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Transversal parameters and tangential flatness

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Introduction

Let X be an algebraic variety over a field k, z a point of X and $R = \mathcal{O}_{X,z}$. Let $\mathbf{t} = \{t_1, \ldots, t_s\}$ be part of a system of parameters of R. Then these parameters are algebraically independent over k, and therefore they define (near z) a projection $f\colon X \to A^s(k)$ to an s-dimensional affine space over k, sending z to the origin $0 \in A^s(k)$. In general, the Hilbert function (resp. multiplicity) of $\mathcal{O}_{f^{-1}(0),z} = R/\mathbf{t}R$ will be worse than that of $R = \mathcal{O}_{X,z}$, and we will call the system $\mathbf{t}$ transversal, if the Hilbert function (resp. multiplicity) of R and $R/\mathbf{t}R$ agree. This gives two notions of transversality: one for Hilbert functions (H-transversal), and a weaker one for multiplicities (e-transversal). For $s = \dim R$ we recover the notion of a transversal system of parameters introduced by Zariski for studying equisingularity problems (see e.g. [**17**]). In the above set-up the numerical characters are defined with respect to the maximal ideal of R, but we will consider this problem for arbitrary ideals I using generalized Hilbert functions of type $H[\mathbf{x}, I, R]$ (see [**7**] and [**10**]).

The first section contains a complete description of H-transversal parameters, generalizing results of B. Singh [**14**], and in the second section we will give the analogous results for the notion of e-transversal. Both types of results will be applied in the next section to derive some consequences for blowing ups with maximal Hilbert functions, resp. multiplicities. Furthermore, we give a numerical criterion for tangential flatness (as introduced in [**6**]). The study of e-transversal parameters was originally motivated by Lipman's characterization of flat families with fibres of constant multiplicity (see [**8**]), where he uses this notion implicitly in the proof. The last section of this paper contains an analogous characterization of flat families with fibres of constant Hilbert function.

1. *H-transversal parameters*

Let (R, M) be a local ring, $I \subset M$ an ideal and $\mathbf{x} = \{x_1, \ldots, x_r\} \subset R$ a system of parameters with respect to I, i.e. the images of $x_1, \ldots, x_r$ in R/I are a system of parameters. Recall [**7**] that the multiplicity symbol $e(\mathbf{x}; —)$ of [**9**] can be used to define generalized Hilbert functions by

$$H^{(0)}[\mathbf{x}, I, R](n) = e(\mathbf{x}; I^n/I^{n+1}),$$

$$H^{(i)}[\mathbf{x}, I, R](n) = \sum_{k=0}^{n} H^{(i-1)}[\mathbf{x}, I, R](k) \quad \text{for} \quad i > 0.$$

We will use the notation $H^{(i)}[I, R]$ if $\mathbf{x} = \varnothing$ and $H^{(i)}[R]$ if $I = M$.

The following Proposition generalizes a result of B. Singh ([**14**], theorem 1):

(1·1) Proposition. *With R, I, $\mathbf{x}$ as above, let $t \in I$ and $\bar{R} = R/tR$, $\bar{I} = I/tR$, $y_i = x_i \bmod tR$ $(i = 1, \ldots, r)$. Then putting*

$$D_n = (I^{n+1}\colon tR)/I^n \quad (n \geqslant 0)$$

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we have

$$H^{(1)}[\mathbf{y}, \bar{I}, \bar{R}](n) = H^{(0)}[\mathbf{x}, I, R](n) + e(\mathbf{x}; D_n) \quad (n \geqslant 0).$$

In particular we have

$$H^{(0)}[\mathbf{x}, I, R] \leqslant H^{(1)}[\mathbf{y}, \bar{I}, \bar{R}].$$

Proof. We proceed by induction on n, the case $n = 0$ being trivial. For $n > 0$, consider the exact sequences

$$0 \to I^{n-1}/I^n \to (I^n : tR)/I^n \to D_{n-1} \to 0, \tag{1}$$

$$0 \to (I^n \cap tR)/(I^{n+1} \cap tR) \to I^n/I^{n+1} \to \bar{I}^n/\bar{I}^{n+1} \to 0, \tag{2}$$

$$0 \to D_n \to (I^n : tR)/I^n \xrightarrow{t} (I^n \cap tR)/(I^{n+1} \cap tR) \to 0. \tag{3}$$

Since $\mathbf{x}$ is a multiplicity system on all these R-modules, additivity of $e(\mathbf{x}; —)$ yields

$$\begin{aligned}
&H^{(1)}[\mathbf{y}, \bar{I}, R](n) - H^{(0)}[\mathbf{x}, I, R](n) \\
&\quad = H^{(1)}[\mathbf{y}, \bar{I}, R](n-1) + H^{(0)}[\mathbf{y}, \bar{I}, R](n) - H^{(0)}[\mathbf{x}, I, R](n) && (4)\\
&\quad = H^{(0)}[\mathbf{x}, I, R](n-1) + e(\mathbf{x}; D_{n-1}) + e(\mathbf{x}; \bar{I}^n/\bar{I}^{n+1}) - e(\mathbf{x}; I^n/I^{n+1}) && (5)\\
&\quad = e(\mathbf{x}; I^{n-1}/I^n) + e(\mathbf{x}; D_{n-1}) - e(\mathbf{x}; (I^n \cap tR)/(I^{n+1} \cap tR)) && (6)\\
&\quad = e(\mathbf{x}; (I^n : tR)/I^n) - e(\mathbf{x}; (I^n \cap tR)/(I^{n+1} \cap tR)) && (7)\\
&\quad = e(\mathbf{x}; D_n) && (8)
\end{aligned}$$

as asserted. Here (4) holds by definition, (5) by inductive assumption, and (6), (7), (8) are consequences of (2), (1), (3) respectively.

(1·2) COROLLARY. *With the notations as above, consider the conditions:*

(i) $H^{(0)}[\mathbf{x}, I, R] = H^{(1)}[\mathbf{y}, \bar{I}, R]$.

(ii) $\mathrm{in}_I(t)$ *is a regular element of degree* 1 *in* $\mathrm{gr}_I(R)$. *Then* (ii) $\Rightarrow$ (i), *and if we assume in addition that*

$$\mathrm{Ass}\,(I^n/I^{n+1}) = \mathrm{Assh}\,(R/I) \quad \textit{for all} \quad n \geqslant 0, \tag{9}$$

then also (i) $\Rightarrow$ (ii). (*Here* $\mathrm{Assh}\,(R/I) = \{P \in \mathrm{Ass}\,(R/I) \mid \dim R/P = \dim R/I\}$.)

Proof. Clearly (ii) $\Rightarrow$ (i) by the Proposition, so assume (i) and (9). If $r = 0$ (in which case assumption (9) is empty), then $e(\mathbf{x}; D_n) = l_R(D_n) = 0$ for all n, and therefore (ii) follows. For general r we have (see [7])

$$H^{(0)}[\mathbf{x}, I, R] = \sum_{P \in \mathrm{Assh}\,(R/I)} e(\mathbf{x}; R/P)\, H^{(0)}[IR_P, R_P]$$

and

$$H^{(1)}[\mathbf{y}, \bar{I}, \bar{R}] = \sum_{\bar{P} \in \mathrm{Assh}\,(\bar{R}/\bar{I})} e(\mathbf{y}; \bar{R}/\bar{P})\, H^{(1)}[\bar{I}\bar{R}_{\bar{P}}, \bar{R}_{\bar{P}}].$$

Therefore (i) and Proposition (1·1) imply

$$H^{(0)}[IR_P, R_P] = H^{(1)}[\bar{I}\bar{R}_{\bar{P}}, \bar{R}_{\bar{P}}] \quad \text{for all} \quad P \in \mathrm{Assh}\,(R/I), \tag{10}$$

where $\bar{P} = P/tR$. Now consider the homomorphism

$$I^n/I^{n+1} \xrightarrow{t} I^{n+1}/I^{n+2}.$$

From (10) and the case $r = 0$ we conclude that $t \otimes R_P$ is injective for all $P \in \mathrm{Assh}\,(R/I)$. Therefore t is injective by assumption (9).

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(1·3) *Definition.* Let R be a local ring and I a proper ideal of R. We call R normally Cohen–Macaulay along I if

$$\operatorname{depth}(I^n/I^{n+1}) = \dim(R/I) \quad \text{for all} \quad n \geqslant 0.$$

(1·4) *Remarks.* (*a*) It is not difficult to see that R is normally Cohen–Macaulay along I if and only if R/I^{n+1} is a Cohen–Macaulay ring for all $n \geqslant 0$ ([7], p. 117). Therefore we have in this case

$$\operatorname{Ass}(I^n/I^{n+1}) \subset \operatorname{Ass}(R/I^{n+1}) = \operatorname{Min}(I^{n+1}) = \operatorname{Min}(I),$$

where $\operatorname{Min}(I)$ denotes the set of minimal primes of I. It follows that (9) is satisfied if R is normally Cohen–Macauley along I.

(*b*) Let $\mathbf{x}$ be a system of parameters with respect to I. By (*a*), R is normally Cohen–Macaulay along I if and only if $\mathbf{x}$ is a regular sequence on R/I^{n+1} for all $n \geqslant 0$. This implies, by Krull's intersection theorem, that $\mathbf{x}$ is a regular sequence in R.

We recall the following result for later reference:

(1·5) THEOREM. *Let R be a local ring, I a proper ideal in R and $\mathbf{x} = \{x_1, \ldots, x_r\}$ a system of parameters with respect to I. Let $I(\mathbf{x}) = I + \mathbf{x}R$. Then we have:*

(*a*) $H^{(r)}[\mathbf{x}, I, R] \leqslant H^{(0)}[I(\mathbf{x}), R]$.

(*b*) $H^{(r)}[\mathbf{x}, I, R] = H^{(0)}[I(\mathbf{x}), R]$ if and only if R is normally Cohen–Macaulay along I.

(*c*) R is normally Cohen–Macaulay along I if and only if the canonical epimorphism

$$\operatorname{gr}_I(R) \otimes_R (R/I(\mathbf{x}))[X_1, \ldots, X_r] \to \operatorname{gr}_{I(\mathbf{x})}(R)$$

is an isomorphism.

Proof. (*a*) and (*b*) were proved in [7], while (*c*) follows from [13], Theorem 1·8 together with Remark (1·4).

Now let $R, I, \mathbf{x}$ be as at the beginning of this section and let $t = \{t_1, \ldots, t_s\}$ be any subset of I. By Proposition (1·1) we know that

$$H^{(0)}[\mathbf{x}, I, R] \leqslant H^{(s)}[\mathbf{y}, I/tR, R/\mathbf{t}R], \tag{11}$$

where $\mathbf{y}$ denotes the image of $\mathbf{x}$ in $R/\mathbf{t}R$. We now want to give a criterion for equality in (11). First we make the following:

(1·6) *Definition.* Let R be a local ring, I a proper ideal of R and $\mathbf{x} = \{x_1, \ldots, x_r\}$ a system of parameters with respect to I. A subset $\mathbf{t} = \{t_1, \ldots, t_s\}$ of I will be called H-transversal for $(\mathbf{x}, I)$, if

$$H^{(0)}[\mathbf{x}, I, R] = H^{(s)}[\mathbf{y}, I/\mathbf{t}R, R/\mathbf{t}R],$$

where $\mathbf{y}$ denotes the image of $\mathbf{x}$ in $R/\mathbf{t}R$. If $\mathbf{x} = \varnothing$, $\mathbf{t}$ will be called H-transversal for I.

(1·7) THEOREM. *Let R be a local ring, I a proper ideal of R, $\mathbf{x} = \{x_1, \ldots, x_r\}$ a system of parameters with respect to I and $\mathbf{t} = \{t_1, \ldots, t_s\}$ a subset of I. Consider the conditions:*

(i) $\mathbf{t}$ *is H-transversal for* $(\mathbf{x}, I)$.

(ii) $\operatorname{in}_I(t_1), \ldots, \operatorname{in}_I(t_s)$ *is a regular sequence of degree* 1 *in* $\operatorname{gr}_I(R)$.

Then (ii) $\Rightarrow$ (i), *and if $R/\mathbf{t}R$ is normally Cohen–Macaulay along $I/\mathbf{t}R$, then both conditions are equivalent.*

Proof. We note that if $\operatorname{in}_I(t_1)$ is regular in $\operatorname{gr}_I(R)$, then

$$\operatorname{gr}_{I/t_1R}(R/t_1R) \simeq \operatorname{gr}_I(R)/(\operatorname{in}_I(t_1))$$

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(compare [16]). Therefore (ii) $\Rightarrow$ (i) by Corollary (1·2) using induction. For the converse, assume in addition that $R/\mathbf{t}R$ is normally Cohen–Macaulay along $I/\mathbf{t}R$. We will use induction on s, and we may assume $s > 0$, since the case $s = 0$ is trivial. Let

$$\bar{R} = R/t_1R + \ldots + t_{s-1}R, \quad \bar{I} = I\bar{R}$$

and $\mathbf{z}$ the image of $\mathbf{x}$ in $\bar{R}$. Then (i) implies

$$H^{(0)}[\mathbf{x}, I, R] = H^{(s-1)}[\mathbf{z}, \bar{I}, \bar{R}] = H^{(s)}[\mathbf{y}, I/\mathbf{t}R, R/\mathbf{t}R], \tag{12}$$

i.e. $\{t_1, \ldots, t_{s-1}\}$ is H-transversal for $(\mathbf{x}, I)$ and the image $\bar{t}_s$ of t_s in $\bar{R}$ is H-transversal for $(\mathbf{z}, \bar{I})$. Let us show that $\bar{R}$ is normally Cohen–Macaulay along $\bar{I}$. For this we put $\bar{I}(\mathbf{z}) = \bar{I} + \mathbf{z}\bar{R}$. Then we have

$$\begin{aligned} H^{(r)}[\mathbf{x}, I, R] &= H^{(r+s-1)}[\mathbf{z}, \bar{I}, \bar{R}] \leqslant H^{(s-1)}[\bar{I}(\mathbf{z}), \bar{R}] \leqslant H^{(s)}[\bar{I}(\mathbf{z})/\bar{t}_s\bar{R}, \bar{R}/\bar{t}_s\bar{R}] \\ &= H^{(r+s)}[\mathbf{y}, I/\mathbf{t}R, R/\mathbf{t}R]; \end{aligned}$$

the last equality holds because $R/\mathbf{t}R$ is normally Cohen–Macaulay along $I/\mathbf{t}R$. We conclude that $H^{(r)}[\mathbf{z}, \bar{I}, \bar{R}] = H^{(0)}[\bar{I}(\mathbf{z}), \bar{R}]$, so that $\bar{R}$ is normally Cohen–Macaulay along $\bar{I}$. Now the inductive assumption implies that $\mathrm{in}_I(t_1), \ldots, \mathrm{in}_I(t_{s-1})$ is a regular sequence of degree 1 in $\mathrm{gr}_I(R)$, and in particular

$$\mathrm{gr}_I(\bar{R}) \simeq \mathrm{gr}_I(R)/(\mathrm{in}_I(t_1), \ldots, \mathrm{in}_I(t_{s-1})).$$

Finally, using Remark (1·4), we know by Corollary (1·2) and (12) that $\mathrm{in}_I(\bar{t}_s)$ is a regular element of degree 1 in $\mathrm{gr}_{\bar{I}}(\bar{R})$, which proves (ii).

(1·8) *Remarks.* (*a*) In the proof of Theorem (1·7) it was shown, without any extra assumption on R, that $R/\mathbf{t}R$ normally Cohen–Macaulay along $I/\mathbf{t}R$ and $\mathbf{t}$ H-transversal for $(\mathbf{x}, I)$ imply that R is normally Cohen–Macaulay along I.

(*b*) The second condition of (1·7) is independent of $\mathbf{x}$, and so is the property of being H-transversal for $(\mathbf{x}, I)$, provided that $R/\mathbf{t}R$ is normally Cohen–Macaulay along $I/\mathbf{t}R$.

As a special case of Theorem (1·7) we obtain the following Corollary:

(1·9) COROLLARY. *Let (R, M) be a local ring and I an M-primary ideal. Then a subset $\mathbf{t} = \{t_1, \ldots, t_s\}$ of I is H-transversal for I if and only if $\mathrm{in}_I(t_1), \ldots, \mathrm{in}_I(t_s)$ is a regular sequence in $\mathrm{gr}_I(R)$ of degree 1.*

(1·10) COROLLARY. *If R is normally Cohen–Maculay along I and $\mathbf{x}$ is any system of parameters with respect to I, then $\mathbf{x}$ is H-transversal for $I(\mathbf{x})$.*

Proof. This follows immediately from Corollary (1·9) and Theorem (1·5), (*c*).

The title of this first section may need some explanation, since it is not clear from the definition (1·6) that an H-transversal subset of I is part of a system of parameters of R, and in fact this may fail to hold in general. But in the situation of Theorem (1·7) it is clear that not only $\{t_1, \ldots, t_s\}$ but even $\{x_1, \ldots, x_r, t_1, \ldots, t_s\}$ is part of a system of parameters of R.

2. *e-transversal parameters*

Throughout this section (R, M) will denote a local ring with infinite residue field (to assure the existence of suitable minimal reductions). Furthermore, since we want to apply some fundamental results of multiplicity theory, we assume for this section that R is quasi-unmixed. From this we conclude that if I is an ideal of R with $\mathrm{ht}\,(I) = l(I)$, then $\mathrm{Assh}\,(R/I) = \mathrm{Min}\,(I)$ as in Remark (1·4), (*a*). We also recall that if $\mathbf{x}$ is any system

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of parameters with respect to I, then the condition $ht(I) = l(I)$ together with the formula

$$H^{(0)}[\mathbf{x}, I, R] = \sum_{P \in \operatorname{Assh}(R/I)} e(\mathbf{x}; R/P)\, H^{(0)}[IR_P, R_P] \tag{13}$$

implies that

$$e(\mathbf{x}, I, R) = \sum_{P \in \operatorname{Assh}(R/I)} e(\mathbf{x}; R/P)\, e(IR_P, R_P), \tag{14}$$

where $e(\mathbf{x}, I, R)$ is the multiplicity associated to the polynomial function $H^{(0)}[\mathbf{x}, I, R]$.

In order to derive results for multiplicities parallel to those for Hilbert functions, we need the following

(2·1) LEMMA. *Let R be a quasi-unmixed local ring, I a proper ideal of R, $\mathbf{x}$ a system of parameters with respect to I and $\mathbf{t} = \{t_1, \ldots, t_s\}$ a subset of I. Assume that $ht(I) = l(I)$ and* $\dim R/\mathbf{t}R = \dim R - s$. *Then, putting $\bar{R} = R/\mathbf{t}R$ and $\bar{I} = I/\mathbf{t}R$, we have*

$$e(\mathbf{y}, \bar{I}, \bar{R}) = \sum_{\bar{P} \in \operatorname{Assh}(\bar{R}/\bar{I})} e(\mathbf{y}; \bar{R}/\bar{P})\, e(\bar{I}\bar{R}_{\bar{P}}, \bar{R}_{\bar{P}}), \tag{15}$$

where $\mathbf{y}$ is the image of $\mathbf{x}$ in R as usual.

Proof. Comparing with (13) (applied to $\bar{R}$), it is clear that the assertion will follow from

$$\operatorname{ht}(\bar{P}) = \operatorname{ht}(\bar{I}) \quad \text{for all} \quad \bar{P} \in \operatorname{Assh}(\bar{R}/\bar{I}). \tag{16}$$

So let $\bar{P} \in \operatorname{Assh}(\bar{R}/\bar{I})$ and let P be the inverse image of P in R. Let Q be any minimal prime of $\mathbf{t}R$ contained in P. Then $\operatorname{ht}(Q) \leqslant s$, but $\dim R/Q \leqslant \dim \bar{R} = \dim R - s$ by assumption, and $\dim R = \dim R/Q + \operatorname{ht}(Q)$ since R is quasi-unmixed. It follows that $\operatorname{ht}(Q) = s = \operatorname{ht}(QR_P)$, showing that $\dim R_P/QR_P = \dim R_P - s$ for all such Q. We conclude that $\operatorname{ht}(\bar{P}) = \operatorname{ht}(P) - s$, which proves (16).

(2·2) COROLLARY. *With the notations and assumptions as above, we have*

$$e(\mathbf{x}, I, R) \leqslant e(\mathbf{y}, I/tR, R/tR). \tag{17}$$

Proof. For $\mathbf{x} = \varnothing$ the assertion follows from Proposition (1·1) and the assumption $\dim R = \dim R/\mathbf{t}R + s$. From this the general case follows by comparing (14) and (15).

Now we want to characterize equality in (17), for which it seems convenient to treat the case $\mathbf{x} = \varnothing$ first. This case, in which I is M-primary, is of special interest, and it will simplify the argument for the general case.

(2·3) *Definition.* Let R be a quasi-unmixed local ring, I a proper ideal in R and $\mathbf{x}$ a system of parameters with respect to I. A subset $\mathbf{t} = \{t_1, \ldots, t_s\}$ will be called *e*-transversal for $(\mathbf{x}, I)$, if

(*a*) $\dim R/\mathbf{t}R = \dim R - s$ and

(*b*) $e(\mathbf{x}, I, R) = e(\mathbf{y}, I/\mathbf{t}R, R/\mathbf{t}R)$,

where $\mathbf{y}$ denotes the image of $\mathbf{x}$ in $R/\mathbf{t}R$. If $\mathbf{x} = \varnothing$, $\mathbf{t}$ will be called *e*-transversal for I.

(2·4) PROPOSITION. *Let (R, M) be a quasi-unmixed local ring (with infinite residue field), let I be an M-primary ideal and let $\mathbf{t} = \{t_1, \ldots, t_s\}$ be a subset of I such that*

$$\dim R/\mathbf{t}R = \dim R - s.$$

Then the following conditions are equivalent:

(i) *$\mathbf{t}$ is e-transversal for I.*

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(ii) *For all* $P \in \operatorname{Min}(\mathbf{t}R)$, R_P *is Cohen–Macaulay, and there are elements*

$$u_{s+1}, \ldots, u_d \in I \quad (d = \dim R)$$

such that $t_1, \ldots, t_s, u_{s+1}, \ldots, u_d$ *generate a minimal reduction of* I.

Proof. (*i*) $\Rightarrow$ (ii). Choose $u_{s+1}, \ldots, u_d \in I$ such that their images in $I/\mathbf{t}R$ generate a minimal reduction of this ideal. Let $J = \mathbf{t}R + \mathbf{u}R$, where $\mathbf{u} = \{u_{s+1}, \ldots, u_d\}$. Then $J/\mathbf{t}R$ is a reduction of $I/\mathbf{t}R$, and therefore

$$e(I, R) \leqslant e(J, R) \leqslant e(J/\mathbf{t}R, R/\mathbf{t}R) = e(I/\mathbf{t}R, R/\mathbf{t}R) = e(I, R)$$

by Corollary (2·2) and (i). Therefore J is a reduction of I by Rees's theorem [12]. Now since $ht(\mathbf{t}R) = s$, we have $\operatorname{Assh}(R/\mathbf{t}R) = \operatorname{Min}(\mathbf{t}R)$, and therefore

$$\begin{aligned} e(I, R) = e(J, R) &= \sum_{P \in \operatorname{Min}(\mathbf{t}R)} e(\mathbf{u}; R/P)\, e(\mathbf{t}R_P, R_P) \\ &\leqslant \sum_{P \in \operatorname{Min}(\mathbf{t}R)} e(\mathbf{u}; R/P)\, l(R_P/\mathbf{t}R_P) \\ &= e(J/\mathbf{t}R, R/\mathbf{t}R) = e(I/\mathbf{t}R, R/\mathbf{t}R). \end{aligned}$$

It follows that $e(\mathbf{t}R_P, R_P) = l(R_P/\mathbf{t}R_P)$ for all $P \in \operatorname{Min}(\mathbf{t}R)$ (Note that $e(\mathbf{u}; R/P) \neq 0$ for all $P \in \operatorname{Min}(\mathbf{t}R)$.) Since $\mathbf{t}$ is a system of parameters in R_P for $P \in \operatorname{Min}(\mathbf{t}R)$, these R_P are Cohen–Macaulay.

(ii) $\Rightarrow$ (i). Let $\mathbf{u} = \{u_{s+1}, \ldots, u_d\}$ and let $\mathbf{v}$ be the image of $\mathbf{u}$ in $R/\mathbf{t}R$. Then by assumption

$$\begin{aligned} e(I, R) &= e(\mathbf{t}R + \mathbf{u}R) \\ &= \sum_{P \in \operatorname{Assh}(R/\mathbf{t}R)} e(\mathbf{u}; R/P)\, e(\mathbf{t}R_P, R_P) \\ &= \sum_{P \in \operatorname{Assh}(R/\mathbf{t}R)} e(\mathbf{u}; R/P)\, l(R_P/\mathbf{t}R_P) \\ &= e(\mathbf{v}; R/\mathbf{t}R) = e(I/\mathbf{t}R, R/\mathbf{t}R). \end{aligned}$$

(2·5) THEOREM. *Let* (R, M) *be a local ring (with infinite residue field),* I *a proper ideal of* R, $\mathbf{x}$ *a system of parameters with respect to* I *and* $\mathbf{t} = \{t_1, \ldots, t_s\}$ *a subset of* I. *Assume that* R *is quasi-unmixed and* $\operatorname{ht}(I) = l(I)$. *Then the following conditions are equivalent:*

(i) $\mathbf{t}$ *is e-transversal for* $(\mathbf{x}, I)$ *and* $\operatorname{ht}(I/\mathbf{t}R) = l(I/\mathbf{t}R)$.

(ii) R_P *is Cohen-Macaulay for all* $P \in \operatorname{Min}(\mathbf{t}R)$, *and* $\mathbf{t}$ *is part of a minimal set of generators of a minimal reduction of* I.

Proof. (i) $\Rightarrow$ (ii). By [3], $\operatorname{ht}(I/\mathbf{t}R) = l(I/\mathbf{t}R)$ implies

$$e(\mathbf{x}, I/\mathbf{t}R, R/\mathbf{t}R) = e(I(\mathbf{x})/\mathbf{t}R, R/\mathbf{t}R),$$

where $I(\mathbf{x}) = I + \mathbf{x}R$, and similarly

$$e(\mathbf{x}, I, R) = e(I(\mathbf{x}), R).$$

Therefore, by Proposition (2·4), R_P is Cohen–Maculay for all $P \in \operatorname{Min}(\mathbf{t}R)$. Using $\operatorname{ht}(I/\mathbf{t}R) = l(I/\mathbf{t}R)$ again we may choose $u_{s+1}, \ldots, u_h \in I$, $h = \operatorname{ht}(I) = s + \operatorname{ht}(I/\mathbf{t}R)$, such that the images in $R/\mathbf{t}R$ generate a minimal reduction of $I/\mathbf{t}R$. Let $\mathbf{u} = \{u_{s+1}, \ldots, u_h\}$

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and $J = \mathbf{t}R + \mathbf{u}R$. Then $\operatorname{Min}(J) = \operatorname{Min}(I) = \operatorname{Assh}(R/J)$, and by Proposition (2·4) we have (with the notation of (2·3))

$$\sum_{P \in \operatorname{Assh}(R/I)} e(\mathbf{x}; R/P)\, e(IR_P, R_P) = e(\mathbf{x}, I, R)$$
$$= e(\mathbf{y}, I/\mathbf{t}R, R/\mathbf{t}R) = e(\mathbf{y}, J/\mathbf{t}R, R/\mathbf{t}R) = e(\mathbf{x}, J, R)$$
$$= \sum_{P \in \operatorname{Assh}(R/J)} e(\mathbf{x}; R/P)\, e(JR_P, R_P).$$

We conclude that $e(IR_P, R_P) = e(JR_P, R_P)$ for all

$$P \in \operatorname{Min}(I) = \operatorname{Assh}(R/I) = \operatorname{Assh}(R/J) = \operatorname{Min}(J),$$

and therefore J is a reduction of I by Böger's extension of Rees's theorem [1].

(ii) $\Rightarrow$ (i). Clearly $\operatorname{ht}(I/\mathbf{t}R) = l(I/\mathbf{t}R)$, and therefore (i) is a direct consequence of Proposition (2·4) and (14).

(2·6) Corollary. *Let R be a quasi-unmixed local ring with infinite residue field, I a proper ideal of R and $\mathbf{x}$ a system of parameters with respect to I. Assume that* $\operatorname{ht}(I) = l(I)$ *and that $\mathbf{x}$ is a regular sequence in R. Then $\mathbf{x}$ is e-transversal for $I(\mathbf{x})$.*

Proof. If $\mathbf{u}$ generates a minimal reduction of I, then $\mathbf{u} \cup \mathbf{x}$ generates a minimal reduction of $I(\mathbf{x})$.

(2·7) *Remark.* We want to give some technical comments on the proof of Theorem (2·5). First we note that condition (ii) is independent of the system $\mathbf{x}$. Furthermore, for proving (ii) $\Rightarrow$ (i), it is enough to assume R_P Cohen–Macaulay for those $P \in \operatorname{Min}(\mathbf{t}R)$, which are contained in some minimal prime ideal of I. For the others it will follow automatically by (i) $\Rightarrow$ (ii). Finally we point out that under the conditions of Theorem (2·5) any elements $u_{s+1}, \ldots, u_h \in I$ generate, together with $\mathbf{t}$, a minimal reduction of I, provided that their images generate a minimal reduction of $I/\mathbf{t}R$.

(2·8) *Remark.* If R is quasi-unmixed, $\operatorname{ht}(I) = l(I)$ and $\dim R/\mathbf{t}R = \dim R - s$, then $\mathbf{t}$ H-transversal for $(\mathbf{x}, I)$ trivially implies $\mathbf{t}$ e-transversal for $(\mathbf{x}, I)$. If R is a hypersurface, i.e. $R = S/fS$ for some regular local ring S, then both notions are equivalent. This follows from the results in [4], where also more general cases are treated, in which H-transversal and e-transversal coincide, namely certain 'strict complete intersections'. If R is a hypersurface and $s = \dim R$, then $\mathbf{t}$ e-transversal for M means that $\mathbf{t}$ is a transversal system of parameters in the sense of Zariski [17], and this is equivalent to saying that $\mathbf{t}$ generates a minimal reduction of M.

(2·9) Proposition. *Let (R, M) be a quasi-unmixed local ring (with infinite residue field) and $\mathbf{t} = \{t_1, \ldots, t_s\}$ a subset of M. Assume that $\mathbf{t}$ is e-transversal for M and that $R/\mathbf{t}R$ is Cohen–Macaulay. Then R is Cohen–Macaulay.*

Proof. By Theorem (2·5) we may choose $\mathbf{u} = \{u_{s+1}, \ldots, u_d\}$, $d = \dim R$, such that $\mathbf{t} \cup \mathbf{u}$ generates a minimal reduction of M. Consequently

$$e(R) = e(\mathbf{t}R + \mathbf{u}R, R) = e(\mathbf{u}(R/\mathbf{t}R), R/\mathbf{t}R) = 1(R/\mathbf{t}R + \mathbf{u}R).$$

(2·10) *Remark.* The same argument as in (2·9) shows that, under the conditions of Theorem (2·5), if P is any minimal prime ideal of I and $R_P/\mathbf{t}R_P$ is Cohen–Macaulay, then R_P is Cohen–Macaulay.

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3. *Tangential flatness, blowing-ups*

Let $f\colon S \to R$ be a local homomorphism of local rings, and let $J \subset S$, $I \subset R$ be ideals such that $f(J) \subset I$. In [6] we have studied the question, under which conditions the induced homomorphism $f^* = \mathrm{gr}\,(f, J, I)\colon \mathrm{gr}_J(S) \to \mathrm{gr}_I(R)$ is flat. In particular we have considered the case in which J is generated by a regular sequence. The results of [6] can be related to H-transversal parameters in the following way:

(3·1) THEOREM. *Let R, S be local rings, $f\colon S \to R$ a flat local homomorphism,*

$$\mathbf{t} = \{t_1, \ldots, t_s\}$$

a regular sequence in S and I an ideal of R such that $f(\mathbf{t}) \subset I$. Let

$$f^* = \mathrm{gr}\,(f, \mathbf{t}S, I)\colon \mathrm{gr}_{\mathbf{t}S}(S) \to \mathrm{gr}_I(R)$$

be the induced homomorphism and assume

(*a*) *$R/\mathbf{t}R$ is normally Cohen–Macaulay along $I/\mathbf{t}R$.*

(*b*) *The canonical homomorphism*

$$S/\mathbf{t}S \to \mathrm{gr}_I(R)/(f^*(\mathrm{in}_{\mathbf{t}S} t_1), \ldots, f^*(\mathrm{in}_{\mathbf{t}S} t_s))$$

is flat.

Then the following conditions are equivalent:

(i) *f^* is flat.*

(ii) *$f^*(\mathrm{in}_{\mathbf{t}S} t_1), \ldots, f^*(\mathrm{in}_{\mathbf{t}S} t_s)$ is a regular sequence in $\mathrm{gr}_I(R)$ (necessarily of degree 1).*

(iii) *For any system of parameters $\mathbf{x}$ with respect to I we have*

$$H^{(0)}[\mathbf{x}, I, R] = H^{(s)}[\mathbf{y}, I/\mathbf{t}R, R/\mathbf{t}R].$$

(*Here we have written $\mathbf{t}R$ for $f(\mathbf{t})\,R$, and $\mathbf{y}$ is the image of $\mathbf{x}$ in $R/\mathbf{t}R$.*)

Proof. (i)$\Leftrightarrow$(ii) is contained in theorem (2·9) of [6], and (ii)$\Leftrightarrow$(iii) is the content of Theorem (1·7).

(3·2) *Remark.* Note that assumption (*a*) of Theorem (3·1) can be omitted if I is primary to the maximal ideal of R.

We will come back to tangential flatness in the next section. For the rest of this section we want to apply the results obtained so far to derive some consequences for blowing-ups with maximal Hilbert functions resp. multiplicities in the sense of the results of [10]. For this purpose we introduce some notation that will be kept fixed for the rest of this section.

(3·3) *Notation.* R is an excellent local ring, I is an ideal of R and $R \to R_1$ is a local homomorphism obtained by blowing up R with centre I. We put $A = \mathrm{gr}_I(R)$, so that $R_1/IR_1 \simeq A_{(Q)}$ for a suitable homogeneous prime ideal Q of A. Let N be the unique homogeneous maximal ideal of A and $M(R_1)$ the maximal ideal of R_1. We put $s = \dim A/Q$ and we fix $t \in I$ such that $IR_1 = tR_1$. Finally, $\mathbf{x} = \{x_1, \ldots, x_r\}$ will denote a system of parameters with respect to I and $I(\mathbf{x}) = I + \mathbf{x}R$.

(3·4) *Remark.* Using Ratliff's theory of quasi-unmixed rings, it can be shown that if E is quasi-unmixed, the same is true for R_1.

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Let R be normally Cohen–Macaulay along I. Then by [**10**] we have the following inequalities:

$$H^{(s+1)}[R_1] \leqslant H^{(s+2)}[R_1/IR_1] = H^{(s+2)}[A_Q] \tag{18}$$

$$\leqslant H^{(1)}[A_N] \tag{19}$$

$$\leqslant H^{(r+1)}[\mathbf{x}, I, R]. \tag{20}$$

(3·5) Proposition. *With the notations of* (3·3), *assume that R is normally Cohen–Macaulay along I. Then $H^{(s+1)}[R_1] = H^{(r+1)}[\mathbf{x}, I, R]$ if and only if the following conditions are satisfied:*

(*a*) $\mathrm{in}_{M(R_1)}t$ *is a regular element of degree* 1 *in* $\mathrm{gr}_{M(R_1)}(R_1)$.

(*b*) Q *is a permissible in* A.

(*c*) R/I *is regular and* $\mathbf{x}$ *is a regular system of parameters with respect to* I.

Proof. By Corollary (1·2), (*a*) is equivalent to t being H-transversal for $M(R_1)$, which means equality in (18). (*b*) is equivalent to equality in (19) by a result of B. Singh [**15**]. For (*c*), let $\mathbf{y}$ be the image of $\mathbf{x}$ in $R/I \subset A$ and let

$$I^* = \bigoplus_{n>0} I^n/I^{n+1} \subset A.$$

Then
$$H^{(r+1)}[\mathbf{x}, I, R] = H^{(r+1)}[\mathbf{y}, I^*A_N, A_N],$$

and therefore equality in (20) means

$$H^{(r+1)}[\mathbf{x}, I^*A_N, A_N] = H^{(1)}[A_N]. \tag{21}$$

Now obviously A_N is normally Cohen–Macaulay along I^*A_N, and therefore by Theorem (1·5) equality holds in (20) if and only if

$$H^{(1)}[\mathbf{y}A_N + I^*A_N, A_N] = H^{(1)}[A_N],$$

i.e. if and only if (*c*) is satisfied.

Next we want to study the same question for multiplicities. We recall from [**10**] that, if R is excellent and quasi-unmixed, the condition $\mathrm{ht}\,(I) = l(I)$ implies the following inequalities:

$$e(R_1) \leqslant e(R_1/IR_1) = e(A_Q) \tag{22}$$

$$\leqslant e(A_N) \tag{23}$$

$$\leqslant e(\mathbf{x}, I, R). \tag{24}$$

(3·6) Proposition. *With the notations of* (3·3), *assume that R is quasi-unmixed and has an infinite residue field, and that* $\mathrm{ht}\,(I) = l(I)$. *Then $e(R_1) = e(\mathbf{x}, I, R)$ if and only if the following conditions are satisfied:*

(*a*) t *belongs to a minimal set of generators of a minimal reduction of* $M(R_1)$.

(*b*) $e(A_Q) = e(A_N)$.

(*c*) $\mathbf{x}(R/I)$ *is a minimal reduction of* M/I, *where* M *denotes the maximal ideal of* R.

Proof. It suffices to show that (*c*) is equivalent to equality in (24). Using the notation of the preceeding proof, this equality is equivalent to

$$e(\mathbf{y}A_N + I^*A_N, A_N) = e(A_N). \tag{25}$$

Since A_N is quasi-unmixed, and since N and $\mathbf{y}A + I^*$ are homogeneous, this means that

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$\mathbf{y}A + I^*$ is a reduction of N (by the theorem of Rees). Finally, taking into account the grading of $A = \mathrm{gr}_I(R)$, we see that (25) is equivalent to (c).

4. *Flat families with constant Hilbert functions*

We now turn to the following situation, which has also been studied by Lipman in [8], and which was named by him a 'family of zero-dimensional ideals': $f: S \to R$ is a local homomorphism of local rings, and I is an ideal of R such that $f^{-1}(I) = (0)$ and R/I is finite over S (via f). In this situation, every $P \in \mathrm{Spec}(S)$ gives rise to a Hilbert function

$$H^{(0)}[f, P, I](n) = \dim_{k(P)} I^n/I^{n+1} \otimes_S k(P),$$

where $k(P) = S_P/PS_P$ is the residue field at P. In analogy to Lipman's result we want to characterize those flat families, for which $H^{(0)}[f, P, I]$ is independent of P. We note the following semicontinuity property of these Hilbert functions: if $P_1 \subset P_2$, then

$$H^{(0)}[f, P_1, I](n) \leqslant H^{(0)}[f, P_2, I](n) \quad \text{for all} \quad n \geqslant 0.$$

Before stating and proving our result, we need some auxiliary computations.

(4·1) LEMMA. *In the situation described above, let* $P \in \mathrm{Spec}(S)$ and $\tilde{R} = R/PR$. *Then for every* $i \geqslant 0$ *we have*

$$H^{(i)}[f, P, I] = \sum_{Q \in V(I),\, f^{-1}(Q) = P} [k(Q): k(P)]\, H^{(i)}[I\tilde{R}_Q, \tilde{R}_Q].$$

Proof. It is enough to treat the case $i = 0$. Since $(R/I) \otimes_S k(P)$ is a finite $k(P)$-module, it is semilocal with maximal ideals in one-to-one correspondence to those $Q \in V(I)$ for which $f^{-1}(Q) = P$. Therefore the sum on the right-hand side is finite. Let

$$\{Q \in V(I)/f^{-1}(Q) = P\} = \{Q_1, \ldots, Q_s\}$$

and $$R_i = R_{Q_i}/(I + PR)\, R_{Q_i} \simeq (R/I \otimes_S k(P)) \otimes_R R_{Q_i} \quad (i = 1, \ldots, s).$$

These are local rings with maximal ideals $Q_i R_i$ and residue fields $k(Q_i)$. Now we have

$$\dim_{k(P)}(I^n/I^{n+1} \otimes_R k(Q_i)) = [k(Q_i): k(P)] \dim_{k(Q_i)}(I^n/I^{n+1} \otimes_R k(Q_i))$$

and $$I^n/I^{n+1} \otimes_S k(P) \simeq \bigoplus_{i=1}^{s} (I^n/I^{n+1} \otimes_R k(Q_i)),$$

which proves the lemma.

(4·2) LEMMA. *Assume in addition that* S *is a domain. Let* $\mathbf{x}$ *be a system of parameters of* S *and let* M *and* N *denote the maximal ideals of* R *and* S *respectively. Then, identifying* $\mathbf{x}$ *with its image in* R, *we have*:

(a) $\mathbf{x}$ *is a system of parameters with respect to* I.

(b) $e(\mathbf{x}; S)\, H^{(0)}[f, (0), I] = [R/M: S/N]\, H^{(0)}[\mathbf{x}, I, R]$.

Proof. (a) Finiteness of R/I over S clearly implies that $\dim R/I = \dim S$ and $I + \mathbf{x}R$ is M-primary, which proves (a).

(b) By the projection formula for multiplicities, applied to the finite extension $S \to R/I$, we have for any $Q \in \mathrm{Min}(I)$ with $f^{-1}(Q) = (0)$

$$e(\mathbf{x}; S)\, [k(Q): k(0)] = [R/M: S/N]\, e(\mathbf{x}(R/Q), R/Q).$$

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Therefore, applying Lemma (4·1) to $P = (0)$, we obtain

$$e(\mathbf{x}; S)\, H^{(0)}[f, (0), I] = \sum_{\substack{Q \in \operatorname{Min}(I) \\ f^{-1}(Q) = (0)}} e(\mathbf{x}; S)\, [k(Q) : k((0))]\, H^{(0)}[IR_Q, R_Q]$$

$$= \sum_{Q \in \operatorname{Min}(I),\, f^{-1}(Q) = (0)} [R/M : S/N]\, e(\mathbf{x}(R/Q), R/Q)\, H^{(0)}[IR_Q, R_Q].$$

For $Q \in \operatorname{Min}(I)$ with $f^{-1}(Q) = (0)$, the injection $S \to R/Q$ defines a finite extension, and therefore $\dim R/Q = \dim S$. It follows that $e(\mathbf{x}; R/Q) \neq 0$ and consequently

$$e(\mathbf{x}; R/Q) = e(\mathbf{x}(R/Q), R/Q).$$

Now the assertion is immediate (compare (13)).

From these Lemmas we can deduce the corresponding formulas for multiplicities (compare Lipman [8]), provided we make some additional assumptions which assure that all the Hilbert functions in the formula have the same degree.

(4·3) LEMMA. *Let* $\mathbf{x}$ *be a system of parameters of* S *and let* M *resp.* N *be the maximal ideal of* R *resp.* S *as before. Assume that the canonical homomorphism* $\bar{S} \to \operatorname{gr}_{\bar{I}}(\bar{R})$ *is flat, where* $\bar{S} = S/\mathbf{x}S$, $\bar{R} = R/\mathbf{x}R$, $\bar{I} = I\bar{R}$. *Then we have*

$$H^{(0)}[\bar{I}, \bar{R}] = l_S(\bar{S})\, H^{(0)}[I(R/NR), R/NR].$$

Proof. We have to show that

$$l_S(S/\mathbf{x}S)\, l_R(I^n + NR/I^{n+1} + NR) = l_R(I^n + \mathbf{x}R/I^{n+1} + \mathbf{x}R)$$

for all $n \geqslant 0$, and this will follow from

$$l_S(S/\mathbf{x}S)\, l_R(R/I^{n+1} + NR) = l_R(R/I^{n+1} + \mathbf{x}R).$$

Let $R_n = R/I^{n+1} + \mathbf{x}R$ and $I_n = IR_n$ $(n \geqslant 0)$. Then

$$\operatorname{gr}_{I_n}(R_n) = \bigoplus_{i \leqslant n} I^i/I^{i+1}$$

is $\bar{S}$-flat, and therefore R_n is $\bar{S}$-flat (see [6], theorem (3·5)). From this we deduce easily (e.g. by induction on $l_S(\bar{S})$) that

$$l(R_n) = l_S(\bar{S})\, l_R(R_n/NR_n).$$

Since $R_n/NR_n = R/I^{n+1} + NR$, the proof is complete.

(4·4) THEOREM. *Let* S *be a local Cohen–Macaulay domain and* $\mathbf{x}$ *a system of parameters of* S. *Let* R *be a local ring,* $f\colon S \to R$ *a flat local homomorphism and* $I \subset R$ *an ideal such that* $f^{-1}(I) = (0)$ *and* R/I *is finite over* S. *Then the following conditions are equivalent:*

(i) $H^{(0)}[f, P, I]$ *is independent of* $P \in \operatorname{Spec}(S)$, *and* $f^* = \operatorname{gr}(f, \mathbf{x}S, I(\mathbf{x}))$ *is flat, where* $I(\mathbf{x}) = I + \mathbf{x}R$.

(ii) $H^{(0)}[f, P, I]$ *is independent of* $P \in \operatorname{Spec}(S)$, *the homomorphism*

$$S/\mathbf{x}S \to \operatorname{gr}_{I(\mathbf{x})/\mathbf{x}R}(R/\mathbf{x}R)$$

induced by f *is flat, and* $f(\mathbf{x})$ *is* H*-transversal for* $I(\mathbf{x})$.

(iii) R *is normally Cohen–Macaulay along* I *and the homomorphism*

$$S/\mathbf{x}S \to \operatorname{gr}_{I(\mathbf{x})/\mathbf{x}R}(R/\mathbf{x}R)$$

is flat.

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Proof. Since $I(\mathbf{x})$ is M-primary, clearly (i) $\Rightarrow$ (ii) by Theorem (3·1). We note that all three conditions imply that the elements $f^*(\mathrm{in}_{\mathbf{x}S} x_i) = \mathrm{in}_{I(\mathbf{x})} f(x_i)$ form a regular sequence of homogeneous elements of degree 1, and consequently

$$\mathrm{gr}_{I(\mathbf{x})/\mathbf{x}R}(R/\mathbf{x}R) \simeq \mathrm{gr}_{I(\mathbf{x})}(R)/(f^*(\mathrm{in}_{\mathbf{x}S} x_1), \ldots, f^*(\mathrm{in}_{\mathbf{x}S} x_r)),$$

$r = \dim S$.

(i) $\Rightarrow$ (iii). We have to show that R is normally Cohen–Macaulay along I. Since S is Cohen–Macaulay, we have $e(\mathbf{x}; S) = l_S(S/\mathbf{x}S)$, and therefore by Lemmas (4·1), (4·2) and (4·3) (using the notation above) we obtain

$$\begin{aligned} e(\mathbf{x}; S)\, H^{(0)}[f, N, I] &= [R/M : S/N]\, H^{(0)}[I(R/NR), R/NR]\, e(\mathbf{x}; S) \\ &= [R/M : S/N]\, H^{(0)}[I(\mathbf{x})/\mathbf{x}R, R/\mathbf{x}R] \end{aligned} \tag{26}$$

and

$$e(\mathbf{x}; S)\, H^{(0)}[f, (0), \mathrm{I}] = [R/M : S/N]\, H^{(0)}[\mathbf{x}, I, R]. \tag{27}$$

Therefore the assumption $H^{(0)}[f, N, I] = H^{(0)}[f, (0), I]$ implies

$$H^{(0)}[I(\mathbf{x})/\mathbf{x}R, R/\mathbf{x}R] = H^{(0)}[\mathbf{x}, I, R].$$

Recall that $r = \dim S$. From $H^{(r)}[I(\mathbf{x})/\mathbf{x}R, R/\mathbf{x}R] = H^{(r)}[\mathbf{x}, I, R]$ and the general inequalities

$$H^{(r)}[\mathbf{x}, I, R] \leqslant H^{(0)}[I(\mathbf{x}), R] \leqslant H^{(r)}[I(\mathbf{x})/\mathbf{x}R, R/\mathbf{x}R]$$

we conclude that R is normally Cohen–Macaulay along I by Theorem (1·5).

(iii) $\Rightarrow$ (i). Flatness of f^* follows from Theorem (3·1), and because of the semi-continuity of $H^{(0)}[f, P, I]$ we only have to show that $H^{(0)}[f, N, I] = H^{(0)}[f, (0), I]$. But this is immediate from (26), (27) and the assumption (in conjunction with Corollary (1·10)).

(4·5) COROLLARY. *Assume in addition that S is regular and $\mathbf{x}$ is a regular system of parameters. Then the following conditions are equivalent:*

(i) *$H^{(0)}[f, P, I]$ is independent of $P \in \mathrm{Spec}\,(S)$ and $f^* = \mathrm{gr}\,(f, N, I + NR)$ is flat.*

(ii) *$H^{(0)}[f, P, I]$ is independent of $P \in \mathrm{Spec}\,(S)$ and $f(\mathbf{x})$ is H-transversal for $I(\mathbf{x})$.*

(iii) *R is normally Cohen–Macaulay along I.*

Note that condition (iii) of the Corollary is independent of the homomorphism f.

(4·6) *Remark.* Assume that R is normally flat along I. Then $H^{(0)}[f, P, I](n)$ is the rank of the free R/I-module I^n/I^{n+1}, which is certainly independent of P. But there are examples for R and I such that R and R/I are Cohen–Macaulay, R is normally Cohen–Macaulay along I but not normally flat along I (I is even prime, see [2]). Therefore normal flatness will not give a necessary condition for the constance of the Hilbert functions of the fibres.

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