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Actions of finite rank: weak rational ergodicity and partial rigidity

ALEXANDRE I. DANILENKO

Institute for Low Temperature Physics & Engineering of National Academy of Sciences of Ukraine, 47 Lenin Ave., Kharkov, 61164, Ukraine
(e-mail: alexandre.danilenko@gmail.com)

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Abstract. A simple proof of the fact that each rank-one infinite measure preserving (i.m.p.) transformation is subsequence weakly rationally ergodic is found. Some classes of funny rank-one i.m.p. actions of Abelian groups are shown to be subsequence weakly rationally ergodic. A constructive definition of finite funny rank for actions of arbitrary infinite countable groups is given. It is shown that the ergodic i.m.p. transformations of balanced finite funny rank are subsequence weakly rationally ergodic. It is shown that the ergodic probability preserving transformations of exact finite rank, the ergodic Bratteli–Vershik maps corresponding to the 'consecutively ordered' Bratteli diagrams of finite rank, some their generalizations and the ergodic interval exchange transformations are partially rigid.

1. *Introduction*

In [**Aa1**], Aaronson (motivated by estimating the asymptotic growth of some special ergodic averages in infinite measure spaces) introduced concepts of *rational* and *weak rational ergodicity* for infinite measure preserving (i.m.p.) transformations. More recently, in [**Aa2**], he introduced a related concept of rational weak mixing and considered more general *subsequence* versions of these concepts. All the examples of systems possessing these properties given in [**Aa1**] and [**Aa2**] have positive Krengel entropy and countable Lebesgue spectrum. Other kind of examples—rank-one i.m.p. transformations—related to the aforementioned concepts were considered in subsequent papers [**Aa3**, **Dai–Si**, **Bo–Wa**]. We recall that the rank-one transformations have zero Krengel entropy and simple spectrum (see [**DaSi**] and references therein).

The main result of [**Bo–Wa**] is that each rank-one i.m.p. transformation is subsequence weakly rationally ergodic. In the present paper we give a short proof of this fact. Moreover, we generalize the concept of subsequence weak rational ergodicity to i.m.p. actions of arbitrary countable discrete groups and construct some families of weakly rationally ergodic (along a Følner sequence) funny rank-one i.m.p. actions for arbitrary countable

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infinite Abelian groups. We obtain as an immediate corollary that all these actions are non-squashable, i.e. every non-singular transformation commuting with this action preserves the measure. This fact (i.e. the non-squashability) was first proved in [**Aa1**] for the weakly rationally ergodic i.m.p. $\mathbb{Z}$-actions, but that proof is valid in the general case of weakly rationally ergodic (along a Følner sequence) i.m.p. group actions.

One of the main results of this paper is that the ergodic i.m.p. transformations of balanced finite rank are subsequence weakly rationally ergodic. We say that an i.m.p. transformation is of *balanced* finite rank if it is of finite rank and the bases of the Rokhlin towers on the nth step of the cutting-and-stacking inductive construction have asymptotically comparable measures as $n \to \infty$. We recall that the transformations of finite rank have zero Krengel entropy and finite spectral multiplicities (see [**DaSi**]).

We give a constructive definition of finite funny rank actions for arbitrary countable infinite groups by developing the (C, F)-construction defined originally in [**dJ**] and [**Da1**] to produce funny rank-one actions. We hope that this generalized (C, F)-construction will find other applications in ergodic theory, especially in the spectral theory, the theory of joinings of dynamical systems, the theory of i.m.p. and non-singular systems, etc. (see the survey [**Da3**] for various applications of the (C, F)-techniques in the rank-one case).

Rosenthal showed in an unpublished paper [**Ro**] that the ergodic (finite measure preserving) transformation of exact finite rank is not mixing. Recently this fact was re-proved under some restrictions in [**Be–So**]. The *exactness* means that the transformation is constructed via the cutting-and-stacking procedure without adding spacers and the corresponding Rokhlin towers do not asymptotically vanish. Since the exact rank-one transformations have pure point spectrum, the transformations of exact rank greater than one can be thought of as 'higher-rank' analogues of systems with pure point spectrum. We refine Rosenthal's result by showing that the ergodic transformations of exact finite rank are partially rigid†. We also extend this assertion to the transformations of quasi-exact finite rank, which means that spacers in the underlying cutting-and-stacking construction are possible but with some uniform (over the indicative steps) bound on their number. It was proved in a recent paper [**Be–So**] that some Vershik transformations associated with the so-called *consecutively ordered* (see [**Du**] for the definition) Bratteli diagrams of finite rank are non-mixing. We show that these transformations (and some generalizations of them) are indeed partially rigid. It is also shown how to deduce from Katok's proof [**Ka**] of non-mixing for the ergodic interval exchange transformations (IETs) that they are partially rigid‡.

The paper is organized as follows. Section 2 is devoted to the (C, F)-construction of funny rank-one actions for discrete countable groups. The construction first appeared in [**dJ**] and [**Da1**] in slightly different versions (see also [**Da3**]). We compare them in the present paper and introduce a third version which is formally more general (in fact, as general as possible in view of Proposition 2.4). We show, however, in Theorem 2.8 that the class of measurable (C, F)-actions in the sense of the third definition (which is exactly the

† When this paper had been already submitted, V. Ryzhikov informed the author about his earlier work [**Ry**]. Although the partial rigidity of the finite-rank transformations was not explicitly asserted there, it was actually proved there.

‡ The fact that the ergodic IETs are partially rigid was also established (implicitly) in [**Ry**].

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class of all funny rank-one actions with an invariant σ-finite measure by Theorem 2.6) is the same as the class of measurable (C, F)-actions in the sense of the definition from [**Da1**] if the actions are considered up to modification on null subsets. Moreover, in the *finite* measure preserving case the three versions of the (C, F)-constructions define the very same class of actions and the acting groups are necessarily amenable. Since every (C, F)-action T in the sense of the definition from [**Da1**] is *strictly ergodic*, i.e. T is a topological action on a locally compact second countable space, T is minimal and T admits a unique (up to scaling) invariant σ-finite Radon measure, we obtain as a byproduct strictly ergodic models for arbitrary funny rank-one σ-finite measure preserving actions of arbitrary discrete countable groups. We recall that strictly ergodic models for the arbitrary (not only rank-one) ergodic finite measure preserving actions of Abelian groups were constructed in [**We1**] (see also the discussion there of earlier results) and strictly ergodic models for the ergodic i.m.p. $\mathbb{Z}$-actions were constructed in [**Yu2**]. We also mention another application of the (C, F)-construction. By [**Zi**], if a discrete countable group G admits a free ergodic probability preserving action whose orbit equivalence relation is hyperfinite then G is amenable. However, each non-amenable group has i.m.p. free actions with hyperfinite orbit equivalence relations (see, for example, [**BeGo**]). The (C, F)-construction provides a simple way to obtain such actions possessing additional properties such as a strict ergodicity (in locally compact spaces) and funny rank one.

In §3, for an arbitrary discrete countable amenable group G, we introduce a concept of weak rational ergodicity along a Følner sequence in G. In the case where $G = \mathbb{Z}$ and the Følner sequence consists of intervals with zero as the left endpoint, our concept coincides with Aaronson's sequence weak rational ergodicity [**Aa2**]. Using the language of the (C, F)-construction, we give a short proof for the main result of the first version of [**Bo–Wa**] that each rank-one i.m.p. transformation is subsequence weakly rationally ergodic (Theorem 3.4)†. Then we extend this result to some classes of funny rank-one i.m.p. actions of arbitrary countable Abelian groups (Corollary 3.7 and Theorem 3.8). The question whether every funny rank-one i.m.p. transformation (or an Abelian group action) is subsequence weakly rationally ergodic remains open.

In §4 we consider σ-finite measure preserving group actions of finite funny rank (see [**Fe2**] for the definition of finite funny rank in the case where the acting group is $\mathbb{Z}$). We pay special attention to ergodic $\mathbb{Z}$-actions of finite rank. It is shown that each ergodic σ-finite measure preserving transformation of finite rank is built over a finite measure preserving transformation of exact finite rank and under a piecewise constant integer-valued function (see Corollary 4.5).

In §5 we generalize the (C, F)-construction in such a way that it yields actions of finite funny rank and that every σ-finite measure preserving action of finite funny rank is isomorphic to a (C, F)-action in the generalized sense (see Theorems 5.7 and 5.8). Thus the (C, F)-construction can be considered as a constructive definition for the actions of finite funny rank. As in the rank-one case, we obtain strictly ergodic models for the ergodic σ-finite measure preserving G-actions of finite funny rank (Theorem 5.9). In the case where $G = \mathbb{Z}$ and the actions are of finite rank, we show how to associate an ordered

† Being informed about our proof of Theorem 3.4, the authors of [**Bo–Wa**] replaced their main result with a stronger one in the final version of [**Bo–Wa**]. The two versions of [**Bo–Wa**] can be found in arXiv.

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Bratteli diagram to (C, F)-data in such a way that the corresponding (C, F)-action of G is the Bratteli–Vershik map associated with the diagram (Remark 5.10). Thus the (C, F)-construction can be viewed as a generalization of Bratteli–Vershik construction from $\mathbb{Z}$-actions to actions of arbitrary discrete countable groups†.

In §6 we show that the ergodic i.m.p. transformations of balanced finite rank are weakly rationally ergodic (Theorem 6.5).

In §7 we consider finite measure preserving transformations of finite rank. We show that each ergodic transformation of exact finite rank is partially rigid (Theorem 7.1). Then in Theorem 7.4 we extend it to the transformations of quasi-exact finite rank. In a similar way we strengthen another result from [**Be–So**] on non-mixing for another class of transformations of finite rank. We prove in Theorem 7.6 that each ergodic transformation of finite rank with consecutive ordering of towers that satisfies a 'non-degeneracy' condition is partially rigid. It is also shown that the ergodic IETs are partially rigid (Proposition 7.8).

The concluding §8 is a list of open problems related to the subject of this paper.

2. (C, F)-construction of funny rank-one actions

Let $T = (T_g)_{g\in G}$ be a measure preserving action of a countable infinite discrete group G on a standard σ-finite measure space $(X, \mathfrak{B}, \mu)$. The following definition was given by J.-P. Thouvenot in the case where $G = \mathbb{Z}$ (see also [**Fe1**, **Fe2**]).

Definition 2.1. If there exist a sequence $(B_n)_{n\geq 0}$ of subsets of finite measure in X and a sequence $(F_n)_{n\geq 0}$ of finite subsets in G such that

(i) for each $n \geq 0$, the subsets $T_g B_n$, $g \in F_n$, are pairwise disjoint, and

(ii) for each subset $B \in \mathfrak{B}$ with $\mu(B) < \infty$,

$$\lim_{n\to\infty} \inf_{F\subset F_n} \mu\Big(B \triangle \bigsqcup_{g\in F} T_g B_n\Big) = 0,$$

then T is called an action *of funny rank one*.

If $G = \mathbb{Z}$ and every F_n is an interval $\{0, 1, \ldots, \#F_n - 1\}$ in $\mathbb{Z}$ then T is called an action *of rank one*. For the constructive definition of rank-one transformations (i.e. $\mathbb{Z}$-actions) using the cutting-and-stacking inductive process we refer to [**Fe2**]. A constructive definition of actions of funny rank-one was given in [**dJ**] and [**Da1**] (see also [**Da2**]). We now recall it.

Let $(F_n)_{n\geq 0}$ and $(C_n)_{n\geq 1}$ be two sequences of finite subsets in G such that, for each $n > 0$:

(I) $F_0 = \{1\}$, $\#C_n > 1$;

(II) $F_n C_{n+1} \subset F_{n+1}$;

(III) $F_n c \cap F_n c' = \emptyset$ if $c, c' \in C_{n+1}$ and $c \neq c'$.

We let $X_n := F_n \times C_{n+1} \times C_{n+2} \times \cdots$ and endow this set with the infinite product topology. Then X_n is a compact Cantor (i.e. totally disconnected perfect metric) space.

† In this paper we consider only (C, F)-actions of finite funny rank. However, the infinite funny rank (C, F)-actions can be defined in a similar way. Remark 5.10 will hold for them as well. This will be done elsewhere.

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The mapping

$$X_n \ni (f_n, c_{n+1}, c_{n+2}, \dots) \mapsto (f_n c_{n+1}, c_{n+2}, \dots) \in X_{n+1}$$

is a topological embedding of X_n into X_{n+1}. Therefore an inductive limit X of the sequence $(X_n)_{n\ge 0}$ furnished with these embeddings is a well-defined locally compact Cantor space. We call it the (C, F)-*space associated with the sequence* $(C_n, F_{n-1})_{n\ge 1}$. It is easy to see that the (C, F)-space is compact if and only if there is $N > 0$ with $F_{n+1} = F_n C_{n+1}$ for all $n > N$. Given a subset $A \subset F_n$, we let

$$[A]_n := \{x = (f_n, c_{n+1}, \dots) \in X_n \mid f_n \in A\}$$

and call this set an n-*cylinder* in X. It is open and compact in X. The collection of all cylinders coincides with the family of all compact open subsets in X. It is easy to see that

$$\begin{aligned} [A]_n \cap [B]_n &= [A \cap B]_n, \\ [A]_n \cup [B]_n &= [A \cup B]_n, \\ [A]_n &= [AC_{n+1}]_{n+1}, \end{aligned}$$

for all $A, B \subset F_n$ and $n \ge 0$. For brevity, we will write $[f]_n$ for $[\{f\}]_n$, $f \in F_n$.

Let $\mathcal{R}$ denote the *tail equivalence relation* on X. This means that the restriction of $\mathcal{R}$ to X_n is the tail equivalence relation on X_n for each $n \ge 0$. We note that $\mathcal{R}$ is *minimal*, i.e. the $\mathcal{R}$-class of every point is dense in X. There exists a unique σ-finite $\mathcal{R}$-invariant Borel measure μ on X such that $\mu(X_0) = 1$. It is a Radon measure, i.e. it is finite on every compact subset. Moreover, μ is strictly positive on every non-empty open subset. We note that the $\mathcal{R}$-*invariance*† of μ is equivalent to the following property:

$$\mu([f]_n) = \mu([f']_n) \quad \text{for all } f, f' \in F_n, n \ge 0.$$

It is easy to see that

$$\mu([A]_n) = \frac{\#A}{\#C_1 \cdots \#C_n} \quad \text{for each subset } A \subset F_n, n > 0.$$

We call μ the (C, F)-*measure associated with* $(C_n, F_{n-1})_{n\ge 1}$. It is finite if and only if‡

$$\lim_{n\to\infty} \frac{\#F_n}{\#C_1 \cdots \#C_n} < \infty. \tag{2.1}$$

It is easy to see that μ is $\mathcal{R}$-*ergodic*, i.e. each Borel $\mathcal{R}$-saturated subset of X is either μ-null or μ-conull. We now define an action of G on X (or, more rigorously, on a subset of X). Given $g \in G$, let

$$X_n^g := \{(f_n, c_{n+1}, c_{n+2} \dots) \in X_n \mid g f_n \in F_n\}.$$

Then X_n^g is a compact open subset of X_n and $X_n^g \subset X_{n+1}^g$. Hence the union $X^g := \bigcup_{n\ge 0} X_n^g$ is a well-defined open subset of X. Let $X^G := \bigcap_{g\in G} X^g$. Then X^G is a G_δ-subset of X. Hence X^G is Polish in the induced topology. Given $x \in X^G$ and $g \in G$, there is $n > 0$ such that $x = (f_n, c_{n+1}, \dots) \in X_n$ and $g f_n \in F_n$. We now let

$$T_g x := (g f_n, c_{n+1}, \dots) \in X_n \subset X.$$

† μ is called $\mathcal{R}$-invariant if μ is invariant under each Borel transformation whose graph is contained in $\mathcal{R}$.
‡ In view of (I)–(III), the sequence $(\#F_n/\#C_1 \cdots \#C_n)_{n=1}^{\infty}$ is non-decreasing and bounded by one from below.

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It is standard to verify that:
(i) $T_g x \in X^G$;
(ii) the map $T_g : X^G \ni x \mapsto T_g x \in X^G$ is a homeomorphism of X^G; and
(iii) $T_g T_{g'} = T_{gg'}$ for all $g, g' \in G$.
Thus $T := (T_g)_{g\in G}$ is a continuous action of G on X^G. We call it *the* (C, F)*-action of* G *associated with the sequence* $(C_n, F_{n-1})_{n\geq 0}$. It is free. It is obvious that X^G is $\mathcal{R}$-invariant and the T-orbit equivalence relation is the restriction of $\mathcal{R}$ to X^G. It follows that T preserves μ.

Given sequences $(F_n)_{n\geq 0}$ and $(C_n)_{n\geq 1}$ satisfying (I)–(III) and a sequence $(z_n)_{n\geq 1}$ of elements of G, we let $C'_n := z_n^{-1} C_n z_{n+1}^{-1}$ and $F'_{n-1} := F_{n-1} z_n$ for each $n \geq 1$. Then the sequences $(F'_n)_{n\geq 0}$ and $(C'_n)_{n\geq 1}$ satisfy (I)–(III). Denote by X', $\mathcal{R}'$ and T' the associated (C, F)-space, (C, F)-equivalence relation and (C, F)-action, respectively. Then there is a canonical homeomorphism $\phi : X \to X'$ that intertwines $\mathcal{R}$ with $\mathcal{R}'$ and T with T'. It is given by

$$\phi(f_n, c_{n+1}, c_{n+2}, \dots) = (f_n z_{n+1}, z_{n+1}^{-1} c_{n+1} z_{n+2}, z_{n+1}^{-1} c_{n+1} z_{n+2}, \dots) \in X'_n \subset X'$$

whenever $(f_n, c_{n+1}, c_{n+2}, \dots) \in X_n \subset X$, for each $n \geq 0$. Choosing $(z_n)_{n=1}^{\infty}$ in an appropriate way, we may always assume without loss of generality† that the condition
(IV) $1 \in \bigcap_{n\geq 0} F_n \cap \bigcap_{n\geq 1} C_n$
is always satisfied in addition to (I)–(III).

Since X^G is $\mathcal{R}$-saturated and μ is $\mathcal{R}$-ergodic, we have either $\mu(X^G) = 0$ or $\mu(X \backslash X^G) = 0$. Each of these two cases can occur. Moreover, X^G may be empty. discuss conditions under which X^G is μ-conull or even equal to X.

PROPOSITION 2.2. *$X^G = X$ if and only if, for each $g \in G$ and $n > 0$, there is $m > n$ such that*

$$g F_n C_{n+1} C_{n+2} \cdots C_m \subset F_m. \tag{2.2}$$

Proof. The 'if' part is trivial. We now prove the 'only if' part. Fix $n \geq 0$ and $g \in G$. The map

$$\phi_m : X_n \ni x = (f_n, c_{n+1}, \dots) \mapsto g f_n c_{n+1} \cdots c_m \in G$$

is continuous, $m > n$. Since $X^g = X$, we obtain that $X_n = \bigcup_{m>n} \phi_m^{-1}(F_m)$. Since X_n is compact and $\phi_{n+1}^{-1}(F_{n+1}) \subset \phi_{n+2}^{-1}(F_{n+2}) \subset \cdots$, it follows that $X_n = \phi_m^{-1}(F_m)$ for some $m > n$. The inclusion (2.2) follows. □

Thus, in this case the (C, F)-action is defined on the entire (locally compact) space X.

Remark 2.3. We note that if X is not compact then T extends to the one-point compactification $X^* = X \sqcup \{\infty\}$ of X by setting $T_g \infty = \infty$ for all $g \in G$. We obtain a continuous action of G on X^*. This action is *almost minimal*, i.e. there is one fixed point and the orbit of any other point is dense. This concept was introduced in [**Da2**] in the case where $G = \mathbb{Z}$. For the (topological) orbit classification of the almost minimal $\mathbb{Z}$-systems, see [**Da2**, **Ma**]. Some natural examples of such systems (subshifts arising from non-primitive substitutions) are given in [**Yu1**].

† This means that we can modify the (C, F)-sequences in such a way that the modified associated (C, F)-action (and the (C, F)-equivalence relation) is isomorphic to the original one.

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PROPOSITION 2.4. *$\mu(X\setminus X^G)=0$ if and only if, for each $g\in G$ and $n>0$,*

$$\lim_{m\to\infty}\frac{\#((gF_nC_{n+1}C_{n+2}\cdots C_m)\cap F_m)}{\#F_n\#C_{n+1}\cdots\#C_m}=1. \tag{2.3}$$

If $\mu(X)<\infty$ then $\mu(X\setminus X^G)=0$ if and only if $(F_n)_{n\ge0}$ is a Følner sequence in G and hence G is amenable†.

Proof. Since $\mu(X\setminus X^G)=0$ if and only if $\mu(X_n\cap X_m^g)/\mu(X_n)\to1$ as $m\to\infty$ for each $g\in G$ and $n>0$, it suffices to note that

$$\frac{\mu(X_n\cap X_m^g)}{\mu(X_n)}=\frac{\#((gF_nC_{n+1}\cdots C_m)\cap F_m)}{\#F_n\#C_{n+1}\cdots\#C_m}.$$

In the case where μ is finite we have $\mu(X\setminus X^G)=0$ if and only if $(\mu(X_n^g))/(\mu(X_n))\to1$ as $n\to\infty$ for each $g\in G$. Since

$$\frac{\mu(X_n^g)}{\mu(X_n)}=\frac{\#(gF_n\cap F_n)}{\#F_n},$$

the second assertion of the lemma follows. □

We note that the condition $\mu(X)<\infty$ from the second claim of Proposition 2.4 cannot be omitted. Indeed, if G is amenable then it is not difficult to construct sequences $(C_n,\,F_{n-1})_{n\ge1}$ such that $(F_n)_{n\ge0}$ is Følner, (I)–(IV) are satisfied but (2.3) is not satisfied. Hence $\mu(X^G)=0$ and $\mu(X)=\infty$.

From now on we assume that (2.3) holds. Then $(X,\,\mu,\,T)$ is a σ-finite measure preserving dynamical system. Of course, it is free, conservative and ergodic. We claim that it is of funny rank one. Indeed, the sequences $([1]_n)_{n\ge0}$ and $(F_n)_{n\ge0}$ satisfy Definition 2.1. We note that

$$T_g[f]_n=[gf]_n\ (\text{up to }\mu\text{-null subset})\quad\text{whenever } f,\,gf\in F_n. \tag{2.4}$$

We summarize the aforementioned results on $(C,\,F)$-actions in the following theorem.

THEOREM 2.5. *Given a sequence $(C_n,\,F_{n-1})_{n\ge1}$ satisfying (I)–(IV), there exist a locally compact Cantor space X and a countable equivalence relation $\mathcal{R}$ on X such that:*

(i) *Every $\mathcal{R}$-class is dense in X.*
(ii) *There is only one (up to scaling) $\mathcal{R}$-invariant non-trivial σ-finite Radon measure μ on X.*
(iii) *μ is finite if and only if (2.1) is satisfied.*
(iv) *There is a free topological G-action T on an $\mathcal{R}$-invariant G_δ-subset X^G of X such that the T-orbit equivalence relation is the restriction of $\mathcal{R}$ to X^G.*
(v) *$X^G=X$ if and only if (2.2) is satisfied.*
(vi) *$\mu(X\setminus X^G)=0$ if and only if (2.3) is satisfied. Moreover, if (2.3) is not satisfied then $\mu(X^G)=0$.*

† Another way to see that G is amenable is to apply a theorem by Zimmer from [**Zi**]: if G has a free probability preserving ergodic action T and the T-orbit equivalence relation is hyperfinite then G is amenable. Of course, the tail equivalence relation is hyperfinite, i.e. it is the union of an increasing sequence of equivalence relations with finite equivalence classes.

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(vii) *If $\mu(X) < \infty$ then (2.3) is equivalent to the fact that $(F_n)_{n\geq 0}$ is a Følner sequence in G.*

(viii) *Under (2.3), the dynamical system (X, μ, T) is ergodic, conservative and of funny rank one.*

The converse to Theorem 2.5(viii) also holds.

THEOREM 2.6. ((C, F)-models for actions of funny rank one) *If T is a σ-finite measure preserving G-action of funny rank one then T is isomorphic to a (C, F)-action of G equipped with the (C, F)-measure.*

Proof. Let $(B_n)_{n\geq 0}$ and $(F_n)_{n\geq 0}$ be as in Definition 2.1. Without loss of generality we may assume that $F_0 = \{1\}$, $1 \in \bigcap_{n\geq 0} F_n$ and, for each $g \in F_n$, the subset $T_g B_n$ is a union of subsets $T_s B_{n+1}$ for some family of $s \in F_{n+1}$. We now define inductively two sequences $(C_n)_{n\geq 1}$ and $(\widetilde{F}_n)_{n\geq 0}$ of finite subsets in G that satisfy (I)–(III) and, for each n,

$$\{T_g B_n \mid g \in F_n\} = \{T_g B_n \mid g \in \widetilde{F}_n\}. \tag{2.5}$$

For that, we first set $\widetilde{F}_0 := F_0$. Suppose now that $\widetilde{F}_n$ is defined and (2.5) holds for some n. There is a finite subset $C_{n+1} \subset F_{n+1}$ such that $B_n = \bigsqcup_{c\in C_{n+1}} T_c B_{n+1}$. Then, in view of (2.5), and the refining property of towers, for each $g \in \widetilde{F}_n$, there is a finite subset $I_g \subset F_{n+1}$ such that

$$\bigsqcup_{c\in C_{n+1}} T_{gc} B_{n+1} = T_g B_n = \bigsqcup_{c\in I_g} T_g B_{n+1}.$$

Of course, the subsets I_g, $g \in \widetilde{F}_n$, are pairwise disjoint. We now set $\widetilde{F}_{n+1} := \widetilde{F}_n C_n \sqcup (F_{n+1} \setminus \bigsqcup_{g\in\widetilde{F}_n} I_g)$. Then, of course, (2.5) is satisfied for $n+1$. Moreover, it is easy to verify that the sequence $(C_n, \widetilde{F}_{n-1})_{n>1}$ satisfies (I)–(III). We denote the corresponding (C, F)-action of G by S. In view of Definition 2.1(ii), the one-to-one correspondence

$$\bigsqcup_{g\in F} T_g B_n \mapsto [F]_n, \quad F \subset \widetilde{F}_n,$$

between

(a) the subsets measurable with respect to some of the partitions $\{T_g B_n \mid g \in F_n\}$, $n > 0$, and

(b) the compact open subsets of the underlying (C, F)-space

generates a Borel isomorphism (mod 0) between the underlying measure spaces. Now (2.4) yields that this isomorphism intertwines T with S. Therefore S is defined almost everywhere and hence (2.3) holds by Proposition 2.4. □

We obtain the following corollary from Theorem 2.6 and Proposition 2.4.

COROLLARY 2.7. *If G has a finite measure preserving action of funny rank one then G is amenable.*

It may seem that the condition (2.3) is essentially more general than (2.2). However, we show that the two conditions determine the same class of (C, F)-actions (modulo measure theoretic isomorphism). First of all, we introduce a technique of passing to a (C, F)-subsequence. Let T be the (C, F)-action associated with a sequence $(C_n, F_{n-1})_{n>0}$.

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Given an increasing sequence $(k_n)_{n\ge 0}$ of non-negative integers with $k_0 = 0$, we let $\widetilde{F}_n := F_{k_n}$ and $\widetilde{C}_n := C_{k_{n-1}+1} C_{k_{n-1}+2} \cdots C_{k_n}$. Since $(C_n, F_{n-1})_{n>1}$ satisfies (I)–(III) and (2.3), the sequence $(\widetilde{C}_n, \widetilde{F}_{n-1})_{n>1}$ also satisfies these conditions. We call the latter sequence a (C, F)-*subsequence* of $(C_n, F_{n-1})_{n>1}$. Denote by $\widetilde{T}$ the (C, F)-action associated with it. Then $\widetilde{T}$ is canonically isomorphic to T. Indeed, let X and $\widetilde{X}$ denote the corresponding (C, F)-spaces. We recall that

$$X = \bigcup_{n\ge 0} X_n = \bigcup_{n\ge 0} X_{k_n} \quad \text{and} \quad \widetilde{X} = \bigcup_{n\ge 0} \widetilde{X}_n,$$

where $X_n = F_n \times C_{n+1} \times \cdots$ and $\widetilde{X}_n = \widetilde{F}_n \times \widetilde{C}_{n+1} \times \cdots$. Then the mappings

$$X_{k_n} \ni (f_{k_n}, c_{k_n+1}, \dots) \mapsto (f_{k_n}, c_{k_n+1} \cdots c_{k_{n+1}}, c_{k_{n+1}+1} \cdots c_{k_{n+2}}, \dots) \in \widetilde{X}_n,$$

$n \ge 0$, define a homeomorphism of X onto $\widetilde{X}$. This homeomorphism intertwines T with $\widetilde{T}$ and the (C, F)-equivalence relation on X with the (C, F)-equivalence relations on $\widetilde{X}$.

THEOREM 2.8. *Let T be the (C, F)-action of G associated with a sequence $(C_n, F_{n-1})_{n\ge 1}$ satisfying* (I)–(III) *and (2.3). Then T is (measure theoretically) isomorphic to the (C, F)-action S of G associated with a sequence $(C'_n, F'_{n-1})_{n\ge 1}$ satisfying* (I)–(III) *and (2.2). Moreover, the sequence $(F'_n)_{n\ge 0}$ is a subsequence of $(F_n)_{n\ge 0}$. In particular, if $G = \mathbb{Z}$ and T is of rank one then S is also of rank one.*

Proof. Let $G = \{g_j \mid j \in \mathbb{N}\}$. Denote by X the (C, F)-space of T. Applying Proposition 2.4 and passing, if necessary, to a (C, F)-subsequence, we may assume without loss of generality that, for each n, there exists a subset $C'_{n+1} \subset C_{n+1}$ such that

$$\#g_j F_n C'_{n+1} \subset F_{n+1}, \quad j = 1, \dots, n, \quad \text{and} \quad \#C'_{n+1}/\#C_{n+1} > 1 - n^2.$$

Then the sequence $(C'_n, F_{n-1})_{n\ge 1}$ satisfies (I)–(III) and (2.2). We denote by S the (C, F)-action of G associated with $(C'_n, F_{n-1})_{n\ge 1}$. Let Y stand for the corresponding (C, F)-space. Then $Y = \bigcup_{n\ge 1} Y_n$, where $Y_n = F_n \times C'_{n+1} \times C'_{n+2} \times \cdots$. By the Borel–Cantelli lemma, for almost all $x \in X$, there is $n > 0$ such that $x = (f_n, c_{n+1}, c_{n+2}, \dots) \in F_n \times C'_{n+1} \times C'_{n+2} \times \cdots \subset X_n$. It is easy to see that the mapping

$$X \ni x \mapsto (f_n, c_{n+1}, c_{n+2}, \dots) \in Y_n \subset Y$$

is a well-defined (mod 0) isomorphism of T with S.

The final claim of the theorem is obvious. □

From Theorems 2.6 and 2.8, Proposition 2.2 and Remark 2.3 we deduce the following corollary.

COROLLARY 2.9. (Minimal and almost minimal uniquely ergodic models for rank-one actions).

(i) *Every funny rank-one σ-finite measure preserving action of G is measure theoretically isomorphic to a strictly ergodic topological G-action on a locally compact Cantor space.*

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(ii) *Every funny rank-one σ-finite measure preserving action of G is measure theoretically isomorphic to an almost minimal uniquely ergodic† topological G-action on a compact Cantor space.*

We note that claims (i) and (ii) of the corollary are equivalent.

Remark 2.10. The authors of [**Dai–Si**] (see also [**Bo–Wa**]) introduced the concept of *normal* rank-one $\mathbb{Z}$-action. This means that (in the cutting-and-stacking construction of the action) at least one spacer is added above the highest subtower for infinitely many cuts. As follows from Theorem 2.8, every rank-one transformation is isomorphic to a normal one. We note, however, that passing to an isomorphic normal copy may 'destroy' some other important properties of the rank-one construction such as the property of 'bounded cuts' (which means that the sequence $(\#C_n)_{n=1}^{\infty}$ is bounded).

We now state without proof (it is easy) a lemma which will be used in the next section.

LEMMA 2.11. *Let $a, b \in F_n$. Then $\mu(T_g[a]_n \cap [b]_n) > 0$ if and only if*

$$g \in \bigcup_{j>n} bC_{n+1}C_{n+2}\cdots C_jC_j^{-1}\cdots C_{n+2}^{-1}C_{n+1}^{-1}a^{-1}.$$

Moreover, $[a]_n \cap T_g^{-1}[b]_n = \bigsqcup [ad_{n+1}\cdots d_j]_j$, where the union is taken over all possible expansions of g into the sum $g = bc_{n+1}c_{n+2}\cdots c_jd_j^{-1}\cdots d_{n+2}^{-1}d_{n+1}^{-1}a^{-1}$ with $c_l, d_l \in C_l$ for each l and $c_j \neq d_j$.

Remark 2.12. The (C, F)-construction of funny rank-one actions was introduced in [**dJ**] and [**Da1**] in slightly different ways. It was assumed in [**dJ**] that the sequences $(C_n, F_{n-1})_{n\geq 1}$ satisfy (I)–(III) plus an additional condition that $(F_n)_{n\geq 0}$ is a Følner sequence. As we showed in Proposition 2.4, this additional condition is equivalent to (2.3) in the case of finite measure preserving (C, F)-actions (only such actions were under consideration in [**dJ**]). In [**Da1**] we assumed that $(C_n, F_{n-1})_{n\geq 1}$ satisfy (I)–(III) and (2.2). An advantage of this approach is that the associated (C, F)-actions are topological actions defined on locally compact spaces. In this paper we have introduced another condition (2.3) which is formally more general than (2.2). However, it is shown in Theorem 2.8 that they define the very same class of the associated (C, F)-actions. Thus the (C, F)-constructions from [**dJ**] and [**Da1**] are equivalent in the finite measure preserving case, while the construction from [**Da1**] is more general than that from [**dJ**] in the i.m.p. case. In particular, for each non-amenable group G, there exist i.m.p. (and only i.m.p.) (C, F)-actions. Thus we obtain a class of free ergodic conservative i.m.p. G-actions whose orbit equivalence relations are hyperfinite (see also [**BeGo**], where another construction of such actions was given).

3. *Weak rational ergodicity and non-squashability of funny rank-one Abelian actions*

Let T be an ergodic conservative measure preserving action of an amenable group G on a σ-finite measure space $(X, \mathfrak{B}, \mu)$. Fix an increasing Følner sequence $(F_n)_{n\geq 0}$ in G.

† An almost minimal G-action is called uniquely ergodic if there is only one (up to scaling) non-atomic σ-finite G-invariant Borel measure that is finite on every compact in the complement to the fixed point of the action.

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Definition 3.1. T is called *weakly rationally ergodic along* $(F_n)_{n\geq 0}$ if there is a subset Y of finite positive measure in X such that

$$\lim_{n\to\infty} \frac{1}{a_n(Y)} \sum_{g\in F_n} \mu(A\cap T_g B) = \mu(A)\mu(B) \quad \text{for all } A,\, B\subset \mathfrak{B}\cap Y,$$

where $a_n(Y) := \sum_{g\in F_n} (\mu(Y\cap T_g Y))/(\mu(Y)^2)$.

We note that in the case where $G=\mathbb{Z}$ and $F_n=\{0,1,\dots,n-1\}$ we obtain the standard definition of weak rational ergodicity [**Aa1**]. In the case where $G=\mathbb{Z}$ but $F_n=\{0,1,\dots,h_n-1\}$ for an increasing sequence $(h_n)_{n=1}^{\infty}$, we obtain the notion of *subsequence weak rational ergodicity* [**Aa2**].

Our purpose in this section is to exhibit a class of funny rank-one actions of Abelian groups that are weakly rationally ergodic.

First of all, we consider the case where $G=\mathbb{Z}$ and give a short proof of the main result from the first version of [**Bo–Wa**] that every rank-one transformation is subsequence weakly rationally ergodic (see Theorem 3.4 below). For that we need two auxiliary lemmas.

LEMMA 3.2. *Let T be the (rank-one) (C,F)-action of $\mathbb{Z}$ associated with a sequence $(C_n, F_{n-1})_{n=1}^{\infty}$ with $F_n=\{0,\dots,h_n-1\}$ for each n and let μ be the (C,F)-measure on the (C,F)-space X such that $\mu([0]_0)=1$. Then for all cylinders A and B in X,*

$$\lim_{n\to\infty} \frac{\sum_{k=0}^{n-1}\mu(A\cap T_k B)}{a_n([0]_0)} = \mu(A)\mu(B). \tag{3.1}$$

Proof. Fix $l\in\mathbb{N}$. Let A, B be l-cylinders, i.e. $A=[A']_l$ and $B=[B']_l$ for some A', $B'\subset F_l$. Then

$$\mu(A\cap T_k B) = \sum_{a\in A', b\in B'} \mu([a]_l\cap T_k[b]_l) = \sum_{a\in A', b\in B'} \mu([0]_l\cap T_{k+b-a}[0_l]). \tag{3.2}$$

In a similar way,

$$\mu([0]_0\cap T_k[0]_0) = \sum_{c,c'\in C_1+\cdots+C_l} \mu([0]_l\cap T_{k+c'-c}[0]_l). \tag{3.3}$$

Since, for each $m\in\mathbb{Z}$, the sequence $\sum_{k=1}^{n}\mu([0]_l\cap T_{k+m}[0]_l)$ of positive reals is equivalent† to the sequence $\sum_{k=1}^{n}\mu([0]_l\cap T_k[0]_l)$ as $n\to\infty$, it follows from (3.2) and (3.3) that

$$\frac{\sum_{k=0}^{n-1}\mu(A\cap T_k B)}{\sum_{k=0}^{n-1}\mu([0]_0\cap T_k[0]_0)} \to \frac{\#A'\#B'}{\#(C_1+\cdots+C_l)^2} = \mu([A']_l)\mu([B']_l),$$

as desired. □

LEMMA 3.3. *Under the condition of the previous lemma, for arbitrary subsets A and B in $[0]_0$,*

$$\frac{\sum_{k=0}^{h_l-1}\mu(A\cap T_k B)}{\sum_{k=0}^{h_l-1}\mu([0]_0\cap T_k[0]_0)} \leq 2\min(\mu(A),\,\mu(B)),$$

where $h_l := \#F_l$.

† The equivalence of two sequences of reals means that the ratio of these sequences goes to one.

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Proof. Let $C := C_1 + \cdots + C_l \subset \mathbb{Z}$. For each subset $J \subset [0]_0$ and an element $c \in C$, we set $[c]_{l,J} := [c]_l \cap J$. Then $J = \bigsqcup_{c\in C}[c]_{l,J}$. We now have

$$\begin{aligned}\sum_{k=0}^{h_l-1}\mu(A\cap T_kB) &\le \sum_{k=0}^{h_l-1}\sum_{c,c'\in C}\mu([c]_{l,A}\cap T_k[c']_l)\\ &= \sum_{c,c'\in C}\sum_{k=0}^{h_l-1}\mu([c]_{l,A}\cap T_{c'}T_k[0]_l)\\ &= \sum_{c,c'\in C}\mu([c]_{l,A}\cap T_{c'}[0]_0)\\ &\le \sum_{c,c'\in C}\mu([c]_{l,A})\\ &= \mu(A)\#C.\end{aligned}$$

In a similar way,

$$\begin{aligned}\sum_{k=0}^{h_l-1}\mu(A\cap T_kB) &\le \sum_{k=0}^{h_l-1}\sum_{c,c'\in C}\mu(T_{c+1-h_l}T_{-k}[h_l-1]_l\cap [c']_{l,B})\\ &= \sum_{c,c'\in C}\mu(T_{c+1-h_l}[0]_0\cap [c']_{l,B})\\ &\le \sum_{c'\in C}\mu([c']_{l,B})\\ &= \mu(B)\#C.\end{aligned}$$

Hence $\sum_{k=0}^{h_l-1}\mu(A\cap T_kB)\le \#C\cdot\min(\mu(A),\mu(B))$. On the other hand,

$$\begin{aligned}\sum_{k=0}^{h_l-1}\mu([0]_0\cap T_k[0]_0) &\ge \sum_{c\ge c'\in C}\sum_{k=0}^{h_l-1}\mu([c]_l\cap T_k[c']_l)\\ &= \sum_{c\ge c'\in C}\sum_{k=0}^{h_l-1}\mu([c-c']_l\cap T_k[0]_l)\\ &= \sum_{c\ge c'\in C}\mu([c-c']_l)\\ &= \frac{\#C+1}{2},\end{aligned}$$

and we are done. □

Approximating arbitrary pairs of Borel subsets in $[0]_0$ with cylinders, we deduce from Lemmas 3.2 and 3.3 the following assertion.

THEOREM 3.4. *For arbitrary Borel subsets A and B in $[0]_0$,*

$$\lim_{l\to\infty}\frac{\sum_{k=0}^{h_l-1}\mu(A\cap T_kB)}{\sum_{k=0}^{h_l-1}\mu([0]_0\cap T_k[0]_0)} = \mu(A)\mu(B),$$

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i.e. T is subsequence weakly rationally ergodic. Hence the rank-one transformations are non-squashable.

We now pass to the general Abelian case. Let G be an Abelian countable infinite discrete group G and let $T=(T_g)_{g\in G}$ be a (C,F)-action of G associated with $(C_n,\,F_{n-1})_{n\geq 1}$ satisfying (I)–(III). In view of Theorem 2.8, we may assume without loss of generality that (2.2) holds. Moreover, we may assume without loss of generality that $0\in\bigcap_{n>0}F_n$.

We now prove an analogue of Lemma 3.2. Given a finite subset D in G, we let $\nu_D:=\sum_{g\in D}\delta_g$, where δ_g is the Kronecker measure supported at $g\in G$.

LEMMA 3.5. *The following are equivalent:*

(i) *For all cylinders $A,\,B\subset[0]_0$,*

$$\lim_{n\to\infty}\frac{\sum_{g\in F_n-F_n}\mu(A\cap T_gB)}{\sum_{g\in F_n-F_n}\mu([0]_0\cap T_g[0]_0)}=\mu(A)\mu(B). \tag{3.4}$$

(ii) *For each $l>0$ and each $h\in F_l-F_l$, we have*

$$\sum_{g\in F_n-F_n+h}\mu([0]_l\cap T_g[0]_l)\underset{n\to\infty}{\sim}\sum_{g\in F_n-F_n}\mu([0]_l\cap T_g[0]_l).$$

(iii) *For each $l>0$ and each $h\in F_l-F_l$, we have*

$$\lim_{n\to\infty}\lim_{L\to\infty}\frac{\nu_{C_{l+1}+\cdots+C_L}*\nu_{-C_{l+1}-\cdots-C_L}(F_n-F_n)}{\nu_{C_{l+1}+\cdots+C_L}*\nu_{-C_{l+1}-\cdots-C_L}(F_n-F_n+h)}=1.$$

Proof. (i) $\Rightarrow$ (ii). If (i) holds then for each $l>0$ and $f,\,v\in F_l$,

$$1=\lim_{n\to\infty}\frac{\sum_{g\in F_n-F_n}\mu([f]_l\cap T_g[v]_l)}{\sum_{g\in F_n-F_n}\mu([0]_l\cap T_g[0]_l)}=\lim_{n\to\infty}\frac{\sum_{g\in F_n-F_n}\mu([0]_l\cap T_{v+g-f}[0]_l)}{\sum_{g\in F_n-F_n}\mu([0]_l\cap T_g[0]_l)}$$

and (ii) follows.

(ii) $\Rightarrow$ (i). This is established in the same way as in the proof of Lemma 3.2.

(ii) $\Leftrightarrow$ (iii). Fix a finite subset $S\subset G$. Since (2.2) holds, for each $l>0$, there is $L>l$ such that

$$S+C_{l+1}+\cdots+C_L\subset F_L.$$

Therefore if $[c]_L\cap T_g[c']_L\neq\emptyset$ for some $c,\,c'\in C_{l+1}+\cdots+C_L$ and $g\in S$ then $c=g+c'$. We now have

$$\begin{aligned}\sum_{g\in S}\mu([0]_l\cap T_g[0]_l)&=\sum_{g\in S}\sum_{c,c'\in C_{l+1}+\cdots+C_L}\mu([c]_L\cap T_g[c']_L)\\&=\sum_{g\in S}\sum_{\{(c,c')|c,c'\in C_{l+1}+\cdots+C_L,\,g=c-c'\}}\mu([c]_L\cap T_g[c']_L)\\&=\sum_{g\in S}\sum_{\{(c,c')|c,c'\in C_{l+1}+\cdots+C_L,\,g=c-c'\}}\frac{1}{\#C_1\cdots\#C_L}\\&=\frac{\nu_{C_{l+1}+\cdots+C_L}*\nu_{-C_{l+1}-\cdots-C_L}(S)}{\#C_1\cdots\#C_L}.\end{aligned}$$

This implies the desired equivalence of (ii) and (iii). □

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Now we establish an analogue of Lemma 3.3.

LEMMA 3.6. *Suppose that there is $K > 0$ such that $F_n - F_n \subset \bigsqcup_{j=1}^{K} d_{n,j} + F_n$ for some elements $d_{n,1}, \ldots, d_{n,K} \in G$. Then for arbitrary subsets A and B in $[0]_0$,*

$$\frac{\sum_{g\in F_n - F_n} \mu(A \cap T_g B)}{\sum_{g\in F_n - F_n} \mu([0]_0 \cap T_g[0]_0)} \le K \min(\mu(A), \mu(B)).$$

Proof. Let $C := C_1 + \cdots + C_n$. For each subset $J \subset [0]_0$ and an element $c \in C$, we set $[c]_{n,J} := [c]_n \cap J$. Then $J = \bigsqcup_{c\in C} [c]_{n,J}$. We now have

$$\begin{aligned}
\sum_{g\in F_n - F_n} \mu(A \cap T_g B) &\le \sum_{g\in F_n - F_n} \sum_{c,c'\in C} \mu([c]_{n,A} \cap T_g[c']_n) \\
&\le \sum_{c,c'\in C} \sum_{g\in F_n} \sum_{j=1}^{K} \mu([c]_{n,A} \cap T_{c'} T_{d_{n,j}+g}[0]_n) \\
&= \sum_{c,c'\in C} \sum_{j=1}^{K} \mu([c]_{n,A} \cap T_{c'} T_{d_{n,j}}[0]_0) \\
&\le K \sum_{c,c'\in C} \mu([c]_{n,A}) \\
&= K\mu(A)\#C.
\end{aligned}$$

By symmetry, $\sum_{g\in F_n - F_n} \mu(A \cap T_g B) \le K\mu(B)\#C$. On the other hand,

$$\begin{aligned}
\sum_{g\in F_n - F_n} \mu([0]_0 \cap T_g[0]_0) &= \sum_{c,c'\in C} \sum_{g\in F_n - F_n} \mu([c]_n \cap T_g[c']_n) \\
&\ge \sum_{c,c'\in C} \sum_{g=c-c'} \mu([c]_n \cap T_g[c']_n) \\
&= \sum_{c,c'\in C} \mu([0]_n) \\
&= \#C,
\end{aligned}$$

and we are done. □

The corollary below follows from Lemmas 3.5 and 3.6 in the very same way as Theorem 3.4 follows from Lemmas 3.2 and 3.3.

COROLLARY 3.7. *If the conditions of Lemmas 3.5 and 3.6 hold then (3.4) is satisfied for all Borel subsets A, $B \subset [0]_0$. Hence T is weakly rationally ergodic along the sequence $(F_n - F_n)_{n\ge 0}$. In particular, T is non-squashable.*

We now give another sufficient condition for the weak rational ergodicity of Abelian (C, F)-actions. Suppose that for each $g \in G$, the (C, F)-sequence $(C_n, F_{n-1})_{n\ge 1}$ satisfies the following condition of 'large holes' in C_{n+1}:

$$(g + F_n + F_n - F_n - F_n) \cap (C_{n+1} - C_{n+1}) = \{0\} \tag{3.5}$$

eventually. Denote by $T = (T_g)_{g\in G}$ the associated (C, F)-action. Without loss of generality we may assume that (2.2) holds.

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THEOREM 3.8. *If (3.5) holds then (3.4) is satisfied for all Borel subsets $A, B \subset [0]_0$. Thus T is weakly rationally ergodic along $(F_n - F_n)_{n\geq 0}$.*

Proof. We proceed in several steps.

Claim 1. Given $g \in G$, we have

$$(g + F_n - F_n) \cap \sum_{j>0}(C_j - C_j) = \sum_{j=1}^{n}(C_j - C_j) \quad \text{eventually.}$$

Indeed, since $F_n + C_{n+1} \subset F_{n+1}$, it follows from (3.5) (with $n+1$ in place of n) that

$$\left(g + \sum_{j=1}^{2}(F_n - F_n) + (C_{n+1} - C_{n+1})\right) \cap (C_{n+2} - C_{n+2}) = \{0\}$$

for all sufficiently large n. From this and (3.5) we deduce that

$$\left(g + \sum_{j=1}^{2}(F_n - F_n)\right) \cap \sum_{j=n+1}^{n+2}(C_j - C_j) = \left(g + \sum_{j=1}^{2}(F_n - F_n)\right) \cap (C_{n+1} - C_{n+1}) = \{0\}.$$

By induction in n, we obtain that $(g + \sum_{j=1}^{2}(F_n - F_n)) \cap \sum_{j=n+1}^{\infty}(C_j - C_j) = \{0\}$ eventually. Since $\sum_{j=1}^{n}(C_j - C_j) \subset F_n - F_n$, we obtain that

$$(g + F_n - F_n) \cap \sum_{j>0}(C_j - C_j) = (g + F_n - F_n) \cap \sum_{j=1}^{n}(C_j - C_j).$$

On the other hand, $-g + \sum_{j=1}^{n} C_j \subset F_n$ eventually in view of (2.2). Claim 1 follows.

Claim 2. For each $g \in G$,

$$\sum_{k\in g+F_n-F_n} \mu([0]_l \cap T_k[0]_l) = \sum_{k\in F_n-F_n} \mu([0]_l \cap T_k[0]_l) \quad \text{eventually.}$$

Indeed, by Lemma 2.11, $\mu([0]_l \cap T_k[0]_l) > 0$ if and only if $k \in \sum_{j>l}(C_j - C_j)$. Claim 1 now yields that

$$(g + F_n - F_n) \cap \sum_{j>l}(C_j - C_j) = \sum_{j=l+1}^{n}(C_j - C_j) = (F_n - F_n) \cap \sum_{j>l}(C_j - C_j)$$

and Claim 2 follows.

Claim 3. Equation (3.4) holds for all cylinders A, B in X.

This follows from Lemma 3.5 and Claim 2.

Claim 4. For arbitrary subsets $A, B \subset [0]_0$,

$$\frac{\sum_{g\in F_n-F_n} \mu(A \cap T_g B)}{\sum_{g\in F_n-F_n} \mu([0]_0 \cap T_g[0]_0)} \leq \min\{\mu(A), \mu(B)\}.$$

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Indeed, let $C := C_1 + \cdots + C_n$ and $[c]_{n,A} := [c]_n \cap A$ for each $c \in C$. It follows from Lemma 2.11 and Claim 1 that

$$\begin{aligned}\sum_{g\in F_n - F_n} \mu(A \cap T_g B) &\le \sum_{g\in F_n - F_n} \sum_{c,c'\in C} \mu([c]_{n,A} \cap T_g[c']_n) \\ &= \sum_{c,c'\in C} \sum_{g\in (F_n - F_n)\cap (c-c'+\sum_{j>n}(C_j - C_j))} \mu([c]_{n,A} \cap T_g[c']_n) \\ &= \sum_{c,c'\in C} \mu([c]_{n,A} \cap T_{c-c'}[c']_n) \\ &= \sum_{c,c'\in C} \mu([c]_{n,A}) \\ &= \mu(A)\#C.\end{aligned}$$

In a similar way, $\sum_{g\in F_n - F_n} \mu(A \cap T_g B) \le \mu(B)\#C$. The same argument yields that $\sum_{g\in F_n - F_n} \mu([0]_0 \cap T_g[0]_0) = \mu([0]_0)\#C = \#C$ and Claim 4 follows.

The assertion of the theorem follows now from Claims 3 and 4. □

4. *Actions of finite funny rank*

Let T be a measure preserving action of a discrete countable infinite group G on a standard σ-finite measure space $(X, \mathfrak{B}, \mu)$. Fix $k > 1$.

Definition 4.1. We say that T is of *funny rank at most k* if there exist k sequences $(B_n^j)_{n\ge 0}$, $j = 1, \ldots, k$, of subsets of finite measure in X and k sequences $(F_n^j)_{n\ge 0}$, $j = 1, \ldots, k$, of finite subsets in G such that

(i) for each $n \ge 0$ and $j \in \{1, \ldots, k\}$, the subsets $T_g B_n^j$, $g \in F_n^j$, are pairwise disjoint and

(ii) for each subset $B \in \mathfrak{B}$ with $\mu(B) < \infty$,

$$\lim_{n\to\infty} \inf_{F^j \subset F_n^j} \mu\left(B \triangle \bigsqcup_{j=1}^{k} \bigsqcup_{g\in F_n^j} T_g B_n^j\right) = 0.$$

Without loss of generality we may assume that $\lim_{n\to\infty} \mu(B_n^j) = 0$ for each j. The collection $\{T_g B_n^j \mid g \in F_n^j\}$ of subsets in X is called the *jth T-tower (of the nth T-castle)*. The subsets $T_g B_b^j$, $g \in F_n^j$, are called *levels* of the jth T-tower. The *nth T-castle* is the collection of jth T-towers when j runs over the set $\{1, \ldots, k\}$. We say that the sequence of castles *refines* if, for each $n > 0$, every level of the nth castle is a union of levels of the $(n+1)$th castle. The union of all levels of the jth T-tower in the nth T-castle is denoted by W_n^j. If $\mu(X \setminus \bigsqcup_{j=1}^k W_n^j) = 0$ for all n, then we say that T is *of funny rank at most k without spacers*. If, moreover, there is $\delta > 0$ such that $\mu(W_n^j) > \delta$ for all n and j then T is of *exact funny rank at most k*. If, in addition, there is $D > 1$ such that $D^{-1} < \mu(B_n^i)/\mu(B_n^j) < D$ for all n and $i, j \in \{1, \ldots, k\}$ then we say that T is *of balanced exact rank at most k*. We say that T is of *finite funny rank* if there is $k \ge 1$ such that T is of funny rank at most k. If $G = \mathbb{Z}$ and $F_n^j = \{0, 1 \ldots, \#F_n^j - 1\}$ then we obtain the standard definition of 'finite rank', 'exact rank', etc. (see [**Fe2**] for details).

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The following lemma is standard. We state it without proof.

LEMMA 4.2. *Let $(B_n^j)_{n\ge 0}$ and $(F_n^j)_{n\ge 0}$ be as in Definition 4.1. Given a sequence $(\epsilon_l)_{l=0}^{\infty}$ of positive reals tending to zero, there is a sequence $(n_l)_{l=0}^{\infty}$ of positive integers increasing to infinity and subsets $\widetilde{B}_l^j \subset B_{n_l}^j$ such that $\mu(B_{n_l}^j \setminus \widetilde{B}_l^j) < \epsilon_l$ for all l and j, the sequences $(\widetilde{B}_l^j)_{l\ge 0}$ and $(F_{n_l}^j)_{l\ge 0}$, $j \in \{1, \dots, k\}$, satisfy Definition 4.1 and the sequence of the nth T-castles corresponding to them refines.*

Thus without loss of generality we may assume that if T is of finite funny rank then the corresponding sequence of T-castles refines.

It is easy to see that every action of finite funny rank is conservative. However, such actions can be non-ergodic. It is obvious that every action of finite funny rank without spacers is defined on a space with finite measure.

We now consider the case of $\mathbb{Z}$-actions of finite rank in more detail.

LEMMA 4.3. *Suppose that T is an ergodic $\mathbb{Z}$-action of rank at most k. If there is a Borel subset $A \subset X$ of finite strictly positive measure and $j \in \{1, \dots, k\}$ such that $\lim_{n\to\infty} \mu(A \cap W_n^j) = 0$ then, for each Borel subset $B \subset X$ of finite measure, $\lim_{n\to\infty} \mu(B \cap W_n^j) = 0$.*

Proof. Let

$$\mathfrak{F} := \Big\{ C \in \mathfrak{B} \mid \lim_{n\to\infty} \mu(C \cap W_n^j) = 0 \Big\}.$$

We claim that if $C \in \mathfrak{F}$ and $g \in \mathbb{Z}$ then $T_g C \in \mathfrak{F}$. Indeed,

$$\mu(T_g C \cap W_n^j) = \mu(C \cap T_{-g} W_n^j) \le \mu(C \cap W_n^j) + \mu(W_n^j \triangle T_{-g} W_n^j).$$

Since $\mu(W_n^j \triangle T_{-g} W_n^j) = 2|g|\mu(B_n) \to 0$, it follows that $T_g C \in \mathfrak{F}$.

Now, given a Borel subset $B \subset X$ of finite measure and $\epsilon > 0$, there is $N > 0$ such that $\mu(B \setminus \bigcup_{m=0}^{N} T_m A) < \epsilon$. Therefore,

$$\mu(B \cap W_n^j) \le \sum_{m=0}^{N} \mu(T_m A \cap W_n^j) + \epsilon.$$

Hence $B \in \mathfrak{F}$. □

We now show that for the ergodic $\mathbb{Z}$-actions of finite rank, one can choose a sequence of approximating T-castles (from Definition 4.1) that satisfy certain additional properties.

PROPOSITION 4.4. *Let T be an ergodic $\mathbb{Z}$-action of rank at most k_1. Then there exist $k \le k_1$ and sequences $(B_n^j)_{n=1}^{\infty}$ and $(F_n^j)_{n=1}^{\infty}$, $1 \le j \le k$, satisfying Definition 4.1 and such that the corresponding sequence of T-castles refines and the following limits exist:*

$$\lim_{n\to\infty} \mu\Big(\Big(\bigsqcup_{i=1}^{k} B_0^i\Big) \cap W_n^j\Big) = \delta_j > 0, \quad j = 1, \dots, k, \tag{4.1}$$

with $\sum_{j=1}^{k} \delta_j = \sum_{j=1}^{k} \mu(B_0^j)$.

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Proof. We say that the sequence $(W_n^j)_{n=1}^{\infty}$ of jth towers *vanishes* if

$$\lim_{n\to\infty} \mu\Bigg(\Bigg(\bigsqcup_{i=1}^{k_1} B_0^i\Bigg) \cap W_n^j\Bigg) = 0.$$

It follows from Lemma 4.3 that the vanishing towers do not really contribute to property (ii) of Definition 4.1. In other words, if we drop all the towers that vanish, the remaining sequences of towers will still satisfy Definition 4.1. Let k be the number of remaining sequences of towers. Since they are non-vanishing, there is a subsequence of the associated T-castles and strictly positive numbers $\delta_1, \ldots, \delta_k$ such that (4.1) is satisfied. Then, of course,

$$\sum_{j=1}^{k} \delta_j = \sum_{j=1}^{k} \mu(B_0^j). \tag{4.2}$$

We need to modify the resulting sequence of T-castles to make it refining. For that we apply Lemma 4.2 to find an increasing sequence $(n_l)_{l\geq 0}$ of positive integers and subsets $(\widetilde{B}_l^j)_{l\geq 0}$ such that:

(a) $\widetilde{B}_l^j \subset B_{n_l}^j$ and $\mu(B_{n_0}^j \setminus \widetilde{B}_0^j) < 0.5k^{-1} \min_{1\leq i\leq k} \delta_i$ for all l and j;

(b) the sequences $(\widetilde{B}_l^j)_{l\geq 0}$ and $(F_{n_l}^j)_{l\geq 0}$, $j \in \{1, \ldots, k\}$, satisfy Definition 4.1; and

(c) the sequence of nth T-castles corresponding to them refines.

Passing to a further subsequence (of course, the subsequence satisfies the properties (a)–(c)), we may assume that there are reals $\widetilde{\delta}_1 \geq 0, \ldots, \widetilde{\delta}_k \geq 0$ such that $\lim_{l\to\infty} \mu((\bigsqcup_{i=1}^k \widetilde{B}_0^i) \cap \widetilde{W}_l^j) = \widetilde{\delta}_j$, where $\widetilde{W}_l^j := \bigsqcup_{g\in F_{n_l}^j} T_g \widetilde{B}_l^j$, $j = 1, \ldots, k$. It remains to show that the reals $\widetilde{\delta}_j$ are all strictly positive. Of course, $\sum_{j=1}^k \widetilde{\delta}_j = \sum_{j=1}^k \mu(\widetilde{B}_0^j)$. If follows from (a) that $\widetilde{\delta}_j \leq \delta_j$ for each j. These inequalities, (4.2) and (a) yield that

$$\sum_{j=1}^{k} \delta_j \geq \sum_{j=1}^{k} \widetilde{\delta}_j \geq \sum_{j=1}^{k} \delta_j - 0.5 \min_{1\leq j\leq k} \delta_j.$$

This implies that $\widetilde{\delta}_j > 0$ for each j, as desired. □

COROLLARY 4.5. *Under the conditions of Proposition 4.4, the induced (finite measure preserving) transformation* $(T_1)_{\bigsqcup_{i=1}^{k_1} B_0^i}$ *is of exact rank at most* k.

Proof. Suppose that the sequence of T-castles refines and satisfies (4.1). Then for each $n \in \mathbb{Z}_+$ and $j \in \{1, \ldots, k\}$, the intersection of W_n^j with the space $\bigsqcup_{i=1}^k B_0^i$ is a tower of the induced transformation $(T_1)_{\bigsqcup_{i=1}^k B_0^i}$. The union of these towers, when j runs over $\{1, \ldots, k\}$, is an nth $(T_1)_{\bigsqcup_{i=1}^k B_0^i}$-castle. The sequence of these castles refines and generates the entire σ-algebra of Borel subsets of $\bigsqcup_{i=1}^k B_0^i$. Now (4.1) yields that $(T_1)_{\bigsqcup_{i=1}^k B_0^i}$ is exact. □

We recall that, given a measure preserving transformation S of a standard probability space (Y, ν), the *Koopman unitary operator* U_S on $L^2(Y, \nu)$ is defined by the formula $U_S f := f \circ S$.

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LEMMA 4.6. *Let S be an ergodic measure preserving transformation of a standard probability space (Y, ν). Let $(A_n)_{n=1}^{\infty}$ be a sequence of Borel subsets of Y such that $\lim_{n\to\infty} \nu(A_n) = \delta > 0$ and $\|U_S 1_{A_n} - 1_{A_n}\|_2 \to 0$. Then for each Borel subset $B \subset Y$, $\nu(B \cap A_n) \to \nu(B)\delta$.*

Proof. Since, by the von Neumann mean ergodic theorem, $(1/N)\sum_{j=1} U_S^j 1_B \to \mu(B)$ in the metric of $L^2(Y, \nu)$, we can find, for each $\epsilon > 0$, a positive integer N such that

$$\epsilon \geq \left|\left\langle \frac{1}{N}\sum_{j=1}^{N} U_S^j 1_B,\, 1_{A_n}\right\rangle - \nu(B)\nu(A_n)\right| = \left|\left\langle 1_B,\, \frac{1}{N}\sum_{j=1}^{N} U_S^{-j} 1_{A_n}\right\rangle - \nu(B)\nu(A_n)\right|$$

for each $n \geq 0$. Passing to the limit when $n \to \infty$ and using the condition of the lemma, we obtain that $\epsilon \geq \limsup_{n\to\infty} |\nu(B \cap A_n) - \nu(B)\delta|$. Hence $\nu(B \cap A_n) \to \nu(B)\delta$. □

Applying Lemma 4.6, we refine Proposition 4.4 in the following way.

COROLLARY 4.7. *Passing to a further subsequence in $(B_n^j, F_n^j)_{n=1}^{\infty}$ from the statement of Proposition 4.4 and normalizing μ such that $\mu(\bigsqcup_{l=1}^{k} B_0^l) = 1$, we may assume without loss of generality that the following limits hold: (4.1) and*

$$\lim_{n\to\infty} \mu\big(A \cap W_n^i \cap W_{n+1}^j\big) = \mu(A)\delta_i\delta_j, \quad i, j = 1, \ldots, k,$$

for each Borel subset $A \subset \bigsqcup_{l=1}^{k} B_0^l$. Moreover, there exist limits

$$\Lambda_i := \lim_{n\to\infty} \frac{\mu(B_n^i)}{\sum_{l=1}^{k} \mu(B_n^l)}$$

and for each level D_n^i of the ith tower of the nth T-castle such that $D_n^i \subset \bigsqcup_{l=1}^{k} B_0^l$,

$$\lim_{n\to\infty} \frac{\mu(D_n^i \cap W_{n+1}^j)}{\mu((\bigsqcup_{l=1}^{k} D_n^l) \cap W_{n+1}^j)} = \Lambda_i, \quad i = 1, \ldots, k.$$

Of course, $\sum_{i=1}^{k} \Lambda_i = 1$.

Proof. It suffices to note that the reals $\delta_1, \ldots, \delta_{k_1}$ do not depend on passing to a subsequence of approximating castles, and that the intersection of W_n^j with the set $\bigsqcup_{l=1}^{k} B_0^l$ is a tower of the ergodic probability preserving induced transformation $(T_1)_{\bigsqcup_{l=1}^{k} B_0^l}$, and to apply Lemma 4.6. □

5. (C, F)-construction of actions of finite funny rank

We now give a constructive definition for actions of G of finite funny rank.

We set $G' := G \times \{1, \ldots, k\}$ and $G'' := \{1, \ldots, k\} \times G \times \{1, \ldots, k\}$. Given $\boldsymbol{c} \in G''$, we denote by c the element in G such that $\boldsymbol{c} = (i, c, j)$ for some $i, j \in \{1, \ldots, k\}$. In a similar way, given $\boldsymbol{a} \in G'$, we denote by a the element in G such that $\boldsymbol{a} = (a, i)$ for some $i \in \{1, \ldots, k\}$. We will consider G' as a left G-space, where G acts by the formula $g \cdot (f, i) := (gf, i)$. Given a subset A of G' and $i \in \{1, \ldots, k\}$, we let $A^i := A \cap (G \times \{i\})$. In a similar way, given a subset C of G'' and $i, j \in \{1, \ldots, k\}$, we denote by $C^{i,j}$ the

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intersection $C \cap (\{i\} \times G \times \{j\})$. Let $(f, i) \in G'$ and $(k, g, l) \in G'$. If $i = k$ we define a 'product' $(f, i) * (k, g, l)$ by setting

$$(f, i) * (k, g, l) := (fg, l) \in G'.$$

For arbitrary subsets $A \subset G'$ and $C \subset G''$, we let $A * C$ be the set of all products $\boldsymbol{a} * \boldsymbol{c}$, where $\boldsymbol{a} \in A$, $\boldsymbol{c} \in C$ and $\boldsymbol{a} * \boldsymbol{c}$ is defined. We reduce the notation $A * \{\boldsymbol{c}\}$ to $A * \boldsymbol{c}$. In a similar way one can define a product of two elements of G'':

$$(i, c, j) * (k, c', k) := (i, cc', k) \in G'' \quad \text{if } j = k.$$

Hence the product $C * C'$ of two subsets C, C' in G'' is also well defined.

Suppose we are given two sequences of finite subsets $(F_n)_{n\geq 0}$ in G' and $(C_n)_{n\geq 1}$ in G'' such that the following conditions hold for each n:

(I) $F_0 = \{1\} \times \{1, \dots, k\}$, $\sum_{j=1}^{k} \#C_n^{i,j} > 1$ and $(1, i) \in F_n$ for each i;

(II) $F_n * C_{n+1} \subset F_{n+1}$;

(III) $F_n * \boldsymbol{c} \cap F_n * \boldsymbol{c}' = \emptyset$ if $\boldsymbol{c} \neq \boldsymbol{c}' \in C_{n+1}$.

We let, for each $n \geq 0$,

$$X_n := \{(\boldsymbol{f}_n, \boldsymbol{c}_{n+1}, \boldsymbol{c}_{n+2} \dots) \in F_n \times C_{n+1} \times C_{n+2} \times \cdots \mid \boldsymbol{f}_n * \boldsymbol{c}_{n+1} * \cdots * \boldsymbol{c}_l \text{ is well defined for each } l > n\}.$$

Then X_n is a perfect subset of the compact Cantor space $F_n \times C_{n+1} \times C_{n+2} \times \cdots$ (endowed with Tikhonov's topology). Hence X_n is itself a compact Cantor space. The map

$$X_n \ni (\boldsymbol{f}_n, \boldsymbol{c}_{n+1}, \boldsymbol{c}_{n+2} \dots) \mapsto (\boldsymbol{f}_n * \boldsymbol{c}_{n+1}, \boldsymbol{c}_{n+2} \dots) \in X_{n+1}$$

is a topological embedding of X_n into X_{n+1}. Therefore we will consider X_n as a (clopen) subset of X_{n+1}. We now define X to be the topological inductive limit of the increasing sequence $X_1 \subset X_2 \subset \cdots$ of compact Cantor spaces. Then X is a locally compact Cantor space†. We call it the (C, F)*-space associated with the sequence* $(C_n, F_{n-1})_{n\geq 1}$. Given a subset A of F_n, we let

$$[A]_n := \{(\boldsymbol{f}_n, \boldsymbol{c}_{n+1}, \dots) \in X_n \mid \boldsymbol{f}_n \in A\} \subset X,$$

and call this set an *n-cylinder*. It is a compact open subset of X. Conversely, every compact open subset of X is a cylinder. The set of all cylinders is a base of the topology in X. It is easy to see that

$$[A]_n \cap [B]_n = [A \cap B]_n, \quad [A]_n \cup [B]_n = [A \cup B]_n \quad \text{and} \quad [A]_n = [A * C_{n+1}]_{n+1}$$

for all A, $B \subset F_n$ and $n \geq 0$. For brevity we will write $[\boldsymbol{f}]_n$ for $[\{\boldsymbol{f}\}]_n$, $\boldsymbol{f} \in F_n$.

Let $\mathcal{R}$ denote the *tail equivalence relation* on X. This means that the restriction of $\mathcal{R}$ to X_n is the tail equivalence relation on X_n for each $n \geq 0$. If, in addition to (I)–(III), the condition

(IV) $\#C_n^{i,j} > 1$, for all $i, j \in \{1, \dots, k\}$ and $n > 0$,

is satisfied, then $\mathcal{R}$ is *minimal*, i.e. the $\mathcal{R}$-class of every point in X is dense in X.

† X is compact if and only if there is $N > 0$ with $F_{n+1} = F_n * C_{n+1}$ for all $n > N$.

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We now investigate the problem of existence and uniqueness of $\mathcal{R}$-invariant Radon measures on X. Of course, such a measure μ is completely determined by its values on the cylinders. In turn, every cylinder is a disjoint union of some 'elementary' cylinders $[\boldsymbol{f}]_n$, where $\boldsymbol{f}$ runs over F_n. Since the elementary cylinders $[\boldsymbol{f}]_n$ and $[\boldsymbol{g}]_n$ are of the same measure whenever $\boldsymbol{f}, \boldsymbol{g} \in F_n^i$ for some $i \in \{1, \dots, k\}$ (this fact is equivalent to the $\mathcal{R}$-invariance of μ), we obtain that μ is determined uniquely by its values

$$\lambda_n^i := \mu([(1, i)]_n), \quad i = 1, \dots, k \text{ for all } n \geq 0. \tag{5.1}$$

Thus, we obtain a sequence of vectors

$$\lambda_n := \begin{pmatrix} \lambda_n^1 \\ \vdots \\ \lambda_n^k \end{pmatrix} \in \mathbb{R}_+^k, \quad n \geq 0.$$

There is a consistency condition for these vectors. Indeed, the property

$$\mu([\boldsymbol{f}]_n) = \sum_{\boldsymbol{c} \in C_{n+1}} \mu([\boldsymbol{f} * \boldsymbol{c}]_{n+1}), \quad \boldsymbol{f} \in F_n,$$

can be rewritten as

$$\lambda_n = r_{n+1} \lambda_{n+1}, \tag{5.2}$$

where $r_n = (r_n^{i,j})_{1 \leq i,j \leq k}$ is a $k \times k$ integer matrix defined by setting $r_n^{i,j} := \#(C_n^{i,j})$. Conversely, given a sequence $(\lambda_n)_{n \geq 0}$ of positive vectors in $\mathbb{R}^k$ satisfying (5.2), there is a unique $\mathcal{R}$-invariant Radon measure μ on X satisfying (5.1). Indeed, we define μ by setting, for each $n \geq 0$ and a subset $A \subset F_n$,

$$\mu([A]_n) := \sum_{i=1}^{k} \#(A^i) \lambda_n^i. \tag{5.3}$$

We call μ the (C, F)-*measures on X associated with the sequence* $(\lambda_n)_{n \geq 0}$. It is easy to see that μ is finite if and only if

$$\lim_{n \to \infty} \sum_{j=1}^{k} \lambda_n^i \#(F_n^i) < \infty. \tag{5.4}$$

Thus we obtain the following proposition.

PROPOSITION 5.1. *Formula (5.3) establishes an affine one-to-one correspondence between the set of $\mathcal{R}$-invariant Radon measures on X and the projective limit of the sequence*

$$\mathbb{R}_+^k \xleftarrow{r_1} \mathbb{R}_+^k \xleftarrow{r_2} \mathbb{R}_+^k \xleftarrow{r_3} \cdots .$$

Unlike the funny rank-one case considered in §2, in the general case there can be several mutually disjoint ergodic $\mathcal{R}$-invariant Radon measures on X. We now provide a simple sufficient condition on the sequence $(r_n)_{n=1}^{\infty}$ under which $\mathcal{R}$ is *uniquely ergodic*, i.e. there exists a unique (up to scaling) ergodic $\mathcal{R}$-invariant non-trivial Radon measure on X.

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PROPOSITION 5.2. *Suppose that there exist non-negative reals* $\Lambda_1, \dots, \Lambda_k$ *such that, for an increasing subsequence of integers* $n_p \to \infty$, *there exist limits*

$$\lim_{p\to\infty} \frac{r_{n_p}^{i,l}}{\sum_{j=1}^{k} r_{n_p}^{j,l}} = \Lambda_i \quad \text{for all } i, l \in \{1, \dots, k\}. \tag{5.5}$$

Then $\mathcal{R}$ *is uniquely ergodic.*

Proof. In view of Proposition 5.1, it suffices to show that the intersections of the cones $r_{n_p}(\mathbb{R}_+^k)$ with the simplex $\Delta := \{(z_1, \dots, z_k) \in \mathbb{R}_+^k \mid z_1 + \cdots + z_k = 1\}$ shrink to a single point as $p \to \infty$. The cone $r_{n_p}(\mathbb{R}_+^k)$ is generated by k rays passing through the following vectors $(r_{n_p}^{1,1}, \dots, r_{n_p}^{k,1}), \dots, (r_{n_p}^{1,k}, \dots, r_{n_p}^{k,k}) \in \mathbb{R}_+^k$. Therefore (5.5) yields that $r_{n_p}(\mathbb{R}_+^k) \cap \Delta \to \{(\Lambda_1, \dots, \Lambda_k)\}$ (in Hausdorff metric) as $p \to \infty$. □

Remark 5.3. We also note that X_0 intersects each $\mathcal{R}$-orbit infinitely many times. The map $\mu \mapsto \mu \restriction X_0$ is an affine isomorphism of the simplex of Radon $\mathcal{R}$-invariant measures on X which equal one on X_0 onto the simplex of $(\mathcal{R} \restriction X_0)$-invariant probability Borel measures on X_0. Indeed, every $(\mathcal{R} \restriction X_0)$-invariant measure extends uniquely to an $\mathcal{R}$-invariant measure on X via (5.3) and (5.1). Therefore this isomorphism maps the ergodic $\mathcal{R}$-invariant Radon measures which equal one on X_0 onto the ergodic $(\mathcal{R} \restriction X_0)$-invariant probability Borel measures. In particular, $\mathcal{R}$ is uniquely ergodic if and only if $\mathcal{R} \restriction X_0$ is as well.

We now define an action of G on X (or, more rigorously, on a subset of X). Given $g \in G$, let

$$X_n^g := \{(\boldsymbol{f}_n, \boldsymbol{c}_{n+1}, \boldsymbol{c}_{n+2} \dots) \in X_n \mid g \cdot \boldsymbol{f}_n \in F_n\}.$$

Then X_n^g is a compact open subset of X_n and $X_n^g \subset X_{n+1}^g$. Hence the union $X^g := \bigcup_{n\geq 0} X_n^g$ is a well-defined open subset of X. Let $X^G := \bigcap_{g\in G} X^g$. Then X^G is a G_δ-subset of X. It is Polish in the induced topology. Given $x \in X^G$ and $g \in G$, there is $n > 0$ such that $x = (\boldsymbol{f}_n, \boldsymbol{c}_{n+1}, \dots) \in X_n$ and $g \cdot \boldsymbol{f}_n \in F_n$. We now let

$$T_g x = (g \cdot \boldsymbol{f}_n, \boldsymbol{c}_{n+1}, \dots) \in X_n \subset X.$$

It is standard to verify that $T_g x \in X^G$, the map $T_g : X^G \ni x \mapsto T_g x \in X^G$ is a homeomorphism of X^G, and $T_g T_{g'} = T_{gg'}$ for all $g, g' \in G$. Thus $T := (T_g)_{g\in G}$ is a continuous action of G on X^G.

Definition 5.4. We call T *the* (C, F)*-action of* G *associated with the sequence* $(C_n, F_{n-1})_{n\geq 0}$.

The (C, F)-actions are free. It is obvious that X^G is $\mathcal{R}$-invariant and the T-orbit equivalence relation is the restriction of $\mathcal{R}$ to X^G. Hence for each ergodic (C, F)-measure μ, either $\mu(X^G) = 0$ or $\mu(X \setminus X^G) = 0$ and T preserves μ.

We now state an analogue of Proposition 2.2.

PROPOSITION 5.5. $X^G = X$ *if and only if, for each* $g \in G$ *and* $n > 0$, *there is* $m > n$ *such that*

$$g \cdot F_n * C_{n+1} * C_{n+2} * \cdots * C_m \subset F_m. \tag{5.6}$$

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Thus, in this case the (C, F)-action is defined on the entire (locally compact) space X. We do not give the proof of this lemma because it is an obvious slight modification of the proof of Proposition 2.2. As in the rank-one case considered in §2, if X is not compact then T extends continuously to the one-point compactification of X. If (IV) is satisfied then the extended action is almost minimal.

The following lemma is a counterpart of Proposition 2.4.

PROPOSITION 5.6. *Let μ be a (C, F)-measure on X associated with a sequence $(\lambda_n)_{n\geq 0}$.*

(i) *$\mu(X\backslash X^G) = 0$ if and only if, for each $g \in G$ and $n > 0$,*

$$\lim_{m\to\infty} \sum_{i=1}^{k} \#(A^i_{m,n})\lambda^i_m = \sum_{i=1}^{k} \#(F^i_n)\lambda^i_n, \tag{5.7}$$

where $A_{m,n} := (g \cdot F_n * C_{n+1} * \cdots * C_m) \cap F_m \subset G'$.

(ii) *If $\mu(X) < \infty$ and there exist limits $\gamma_i := \lim_{n\to\infty} (\lambda^i_n \#F^i_n)/(\sum_{j=1}^{k} \lambda^j_n \#F^j_n)$, $i = 1, \ldots, k$†, then $\mu(X\backslash X^G) = 0$ if and only if the sequence $(F^i_n)_{n=1}^{\infty}$ is Følner in G for each i such that $\gamma_i \neq 0$. In particular, G is amenable.*

Proof. We only prove (ii). We note that $\mu(X\backslash X^G) = 0$ if and only if, for each $g \in G$, $\mu(X^g_n)/\mu(X_n) \to 1$. Since

$$\frac{\mu(X^g_n)}{\mu(X_n)} = \frac{\sum_{i=1}^{k} \#(g \cdot F_n \cap F_n)^i \lambda^i_n}{\sum_{i=1}^{k} \#F^i_n \lambda^i_n} = \sum_{i=1}^{k} \frac{\#(g \cdot F_n \cap F_n)^i}{\#F^i_n} \cdot \frac{\lambda^i_n \#F^i_n}{\sum_{j=1}^{k} \lambda^j_n \#F^j_n},$$

it follows from this and the condition of the proposition that

$$\sum_{i=1}^{k} \frac{\#((g \cdot F_n)^i \cap F^i_n)}{\#F^i_n} \gamma_i \to 1. \tag{5.8}$$

Since $\gamma_i \geq 0$ for each i and $\sum_{i=1}^{k} \gamma_i = 1$, it follows that (5.8) is equivalent to the claim that

$$\frac{\#((g \cdot F_n)^i \cap F^i_n)}{\#F^i_n} \to 1 \quad \text{for each } i \text{ such that } \gamma_i \neq 0. \qquad \square$$

We note that if G is amenable, $\mu(X) < \infty$ and $(F^i_n)_{n\geq 0}$ is a Følner sequence in G for each $i \in \{1, \ldots, k\}$, then (5.7) holds.

From now on we assume that (5.7) is satisfied. Then (X, μ, T) is a (well-defined) measure preserving dynamical system. It is easy to see that it is conservative.

By analogy with the case of (C, F)-actions of rank one, we may assume without loss of generality that the sequences $(F_n)_{n\geq 0}$ and $(C_n)_{n\geq 1}$ satisfy the condition

(V) $\{(1, i) \mid 1 \leq i \leq k\} \subset \bigcap_{n=0}^{\infty} F_n$ and $\{(i, 1, i) \mid 1 \leq i \leq k\} \subset \bigcap_{n=1}^{\infty} C_n$,

in addition to (I)–(III).

† Passing to a (C, F)-subsequence, we can assume without loss of generality that the latter condition always holds.

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We claim that each (C, F)-action (defined in this section) is of funny rank at most k. Indeed, the sequences $([(1, 1)]_n)_{n\geq 0}, \ldots, ([(1, k)]_n)_{n\geq 0}$ and $(F_n^1)_{n\geq 0}, \ldots, (F_n^k)_{n\geq 0}$ satisfy Definition 4.1. We also note that $T_g[\boldsymbol{f}]_n = [g \cdot \boldsymbol{f}]_n$ (up to a μ-null subset) whenever $\boldsymbol{f}, g \cdot \boldsymbol{f} \in F_n$.

We gather together some of the above results on (C, F)-actions of finite funny rank in the following theorem.

THEOREM 5.7. *Given a sequence $(C_n, F_{n-1})_{n\geq 1}$ satisfying (I)–(V) and (5.5), there exist a locally compact Cantor space X and a countable equivalence relation $\mathcal{R}$ on X such that the following statements hold:*

(i) *Every $\mathcal{R}$-class is dense in X.*
(ii) *There is only one (up to scaling) $\mathcal{R}$-invariant non-trivial σ-finite Radon measure μ on X.*
(iii) *μ is finite if and only if (5.4) and (5.1) are satisfied.*
(iv) *There is a free topological G-action T on an $\mathcal{R}$-invariant G_δ-subset X^G of X such that the T-orbit equivalence relation is the restriction of $\mathcal{R}$ to X^G.*
(v) *$X^G = X$ if and only if (5.6) is satisfied.*
(vi) *$\mu(X \backslash X^G) = 0$ if and only if (5.7) is satisfied. If (5.7) is not satisfied then $\mu(X^G) = 0$.*
(vii) *If $\mu(X) < \infty$ and $(F_n^j)_{n\geq 0}$ is a Følner sequence in G for each $j = 1, \ldots, k$, then (5.7) is satisfied.*
(viii) *Under (5.7), the dynamical system (X, μ, T) is ergodic, conservative and of funny rank at most k.*

The converse to Theorem 5.7(viii) also holds.

THEOREM 5.8. ((C, F)-models for finite funny rank actions) *If T is a G-action of funny rank at most k on a standard σ-finite measure space $(X, \mathfrak{B}, \mu)$ then T is isomorphic to a (C, F)-action of G and the corresponding isomorphism maps μ to a (C, F)-measure.*

We do not give a proof of this theorem because it is an obvious modification of the proof of Theorem 2.6.

Let T be the (C, F)-action associated with a sequence $(C_n, F_{n-1})_{n>0}$ satisfying (I)–(III) and (5.7). Given an increasing sequence $(k_n)_{n\geq 0}$ of non-negative integers with $k_0 = 0$, we let $\widetilde{F}_n := F_{k_n}$ and $\widetilde{C}_n := C_{k_{n-1}+1} * C_{k_{n-1}+2} * \cdots * C_{k_n}$. Then the sequence $(\widetilde{C}_n, \widetilde{F}_{n-1})_{n>1}$ also satisfies (I)–(III) and (5.7). By analogy with the rank-one (C, F)-actions, we call $(\widetilde{C}_n, \widetilde{F}_{n-1})_{n>1}$ a (C, F)-*subsequence* of $(C_n, F_{n-1})_{n>1}$. The (C, F)-action associated with it is canonically isomorphic to T.

We also state a higher-rank analogue of Theorem 2.8.

THEOREM 5.9. *Let T be the (C, F)-action of G associated with a sequence $(C_n, F_{n-1})_{n\geq 1}$ satisfying* (I)–(III) *and (5.7). Then T is (measure theoretically) isomorphic to the (C, F)-action S of G associated with a sequence $(C'_n, F'_{n-1})_{n\geq 1}$ satisfying* (I)–(III) *and (5.6). Moreover, the sequence $(F'_n)_{n\geq 0}$ is a subsequence of $(F_n)_{n\geq 0}$. In particular, if $G = \mathbb{Z}$ and T is of rank at most k then S is also of rank at most k.*

We omit the proof of this theorem because it is a slight modification of the proof of Theorem 2.8.

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Remark 5.10. In the case where $G=\mathbb{Z}$, $F_n^i=\{0,\dots,h_n^i-1\}$ and the (C,F)-space X is compact (i.e. when the associated (C,F)-action T has rank at most k without spacers and the corresponding sequence of T-castles refines) we can associate an ordered Bratteli diagram with T. We refer to [**Du**, **Be–So**] for the definitions related to Bratteli diagrams and Bratteli–Vershik systems. We define a graded vertex set $V=(V_n)_{n\ge -1}$ and a graded edge set $E:=(E_n)_{n\ge 0}$ in the following way. We let $V_{-1}:=\{0\}$, $V_n:=\{1,\dots,k\}$ for $n\ge 0$, and $E_0:=F_0$ and $E_n:=C_n$ for $n\ge 1$. More precisely, we consider $C_n^{i,j}$ as the set of edges connecting the vertex $i\in V_{n-1}$ with the vertex $j\in V_n$, $n\ge 1$. The corresponding graded graph (V,E) is called a *Bratteli diagram*. We now define an order relation on it. For that we define, for each $n\ge 1$ and $j\in V_n$, a linear order $\succ$ on the set $\bigsqcup_{i=1}^k C_n^{i,j}$ by setting: $\boldsymbol{c}\succ\boldsymbol{d}$ if $c\ge d$. Thus, we obtain an ordered Bratteli diagram $(V,E,\succ)$. It is straightforward to verify that the map $\phi: X_0\ni(f_0,c_1,c_2,\dots)\mapsto(f_0,c_1,c_2,\dots)$ identifies X_0 with the Bratteli compactum Y associated with (V,E). Moreover, ϕ intertwines T with the Bratteli–Vershik $\mathbb{Z}$-action associated with $(V,E,\succ)$ and $\phi\times\phi$ maps bijectively the (C,F)-equivalence relation on X onto the tail equivalence relation on Y.

6. *Weak rational ergodicity of transformations of balanced finite rank*

Let T be an ergodic i.m.p. $\mathbb{Z}$-action of finite rank. Let (X,μ) stand for the space of this action. By Theorem 5.8, T is isomorphic to a (C,F)-action of $\mathbb{Z}$ associated with sequences $(C_n)_{n>0}$ and $(F_n)_{n\ge 0}$ satisfying (I)–(III) and (V) from §5. Thus we consider X as a (C,F)-space and μ as a (C,F)-measure associated with a sequence $(\lambda_n)_{n=1}^\infty$. Since T is ergodic, we may apply Proposition 4.4 and Corollary 4.7. 'Translating' their assertions into the language of (C,F)-systems, we may assume without loss of generality that:

(α) $\mu(X_0)=1$ and the induced transformation $(T_1)_{X_0}$ is an ergodic probability preserving transformation of exact rank at most k;

(β) there exist limits $\lim_{n\to\infty}\mu(X_0\cap[F_n^j]_n)=\delta_j>0$ for each $j\in\{1,\dots,k\}$, and $\sum_{j=1}^k\delta_j=1$;

(γ) there exist limits

$$\lim_{n\to\infty}\frac{\lambda_n^i}{\sum_{j=1}^k\lambda_n^j}=\lim_{n\to\infty}\frac{r_n^{i,j}}{\sum_{l=1}^k r_n^{l,j}}=\Lambda_i\ge 0$$

for all $i,j\in\{1,\dots,k\}$ and $\sum_{i=1}^k\Lambda_i=1$.

We note that if $\Lambda_i>0$ for all i then (IV) from §5 is satisfied. Hence T is a minimal (C,F)-action.

It is well known that each transformation of rank one is isomorphic to a *tower transformation*† built under a certain 'spacer map' over an ergodic transformation with pure point spectrum. We now extend this result to the transformations of finite rank. For that we first introduce a sequence $(s_n)_{n\ge 0}$ of auxiliary mappings from X_0 to $\mathbb{Z}_+$ by setting

$$s_n(\boldsymbol{f}_0,\boldsymbol{c}_1,\boldsymbol{c}_2,\dots):=\max\{t\ge 0\mid t\cdot\boldsymbol{f}_0*\boldsymbol{c}_1*\cdots*\boldsymbol{c}_n\in F_n\backslash(F_0*C_1*\cdots*C_n)\},$$

† We recall a standard definition. Given a measure preserving transformation S on a standard measure space (Y,ν) and a Borel map $f:Y\to\mathbb{Z}_+$, we define a new dynamical system (X,μ,T) by setting $X=\{(y,i)\in Y\times\mathbb{Z}\mid 0\le i<f(y)\}$, $d\mu(y,i)=d\mu(x)$ and $T(y,i)=(y,i+1)$ if $i+1<f(y)$ and $T(y,i)=(Sy,i)$ if $i+1=f(y)$. Then T is called the transformation built under f over S.

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where $\boldsymbol{f}_0 \in F_0$ and $\boldsymbol{c}_j \in C_j$, $n \geq 0$. Of course, s_n is continuous for each $n \geq 0$ and $s_0 \leq s_1 \leq \cdots$. We let

$$C_n^{\max} := \left\{ \boldsymbol{c} \in \bigsqcup_{i=1}^{k} C_n^{i,j} \;\middle|\; c = \max_{\boldsymbol{d} \in \bigsqcup_{i=1}^{k} C_n^{i,j}} d,\ j = 1, \ldots, k \right\} \subset C_n.$$

It is easy to see that

$$s_n(x) = s_{n+1}(x) = \cdots \quad \text{whenever } x \notin [F_0 * C_1 * \cdots * C_{n-1} * C_n^{\max}]_n. \tag{6.1}$$

Let $Y_n := \{(x, t) \in X_0 \times \mathbb{Z}_+ \mid 0 \leq t < s_n(x)\}$. Then the map

$$\phi_n : Y_n \ni ((\boldsymbol{f}_0, \boldsymbol{c}_1, \boldsymbol{c}_2, \ldots), t) \mapsto (t \cdot \boldsymbol{f}_0 * \boldsymbol{c}_1 * \cdots * \boldsymbol{c}_{n-1}, \boldsymbol{c}_n, \boldsymbol{c}_{n+1}, \ldots) \in X_n$$

is a homeomorphism of Y_n on X_n for each $n \geq 0$. Of course, $Y_0 \subset Y_1 \subset \cdots$. We now let

$$s := \sup_{n \geq 0} s_n \quad \text{and} \quad Y := \{(x, t) \in X_0 \times \mathbb{Z}_+ \mid 0 \leq t < s(x)\}.$$

We call s the *spacer map*. It takes values in $\mathbb{Z}_+ \cup \{+\infty\}$ and it is lower semicontinuous. Denote by $\mathcal{D}$ the subset of all $x = (f_0, c_1, c_2, \ldots) \in X_0$ such that $c_n \in C_n^{\max}$ eventually. Then $\mathcal{D}$ is an F_σ-subset of X_0 and $\mu(\mathcal{D}) = 0$. In fact, it is easy to verify that $\mathcal{D}$ is countable†. It follows from (6.1) that the spacer map is continuous when restricted to $X_0 \backslash \mathcal{D}$. The map

$$\phi : Y \supset Y_n \ni y \mapsto \phi_n(y) \in X_n \subset X, \quad n \geq 0,$$

is a homeomorphism of Y onto X. We define an equivalence relation $\mathcal{Y}$ on Y by setting $(y, i) \sim_{\mathcal{Y}} (y', i')$ if $(y, y') \in \mathcal{R} \cap (X_0 \times X_0)$, where $\mathcal{R}$ stands for the (C, R)-equivalence relation on X. Then $\phi \times \phi$ maps $\mathcal{Y}$ onto $\mathcal{R}$. We now define a measure ν on Y by setting $d\nu(x, i) := d\mu(x)$, $(x, i) \in Y$. It is easy to verify that $\mu \circ \phi = \nu$. Hence μ is finite if and only if $\int_{X_0} s \, d\mu < \infty$. By (6.1), if $s(x) = +\infty$ then $x \in \mathcal{D}$. Hence the spacer map is finite μ-almost everywhere on X_0. We also note that

$$\phi^{-1} T_1 \phi(x, t) = \begin{cases} (x, t+1) & \text{if } t + 1 < s(x), \\ ((T_1)_{X_0} x, t) & \text{otherwise.} \end{cases}$$

Thus we have shown the following proposition.

PROPOSITION 6.1. *T_1 is a tower transformation built under the spacer map over the base $(T_1)_{X_0}$ which is an ergodic transformation of exact finite rank.*

It remains to note that we consider the transformations of exact rank as 'higher-rank' analogues of the transformations with pure point spectrum.

Definition 6.2. We say that T is of *balanced* finite rank it is isomorphic to a (C, F)-action such that (α)–(γ) are satisfied and $\Lambda_i > 0$ for each $i \in \{1, \ldots, k\}$.

In other words, T is of balanced finite rank if there is a refining approximating sequence of T-castles such that the induced (finite measure preserving) transformation $T_1 \restriction (\bigsqcup_{j=1}^{k} B_0^j)$ is of balanced exact finite rank (see Definition 4.1).

We now prove a higher-rank analogue of Lemma 3.2.

† This follows from the fact that there are only finitely many points $(f_0, c_1, c_2, \ldots) \in X_0$ with $c_n \in C_n^{\max}$ for each $n > 0$.

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PROPOSITION 6.3. *If T is of balanced finite rank then for each pair of cylinders $[A]_l$, $[B]_l$ in X, we have*

$$\lim_{n\to\infty}\frac{\sum_{m=0}^{n-1}\mu([A]_l\cap T_m[B]_l)}{\sum_{m=0}^{n-1}\mu([F_0]_0\cap T_m[F_0]_0)}=\mu([A]_l)\mu([B]_l).$$

Proof. We first note that for each pair $i,\ j\in\{1,\dots,k\}$,

$$\begin{aligned}\sum_{m=0}^{n-1}\mu([F_0^i]_l\cap T_m[F_0^j]_l)&=\sum_{a,b=1}^{k}\sum_{m=0}^{n-1}\mu([(0,i)*C_{l+1}^{i,a}]_{l+1}\cap T_m[(0,j)*C_{l+1}^{j,b}]_{l+1})\\&=\sum_{a,b=1}^{k}\sum_{\boldsymbol{c}\in C_{l+1}^{i,a}}\sum_{\boldsymbol{d}\in C_{l+1}^{j,b}}\sum_{m=0}^{n-1}\mu([(0,a)]_{l+1}\cap T_{d+m-c}[(0,b)]_{l+1}).\end{aligned}$$

It is easy to see that for all $a,\ b\in\{1,\dots,k\}$, and $s\in\mathbb{Z}$,

$$\sum_{m=0}^{n-1}\mu([(0,a)]_{l+1}\cap T_{m+s}[(0,b)]_{l+1})\underset{n\to\infty}{\sim}\sum_{m=0}^{n-1}\mu([(0,a)]_{l+1}\cap T_m[(0,b)]_{l+1}).$$

Since $\#C_{l+1}^{i,a}=r_{l+1}^{i,a}$ and $\{(0,a)\}=F_0^a$ for all $i,\ a\in\{1,\dots,k\}$, we obtain the equivalence

$$\sum_{m=0}^{n-1}\mu([F_0^i]_l\cap T_m[F_0^j]_l)\underset{n\to\infty}{\sim}\sum_{a,b=1}^{k}r_{l+1}^{i,a}r_{l+1}^{j,b}\sum_{m=0}^{n-1}\mu([F_0^a]_{l+1}\cap T_m[F_0^b]_{l+1}).\tag{6.2}$$

Let $C:=F_0*C_1*\cdots*C_l$. Then we have

$$\begin{aligned}\frac{\sum_{m=0}^{n-1}\mu([A]_l\cap T_m[B]_l)}{\sum_{m=0}^{n-1}\mu([F_0]_0\cap T_m[F_0]_0)}&=\frac{\sum_{i,j=1}^{k}\sum_{\boldsymbol{a}\in A^i,\boldsymbol{b}\in B^j}\sum_{m=0}^{n-1}\mu([\boldsymbol{a}]_l\cap T_m[\boldsymbol{b}]_l)}{\sum_{i,j=1}^{k}\sum_{\boldsymbol{c}\in C^i,\boldsymbol{d}\in C^j}\sum_{m=0}^{n-1}\mu([\boldsymbol{c}]_l\cap T_m[\boldsymbol{d}]_l)}\\&=\frac{\sum_{i,j=1}^{k}\sum_{\boldsymbol{a}\in A^i,\boldsymbol{b}\in B^j}\sum_{m=0}^{n-1}\mu([F_0^i]_l\cap T_{b+m-a}[F_0^j]_l)}{\sum_{i,j=1}^{k}\sum_{\boldsymbol{c}\in C^i,\boldsymbol{d}\in C^j}\sum_{m=0}^{n-1}\mu([F_0^i]_l\cap T_{d+m-c}[F_0^j]_l)}.\end{aligned}$$

It follows from this and (6.2) that there is a limit

$$\lim_{n\to\infty}\frac{\sum_{m=0}^{n-1}\mu([A]_l\cap T_m[B]_l)}{\sum_{m=0}^{n-1}\mu([F_0]_0\cap T_m[F_0]_0)}=\frac{\sum_{i,j=1}^{k}\#A^i\#B^j\sum_{p,q=1}^{k}r_{l+1}^{i,p}r_{l+1}^{j,q}}{\sum_{i,j=1}^{k}\#C^i\#C^j\sum_{p,q=1}^{k}r_{l+1}^{i,p}r_{l+1}^{j,q}}.\tag{6.3}$$

Since T is of balanced finite rank, there is $D>0$ such that

$$\max_{l\ge1}\max_{1\le i\le k}\frac{\sum\lambda_l^i}{\lambda_l^j}<D.$$

It now follows from this inequality, (γ) and (6.3) that for each $\epsilon>0$, there is L such that if $l>L$ then

$$\begin{aligned}\lim_{n\to\infty}\frac{\sum_{m=0}^{n-1}\mu([A]_l\cap T_m[B]_l)}{\sum_{m=0}^{n-1}\mu([F_0]_0\cap T_m[F_0]_0)}&=\frac{\sum_{i,j=1}^{k}\#A^i\#B^j\sum_{p,q=1}^{k}(\Lambda_i\pm\epsilon)(\Lambda_j\pm\epsilon)}{\sum_{i,j=1}^{k}\#C^i\#C^j\sum_{p,q=1}^{k}(\Lambda_i\pm\epsilon)(\Lambda_j\pm\epsilon)}\\&=\frac{\sum_{i,j=1}^{k}\#A^i\#B^j\lambda_l^i\lambda_l^j(1\pm2\epsilon D)^2}{\sum_{i,j=1}^{k}\#C^i\#C^j\lambda_l^i\lambda_l^j(1\pm2\epsilon D)^2}\end{aligned}$$

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$$= \frac{\sum_{i,j=1}^{k} \mu([A^i]_l)\mu([B^j]_l)(1 \pm 2\epsilon D)^2}{\sum_{i,j=1}^{k} \mu([C^i]_l)\mu([C^j]_l)(1 \pm 2\epsilon D)^2}$$
$$= \mu([A]_l)\mu([B]_l)(1 \pm 10\epsilon D)^2.$$

Since every cylinder J in X can be presented as $J = [J_l]_l$ for each sufficiently large l and some finite subset $J_l \subset \mathbb{Z}$, we are done. □

We now let $h_l := \min_{1 \le i \le k} \#F_l^i$ for each $l \ge 1$. The following assertion is a higher-rank analogue of Lemma 3.3.

LEMMA 6.4. *Under the condition of Proposition 6.3, for arbitrary subsets A and B in X_0,*

$$\frac{\sum_{m=0}^{h_l-1} \mu(A \cap T_m B)}{\sum_{m=0}^{h_l-1} \mu(X_0 \cap T_m X_0)} \le \frac{4k \min(\mu(A), \mu(B))}{\min_{1 \le j \le k} \delta_j}. \tag{6.4}$$

Proof. Let $C := F_0 * C_1 * \cdots * C_l$. For each subset $J \subset X_0 = [F_0]_0$ and element $\boldsymbol{c} \in C$, we set $[\boldsymbol{c}]_{l,J} := [\boldsymbol{c}]_l \cap J$. Then $J = \bigsqcup_{\boldsymbol{c} \in C} [\boldsymbol{c}]_{l,J}$. We now have

$$\begin{aligned}
\sum_{m=0}^{h_l-1} \mu(A \cap T_m B) &\le \sum_{m=0}^{h_l-1} \sum_{\boldsymbol{c},\boldsymbol{d} \in C} \mu([\boldsymbol{c}]_{l,A} \cap T_m [\boldsymbol{d}]_l) \\
&= \sum_{\boldsymbol{c} \in C} \sum_{j=1}^{k} \sum_{\boldsymbol{d} \in C^j} \sum_{m=0}^{h_l-1} \mu([\boldsymbol{c}]_{l,A} \cap T_d[(m, j)]_l) \\
&\le \sum_{\boldsymbol{c} \in C} \sum_{j=1}^{k} \sum_{\boldsymbol{d} \in C^j} \mu([\boldsymbol{c}]_{l,A} \cap T_d [F_l^j]_l) \\
&\le k \max_{1 \le j \le k} \#C^j \sum_{\boldsymbol{c} \in C} \mu([\boldsymbol{c}]_{l,A}) \\
&= k\mu(A) \max_{1 \le j \le k} \#C^j.
\end{aligned}$$

In a similar way,

$$\begin{aligned}
\sum_{m=0}^{h_l-1} \mu(A \cap T_m B) &\le \sum_{\boldsymbol{d} \in C} \sum_{j=1}^{k} \sum_{\boldsymbol{c} \in C^j} \sum_{m=0}^{h_l-1} \mu(T_{c+1-h_l}[(h_l - 1 - m, j)]_l \cap [\boldsymbol{d}]_{l,B}) \\
&\le \sum_{\boldsymbol{d} \in C} \sum_{j=1}^{k} \sum_{\boldsymbol{c} \in C^j} \mu(T_{c+1-h_l}[F_l^j]_l \cap [\boldsymbol{d}]_{l,B}) \\
&\le k \max_{1 \le j \le k} \#C^j \sum_{\boldsymbol{d} \in C} \mu([\boldsymbol{d}]_{l,B}) \\
&\le k\mu(B) \max_{1 \le j \le k} \#C^j.
\end{aligned}$$

Hence

$$\sum_{m=0}^{h_l-1} \mu(A \cap T_m B) \le k \min(\mu(A), \mu(B)) \max_{1 \le j \le k} \#C^j. \tag{6.5}$$

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Choose j_l such that $h_l = \#F_l^{j_l}$, $l \geq 1$. Then we have

$$\begin{aligned}
\sum_{m=0}^{h_l-1} \mu([F_0]_0 \cap T_m[F_0]_0) &\geq \sum_{i=1}^{k} \sum_{m=0}^{h_l-1} \mu([C^i]_l \cap T_m[C^i]_l) \\
&\geq \sum_{i=1}^{k} \sum_{c,d\in C^i, \boldsymbol{c}\geq \boldsymbol{d}} \sum_{m=0}^{h_l-1} \mu([c]_l \cap T_m[d]_l) \\
&\geq \sum_{i=1}^{k} \sum_{c,d\in C^i, \boldsymbol{c}\geq \boldsymbol{d}} \mu([c]_l \cap T_{\boldsymbol{d}}[F_l^{j_l}]_l) \\
&\geq \sum_{i=1}^{k} \sum_{c,d\in C^i, \boldsymbol{c}\geq \boldsymbol{d}} \mu([c]_l) \\
&= \sum_{i=1}^{k} \frac{\#C^i + 1}{2} \#C^i \lambda_l^i \\
&\geq \sum_{i=1}^{k} \frac{\#C^i \mu([C^i]_l)}{2}. \qquad (6.6)
\end{aligned}$$

We note that $[C^i]_l = X_0 \cap [F_l^i]_l$. Hence (β) yields that $\mu([C^i]_l) \geq \delta_i/2$ whenever l is large enough. This inequality, (6.5) and (6.6) imply (6.4). □

We now state the main result of this section. It is a generalization of Theorem 3.4 to the transformations of balanced finite rank.

THEOREM 6.5. *Let T be an ergodic i.m.p. $\mathbb{Z}$-action of balanced finite rank. Let $h_l := \min_{1\leq i\leq k} \#F_l^i$ for each $l \geq 1$. Then for arbitrary Borel subsets A and B in X_0,*

$$\lim_{l\to\infty} \frac{\sum_{m=0}^{h_l-1} \mu(A \cap T_m B)}{\sum_{m=0}^{h_l-1} \mu(X_0 \cap T_m X_0)} = \mu(A)\mu(B),$$

i.e. T is subsequence weakly rationally ergodic. Hence the ergodic transformations of balanced finite rank are non-squashable.

Proof. The theorem is proved in the very same way as Theorem 3.4, but one needs to apply Proposition 6.3 and Lemma 6.4 instead of Lemma 3.2 and Lemma 3.3, respectively. □

7. *Partial rigidity for transformations of finite rank*

In this section we consider only finite measure preserving transformations ($\mathbb{Z}$-actions) of finite rank.

We recall that a measure preserving transformation S of a probability space $(Y, \mathfrak{C}, \nu)$ is called *partially rigid* if there exist $\delta > 0$ and an increasing sequence of integers $(m_n)_{n=1}^{\infty}$ such that $\mu(A \cap S^{m_n} A) \geq \delta\mu(A)$ for each subset $A \in \mathfrak{C}$.

The following theorem refines an unpublished result of Rosenthal [**Ro**] that the transformations of exact finite rank are not mixing (cf. [**Be–So**]).

THEOREM 7.1. *Let $T = (T_m)_{m\in\mathbb{Z}}$ be an ergodic $\mathbb{Z}$-action of an exact finite rank. Then T is partially rigid.*

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Proof†. Let T be of exact rank at most k. Let (X, μ) stand for the space of T. We will use the notation from Definition 4.1. Denote by $h_{n,j}$ the height of the jth T-tower of the nth T-castle. Thus we have $F_n^j = \{0, 1, \ldots, h_{n,j} - 1\}$. Fix $n > 0$. Select $j_0 \in \{1, \ldots, k\}$ such that $\mu(B_n^{j_0}) = \max_{1 \le j \le k} \mu(B_n^j)$. Since $T_{h_{n,j_0}} B_n^{j_0} \subset \bigsqcup_{j=1}^k B_n^j$, there is $j_1 \in \{1, \ldots, k\}$ such that

$$\mu(B_n^{j_1} \cap T_{h_{n,j_0}} B_n^{j_0}) = \max_{1 \le j \le k} \mu(B_n^j \cap T_{h_{n,j_0}} B_n^{j_0}) \ge \frac{\mu(B_n^{j_0})}{k}.$$

Since $T_{h_{n,j_1}} B_n^{j_1} \subset \bigsqcup_{j=1}^k B_n^j$, there is $j_2 \in \{1, \ldots, k\}$ such that

$$\mu(B_n^{j_2} \cap T_{h_{n,j_1}}(B_n^{j_1} \cap T_{h_{n,j_0}} B_n^{j_0})) = \max_{1 \le j \le k} \mu(B_n^j \cap T_{h_{n,j_1}}(B_n^{j_1} \cap T_{h_{n,j_0}} B_n^{j_0})) \ge \frac{\mu(B_n^{j_0})}{k^2}.$$

Continuing this process k times, we define integers $j_0, \ldots, j_k \in \{1, \ldots, k\}$. Hence there are integers a, b such that $0 \le a < b \le k$ and $j_a = j_b$. Relabelling the towers of the nth T-castle, we may assume without loss of generality that $j_a = 1$. We now let $m_n := h_{n,j_a} + \cdots + h_{n,j_b-1}$. Then

$$\mu(T_{m_n} B_n^1 \cap B_n^1) \ge \frac{\mu(B_n^{j_0})}{k^k} \ge \frac{\mu(B_n^1)}{k^k}. \tag{7.1}$$

Let $A \subset X$ be a union of levels of the nth T-castle. Then there is a subset $J \subset F_n^1$ such that $A \cap W_n^1 = \bigsqcup_{j \in J} T_j B_n^1$. Then

$$\begin{aligned} \mu(T_{m_n} A \cap A) &\ge \mu(T_{m_n}(A \cap W_n^1) \cap (A \cap W_n^1)) \\ &\ge \sum_{j \in J} \mu(T_{m_n}(A \cap T_j B_n^1) \cap (A \cap T_j B_n^1)) \\ &= \sum_{j \in J} \mu(T_{m_n} B_n^1 \cap B_n^1). \end{aligned} \tag{7.2}$$

Since T is of exact rank at most k, there is $\delta > 0$ such that $\liminf_{n \to \infty} \mu(W_n^1) = \delta$. It follows from Lemma 4.6 that $\liminf_{n \to \infty} \mu(A \cap W_n^1) = \delta \mu(A)$. This and (7.1) with (7.2) yield that $\liminf_{n \to \infty} \mu(T_{m_n} A \cap A) \ge \mu(A)/k^k$. It follows from the standard lemma below that T is partially rigid. □

LEMMA 7.2. *Let T be a probability preserving $\mathbb{Z}$-action of funny rank at most k. Let there exist a sequence $(d_n)_{n=1}^\infty$ of positive integers and $\eta > 0$ such that* $\liminf_{n \to \infty} \mu(J \cap T_{d_n} J) \ge \eta \mu(J)$ *for each level J of the lth T-castle for each $l \ge 0$. Then T is partially rigid.*

We now generalize the concept of exact finite rank.

Definition 7.3. Let T be an ergodic probability preserving $\mathbb{Z}$-action. We say that T is of *quasi-exact* rank at most k if there exist a refining sequence of T-castles (as in Definition 4.1), $\delta > 0$ and $R > 0$ such that for each $n > 0$ and each $j \in \{1, \ldots, k\}$, the number of levels (spacers) between two consecutive copies of T-towers from the nth T-castle in the jth tower of the $(n+1)$th T-castle is uniformly bounded by R and $\inf_{n \ge 0} \min_{1 \le i \le k} \mu(W_n^i) > \delta$.

† Since our original proof of Theorem 6.1 is rather long we replace it here with a more elegant (and slightly modified) proof reconstructed from [**Ry**].

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Slightly modifying the proof of Theorem 7.1†, we obtain the following theorem.

THEOREM 7.4. *Let T be an ergodic $\mathbb{Z}$-action of quasi-exact finite rank. Then T is partially rigid.*

We now adapt the definition of consecutive ordering from the theory of Bratteli–Vershik systems (see [**Du**]) to the context of measurable systems of finite rank.

Definition 7.5. Let T be a $\mathbb{Z}$-action of rank at most k without spacers and such that the corresponding sequence of T-castles (see Definition 4.1) refines. We say that T *satisfies the CO-condition* if, given arbitrary $n \geq 0$ and $i, j, l \in \{1, \dots, k\}$, if a copy of the lth T-tower from the nth T-castle is between two copies of the ith T-tower inside the jth T-tower of the $(n+1)$th T-castle then $l = i$.

We note that the Bratteli–Vershik maps corresponding to the minimal IETs satisfy the CO-condition [**GjJo**].

It was shown in [**Be–So**] that if T satisfies the CO-condition and an additional 'non-degeneracy' condition (as in Theorem 7.6 below) then T is not mixing. We prove a stronger result.

THEOREM 7.6. *Let T be an ergodic $\mathbb{Z}$-action of rank at most k without spacers. Let T satisfy the CO-condition. Suppose also that, for some $n \geq 0$ and $i, j \in \{1, \dots, k\}$, if a copy of the ith T-tower from the nth T-castle is contained inside the jth T-tower from the $(n+1)$th T-castle then at least one more copy of the ith T-tower is contained inside the jth T-tower. Then T is partially rigid.*

Proof. Let (X, μ) stand for the space of T. Since $\sum_{i=1}^{k} \mu(W_n^i) = 1$ for each $n \geq 0$, there exist a subsequence $(n_m)_{m=1}^{\infty}$ of positive integers and an integer $p \in \{1, \dots, k\}$ such that $\lim_{m\to\infty} \mu(W_{n_m}^p) = \delta > 1/k$. For each $l \geq 0$ and each level I of the lth T-castle, the intersection of I with the pth tower of the n_mth T-castle is the union of some levels of this tower for all sufficiently large m. Let h_m stand for the height of the pth tower of the n_mth T-castle. Passing to a further subsequence, we may assume without loss of generality that there is $q \in \{1, \dots, k\}$ such that $\mu(W_{n_m}^p \cap W_{n_m+1}^q) \geq \mu(W_{n_m}^p)/k$ for all m. Denote by V_m the union of all copies but the top one of the pth tower of the n_mth T-castle in the qth tower of the (n_m+1)th T-castle. It follows from the condition of the theorem that T^{h_m} moves each copy of the pth T-tower in V_m onto the adjacent (from above) copy of the same T-tower inside the qth T-tower from the (n_m+1)th T-castle. Therefore $T_{h_m}(I \cap W_{n_m}^p \cap V_m) \subset I \cap W_{n_m}^p$. Since there are no less than two such copies inside the qth T-tower, we obtain that

$$\begin{aligned}
\mu(T_{h_m} I \cap I) &\geq \mu(T_{h_m}(I \cap W_{n_m}^p \cap V_m) \cap I) \\
&= \mu(I \cap W_{n_m}^p \cap V_m) \\
&\geq \frac{1}{2}\mu(I \cap W_{n_m}^p \cap W_{n_m+1}^q) \\
&\geq \frac{1}{2k}\mu(I \cap W_{n_m}^p).
\end{aligned}$$

† Just take into account that we now have the inclusion $T_{h_{n,j}} B_n^j \subset \bigcup_{r=0}^{R} T_{-r}(\bigsqcup_{j=1}^{k} B_n^j)$ instead of $T_{h_{n,j}} B_n^j \subset \bigsqcup_{j=1}^{k} B_n^j$, $j = 1, \dots, k$.

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It now follows from Lemma 4.6 that $\mu(T_{h_m} I \cap I) \geq \delta\mu(I)/3k$ eventually in m. It remains to apply Lemma 7.2. □

Remark 7.7. It is possible to generalize Theorem 7.6 (with only a slight modification of the proof) in the following way: drop the assumption that T is constructed without spacers and replace the CO-condition in the statement of the theorem with the following two conditions.

CO_1 There is $R > 0$ such that the number of levels (spacers) between two neighbouring copies of T-towers from the nth T-castle in every tower of the $(n+1)$th T-castle is uniformly bounded by R.

CO_2 There is $L > 0$ such that the number of copies of T-towers from the nth T castle between every two consecutive copies of the ith T-tower inside each T-tower of the $(n+1)$th T-castle, $i = 1, \ldots, k$, is uniformly bounded by L.

We leave details of the proof to the reader.

Katok proved in [**Ka**] that the ergodic IETs are not mixing. We now show how to deduce from that proof (or, rather a slight modification of that proof due to [**KSF**]) that they are partially rigid†.

PROPOSITION 7.8. *Let T be an ergodic IET. Then T is partially rigid.*

Proof‡. Indeed, by [**KSF**, Chapter, §3, Lemma 1], there are integers $k > 0$, $r_j^{(n)} > 0$, $j = 1, \ldots, k$, and measurable partitions $(A_1^{(n)}, \ldots, A_k^{(n)})$ of X, $n \in \mathbb{N}$, such that:

(i) $\min_{1\leq j\leq k} r_j^{(n)} \to \infty$ as $n \to \infty$;

(ii) $\mathrm{Leb}(B \triangle (\bigcup_{j=1}^k T^{r_j^{(n)}}(A_j^{(n)} \cap B))) \to 0$ as $n \to \infty$ for each Borel subset $B \subset [0, 1)$; and

(iii) $\mathrm{Leb}(TA_j^{(n)} \triangle A_j^{(n)}) \to 0$ as $n \to \infty$ for each $j = 1, \ldots, k$.

Passing to a subsequence, we may assume without loss of generality that there exist integers $j_n \in \{1, \ldots, k\}$ and a limit

(iv) $\lim_{n\to\infty} \mathrm{Leb}(A_{j_n}^{(n)}) =: \delta \geq 1/k$.

It follows from (ii) that $\sum_{j=1}^k \int_{A_j^{(n)}} (U_T^{r_j^{(n)}} 1_B - 1_B)^2\, dx \to 0$ as $n \to \infty$. Therefore

$$\int_{A_{j_n}^{(n)}} (U_T^{r_{j_n}^{(n)}} 1_B - 1_B)^2\, dx = \int_{A_{j_n}^{(n)}} (U_T^{r_{j_n}^{(n)}} 1_B - 2U_T^{r_{j_n}^{(n)}} 1_B 1_B + 1_B)\, dx \to 0$$

as $n \to \infty$. Applying (i), (iii), (iv) and Lemma 4.6, we obtain that

$$\int_{A_{j_n}^{(n)}} (2U_T^{r_{j_n}^{(n)}} 1_B 1_B - U_T^{r_{j_n}^{(n)}} 1_B)\, dx \to \delta\mu(B) \tag{7.3}$$

as $n \to \infty$. Since

$$\int_{[0,1)} 2U_T^{r_{j_n}^{(n)}} 1_B 1_B\, dx \geq \int_{A_{j_n}^{(n)}} 2U_T^{r_{j_n}^{(n)}} 1_B 1_B\, dx \geq \int_{A_{j_n}^{(n)}} (2U_T^{r_{j_n}^{(n)}} 1_B 1_B - U_T^{r_{j_n}^{(n)}} 1_B)\, dx,$$

we deduce from (7.3) that $\liminf_{n\to\infty} \mu(T^{r_{j_n}^{(n)}} B \cap B) \geq (\delta/2)\mu(B) \geq (1/2k)\mu(B)$. □

† We cannot apply Theorem 7.1 to the ergodic IETs because we do not know whether they are of exact finite rank.

‡ See also [**Ry**] for an alternative proof.

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8. *Open problems*

(1) Given an infinite countable amenable discrete group G, is there a finite measure preserving free action of G which is of funny rank one? We note that this question is related closely to a basic problem in the theory of amenable groups: whether every amenable group has a Følner sequence consisting of monotiles [**We2**].
(2) Is each funny rank-one i.m.p. action of an Abelian countable discrete group G weakly rationally ergodic? The question is especially interesting in the case where $G = \mathbb{Z}$.
(3) Is the squashability of ergodic i.m.p. actions of Abelian groups G a spectral property? Is there an ergodic squashable i.m.p. action of G whose spectrum is of finite multiplicity?
(4) Given an amenable group G, consider two classes of actions of G: the class of all possible i.m.p. (C, F)-actions of funny rank one and the class of i.m.p. (C, F)-actions of funny rank one such that the corresponding sequences $(F_n)_{n\geq 0}$ are Følner. Do these classes coincide?
(5) Let $\mathfrak{F}$ be a factor (i.e. an invariant σ-finite σ-subalgebra) of an ergodic i.m.p. transformation T. Let $T \restriction \mathfrak{F}$ be non-squashable. Is T non-squashable too?
(6) Let $\mathfrak{F}$ be a factor of an ergodic i.m.p. transformation T and let $T \restriction \mathfrak{F}$ be (subsequence) weakly rationally ergodic. Is T (subsequence) weakly rationally ergodic?
(7) Given an ergodic transformation of exact finite rank at least two, is it possible to find a refining sequence of approximating castles such that each castle fills the entire space (i.e. the union of all levels in the castle equals the entire space)? Of course, the answer is affirmative for the transformations of exact rank one.
(8) Give examples of ergodic transformations which are of exact finite rank but not of balanced exact finite rank.
(9) Are there ergodic i.m.p. transformations of finite rank but not of balanced finite rank? If yes, are they subsequence weakly rationally ergodic?
(10) Are there ergodic IETs which are not of exact finite rank?

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