_{0,\Gamma} = \left(\int_\Gamma |\psi(s)|^2 \, \mathrm{d}\Gamma \right)^{\frac{1}{2}}.$$

We denote by $H^{\frac{1}{2}}_{00}(\Gamma)$ the subspace of $H^{\frac{1}{2}}(\Gamma)$ given in Lions and Magenes [26] or in Kikuchi and Oden [24] (see Chap. 5, p. 84). The topological dual space of $H^{\frac{1}{2}}(\Gamma)$ (respectively $H^{\frac{1}{2}}_{00}(\Gamma)$) is denoted $H^{-\frac{1}{2}}(\Gamma)$ (respectively $H^{-\frac{1}{2}}_*(\Gamma)$) and we denote by $\langle \cdot, \cdot \rangle$ the duality pairing. The norm in $H^{-\frac{1}{2}}(\Gamma)$ is defined by

$$||\mu||_{-\frac{1}{2},\Gamma} = \sup_{\varphi \in H^{\frac{1}{2}}(\Gamma), \varphi \neq 0} \frac{\langle \mu, \varphi \rangle}{||\varphi||_{\frac{1}{2},\Gamma}}, \qquad \forall \mu \in H^{-\frac{1}{2}}(\Gamma).$$

If there is no confusion we use the same notation for $H^{-\frac{1}{2}}_*(\Gamma)$ norm, *i.e.*

$$||\mu||_{-\frac{1}{2},\Gamma} = \sup_{\varphi \in H^{\frac{1}{2}}_{00}(\Gamma), \varphi \neq 0} \frac{\langle \mu, \varphi \rangle}{||\varphi||_{\frac{1}{2},\Gamma}}, \qquad \forall \mu \in H^{-\frac{1}{2}}_*(\Gamma).$$

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The Cartesian product of d previous spaces and their elements are denoted by bold character. The respective norms are introduced as follows:

$$||\mathbf{v}||_m = \left(\sum_{i=1}^{d} ||v_i||_m^2\right)^{1/2} \qquad \mathbf{v} = (v_1, \ldots, v_d) \in \mathbf{H}^m(\Omega),$$

$$||\mathbf{w}||_{\frac{1}{2},\Gamma} = \left(\sum_{i=1}^{d} ||w_i||_{\frac{1}{2},\Gamma}^2\right)^{1/2} \qquad \mathbf{w} = (w_1, \ldots, w_d) \in \mathbf{H}^{\frac{1}{2}}(\Gamma),$$

$$||\boldsymbol{\mu}||_{-\frac{1}{2},\Gamma} = \left(\sum_{i=1}^{d} ||\mu_i||_{-\frac{1}{2},\Gamma}^2\right)^{1/2} \qquad \boldsymbol{\mu} = (\mu_1, \ldots, \mu_d) \in \mathbf{H}^{-\frac{1}{2}}(\Gamma).$$

Let $\mathcal{X} \subset H^1(\Omega)$ be a subspace of functions vanishing on an open non-empty portion $\Gamma_0 \subset \partial\Omega$, meas$(\Gamma_0) > 0$:

$$\mathcal{X} = \left\{ v \in H^1(\Omega) \text{ such that } v_{|\Gamma_0} = 0 \right\}.$$

Let $d = 2$. We introduce the energetic norm $|||\cdot|||_1$ in $\boldsymbol{\mathcal{X}}$ corresponding to the scalar product

$$(\mathbf{u}, \mathbf{v})_1 = \int_\Omega \sum_{i,j=1}^{d} \varepsilon_{ij}(\mathbf{u})\varepsilon_{ij}(\mathbf{v}) \mathrm{d}\Omega,$$

where $\varepsilon_{ij}(\mathbf{u})$ is the ijth component of the linearized strain rate tensor

$$\varepsilon(\mathbf{u}) = \frac{1}{2}\left(\nabla\mathbf{u} + \nabla^t\mathbf{u}\right).$$

From the Korn's inequality it follows that $||\cdot||_1$ and $|||\cdot|||_1$ are equivalent in $\boldsymbol{\mathcal{X}}$.

We denote by $\mathbf{n}$ (resp. $\mathbf{t}$) the outward unit normal vector (resp. unit tangent vector) to $\partial\Omega$ and u_n, respectively u_t, the normal, respectively the tangential, component of $\mathbf{u}$.

The stress vector is equal to $\boldsymbol{\sigma}\mathbf{n}$ where $\boldsymbol{\sigma}$ is the Cauchy stress tensor defined by:

$$\boldsymbol{\sigma} = 2\nu\varepsilon(\mathbf{u}) - p\boldsymbol{\delta},$$

where p is the hydrostatic pressure, $\boldsymbol{\delta}$ is the identity tensor and ν is the kinematic fluid viscosity. We denote by σ_n (resp. σ_t) the normal (resp. tangential) component of $\boldsymbol{\sigma}\mathbf{n}$, *i.e.* $\sigma_n = \boldsymbol{\sigma}\mathbf{n}.\mathbf{n}$ and $\sigma_t = \boldsymbol{\sigma}\mathbf{n}.\mathbf{t}$.

Remark 2.1. We use the notation u_t and σ_t for the tangential components of the velocity and the stress vector since in the two-dimensional case ($d = 2$), they are scalar quantities.

3. Setting the stokes problem with nonlinear boundary conditions

Let us consider a convex open bounded set $\Omega \subset \mathbb{R}^2$. The boundary $\partial\Omega$ is the union of two non overlapping portions Γ_0 and Γ such that $\overline{\Gamma} \cap \overline{\Gamma}_0 = \varnothing$. No-slip boundary conditions are prescribed on Γ_0 while Γ is where the fluid may slip. The formulation of the Stokes problem with nonlinear boundary conditions of Tresca friction type consists in finding a velocity vector field $\mathbf{u}$ and a pressure p satisfying the following equations and conditions:

$$\begin{cases} -2\mathbf{div}(\nu\varepsilon(\mathbf{u})) + \nabla p = \mathbf{f} & \text{in } \Omega, \\ \mathrm{div}(\mathbf{u}) = 0 & \text{in } \Omega, \\ \mathbf{u} = 0 & \text{on } \Gamma_0, \\ u_n = 0 & \text{on } \Gamma, \\ |\sigma_t| \leq g & \text{on } \Gamma, \\ |\sigma_t| < g \Rightarrow \quad u_t = 0 & \text{on } \Gamma, \\ |\sigma_t| = g \Rightarrow \exists k \geq 0 \text{ such that } \quad u_t = -k\sigma_t & \text{on } \Gamma. \end{cases} \tag{3.1}$$

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Here $\mathbf{f} \in \mathbf{L}^2(\Omega)$ is the external volume force acting on the fluid, and $g \in L^\infty(\Gamma)$, $g \geq 0$ a.e., is the threshold function.

The next result is needed to derive the variational problem (see [5]).

Proposition 3.1. *The following equivalence holds on Γ:*

$$\begin{cases} |\sigma_t| < g \Rightarrow u_t = 0 \\ |\sigma_t| = g \Rightarrow \exists k \geq 0 \ s.t. \ u_t = -k\sigma_t \end{cases} \iff \begin{cases} |\sigma_t| \leq g \\ \sigma_t \, u_t + g|u_t| = 0. \end{cases} \tag{3.2}$$

Then, using (3.2), one can derive the variational formulation of (3.1):

$$\begin{cases} \text{Find } (\mathbf{u}, p) \in \mathbf{V}_{\text{div}}(\Omega) \times L^2_0(\Omega) \text{ such that } \forall \, \mathbf{v} \in \mathbf{V}(\Omega) \\ a(\mathbf{u}, \mathbf{v} - \mathbf{u}) - (p, \text{div}(\mathbf{v})) + j(\mathbf{v}) - j(\mathbf{u}) \geq L(\mathbf{v} - \mathbf{u}), \end{cases} \tag{3.3}$$

with

$$\mathbf{V}(\Omega) = \{\mathbf{v} \in \mathbf{H}^1(\Omega), \ \mathbf{v}_{|\Gamma_0} = 0, \mathbf{v}.\mathbf{n}_{|\Gamma} = 0\},$$

$$\mathbf{V}_{\text{div}}(\Omega) = \{\mathbf{v} \in \mathbf{V}(\Omega) \ , \ \text{div}(\mathbf{v}) = 0 \text{ in } \Omega\},$$

$$a(\mathbf{u}, \mathbf{v}) = \int_\Omega 2\nu\varepsilon(\mathbf{u}) \colon \varepsilon(\mathbf{v}) \, \mathrm{d}\Omega,$$

$$L(\mathbf{v}) = \int_\Omega \mathbf{f} \, . \, \mathbf{v} \, \mathrm{d}\Omega,$$

and

$$j(\mathbf{v}) = \int_\Gamma g|v_t| \, \mathrm{d}\Gamma.$$

The existence and uniqueness of a solution to (3.3) was established in [16] while a regularity result was proved in [30,31]:

Theorem 3.2. *Problem (3.3) has a solution $(\mathbf{u} \, , \, p) \in \mathbf{V}_{\text{div}}(\Omega) \times L^2(\Omega)$. Moreover, $\mathbf{u}$ is unique.*

In addition, if $\displaystyle\int_\Omega p \, \mathrm{d}\Omega = 0$, i.e. *$p \in L^2_0(\Omega)$, then, it comes that p is unique.*

In addition, if Γ_0 and Γ are connected components of class $\mathscr{C}^2$ and $\mathscr{C}^3$, respectively, and $g \in L^\infty(\Gamma) \cap H^1(\Gamma)$ then it comes that $(\mathbf{u} \, , \, p) \in \mathbf{H}^2(\Omega) \times H^1(\Omega)$ with

$$||\mathbf{u}||_2 + ||p||_1 \leq C \, ||\boldsymbol{f}||_0 + ||g||_{1,\Gamma}.$$

Since the bilinear form $a(\cdot, \cdot)$ is symmetric the variational problem (3.3) is equivalent to the following constrained non-differentiable minimization problem:

$$\begin{cases} \text{Find } \mathbf{u} \in \mathbf{V}_{\text{div}}(\Omega) \text{ such that} \\ \mathcal{J}(\mathbf{u}) \leq \mathcal{J}(\mathbf{v}) \quad \forall \, \mathbf{v} \in \mathbf{V}_{\text{div}}(\Omega), \end{cases} \tag{3.4}$$

where $\mathcal{J}(\mathbf{v}) = \dfrac{1}{2} \, a(\mathbf{v}, \mathbf{v}) + j(\mathbf{v}) - L(\mathbf{v})$.

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4. Mixed formulation

In one hand, in order to solve (3.4) a Lagrange multiplier q is needed to enforce the condition $\mathrm{div}(\mathbf{u}) = 0$ in Ω, which can be identified with the pressure. In the other hand, Fujita proved in [17] that (3.3) is equivalent to

$$\begin{cases} \text{Find } \sigma_t \in H^{-\frac{1}{2}}(\Gamma) \text{ such that } |\sigma_t| \leq g \text{ on } \Gamma \text{ and} \\ \displaystyle\int_\Gamma \sigma_t(v_t - u_t)\mathrm{d}\Gamma + j(\mathbf{v}) - j(\mathbf{u}) \geq 0 \quad \forall\, \mathbf{v} \in \mathbf{V}(\Omega). \end{cases} \tag{4.1}$$

The unknown function σ_t is seen as a Lagrange multiplier and can be identified with the shear stress on Γ. Then, the minimization problem (3.4) is equivalent to the following saddle-point formulation:

$$\begin{cases} \text{Find } (\mathbf{u}, (p, \lambda)) \in \mathscr{H} \quad \text{such that:} \\ \mathscr{L}(\mathbf{u}, (q, \mu)) \leq \mathscr{L}(\mathbf{u}, (p, \lambda)) \leq \mathscr{L}(\mathbf{v}, (p, \lambda)) \quad \forall (\mathbf{v}, (q, \mu)) \in \mathscr{H}, \end{cases} \tag{4.2}$$

where

$$\mathscr{L}(\mathbf{v}, (q, \mu)) = \frac{1}{2}\, a(\mathbf{v}, \mathbf{v}) - \int_\Omega q\, \mathrm{div}(\mathbf{v})\, \mathrm{d}\Omega + \int_\Gamma \mu\, v_t\, \mathrm{d}\Gamma - L(\mathbf{v}),$$

$$\mathscr{H} = \mathbf{V}(\Omega) \times L^2_0(\Omega) \times \mathcal{Q},$$

and

$$\mathcal{Q} = \left\{ \mu \in L^2(\Gamma),\ |\mu| \leq g \text{ a.e.} \right\},$$

endowed with the $H_*^{-\frac{1}{2}}(\Gamma)$-norm. According to [23], we have

$$\mathcal{Q} = \left\{ \mu \in L^2(\Gamma),\ \int_\Gamma \mu\psi\, \mathrm{d}\Gamma - \int_\Gamma g|\psi|\, \mathrm{d}\Gamma \leq 0 \ \ \forall \psi \in L^2(\Gamma) \right\}.$$

The solution of problem (4.2) is characterized by (see [2, 24]):

$$\begin{cases} \text{Find } (\mathbf{u}, (p, \lambda)) \in \mathbf{V}(\Omega) \times \Lambda \quad \text{such that} \\ \qquad\qquad a(\mathbf{u}, \mathbf{v}) + b((p, \lambda), \mathbf{v}) = L(\mathbf{v}) \ \forall \mathbf{v} \in \mathbf{V}(\Omega) \\ \qquad\qquad b((q - p, \mu - \lambda), \mathbf{u}) \leq 0 \qquad \forall (q, \mu) \in \Lambda, \end{cases} \tag{4.3}$$

with

$$b((p, \lambda), \mathbf{v}) = -(p, \mathrm{div}(\mathbf{v})) + \int_\Gamma \lambda\, v_t\, \mathrm{d}\Gamma, \tag{4.4}$$

and $\Lambda = L^2_0(\Omega) \times \mathcal{Q}$, which is a closed convex of $\mathcal{M} = L^2_0(\Omega) \times L^2(\Gamma)$.

To ensure the existence and uniqueness of a solution to (4.3), the following inf-sup condition is needed (see for example [6]):

Lemma 4.1. *If $\partial\Omega$ is regular enough, then there exists a constant $\alpha > 0$ such that: $\forall (q, \mu) \in \mathcal{M}$*

$$\sup_{\mathbf{v} \in \mathbf{V}(\Omega)} \frac{b((q, \mu), \mathbf{v})}{||\mathbf{v}||_1} \geq \alpha \left(||q||_0 + ||\mu||_{-\frac{1}{2}, \Gamma} \right). \tag{4.5}$$

Proof. We note that, similarly as in the study of a Signorini's problem for incompressible materials (see [24]), in order to obtain (4.5) it would be enough to prove that for all $(q, \mu) \in \mathcal{M}$ there exists $\mathbf{u} \in \mathbf{V}(\Omega)$ such that:

$$\begin{cases} \mathrm{div}\, \mathbf{u} = q & \text{in } \Omega, \\ u_t = \mathsf{h}^{-1}(\mu) & \text{on } \Gamma, \end{cases} \tag{4.6}$$

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which satisfies

$$||\mathbf{u}||_1 \leq C \left(||q||_0 + ||\mu||_{-\frac{1}{2},\Gamma} \right), \tag{4.7}$$

where $\mathsf{h}^{-1}(\cdot)$ is the inverse of the Riesz operator $\mathsf{h} : H_{00}^{\frac{1}{2}}(\Gamma) \to H_*^{-\frac{1}{2}}(\Gamma)$.

Firstly, let $q \in L_0^2(\Omega)$ then there exists $\hat{\mathbf{u}} \in \mathbf{H}_0^1(\Omega)$ such that (see, for example, [13])

$$\mathrm{div}\hat{\mathbf{u}} = q, \quad \text{in } \Omega, \quad \text{and} \quad ||\hat{\mathbf{u}}||_1 \leq C_1 ||q||_0. \tag{4.8}$$

with $C_1 > 0$ independent of p and $\hat{\mathbf{u}}$.

Secondly, let $\mu \in L^2(\Gamma)$ and let ψ be the unique solution to the following biharmonic problem:

$$\begin{cases} \Delta^2 \psi = 0 & \text{in } \Omega, \\ \psi = 0 & \text{on } \partial\Omega, \\ \dfrac{\partial \psi}{\partial n} = \chi & \text{on } \partial\Omega, \end{cases} \tag{4.9}$$

where χ is defined as the extension by zero of $\mathsf{h}^{-1}(\mu)$

$$\chi = \begin{cases} 0 & \text{on } \Gamma_0, \\ -\mathsf{h}^{-1}(\mu) & \text{on } \Gamma. \end{cases}$$

Since $\mathsf{h}^{-1}(\mu) \in H_{00}^{\frac{1}{2}}(\Gamma)$, it is easy to see that $\chi \in H^{\frac{1}{2}}(\partial\Omega)$. From [20], since $\partial\Omega$ is regular enough[3], it holds

$$\psi \in H^2(\Omega) \qquad \text{and} \qquad ||\psi||_2 \leq C_2 ||\chi||_{\frac{1}{2},\Gamma}. \tag{4.10}$$

where C_2 is a constant independent of ψ and χ.

Finally, we set $\mathbf{u} = \hat{\mathbf{u}} + curl\, \psi$ with

$$curl\, \psi = \begin{bmatrix} \dfrac{\partial \psi}{\partial x_2} \\ -\dfrac{\partial \psi}{\partial x_1} \end{bmatrix}.$$

Hence, by the properties of $\hat{\mathbf{u}}$ and ψ, it comes that $\mathbf{u} \in \mathbf{V}(\Omega)$, $u_t = \mathsf{h}^{-1}(\mu)$ on Γ and $\mathrm{div}\,\mathbf{u} = q$ in Ω, that is (4.6) holds.

Furthermore, from (4.8) and (4.10) we obtain:

$$\begin{aligned} ||\mathbf{u}||_1 &\leq ||\hat{\mathbf{u}}||_1 + ||curl \psi||_1 \\ &\leq C_1 ||q||_0 + ||\psi||_2 \\ &\leq C_1 ||q||_0 + C_2 ||\chi||_{\frac{1}{2},\Gamma}. \end{aligned} \tag{4.11}$$

Finally, using the continuity of h^{-1} in (4.11) we obtain (4.7) with $C = \max(C_1, C_2 C_3) > 0$ independent of q and μ (here, C_3 stands for the continuity constant of h^{-1}). Hence, the inf-sup condition is satisfied with $\alpha = (2C)^{-1}$. □

On the basis of the previous inf-sup condition (4.5) we have

[3]Following [21] a $\mathscr{C}^{1,1}$ class boundary (or a $\mathscr{C}^{1,1}$ curvilinear polygon boundary) would be enough, in the 2 dimensional case, to have the existence of a solution in H^2 to the biharmonic problem with non-homogeneous boundary conditions.

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Theorem 4.2. *Under the hypothesis of Lemma 4.1, there exists a unique solution* $(\mathbf{u},(p,\lambda))$ *of the mixed problem* (4.3). *Moreover,* $(\mathbf{u},(p,\lambda))$ *is also the unique solution of the saddle-point problem* (4.2).

Remark 4.3. Let $(\mathbf{u},(p,\lambda))$ be the solution of (4.3). Then

$$\sigma_t = -\lambda. \tag{4.12}$$

5. Finite element approximation

This section is devoted to present a finite element approximation of the saddle-point problem (4.2). The key point lies in a finite element discretization of the closed convex $\mathcal{Q}$ of the Lagrange multipliers which leads to a well-posed discrete problem and gives a good convergence rate for the approximate solution.

We use classical *P1 bubble-P1* finite element to disretize $(\mathbf{u},p)$ and *P1* finite element on Γ for the Lagrange multiplier λ, see Section 5.1 below. This choice is motivated by the results presented in Arnold, Brezzi and Fortin [1] and Baillet and Sassi [3]. In Section 5.2, we establish the discrete inf-sup condition related to the discrete weak formulation of (4.3).

5.1. Discretization

The domain $\Omega \subset \mathbb{R}^2$ is supposed to be polygonal and convex. For the sake of simplicity, Γ is assumed to be a straight line[4]. Let $\mathcal{T}_h$ be a regular partition of $\overline{\Omega}$ with triangles in the sense of [8]. We denote by $\mathbb{P}_n(\kappa)$ the space of polynomials of degree less or equal to $n \in \mathbb{N}$ defined on $\kappa \in \mathcal{T}_h$. We denote by $\mathcal{B}_\kappa$ the space of bubble functions defined on κ which is a sub-space of $H_0^1(\kappa)$. Then we can define the following discrete spaces:

$$\mathbb{B} = \bigoplus_{\kappa\in\mathcal{T}_h} \mathcal{B}_\kappa, \qquad \mathcal{V}_h = \left\{\mathbf{v}_h \in \mathcal{C}^0(\overline{\Omega}); \mathbf{v}_{h|\kappa} \in \mathbb{P}_1(\kappa) \ \ \forall \kappa \in \mathcal{T}_h, \ \mathbf{v}_{h|\Gamma_0} = 0, \text{ and } \mathbf{v}_h.\mathbf{n}_{|\Gamma} = 0\right\},$$

$$\mathbf{V}_h = [\mathcal{V}_h + \mathbb{B}]^2, \qquad \mathcal{W}_h = \left\{v_{h|\Gamma}, \quad v_h \in \mathcal{V}_h\right\},$$

$$\mathbb{L}_h = \left\{q_h \in \mathcal{C}^0(\overline{\Omega}); q_{h|\kappa} \in \mathbb{P}_1(\kappa) \ \ \forall \kappa \in \mathcal{T}_h, \ \int_\Omega q_h \, \mathrm{d}\Omega = 0\right\},$$

$$\mathcal{Q}_h = \left\{\mu_h \in \mathcal{W}_h, \int_\Gamma \mu_h \psi_h \, \mathrm{d}\Gamma - \int_\Gamma g|\psi_h| \, \mathrm{d}\Gamma \le 0 \ \ \forall \psi_h \in \mathcal{W}_h\right\},$$

$$\mathcal{M}_h = \mathbb{L}_h \times \mathcal{W}_h, \qquad \Lambda_h = \mathbb{L}_h \times \mathcal{Q}_h.$$

In what follows, we present some classical approximation results involving interpolation and projection operators between the continuous spaces and the above discrete spaces, that will be used in the error estimates. First, let $\mathcal{I}_h$ and i_h be the Lagrange interpolation operators on $\mathbf{V}_h$ and $\mathcal{W}_h$ respectively. From [8], there exists a positive constant C which does not depend on h and such that for all $\mathbf{v} \in \mathbf{H}^2(\Omega)$ and for all $\psi \in H^{\frac{3}{2}}(\Gamma)$ it holds:

$$||\mathbf{v} - \mathcal{I}_h\mathbf{v}||_1 \le Ch||\mathbf{v}||_2, \qquad ||\psi - i_h\psi||_{0,\Gamma} \le Ch^{\frac{3}{2}}||\psi||_{\frac{3}{2},\Gamma}. \tag{5.1}$$

Next, let Π_h be the Clément's projection operator from $L^2(\Omega)$ on $\mathbb{L}_h$ introduced in [9]. The operator Π_h verifies the following two properties:

$$\Pi_h p \in \mathbb{L}_h \quad \text{and} \quad \int_\Omega (\Pi_h p - p)\, q_h \, \mathrm{d}\Omega = 0 \quad \forall\, q_h \in \mathbb{L}_h. \tag{5.2}$$

[4]More general cases need higher technicalities, which are beyond the scope of this work.

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There exists a positive constant C independent of h such that for all $p \in H^1(\Omega) \cap L^2_0(\Omega)$

$$||p - \Pi_h p||_0 \leq Ch||p||_1. \tag{5.3}$$

Now, we introduce the projection operator π_h from $L^2(\Gamma)$ on $\mathcal{W}_h$ defined by:

$$\pi_h \psi \in \mathcal{W}_h, \qquad \int_\Gamma (\pi_h \psi - \psi)\, \mu_h \, \mathrm{d}\Gamma = 0 \quad \forall\, \mu_h \in \mathcal{W}_h. \tag{5.4}$$

The operator π_h has the following approximation property: there exists a positive constant independent of h such that for all $\psi \in H^{\frac{1}{2}+\tau}(\Gamma)$, for all $\tau \in [0,1]$ and any $\eta \in [0, \tau + \frac{1}{2}]$ one has (see [4,11]):

$$h^{-\frac{1}{2}}||\psi - \pi_h \psi||_{-\frac{1}{2},\Gamma} + h^\eta ||\psi - \pi_h \psi||_{\eta,\Gamma} \leq Ch^{\tau+\frac{1}{2}}||\psi||_{\frac{1}{2}+\tau,\Gamma}. \tag{5.5}$$

Finally, if $(\mathbf{u}, (p, \lambda))$, the solution of (4.3), is such that $\mathbf{u} \in \mathbf{H}^2(\Omega)$, then the trace theorem and (4.12) imply

$$||\lambda||_{\frac{1}{2},\Gamma} \leq C||\mathbf{u}||_2. \tag{5.6}$$

Remark 5.1. Note that $\mathbf{V}_h \subset \mathbf{V}(\Omega)$ and $\mathcal{W}_h \subset \mathcal{W}$ while $\mathcal{Q}_h$ is an external approximation of $\mathcal{Q}$, $\mathcal{Q}_h \not\subset \mathcal{Q}$, so the discretization is non-conforming and would weaken its convergence rate.

Discretizing (4.3) we obtain

$$\begin{cases} \text{Find } (\mathbf{u}_h, (p_h, \lambda_h)) \in \mathbf{V}_h \times \Lambda_h \text{ such that:} \\ a(\mathbf{u}_h, \mathbf{v}_h) + b((p_h, \lambda_h), \mathbf{v}_h) = L(\mathbf{v}_h)\ \forall \mathbf{v}_h \in \mathbf{V}_h, \\ b((q_h - p_h, \mu_h - \lambda_h), \mathbf{u}_h) \leq 0 \qquad \forall (q_h, \mu_h) \in \Lambda_h. \end{cases} \tag{5.7}$$

5.2. Discrete inf-sup condition

First, we introduce an equivalent discrete formulation for (5.7) by eliminating the velocity with non-zero tangential component on Γ. Define

$$\mathbf{V}_h^0 = \{\mathbf{v}_h \in \mathbf{V}_h,\ \langle \mu_h, v_{ht} \rangle = 0 \quad \forall\, \mu_h \in \mathcal{W}_h\}.$$

Then we can formulate problem (5.7) as

$$\begin{cases} \text{Find } (\mathbf{u}_h, p_h) \in \mathbf{V}_h^0 \times \mathbb{L}_h \text{ such that:} \\ a(\mathbf{u}_h, \mathbf{v}_h) + (p_h, \mathrm{div}\mathbf{v}_h) = L(\mathbf{v}_h)\ \forall \mathbf{v}_h \in \mathbf{V}_h^0, \\ (q_h, \mathrm{div}\mathbf{u}_h) = 0 \qquad \forall q_h \in \mathbb{L}_h. \end{cases} \tag{5.8}$$

Now we define

$$\mathbf{Z}_h = \{\mathbf{v}_h \in \mathbf{V}_h,\ (q_h, \mathrm{div}\mathbf{v}_h) = 0 \quad \forall\, q_h \in \mathbb{L}_h\}.$$

Then we can also formulate problem (5.7) as

$$\begin{cases} \text{Find } (\mathbf{u}_h, \lambda_h) \in \mathbf{Z}_h \times \mathbb{Q}_h \text{ such that:} \\ a(\mathbf{u}_h, \mathbf{v}_h) + \langle \lambda_h, v_{ht} \rangle = L(\mathbf{v}_h)\ \forall \mathbf{v}_h \in \mathbf{Z}_h, \\ \langle \mu_h - \lambda_h, u_{ht} \rangle \leq 0 \qquad \forall \mu_h \in \mathbb{Q}_h. \end{cases} \tag{5.9}$$

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Note that $\{\mathbf{v}_h \in \mathbf{V}_h,\ \mathbf{v}_h|_{\partial\Omega} = 0\} \subset \mathbf{V}_h^0$. From the usual discrete inf-sup condition for the Stokes problem (see for example [13,20]) we have the following inf-sup condition associated to problem (5.8).

Lemma 5.2. *There is a constant $\beta_1 > 0$ independent of h such that*

$$\sup_{\mathbf{v}_h \in \mathbf{V}_h^0} \frac{(q_h, \mathrm{div}\mathbf{v}_h)}{||\mathbf{v}_h||_1} \geq \beta_1 ||q_h||_0 \qquad \forall\, q_h \in \mathbb{L}_h. \tag{5.10}$$

Lemma 5.2, the coercivity of the bilinear form a on $\mathbf{V}_h$ and the fact that $\mathbf{V}_h^0 \subset \mathbf{V}_h$ guarantee the stability of problem (5.8)

To see that the discrete formulation (5.9) is stable, the next Lemma (proved in the Appendix) shows that the inf-sup condition between spaces $\mathbf{Z}_h$ and $\mathcal{W}_h$ holds.

Lemma 5.3. *There is a constant $\beta_2 > 0$ independent of h such that*

$$\sup_{\mathbf{v}_h \in \mathbf{Z}_h} \frac{\langle \mu_h, v_{ht} \rangle}{||\mathbf{v}_h||_1} \geq \beta_2 ||\mu_h||_{-\frac{1}{2},\Gamma} \qquad \forall\, \mu_h \in \mathcal{W}_h. \tag{5.11}$$

From Lemmas 5.2 and 5.3 we can show the following coupled discrete inf-sup condition

Proposition 5.4. *There is a constant $\beta > 0$ independent of h, such that*

$$\sup_{\mathbf{v}_h \in \mathbf{V}_h} \frac{b\left((q_h, \mu_h), \mathbf{v}_h\right)}{||\mathbf{v}_h||_1} \geq \beta \left(||q_h||_0 + ||\mu_h||_{-\frac{1}{2},\Gamma} \right) \qquad \forall\, (q_h, \mu_h) \in \mathbb{L}_h \times \mathcal{W}_h. \tag{5.12}$$

Proof. Given $(q_h, \mu_h) \in \mathbb{L}_h \times \mathcal{W}_h$, from Lemma 5.2 there exists $\hat{\mathbf{v}}_h \in \mathbf{V}_h^0$ such that

$$\frac{(q_h, \mathrm{div}\hat{\mathbf{v}}_h)}{||\hat{\mathbf{v}}_h||_1} \geq \beta_1 ||q_h||_0.$$

From Lemma 5.3 there exists $\mathbf{z}_h \in \mathbf{Z}_h$ such that

$$\frac{\langle \mu_h, z_{ht} \rangle}{||z_h||_1} \geq \beta_2 ||\mu_h||_{-\frac{1}{2},\Gamma}.$$

Observe that

$$\sup_{\mathbf{v}_h \in \mathbf{V}_h} \frac{b\left((q_h, \mu_h), \mathbf{v}_h\right)}{||\mathbf{v}_h||_1} \geq \frac{b\left((q_h, \mu_h), \hat{\mathbf{v}}_h\right)}{||\hat{\mathbf{v}}_h||_1} = \frac{(q_h, \mathrm{div}\hat{\mathbf{v}}_h)}{||\hat{\mathbf{v}}_h||_1} \geq \beta_1 ||q_h||_0.$$

Analogously

$$\sup_{\mathbf{v}_h \in \mathbf{V}_h} \frac{b\left((q_h, \mu_h), \mathbf{v}_h\right)}{||\mathbf{v}_h||_1} \geq \frac{\langle \mu_h, z_{ht} \rangle}{||\mathbf{z}_h||_1} \geq \beta_2 ||\mu_h||_{-\frac{1}{2},\Gamma}.$$

Then

$$\sup_{\mathbf{v}_h \in \mathbf{V}_h} \frac{b\left((q_h, \mu_h), \mathbf{v}_h\right)}{||\mathbf{v}_h||_1} \geq \frac{\min(\beta_1, \beta_2)}{2} \left(||q_h||_0 + ||\mu_h||_{-\frac{1}{2},\Gamma} \right). \qquad \square$$

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The inf-sup condition (5.12), together with the fact that a is $\mathbf{V}_h$-elliptic guarantees the existence and uniqueness of a solution to (5.7) (see [7, 20]).

6. Error estimates

In this section we will establish *a priori* error estimates for the mixed finite element approximation (5.7). To this end, we note that Lemma 4.1, and therefore Theorem 4.2, is still valid in the case of a polygonal domain in which Γ is supposed to be a straight line[5], that is, we have existence and uniqueness of a solution to problem (4.3) for this boundary regularity. We start with the following lemma inspired of [3, 23].

Lemma 6.1. *Let* $(\mathbf{u}, p, \lambda)$ *and* $(\mathbf{u}_h, p_h, \lambda_h)$ *be solutions to* (4.3) *and* (5.7)*, respectively. Then for any* $(\mathbf{v}_h, q_h, \mu_h) \in \mathbf{V}_h \times \Lambda_h$ *it holds:*

$$\begin{aligned} a(\mathbf{u}-\mathbf{u}_h, \mathbf{u}-\mathbf{u}_h) \leq\ & a(\mathbf{u}-\mathbf{u}_h, \mathbf{u}-\mathbf{v}_h) + b((p,\lambda)-(q_h,\mu_h), \mathbf{u}_h-\mathbf{u}) + b((p,\lambda)-(p_h,\lambda_h), \mathbf{u}-\mathbf{v}_h) \\ & + b((p,\lambda)-(q_h,\mu_h), \mathbf{u}) + b((p_h,\lambda_h)-(p,\lambda), \mathbf{u}). \end{aligned} \tag{6.13}$$

Proof. Let $\mathbf{v}_h$ be an element of $\mathbf{V}_h$. It follows that:

$$a(\mathbf{u}-\mathbf{u}_h, \mathbf{u}-\mathbf{u}_h) = a(\mathbf{u}-\mathbf{u}_h, \mathbf{u}-\mathbf{v}_h) + a(\mathbf{u}-\mathbf{u}_h, \mathbf{v}_h-\mathbf{u}_h).$$

Using the first equations of (4.3) and of (5.7), we obtain:

$$\begin{aligned} a(\mathbf{u}-\mathbf{u}_h, \mathbf{v}_h-\mathbf{u}_h) &= a(\mathbf{u}, \mathbf{v}_h-\mathbf{u}_h) - a(\mathbf{u}_h, \mathbf{v}_h-\mathbf{u}_h), \\ &= L(\mathbf{v}_h-\mathbf{u}_h) - b((p,\lambda), \mathbf{v}_h-\mathbf{u}_h) - L(\mathbf{v}_h-\mathbf{u}_h) + b((p_h,\lambda_h), \mathbf{v}_h-\mathbf{u}_h), \\ &= b((p,\lambda), \mathbf{u}_h-\mathbf{v}_h) + b((p_h,\lambda_h), \mathbf{v}_h-\mathbf{u}_h). \end{aligned}$$

Then we deduce

$$a(\mathbf{u}-\mathbf{u}_h, \mathbf{u}-\mathbf{u}_h) = a(\mathbf{u}-\mathbf{u}_h, \mathbf{u}-\mathbf{v}_h) + b((p,\lambda), \mathbf{u}_h-\mathbf{v}_h) + b((p_h,\lambda_h), \mathbf{v}_h-\mathbf{u}_h).$$

Finally we have

$$\begin{aligned} a(\mathbf{u}-\mathbf{u}_h, \mathbf{u}-\mathbf{u}_h) =\ & a(\mathbf{u}-\mathbf{u}_h, \mathbf{u}-\mathbf{v}_h) + b((p,\lambda)-(q_h,\mu_h), \mathbf{u}_h-\mathbf{u}) + b((p,\lambda)-(p_h,\lambda_h), \mathbf{u}-\mathbf{v}_h) \\ & + b((p,\lambda)-(q_h,\mu_h), \mathbf{u}) + b((p_h,\lambda_h)-(p,\lambda), \mathbf{u}) \\ & + b((q_h,\mu_h)-(p_h,\lambda_h), \mathbf{u}_h). \end{aligned}$$

The inequality in (5.7) implies that $b((q_h,\mu_h)-(p_h,\lambda_h), \mathbf{u}_h) \leq 0$ for all $(q_h,\mu_h) \in \Lambda_h$, and then, the error estimate (6.13) follows. □

By deriving an upper bound of the terms involved in the right hand side of (6.13), for a particular $(\mathbf{v}_h, p_h, \mu_h)$, we will prove the following

[5]In this case we can verify that the boundary data in the biharmonic problem (4.9) satisfy the compatibility condition in ([19], Thm. 3 p. 359), since the tangential vector on Γ is now a constant vector. Hence, the boundary data can be lifted by a $H^2(\Omega)$ function.

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Lemma 6.2. *Let* $(\mathbf{u}, p, \lambda)$, $(\mathbf{u}_h, p_h, \lambda_h)$ *be the solution to* (4.3) *and* (5.7), *respectively. Suppose that* $\mathbf{u} \in \mathbf{H}^2(\Omega)$ *and* $p \in H^1(\Omega)$. *Then*

$$||\mathbf{u} - \mathbf{u}_h||_1^2 \leq C(\left(h^{\frac{3}{2}} + h||\lambda - \lambda_h||_{-\frac{1}{2},\Gamma} + h||p - p_h||_0\right), \tag{6.14}$$

where C *is a positive constant depending only on* $||\mathbf{u}||_2$, $||p||_1$ *and* $||g||_{0,\Gamma}$.

Proof. We will show that there exists $(\mathbf{v}_h, (q_h, \mu_h)) \in \mathbf{V}_h \times \Lambda_h$ satisfying:

$$\begin{cases} a(\mathbf{u} - \mathbf{u}_h, \mathbf{u} - \mathbf{v}_h) & \leq C(\mathbf{u})h||\mathbf{u} - \mathbf{u}_h||_1, \\ b((p, \lambda) - (q_h, \mu_h), \mathbf{u}_h - \mathbf{u}) & \leq C(\mathbf{u}, p)h||\mathbf{u} - \mathbf{u}_h||_1, \\ b((p_h, \lambda_h) - (p, \lambda), \mathbf{u} - \mathbf{v}_h) & \leq Ch\left\{||\mathbf{u} - \mathbf{u}_h||_1 + C(p)h + C(\mathbf{u})h\right\}, \\ b((p, \lambda) - (q_h, \mu_h), \mathbf{u}) & \leq C(\mathbf{u})^2h^2, \\ b((p_h, \lambda_h) - (p, \lambda), \mathbf{u}) & \leq C(\mathbf{u})\left(h||\lambda - \lambda_h||_{-\frac{1}{2},\Gamma} + C(\mathbf{u})h^{\frac{3}{2}} + C(g)h^{\frac{3}{2}}\right). \end{cases} \tag{6.15}$$

In what follows we choose $\mathbf{v}_h = \mathcal{I}_h\mathbf{u}$, $q_h = \Pi_h p$ and $\mu_h = \pi_h\lambda$. From the definition of $\mathcal{Q}$, $\mathcal{Q}_h$ and (5.4), it is readily seen that $\pi_h\lambda \in \mathcal{Q}_h$:

$$\forall\, \psi_h \in \mathcal{W}_h \qquad \int_\Gamma (\pi_h\lambda)\,\psi_h \,\mathrm{d}\Gamma = \int_\Gamma \lambda\psi_h \,\mathrm{d}\Gamma \leq \int_\Gamma g\psi_h \,\mathrm{d}\Gamma \leq \int_\Gamma g|\psi_h| \,\mathrm{d}\Gamma.$$

(i) The first term of (6.15) is obtained by using the continuity of $a(\cdot,\cdot)$ and the property (5.1)

$$\begin{aligned} a(\mathbf{u} - \mathbf{u}_h, \mathbf{u} - \mathbf{v}_h) &\leq C||\mathbf{u} - \mathbf{u}_h||_1\,||\mathbf{u} - \mathbf{v}_h||_1 \\ &\leq C(\mathbf{u})h||\mathbf{u} - \mathbf{u}_h||_1. \end{aligned}$$

(ii) Using (5.3) and (5.5) we obtain the second inequality of (6.15):

$$\begin{aligned} b((p, \lambda) - (q_h, \mu_h), \mathbf{u}_h - \mathbf{u}) &= -(p - q_h, \mathrm{div}(\mathbf{u}_h - \mathbf{u})) + \langle \lambda - \mu_h, u_{ht} - u_t\rangle \\ &\leq C\left\{||p - q_h||_0 + ||\lambda - \mu_h||_{-\frac{1}{2},\Gamma}\right\}||\mathbf{u} - \mathbf{u}_h||_1 \\ &\leq C(\mathbf{u}, p)h||\mathbf{u} - \mathbf{u}_h||_1. \end{aligned} \tag{6.16}$$

(iii) Further, using again (5.1), there holds that

$$\begin{aligned} b((p, \lambda) - (p_h, \lambda_h), \mathbf{u} - \mathbf{v}_h) &= -(p - p_h, \mathrm{div}(\mathbf{u} - \mathbf{v}_h)) + \langle \lambda - \lambda_h, u_t - v_{ht}\rangle, \\ &\leq C\left\{||p - p_h||_0 + ||\lambda - \lambda_h||_{-\frac{1}{2},\Gamma}\right\}||\mathbf{u} - \mathbf{v}_h||_1 \\ &\leq Ch\left\{||p - p_h||_0 + ||\lambda - \lambda_h||_{-\frac{1}{2},\Gamma}\right\}. \end{aligned} \tag{6.17}$$

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(iv) To estimate the fourth term of (6.15) we invoke the definition of the L^2-projection operator on Γ (5.4) and the approximation property (5.5):

$$\begin{aligned} b((p,\lambda)-(q_h,\mu_h),\mathbf{u}) &= \int_\Gamma (\lambda-\mu_h)u_t \,\mathrm{d}\Gamma \\ &= \int_\Gamma (\lambda-\pi_h\lambda)u_t \,\mathrm{d}\Gamma, \\ &= \int_\Gamma (\lambda-\pi_h\lambda)u_t \,\mathrm{d}\Gamma - \int_\Gamma (\lambda-\pi_h\lambda)\,\pi_h u_t \,\mathrm{d}\Gamma, \\ &= \int_\Gamma (\lambda-\pi_h\lambda)(u_t-\pi_h u_t) \,\mathrm{d}\Gamma, \\ &\le ||\lambda-\pi_h\lambda||_{0,\Gamma}||u_t-\pi_h u_t||_{0,\Gamma}, \\ &\le Ch^2. \end{aligned} \tag{6.18}$$

(v) Now we shall estimate the fifth term of (6.15). Noticing that, since $\lambda_h \in Q_h$ then

$$\int_\Gamma (\lambda_h\, i_h(u_t) - g|i_h(u_t)|)\,\mathrm{d}\Gamma \le 0,$$

and that, from (3.2) and (4.12), we have

$$\lambda\, u_t - g|u_t| = 0 \qquad \text{a.e. on } \Gamma.$$

Therefore, combining this with (5.1) and (5.6), we have

$$\begin{aligned} b((p_h,\lambda_h)-(p,\lambda),\mathbf{u}) &= \int_\Gamma (\lambda_h-\lambda)u_t \,\mathrm{d}\Gamma, \\ &= \int_\Gamma (\lambda_h-\lambda)\,(u_t-i_h(u_t)) \,\mathrm{d}\Gamma + \int_\Gamma (\lambda_h-\lambda)\,i_h(u_t)\,\mathrm{d}\Gamma + \int_\Gamma (\lambda u_t - g|u_t|) \,\mathrm{d}\Gamma, \\ &= \int_\Gamma (\lambda_h-\lambda)\,(u_t-i_h(u_t)) \,\mathrm{d}\Gamma + \int_\Gamma (\lambda_h\, i_h(u_t) - g|u_t|) \,\mathrm{d}\Gamma + \int_\Gamma \lambda\,(u_t-i_h(u_t)) \,\mathrm{d}\Gamma, \\ &\le \int_\Gamma (\lambda_h-\lambda)\,(u_t-i_h(u_t)) \,\mathrm{d}\Gamma + \int_\Gamma g\,(|i_h(u_t)|-|u_t|) \,\mathrm{d}\Gamma + \int_\Gamma \lambda\,(u_t-i_h(u_t)) \,\mathrm{d}\Gamma, \\ &\le \int_\Gamma (\lambda_h-\lambda)\,(u_t-i_h(u_t)) \,\mathrm{d}\Gamma + \int_\Gamma g|i_h(u_t)-u_t| \,\mathrm{d}\Gamma + \int_\Gamma \lambda\,(u_t-i_h(u_t)) \,\mathrm{d}\Gamma, \\ &\le ||\lambda_h-\lambda||_{-\frac{1}{2},\Gamma}||u_t-i_h(u_t)||_{\frac{1}{2},\Gamma} + ||g||_{0,\Gamma}||u_t-i_h(u_t)||_{0,\Gamma} + ||\lambda||_{0,\Gamma}||u_t-i_h(u_t)||_{0,\Gamma}, \\ &\le C(\mathbf{u})h||\lambda-\lambda_h||_{-\frac{1}{2},\Gamma} + C(\mathbf{u})^2 h^{\frac{3}{2}} + C(g)C(\mathbf{u})h^{\frac{3}{2}}, \\ &\le C\left\{h||\lambda-\lambda_h||_{-\frac{1}{2},\Gamma} + h^{\frac{3}{2}}\right\}. \end{aligned}$$

Assembling the estimates (i)–(v) and using Lemma 6.1 and the $\mathbf{V}$-ellipticity of the bilinear form $a(\cdot,\cdot)$, we finally arrive at the following estimate

$$||\mathbf{u}-\mathbf{u}_h||_1^2 \le C\left(h||\lambda-\lambda_h||_{-\frac{1}{2},\Gamma} + h||\mathbf{u}-\mathbf{u}_h||_1 + h||p-p_h||_0\right) + C\,h^{\frac{3}{2}},$$

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then, using the Young's inequality, we can write for every constant $A > 0$

$$Ch||\mathbf{u} - \mathbf{u}_h||_1 \leq C\left(Ah^2 + \frac{1}{A}||\mathbf{u} - \mathbf{u}_h||_1^2\right).$$

Finally, if we take A such that $\dfrac{C}{A} < 1$, then it leads to the desired result. □

The next two lemmas will give us error estimates for the pressure and the Lagrange multiplier λ in terms of h and the velocity error.

Lemma 6.3. *Let* $(\mathbf{u}, p, \lambda)$ *and* $(\mathbf{u}_h, p_h, \lambda_h)$ *be solutions to* (4.3) *and* (5.7)*, respectively. Suppose that* $\mathbf{u} \in \mathbf{H}^2(\Omega)$ *and* $p \in H^1(\Omega)$*. Then*

$$||p - p_h||_0 \leq C\left\{h + ||\mathbf{u} - \mathbf{u}_h||_1\right\}, \tag{6.19}$$

where C *is a positive constant depending only on* $||\mathbf{u}||_2$ *and* $||p||_1$*.*

Proof. Since $(\mathbf{u}, p, \lambda)$ and $(\mathbf{u}_h, p_h, \lambda_h)$ are solutions to (4.3) and (5.7), respectively, then for $\mathbf{v}_h \in \mathbf{V}_h$

$$a(\mathbf{u}, \mathbf{v}_h) - (p, \mathrm{div}(\mathbf{v}_h)) + \langle \lambda, v_{th} \rangle = L(\mathbf{v}_h),$$

$$a(\mathbf{u}_h, \mathbf{v}_h) - (p_h, \mathrm{div}(\mathbf{v}_h)) + \langle \lambda_h, v_{th} \rangle = L(\mathbf{v}_h),$$

by subtracting these two equations we obtain

$$a(\mathbf{u} - \mathbf{u}_h, \mathbf{v}_h) - (p - p_h, \mathrm{div}(\mathbf{v}_h)) + \langle \lambda - \lambda_h, v_{th} \rangle = 0 \quad \forall\, \mathbf{v}_h \in \mathbf{V}_h, \tag{6.20}$$

and taking $\mathbf{v}_h \in \mathbf{V}_h^0$, (6.20) becomes

$$a(\mathbf{u} - \mathbf{u}_h, \mathbf{v}_h) + (p - p_h, \mathrm{div}(\mathbf{v}_h)) = 0 \quad \forall \mathbf{v}_h \in \mathbf{V}_h^0.$$

Therefore

$$(p_h - \Pi_h p, \mathrm{div}(\mathbf{v}_h)) = (p_h - p, \mathrm{div}(\mathbf{v}_h)) + (p - \Pi_h p, \mathrm{div}(\mathbf{v}_h)) = a(\mathbf{u} - \mathbf{u}_h, \mathbf{v}_h) + (p - \Pi_h p, \mathrm{div}(\mathbf{v}_h)),$$

and, using (5.3),

$$(p_h - \Pi_h p, \mathrm{div}(\mathbf{v}_h)) \leq C\,||\mathbf{u} - \mathbf{u}_h||_1\,||\mathbf{v}_h||_1 + Ch||p||_1\,||\mathbf{v}_h||_1 \qquad \forall \mathbf{v}_h \in \mathbf{V}_h^0.$$

The *inf-sup* condition (5.10) yields to

$$\beta_1||p_h - \Pi_h p||_0 \leq \sup_{\mathbf{v}_h \in \mathbf{V}_h^0} \frac{(p_h - \Pi_h p, \mathrm{div}(\mathbf{v}_h))}{||\mathbf{v}_h||_1} \leq C||\mathbf{u} - \mathbf{u}_h||_1 + Ch||p||_1. \tag{6.21}$$

The triangle inequality

$$||p - p_h||_0 \leq ||p - \Pi_h p||_0 + ||\Pi_h p - p_h||_0,$$

together with (6.21) and (5.3) proves (6.19). □

Lemma 6.4. *Let* $(\mathbf{u}, p, \lambda)$ *and* $(\mathbf{u}_h, p_h, \lambda_h)$ *be solutions to* (4.3) *and* (5.7) *respectively. Suppose that* $\mathbf{u} \in \mathbf{H}^2(\Omega)$ *and* $p \in H^1(\Omega)$*. Then*

$$||\lambda - \lambda_h||_{-\frac{1}{2},\Gamma} \leq C\left\{h + ||\mathbf{u} - \mathbf{u}_h||_1\right\}, \tag{6.22}$$

where C *is a positive constant depending only on* $||\mathbf{u}||_2$*.*

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Proof. Introducing $\pi_h\lambda$ in (6.20) we have:

$$\langle \pi_h\lambda - \lambda_h, v_{ht}\rangle \;=\; a(\mathbf{u}_h - \mathbf{u}, \mathbf{v}_h) + (p - p_h, \mathrm{div}(\mathbf{v}_h)) - \langle \lambda - \pi_h\lambda, v_{ht}\rangle \quad \forall\, \mathbf{v}_h \in \mathbf{V}_h.$$

Therefore

$$\langle \pi_h\lambda - \lambda_h, v_{ht}\rangle \;\leq\; C||\mathbf{u} - \mathbf{u}_h||_1||\mathbf{v}_h||_1 + C||p - p_h||_0||\mathbf{v}_h||_1 + Ch||\lambda||_{\frac{1}{2},\Gamma}||\mathbf{v}_h||_1,$$

for all $\mathbf{v}_h \in \mathbf{V}_h$, and in particular for all $\mathbf{v}_h \in \mathbf{Z}_h$. Using the *inf-sup* condition (5.11) combined with (6.19) we obtain

$$\begin{aligned} \beta_2||\pi_h\lambda - \lambda_h||_{-\frac{1}{2},\Gamma} &\leq \sup_{\mathbf{v}_h\in\mathbf{Z}_h} \frac{\langle \pi_h\lambda - \lambda_h, v_{ht}\rangle}{||\mathbf{v}_h||_1} \\ &\leq C\left(||\mathbf{u} - \mathbf{u}_h||_1 + ||p - p_h||_0 + h\right) \\ &\leq C\left(||\mathbf{u} - \mathbf{u}_h||_1 + h\right). \end{aligned} \tag{6.23}$$

Inserting the estimates (5.5) and (6.23) in the triangle inequality

$$||\lambda - \lambda_h||_{-\frac{1}{2},\Gamma} \leq ||\lambda - \pi_h\lambda||_{-\frac{1}{2},\Gamma} + ||\pi_h\lambda - \lambda_h||_{-\frac{1}{2},\Gamma}$$

we finally arrive at the estimate (6.22). □

Using Lemmas 6.2, 6.3 and 6.4, we are now able to prove our main *a priori* error estimate result:

Theorem 6.5. *Let* $(\mathbf{u}, p, \lambda)$ *and* $(\mathbf{u}_h, p_h, \lambda_h)$ *be solutions to* (4.3) *and* (5.7) *respectively. Suppose that* $\mathbf{u} \in \mathbf{H}^2(\Omega)$ *and* $p \in H^1(\Omega)$. *Then*

$$||\mathbf{u} - \mathbf{u}_h||_1 + ||p - p_h||_0 + ||\lambda - \lambda_h||_{-\frac{1}{2},\Gamma} \leq Ch^{\frac{3}{4}},$$

where C *is a positive constant depending only on* $||\mathbf{u}||_2$, $||p||_1$ *and* $||g||_{0,\Gamma}$.

Proof. By using (6.19) and (6.22) in (6.14), we can write:

$$\begin{aligned} ||\mathbf{u} - \mathbf{u}_h||_1^2 &\leq C\left\{h||\lambda - \lambda_h||_{-\frac{1}{2},\Gamma} + h||p - p_h||_0 + h^{\frac{3}{2}}\right\}, \\ &\leq Ch\left\{h + ||\mathbf{u} - \mathbf{u}_h||_1\right\} + Ch^{\frac{3}{2}}, \\ &\leq Ch^2 + Ch||\mathbf{u} - \mathbf{u}_h||_1 + Ch^{\frac{3}{2}}, \end{aligned}$$

hence, using again Young's inequality,

$$||\mathbf{u} - \mathbf{u}_h||_1 \leq Ch^{\frac{3}{4}}.$$

Consequently, (6.19) and (6.22) become

$$||p - p_h||_0 \leq Ch^{\frac{3}{4}} \qquad \text{and} \quad ||\lambda - \lambda_h||_{-\frac{1}{2},\Gamma} \leq Ch^{\frac{3}{4}},$$

which leads to the desired result. □

Remark 6.6. We recall that a $\mathbf{H}^2 \times H^1$ regularity result for the Stokes problem with slip boundary conditions is obtained under the hypotheses of Theorem 3.2 (see [30,31]). When Γ and Γ_D are not connected components and $-$ or $-$ have less regularity (*e.g.* only Lipschitz continuous boundaries), an analogous regularity result is, as far as we known, an open problem.

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7. Conclusion

A three field mixed formulation of the Stokes problem with Tresca boundary conditions has been introduced and studied. The convergence analysis and *a priori* error estimates of the discrete corresponding problem have been established. In particular, we show an optimal error estimate of order $h^{\frac{3}{4}}$ for the velocity when it is approximated by classical *P1 bubble* finite element. The implementation of the mixed finite element formulation proposed in Section 5 will be presented in a forthcoming paper.

Appendix A

Proof of Lemma 5.3. The proof presented here follows the same ideas as in Proposition 3.3 from [10]. The main difference is that we are working with the space $\mathbf{Z}_h$ and with a Stokes extension operator $\mathbf{R}_h$.

Let $\mu_h \in \mathcal{W}_h$. There is $\psi \in H^{\frac{1}{2}}(\Gamma)$ with $||\psi||_{\frac{1}{2},\Gamma} = 1$, so that

$$\langle \mu_h, \psi \rangle = \int_\Gamma \mu_h \, \psi \, \mathrm{d}\Gamma = ||\mu_h||_{-\frac{1}{2},\Gamma}.$$

The definition of π_h in (5.4) leads to

$$\int_\Gamma \mu_h \, \psi \, \mathrm{d}\Gamma = \int_\Gamma \mu_h \, \pi_h\psi \, \mathrm{d}\Gamma.$$

For any $\psi_h \in \mathcal{W}_h$, define $\mathbf{R}_h(\psi_h) \in \mathbf{V}_h$ as the discrete velocity solution of the following problem (the discrete Stokes extension):

$$\begin{cases} a(\mathbf{R}_h(\psi_h), \mathbf{v}_h) + (p_h, \mathrm{div}\mathbf{v}_h) = 0 \quad \forall \mathbf{v}_h \in \mathbf{V}_h, \text{such that } v_{ht} = 0 \text{ on } \Gamma, \\ (q_h, \mathrm{div}\mathbf{R}_h(\psi_h)) = 0 \quad \forall q_h \in \mathbb{L}_h, \\ (\mathbf{R}_h(\psi_h)).\mathbf{t} = \psi_h \text{ on } \Gamma. \end{cases} \tag{A.1}$$

Therefore using [28], we have

$$||\mathbf{R}_h(\psi_h)||_1 \leq C_1 ||\psi_h||_{\frac{1}{2},\Gamma}.$$

Take $\mu_h \in \mathcal{W}_h$ and let $\mathbf{w}_h = \mathbf{R}_h(\pi_h\psi) \in \mathbf{V}_h$ be the velocity solution of (A.1). Then $\mathbf{w}_h \in \mathbf{Z}_h$ and

$$||\mathbf{w}_h||_1 \leq C_1 ||\pi_h\psi||_{\frac{1}{2},\Gamma} \leq C_2 ||\psi||_{\frac{1}{2},\Gamma}.$$

Taking into account that $||\psi||_{\frac{1}{2},\Gamma} = 1$, we finally come to the conclusion that there is a positive constant C_2 independent of h (and of μ_h) such that

$$\begin{aligned} \frac{1}{C_2}||\mu_h||_{-\frac{1}{2},\Gamma} \leq \frac{||\mu_h||_{-\frac{1}{2},\Gamma}}{||\mathbf{w}_h||_1} &= \frac{\displaystyle\int_\Gamma \mu_h \, w_{ht} \, \mathrm{d}\Gamma}{||\mathbf{w}_h||_1} \\ &= \frac{\langle \mu_h, w_{ht} \rangle}{||\mathbf{w}_h||_1} \\ &\leq \sup_{\mathbf{v}_h \in \mathbf{Z}_h} \frac{\langle \mu_h, v_{ht} \rangle}{||\mathbf{v}_h||_1}. \end{aligned}$$

Then the inf-sup condition (5.11) is proved to be true. □

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