Feedback

On *Identities connecting the Chebyshev polynomials*: GCHQ Problem Solving Group write: In his article [1], Jonny Griffiths proposed the conjecture:

> With the exception of [2, 3, 5, 7, 17], no integer H-sequence contains more than 4 elements.

We offer the following proof of this conjecture.

Although not mentioned explicitly, we assume that such sequences are limited to positive integers, otherwise, as well as the sign-change/permutation variations mentioned in the article, there are infinite H-sequences, such as $[N,\ 1,\ -1,\ 1,\ -1,\ \ldots\ ]$.

It is clear that, if an element of an H-sequence is less than the previous element, the element following will be negative. Also, if any element is a 1, its successor will be -1. Thus a 1 or a decrease can only happen at the very end of the sequence.

Consider an integer H-sequence quartet $[a, b, c, d]$. Because of the above, we will assume that $2 \leqslant a < b < c$. We first show that, for some integer $n \geqslant 2$,

$$(n+1)b^2 = (n-1)c^2 + 2, a = \frac{(n+1)b - (n-1)c}{2}, d = \frac{(n+1)b + (n-1)c}{2}. \quad (1)$$

Since $a < b < c$, we have that

$$c = \frac{ab-1}{b-a} \Rightarrow bc - ac = ab - 1 \Rightarrow a = \frac{bc+1}{c+b} \Rightarrow b - a = \frac{b^2-1}{c+b}$$

$$\Rightarrow (c+b) \mid \left(b^2-1\right)$$

$$d = \frac{bc-1}{c-b} \Rightarrow d - b = \frac{b^2-1}{c-b} \Rightarrow (c-b) \mid \left(b^2-1\right).$$

Now, b and c must be coprime, since if $p > 1$ and $p \mid b$ and $p \mid c$, we have that $p \mid (c \pm b)$ but $p \nmid (bc \pm 1)$, so a and d would not be integers.

Now let $q = (c + b, c - b)$, then $q \mid 2b$ and $q \mid 2c$, so $q = 1$ or 2. If $q = 1$, then

$$(c + b) \mid \left(b^2 - 1\right), (c - b) \mid \left(b^2 - 1\right) \Rightarrow \left(c^2 - b^2\right) \mid \left(b^2 - 1\right).$$

If $q = 2$, let $c + b = 2k$, $c - b = 2l$, then $(k, l) = 1$ and $c = k + l$, $b = k - l$ and we have that

$$2k \mid \left(b^2 - 1\right), 2l \mid \left(b^2 - 1\right) \Rightarrow 2kl \mid \left(b^2 - 1\right) \Rightarrow \left(c^2 - b^2\right) \mid 2\left(b^2 - 1\right).$$

Therefore, regardless of whether $q = 1$ or 2, we have that $\left(c^2 - b^2\right) \mid 2\left(b^2 - 1\right)$. Noting that both expressions are positive and denoting their ratio by $n - 1$, where $n \geqslant 2$, we have that

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$$2\left(b^2 - 1\right) = (n-1)\left(c^2 - b^2\right) \Rightarrow (n+1)b^2 = (n-1)c^2 + 2.$$

We also have that

$$2bc + 2 = 2bc + (n+1)b^2 - (n-1)c^2 = \left((n+1)b - (n-1)c\right)(b+c)$$

$$\Rightarrow a = \frac{bc+1}{c+b} = \frac{(n+1)b - (n-1)c}{2},$$

$$2bc - 2 = 2bc + (n-1)c^2 - (n+1)b^2 = \left((n+1)b + (n-1)c\right)(c-b)$$

$$\Rightarrow d = \frac{bc-1}{c-b} = \frac{(n+1)b + (n-1)c}{2}.$$

This completes the proof of (1).

For an integer H-sequence quintet $[a, b, c, d, e]$, we can apply the above reasoning to the quartet $[b, c, d, e]$ and so, for some integer $m \geqslant 2$, would also need $b = \frac{1}{2}\left((m+1)c - (m-1)d\right)$. Eliminating d between this and the equation just above yields that $(mn + m + 3 - n)b = (3m + n + 1 - mn)c$ and plugging this into the Pellian equation gives that

$$(n+1)b^2 = (n-1)\left(\frac{mn + m + 3 - n}{3m + n + 1 - mn}\right)^2 b^2 + 2$$

$$\Rightarrow \left(5 + 6(m-n) + \left(5m^2 + 4mn + 5n^2\right) + 2mn(m-n) - 3m^2n^2\right)b^2$$

$$= (3m + n + 1 - mn)^2. \tag{2}$$

Therefore the following function must be a square and hence non-negative.

$$f_m(n) = 5 + 6(m-n) + \left(5m^2 + 4mn + 5n^2\right) + 2mn(m-n) - 3m^2n^2$$

$$\Rightarrow f'_m(n) = -6 + 4m + 10n + 2m^2 - 4mn - 6m^2n$$

$$\Rightarrow f''_m(n) = 10 - 4m - 6m^2 = -2(m-1)(3m+5).$$

Since $m \geqslant 2, f''_m(n) < 0$ so $f'_m(n)$ is decreasing. But $f'_m(1) = 4 - 4m^2 < 0$ and so $f'_m(n) < 0$ for $n \geqslant 1$. Thus $f_m(n)$ is decreasing with n. But $f_m(3) = -16\left(m^2 - 2\right) < 0$, hence $f_m(n) < 0$ for all $n \geqslant 3$.

That just leaves $n = 2$ to consider. Here

$$f_m(2) = 13 + 6m - 3m^2 = 16 - 3(m-1)^2 < 0$$

if $m > 3$, so there are just 2 cases: $(m, n) = (2, 2)$ and $(3, 2)$.

If $(m, n) = (2, 2)$, then (2) becomes $13b^2 = 5^2$ which has no solutions.

If $(m, n) = (3, 2)$, then (2) becomes $4b^2 = 6^2$ with the solution $b = 3$. The original Pellian equation then yields that $c = 5$ which corresponds to the quintet [2, 3, 5, 7, 17]. Thus this is the only quintet consisting of positive integers.

Reference

1. Jonny Griffiths, Identities connecting the Chebyshev polynomials, *Maths. Gaz.* **100** (November 2016) pp. 450-459.

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On: Properties of bisect-diagonal quadrilaterals [1]: Martin Josefsson writes: There was an unfortunate misprint in the previous issue of this *Gazette*. The factor 4 was missing in the denominator of the formula for the area of a bisect-diagonal quadrilateral, which should read

$$K = \frac{Q_1 Q_2}{4\left|a^2 - b^2 - c^2 + d^2\right|}$$

where Q_1 and Q_2 are correctly stated in [1, p. 220].

Reference

1. M. Josefsson, Properties of bisect-diagonal quadrilaterals, *Math. Gaz.* **101** (July 2017) pp. 214-226.

On: Some alternative derivations of Craig's formula [1]: Bob Pawula writes: The earliest known version of Craig's formula was discovered by A. H. Nuttall, in 1972 in a table of sonar integrals [2, eq. (75) p. 23].

References

1. Seán M. Stewart, Some alternative derivations of Craig's formula, *Math. Gaz.* **101** (July 2017) pp. 268-279.
2. A. H. Nuttall, Some integrals involving the Q-function, *Naval Underwater Systems Center*, Newport, RI, USA, Tech. Rep. TR 4297. This report is available at no cost on Google.

On 101.06: Graham Jameson writes: The first example in this Note is the case $a = -\frac{1}{2}$ of the following more general integral. For $-1 < x < 1$, let

$$I(a) = \int \frac{1}{(1-x)^{1-a}(1+x)^{1+a}}\,dx,$$

which we can rewrite as

$$\int \left(\frac{1-x}{1+x}\right)^{a-1} \frac{1}{(1+x)^2}\,dx.$$

The substitution $u = \dfrac{1-x}{1+x}$ gives $du = \dfrac{-2}{(1+x)^2}\,dx$, hence

$$\begin{aligned} I(a) &= -\frac{1}{2}\int u^{a-1}\,du \\ &= -\frac{u^a}{2a} + c \\ &= -\frac{1}{2a}\left(\frac{1-x}{1+x}\right)^a + c \end{aligned}$$

for all $a \neq 0$. This is not entirely trivial even for integer values of a. Trigonometric substitution is not an option except in cases like $a = \pm\frac{1}{2}$.

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On 101.07: Graham Jameson writes: The author establishes that

$$A^{1/3}G^{2/3} \leqslant L(a, b) \leqslant \tfrac{1}{3}A + \tfrac{2}{3}G, \tag{1}$$

where $A = A(a, b)$ and $G = G(a, b)$. It is a rather neat fact that this pair of inequalities contains, almost for free, the further information that the number $\frac{1}{3}$ is optimal on both sides. In other words, for any $r > \frac{1}{3}$, one can find a and b such that $L(a, b) < A^rG^{1-r}$, and for any $s < \frac{1}{3}$, one can find a and b such that $L(a, b) > sA + (1 - s)G$.

To show this, recall first, as stated in the Note, that for positive x, y and $0 < r < 1$, we have $x^ry^{1-r} \leqslant rx + (1 - r)y$. What we need is the following partial converse, showing that we cannot reduce the r on the right-hand side: if $0 < s < r < 1$, then there exists $\delta > 0$ such that whenever $1 < \frac{x}{y} < 1 + \delta$, we have

$$x^ry^{1-r} > sx + (1 - s)y. \tag{2}$$

To prove this, first take the case $y = 1$. Let $f(x) = x^r - sx - (1 - s)$. Then $f(1) = 0$ and $f'(1) = r - s > 0$. So for some $\delta > 0$, we have $f(x) > 0$, hence $x^r > sx + (1 - s)$, when $1 < x < 1 + \delta$. Statement (2) now follows by applying this to $\frac{x}{y}$.

Note next that if $a < b$, then $a < G < A < b$, so $1 < \dfrac{A}{G} < \dfrac{b}{a}$.

Now given $r > \frac{1}{3}$, apply (2) with $s = \frac{1}{3}$ and the right-hand side of (1): if $1 < \frac{b}{a} < 1 + \delta$ then $\frac{A}{G} < 1 + \delta$, so $A^rG^{1-r} > \frac{1}{3}A + \frac{2}{3}G \geqslant L(a, b)$. Similarly, given $s < \frac{1}{3}$, apply (2) with $r = \frac{1}{3}$ and the left-hand side of (1): if $1 < \frac{b}{a} < 1 + \delta$, then $sA + (1 - s)G < A^{1/3}G^{2/3} \leqslant L(a, b)$.

Maryam Mirzakhani (1977 – 2017)

Maryam Mirzakhani, who was born in Tehran, Iran, was, in 2014, the first woman to win the prestigious Fields Medal ‘for her outstanding contributions to the dynamics and geometry of Riemann surfaces and their moduli spaces’. Mourned by the mathematical community, she died of cancer on 14 July 2017 in hospital in California.

Mirzakhani took her bachelor's degree at Tehran's prestigious Sharif University and moved to Harvard in 1999. In 2004 she presented her doctoral thesis which secured her widespread recognition and a fellowship at the Clay Mathematics Institute. She rose rapidly from assistant professor to professor at Princeton University from 2004 to 2008 and then became professor at Stanford University.

For those interested in how young minds develop, Mirzakhani admitted that as a child she did rather poorly at mathematics and aspired to be a writer. Initially, it was an elder brother who fired her interest in mathematics and this was sustained by inspiring teachers at secondary school. Representing Iran, she won gold medals in the International Mathematical Olympiads of 1994 and 1995 with a perfect score in the latter.

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