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RESEARCH ARTICLE

Brown measures of deformed L^∞-valued circular elements

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Abstract

We consider the Brown measure of $a + \mathfrak{c}$, where a lies in a commutative tracial von Neumann algebra $\mathcal{B}$ and $\mathfrak{c}$ is a $\mathcal{B}$-valued circular element. Under certain regularity conditions on a and the covariance of $\mathfrak{c}$ this Brown measure has a density with respect to the Lebesgue measure on the complex plane which is real analytic apart from jump discontinuities at the boundary of its support. With the exception of finitely many singularities this one-dimensional spectral edge is real analytic. We provide a full description of all possible edge singularities as well as all points in the interior, where the density vanishes. The edge singularities are classified in terms of their local edge shape while internal zeros of the density are classified in terms of the shape of the density locally around these points. We also show that each of these countably infinitely many singularity types occurs for an appropriate choice of a when $\mathfrak{c}$ is a standard circular element. The Brown measure of $a + \mathfrak{c}$ arises as the empirical spectral distribution of a diagonally deformed non-Hermitian random matrix with independent entries when its dimension tends to infinity.

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1. Introduction

The empirical spectral distribution of a non-Hermitian random matrix typically converges to a nonrandom probability distribution σ, the limiting spectral measure, on the complex plane as its dimension tends to infinity. The most prominent instance of this phenomenon is the circular law, stating that the empirical eigenvalue distributions of a suitably normalised sequence of matrices $X_n \in \mathbb{C}^{n\times n}$ with centred i.i.d. entries converge to the uniform distribution on the complex unit disk $\mathbb{D}_1$ in the limit $n \to \infty$ [35, 12] (see [57] for optimal moment conditions and [19] for a review).

For the purpose of identifying the limiting spectral measure and computing mixed moments of X_n and X_n^* as $n \to \infty$, according to free probability theory the limit of the sequence X_n is appropriately described by a circular element $\mathfrak{c}$. This $\mathfrak{c}$ is an infinite-dimensional operator in a W^*-probability space $\mathcal{A}$ with faithful, tracial state $\langle\cdot\rangle$ and satisfies $\lim_{n\to\infty} \frac{1}{n} \operatorname{Tr} p(X_n, X_n^*) = \langle p(\mathfrak{c}, \mathfrak{c}^*)\rangle$ for all polynomials p in two noncommutative variables. In agreement with the circular law, the uniform distribution on $\mathbb{D}_1$ is the Brown measure of $\mathfrak{c}$, where the Brown measure is a generalisation of the spectral measure to non-normal operators [22, 36]. When a deterministic deformation matrix $A = A_n$ is added to $X = X_n$, the associated spectral distribution of the random matrix $A + X$ is asymptotically close to the Brown measure $\sigma_{A+\mathfrak{c}}$ of the sum of an embedding of A into the W^*-probability space and a circular element $\mathfrak{c}$ that is $*$-free from A [52]. In this case, the support of $\sigma_{A+\mathfrak{c}}$ coincides with the closure $\overline{\mathbb{S}}_A$, where $\mathbb{S}_a := \{\zeta \in \mathbb{C} : \langle(a-\zeta)^{-1}(a^*-\overline{\zeta})^{-1}\rangle > 1\}$. This observation goes back to [42].

That $\overline{\mathbb{S}}_a$ is the support of the Brown measure of $a + \mathfrak{c}$ for a deformation $a \in \mathcal{A}$, which is $*$-free from $\mathfrak{c}$, was shown in [18] for normal deformations and extended to general deformations in [17, 64]. Subsequently, the regularity of $\sigma_{a+\mathfrak{c}}$ has been analysed. The measure is absolutely continuous with respect to the Lebesgue measure on $\mathbb{C}$ [13] and the density is strictly positive and real analytic on $\mathbb{S}$ [64, 39]. In [32] it has been shown that at the edge of $\mathbb{S}_a$, the density typically possesses a jump discontinuity and that around its zeros within its support $\overline{\mathbb{S}}_a$ it grows at most quadratically with a corresponding quadratic lower bound either on a two-sided cone or in a whole neighbourhood.

In the present work, we refine the distinction between these three possibilities by establishing a full classification of the points of vanishing density within $\overline{\mathbb{S}}_a$. The edge singularities, that is, the zeros of the density in $\partial\overline{\mathbb{S}}_a$ are classified in terms of the local shape of $\partial\overline{\mathbb{S}}_a$, while zeros in the interior of $\overline{\mathbb{S}}_a$ are classified in terms of the local growth of the density. For both cases countably infinite singularity types are possible. Finally, we prove that each case of these possible singularity types does exist in the Brown measure $\sigma_{a+\mathfrak{c}}$ for a suitable choice of a.

In addition to providing a clear picture of the regularity of the Brown measure this full classification of the singularity types is important because of their potential to distinguish the universality classes of the local eigenvalue point processes near these singularities. To that end, it is insightful to review some known results in the analogous Hermitian model, the deformed Wigner matrices $A + H$, where A is a deterministic real diagonal deformation to a sequence of $n \times n$-Wigner matrices H with i.i.d. entries, up to Hermitian symmetry constraints. The spectral density for such matrices is well approximated for large dimensions by the spectral measure of $A + \mathfrak{s}$, where $\mathfrak{s}$ is a semicircular element, $*$-free of A. This follows from [60] and [61] if H has Gaussian entries and for a general Wigner matrix from [29]. The spectral measure of $A + \mathfrak{s}$ has a density ρ with respect to the Lebesgue measure on the real line which is real analytic wherever it is positive. Under some regularity assumptions on the deformation, the singularities of ρ, that form when the density vanishes, can only be of algebraic degree two or three [2]. More precisely, they exhibit a one-sided square root growth at the spectral edge and a two-sided cubic root growth for singularities in the interior of the support of ρ. These singularity types determine

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the corresponding universality classes of the local spectral statistics that are associated with the Airy [43] and Pearcey process [34, 25], respectively. For Gaussian entries this correspondence between the singularity type of the density and the local statistics had already been established for regular square root edges without deformation in [58, 59] and subsequently with deformation in [24], as well as for the cubic root cusp in [20, 21, 1, 24].

Allowing the entries in the randomness $H = (h_{ij})_{i,j=1}^n$ to have differing distributions, introduces an additional structure designed to model systems with nontrivial underlying geometry. For the deformed Wigner type matrices $A + H$ the asymptotic spectral density ρ depends on the variance profile $R = (\mathbb{E}|h_{ij}|^2)_{i,j=1}^n$. To define the limiting object that replaces the semicircular element from the i.i.d. case in this setting, the additional structure of a conditional expectation $E : \mathcal{A} \to \mathcal{B}$ with values in the commutative subalgebra $\mathcal{B} = \mathbb{C}^n \subset \mathcal{A}$ with entrywise multiplication is introduced on the W^*-probability space. This is the setup of $\mathcal{B}$-valued free probability theory [62] and the operator $\mathfrak{s}$, for which ρ is now the spectral density of $A + \mathfrak{s}$, is a $\mathcal{B}$-valued circular element whose covariance encodes the variance profile R through the identity $Rb = E[\mathfrak{s}b\mathfrak{s}]$ for $b \in \mathcal{B}$. The classification of singularities of ρ in terms of the edge - cusp - dichotomy, as well as the associated appearance of the Airy - and Pearcey - universality classes, persists in this more general model [8, 34, 25].

Introducing variance profiles in the non-Hermitian setup requires the same extension to a $\mathcal{B}$-valued probability space $(\mathcal{A}, \mathcal{B}, E)$ with $\mathcal{B} = \mathbb{C}^n$. The empirical spectral measure of a random matrix $X = (x_{ij})_{i,j=1}^n$ with centred independent entries of variances $s_{ij} := \mathbb{E}|x_{ij}|^2$ is well approximated by the Brown measure of a non-normal operator $\mathfrak{c}$ that is a $\mathbb{C}^n$-valued circular element and covariance $S = (s_{ij})_{i,j=1}^n$ [10]. Many entries may have variance zero, which is of particular interest when studying non-Hermitian band matrices [40, 41]. Here the Brown measure is supported on a disk, radially symmetric and its density has a jump at the edge, but is nonconstant on its support in general [27, 5]. Adding a diagonal deformation $A = (a_i\delta_{ij})_{i,j=1}^n$ to X_n breaks the asymptotic radial symmetry of the Brown measure of $A + \mathfrak{c}$ and the spectrum concentrates on an area in the complex plane that is contained in the asymptotic pseudospectrum, whose outer boundary was identified in [7]. In the companion paper [10] we show that the spectrum asymptotically fills the entire area enclosed by this boundary, that is, for deformed random matrices with variance profile the support of the limiting spectral measure and the asymptotic pseudospectrum coincide.

Recently, it has been shown that for non-Hermitian matrices $A + X$, where $X \in \mathbb{C}^{n\times n}$ has i.i.d. entries and A is deterministic, the local universality classes correspond to the local shape of the limiting spectral density σ as well in two cases. In the bulk, that is when σ is positive, the local eigenvalue statistics coincides with the Ginibre bulk statistics. For Gaussian X this was shown in [63] and for non-Gaussian entries and $A = 0$ under the four moment matching assumption in [56] and without this assumption in [45, 48, 28]. At spectral edge points with a jump discontinuity of σ, the Ginibre edge statistic emerges, which has been proved first for Gaussian matrices in [44] and then for non-Gaussian entries in [26] for $A = 0$ and in [23] for nonzero deformations. This phenomenon is expected to generalise to matrices X with variance profile. In contrast to the Hermitian setting the class of singularities that emerge as points where the limiting spectral density vanishes, or equivalently the zeros of the density of the Brown measure of $A + \mathfrak{c}$ with a $\mathbb{C}^n$-valued circular element $\mathfrak{c}$ inside its support, is much more rich. As we show in this work, it contains an infinite family of such possible singularity types both at the spectral edge as well as in the interior of the support. We conjecture that each local singularity type corresponds to a specific universality class for the local statistics of the corresponding random matrix ensemble.

Generalising from the setting of finite dimensional $\mathcal{B} = \mathbb{C}^n$, in this work we consider the general framework of deformed $\mathcal{B}$-valued circular elements $a+\mathfrak{c}$, where $a \in \mathcal{B}$ is the deformation, $\mathfrak{c}$ the $\mathcal{B}$-valued circular element, $\mathcal{B} \subset \mathcal{A}$ is a commutative subalgebra of a W^*-probability space $\mathcal{A}$ and $E\colon \mathcal{A} \to \mathcal{B}$ is a conditional expectation. The covariance $S\colon \mathcal{B} \to \mathcal{B}$ of $\mathfrak{c}$ is defined through $S[b] = E[\mathfrak{c}b\mathfrak{c}^*]$ for all $b \in \mathcal{B}$ and $E[\mathfrak{c}b\mathfrak{c}] = 0$. Higher order $\mathcal{B}$-valued free cumulants of $\mathfrak{c}$ and $\mathfrak{c}^*$ vanish, making it a generalized circular element as introduced in [53]. $\mathcal{B}$-valued circular elements were previously analysed in [30] under the name $\mathcal{B}$-circular elements and were introduced in [31]. For the Brown measure σ of $\mathfrak{c}+a$ we provide a classification in terms of the solution to a nonlinear equation on $\mathcal{B}$. Furthermore, we determine the

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regularity properties of this measure and describe its singularities. We show that under some regularity assumptions on the variance profile the Brown measure σ admits a bounded probability density on the complex plane, which is real analytic and strictly positive on an open domain $\mathbb{S} := \{\beta < 0\} \subset \mathbb{C}$ with boundary $\partial\mathbb{S} = \{\beta = 0\}$, where $\beta : \mathbb{C} \to \mathbb{R}$ is a continuous function that is real analytic in a neighbourhood of $\partial\mathbb{S}$. From this we obtain that $\partial\mathbb{S}$ is a real analytic variety of dimension at most 1. The density of the Brown measure, also denoted by σ, vanishes outside the closure of $\mathbb{S}$ and typically has a jump discontinuity at the spectral edge $\partial\mathbb{S}$, except at critical points of β, where it vanishes continuously. Consequently, we have generalised the previously known results [64, 13, 32] about the density $\sigma_{a+\mathfrak{c}}$, when $\mathfrak{c}$ is a standard circular element, that is, $S = \langle \cdot \rangle$ is the trace of the argument multiplied by the unit in $\mathcal{B}$. In this case our choice of β simplifies to $\beta(\zeta) = \langle |a - \zeta|^{-2} \rangle$, which coincides with the analogous quantity in [42].

Near the zeros or singularities of the density, we prove the following behaviour. A zero ζ lies either at the boundary of the support $\overline{\mathbb{S}}_a$ of the Brown measure, that is, at the spectral edge, or in the interior of $\overline{\mathbb{S}}_a$. In the first case we analyse the local behaviour of the level sets of β around ζ to identify the shape of the edge $\partial\mathbb{S}$ in a neighbourhood of ζ. In appropriately chosen coordinates (x, y) of the plane this shape is of the form $x^2 = y^{k+1}$, forming an edge singularity of order $k \in \mathbb{N}$. In the latter case we classify the singularities in terms of the local shape of the density around its zero ζ. At these internal singularities the density has the form $x^2 + y^{2k}$ with order $k \in \mathbb{N}$ in appropriately chosen coordinates (x, y).

Proof ideas:

All properties of the Brown measure σ and its relationship to β established in this paper in the general setup are ultimately obtained by our analysis of a ζ-dependent system of two coupled $\mathcal{B}$-valued equations, called *Dyson equation*. It describes the diagonal entries $v_1(\zeta)$ and $v_2(\zeta)$ of the matrix in $\mathcal{B}^{2\times 2}$ obtained by applying the conditional expectation E entrywise to the resolvent of the Hermitisation of $a + \mathfrak{c}$. In the random matrix setup, the Hermitisation idea goes back to [35]. See, for example, [14] for its use in the analysis of Brown measures. More explicitly, there are two unique[1] positive functions $v_1, v_2 : \mathbb{S} \to \mathcal{B}$, which satisfy

$$\frac{1}{v_1(\zeta)} = Sv_2(\zeta) + \frac{|\zeta - a|^2}{S^* v_1(\zeta)}, \qquad \frac{1}{v_2(\zeta)} = S^* v_1(\zeta) + \frac{|\zeta - a|^2}{S v_2(\zeta)}, \tag{1.1}$$

for each $\zeta \in \mathbb{S}$. As ζ approaches $\partial\mathbb{S}$ the solution $(v_1(\zeta), v_2(\zeta))$ vanishes. If $\mathfrak{c}$ is a standard circular element, then the $\mathcal{B}$-valued system (1.1) simplifies to a scalar-valued equation. In either case, from v_1 the probability density σ inside $\mathbb{S}$ is derived through

$$\sigma(\zeta) := -\partial_\zeta \left\langle \frac{v_1(\zeta)(a - \zeta)}{\pi S^* v_1(\zeta)} \right\rangle. \tag{1.2}$$

Taking the derivative in (1.2) yields a quadratic form of a nonsymmetric operator. The main idea for the proof of strict positivity of σ in the bulk regime, that is, on $\mathbb{S}$, is to transform the formula for σ into the quadratic form of a strictly positive operator. Near the spectral edge $\partial\mathbb{S}$, the behaviour of σ is governed by the quantity β from the definition of $\mathbb{S}$. In fact, $\beta(\zeta)$ coincides locally around the spectral edge with the isolated eigenvalue of the nonsymmetric operator B_ζ that is closest to zero, where $B_\zeta : \mathcal{B} \to \mathcal{B}$ is defined through $B_\zeta f := |a - \zeta|^2 f - Sf$. A consequence that we derive from this insight is that the jump height of the edge discontinuity of σ at the spectral edge is proportional to $|\partial_\zeta \beta|^2$. This requires a careful singular expansion of v_1, v_2 at the spectral edge, where the Dyson equation (1.1) is unstable. A signature of this instability is that B_ζ is singular for $\zeta \in \partial\mathbb{S}$ and the main contributions to v_1 and v_2 near $\partial\mathbb{S}$ point into the singular eigendirections of B_ζ. Owing to the dependence of v_1, v_2 and β on S, treating nonconstant S and a is a recurring challenge for the analysis in both regimes.

[1]The existence and uniqueness of v_1 and v_2 in (1.1) are proved in Corollary 5.17 below.

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Finally, we employ an iterative scheme to show that for each $k \in \mathbb{N}$ a finite-dimensional commutative subalgebra $\mathcal{B} \subset \mathcal{A}$ and a deformation $a \in \mathcal{B}$ exist, such that the Brown measure of the deformed standard circular element $a + \mathfrak{c}$ exhibits an edge or internal singularity of order k. To perform an iteration step within this scheme, we increase the dimension of $\mathcal{B}$ and add a small carefully chosen perturbation h to a deformation a within the additional dimensions, where a generated a singularity in the Brown measure of $a+\mathfrak{c}$ at the origin. Then by tuning the parameters that determine a and changing it to $\widehat{a} \approx a$ we ensure that $\widehat{a} + h$ now generates a singularity at the origin with a higher order than the one generated by a.

2. Main results

We consider an operator-valued probability space $(\mathcal{A}, E, \mathcal{B})$. That is, $\mathcal{A}$ is a unital von Neumann algebra, $\mathcal{B} \subset \mathcal{A}$ is a von Neumann subalgebra with the same unit and $E\colon \mathcal{A} \to \mathcal{B}$ is a positive conditional expectation, that is, $E[b] = b$ for all $b \in \mathcal{B}$, $E[b_1 \mathfrak{a} b_2] = b_1 E[\mathfrak{a}] b_2$ for all $\mathfrak{a} \in \mathcal{A}$ and $b_1, b_2 \in \mathcal{B}$ as well as $E[\mathfrak{a}]$ is positive for all positive $\mathfrak{a} \in \mathcal{A}$. We also assume that $\mathcal{B}$ is equipped with a faithful tracial state $\langle \cdot \rangle$ such that $(\mathcal{A}, \langle E[\,\cdot\,]\rangle)$ is a tracial W^*-probability space. We call an element $\mathfrak{c} \in \mathcal{A}$ a *$\mathcal{B}$-valued circular element* if it is centred, that is, $E[\mathfrak{c}] = 0$, all mixed free $\mathcal{B}$-cumulants[2] of $\mathfrak{c}$ and $\mathfrak{c}^*$ of any order larger than 2 vanish and if the second order cumulants satisfy

$$E[\mathfrak{c} b \mathfrak{c}] = 0\,, \qquad E[\mathfrak{c}^* b \mathfrak{c}^*] = 0 \tag{2.1}$$

for all $b \in \mathcal{B}$. Define the operators $S\colon \mathcal{B} \to \mathcal{B}$ and $S^*\colon \mathcal{B} \to \mathcal{B}$ through

$$Sb := E[\mathfrak{c} b \mathfrak{c}^*], \qquad S^* b := E[\mathfrak{c}^* b \mathfrak{c}] \tag{2.2}$$

for all $b \in \mathcal{B}$. Note that $S^* b = (S(b^*))^*$ for all $b \in \mathcal{B}$. We call $a + \mathfrak{c}$ with some $a \in \mathcal{B}$ a *deformed $\mathcal{B}$-valued circular element* with deformation a and covariance S.

For $\mathfrak{a} \in \mathcal{A}$ the *Brown measure* $\sigma_{\mathfrak{a}}$ of $\mathfrak{a}$ is the unique compactly supported probability measure on $\mathbb{C}$ with

$$\int_{\mathbb{C}} \log|\zeta - \xi| \sigma_{\mathfrak{a}}(\mathrm{d}\xi) = \log D(\mathfrak{a} - \zeta) \tag{2.3}$$

for all $\zeta \in \mathbb{C}$. Here, $D(\mathfrak{a} - \zeta)$ denotes the *Fuglede-Kadison determinant* of $\mathfrak{a} - \zeta$. The Fuglede-Kadison determinant of an arbitrary $\mathfrak{b} \in \mathcal{A}$ is defined by

$$D(\mathfrak{b}) := \lim_{\varepsilon \downarrow 0} \exp(\langle E[\log(\mathfrak{b}^* \mathfrak{b} + \varepsilon)^{1/2}]\rangle) \in [0, \infty). \tag{2.4}$$

Originally introduced in [22], the Brown measure was revived in [36]. An introduction to the Brown measure and the Fuglede-Kadison determinant can be found for example in [47, Chapter 11].

In the following we will consider the case when $\mathcal{B}$ is commutative. Therefore we may assume that $\mathcal{B} = L^\infty(\mathfrak{X}, \Sigma, \mu)$ is the space of μ-almost everywhere bounded functions with respect to a probability measure μ on $(\mathfrak{X}, \Sigma)$ and $\langle b \rangle = \int_{\mathfrak{X}} b \,\mathrm{d}\mu$ is the trace of $b \in \mathcal{B}$ [55, Chapter III, Theorem 1.18].

We study the Brown measure of a deformed $\mathcal{B}$-valued circular element. To that end, fix $a \in \mathcal{B}$ and let $\mathfrak{c} \in \mathcal{A}$ be a $\mathcal{B}$-valued circular element that satisfies (2.1) and (2.2). Throughout this work we will assume that S and S^* from (2.2) are represented by an integral kernel, that is, that there is a measurable function $s\colon \mathfrak{X} \times \mathfrak{X} \to [0, \infty)$ such that

$$(Su)(x) = \int_{\mathfrak{X}} s(x, y) u(y) \mu(\mathrm{d}y), \qquad (S^* u)(x) = \int_{\mathfrak{X}} s(y, x) u(y) \mu(\mathrm{d}y) \tag{2.5}$$

[2]For the definition of the mixed free $\mathcal{B}$-cumulants, we refer to [47, Definition 7, Chapter 9].

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for $x \in \mathfrak{X}$ and $u \in \mathcal{B}$. Since S and S^* are positive operators and therefore, bounded, after possibly changing s on a $\mu \otimes \mu$-nullset, we assume

$$\sup_{x\in\mathfrak{X}} \int_{\mathfrak{X}} s(x,y)\mu(\mathrm{d}y) < \infty, \qquad \sup_{y\in\mathfrak{X}} \int_{\mathfrak{X}} s(x,y)\mu(\mathrm{d}x) < \infty. \tag{2.6}$$

Under the following regularity assumptions on a and s, we prove detailed information about the Brown measure $\sigma = \sigma_{a+\mathfrak{c}}$ of the deformed $\mathcal{B}$-valued circular operator $a + \mathfrak{c}$. From here on and throughout the paper, we will impose some of the following assumptions.

A1 *Block-primitivity of S*: There is a constant $C > 0$, a primitive matrix $Z = (z_{ij})_{i,j=1}^K \in \{0,1\}^{K\times K}$ with $z_{ii} = 1$ for each i and a measurable partition $(I_i)_{i=1}^K$ of $\mathfrak{X}$ with $\min_i \mu(I_i) \geq \frac{1}{C}$, such that

$$\frac{1}{C} s(x,y) \leq \sum_{i,j=1}^K z_{ij} \mathbb{1}_{I_i\times I_j}(x,y) \leq C s(x,y)$$

for all $x, y \in \mathfrak{X}$.

We recall that a matrix $Z \in [0,\infty)^{K\times K}$ with nonnegative entries is called *primitive* if there is an $L \in \mathbb{N}$ such that all entries of the power Z^L are strictly positive. For a and s as above, we define the function $\Gamma_{a,s}\colon (0,\infty) \to (0,\infty)$ similarly to [6, Section 9] as

$$\Gamma_{a,s}(\tau) := \left(\operatorname{ess\,inf}_{x\in\mathfrak{X}} \int_{\mathfrak{X}} \frac{1}{(\tau^{-1} + |a(x) - a(y)| + d_s(x,y))^2} \mu(\mathrm{d}y)\right)^{1/2}, \tag{2.7}$$

where $d_s(x,y) := \left(\int_{\mathfrak{X}} (|s(x,q) - s(y,q)|^2 + |s(q,x) - s(q,y)|^2)\mu(\mathrm{d}q)\right)^{1/2}$. Note that $\Gamma_{a,s}$ is strictly monotonically increasing.

A2 *Data regularity:* The data a and s satisfy the regularity assumption

$$\lim_{\tau\to\infty} \Gamma_{a,s}(\tau) = \infty\,.$$

Remark 2.1. The following list provides simple conditions that imply our assumptions A1 or A2.

(i) The condition A1 is for example satisfied if S is bounded from below, that is, when

$$\frac{1}{C} \leq s(x,y) \leq C$$

for some constant $C > 0$ and for all $x, y \in \mathfrak{X}$.

(ii) If $\mathfrak{X}$ is a finite set then A2 holds for any a and s.

(iii) In the case $\mathfrak{X} = [0,1]$ and μ the Lebesgue-measure on $[0,1]$, Assumption A2 holds, for example, when s and a are Hölder-continuous with Hölder-exponent $\frac{1}{2}$.

(iv) In fact it suffices that Hölder-continuity of a and s holds piecewise. Let $I_1, \ldots, I_K$ be disjoint intervals in $[0,1]$ such that $I_1 \cup \ldots \cup I_K = [0,1]$. If $s\colon [0,1]\times[0,1] \to [0,\infty)$, $a\colon [0,1] \to \mathbb{C}$ are such that $s|_{I_l\times I_k}$ and $a|_{I_l}$ are $\frac{1}{2}$-Hölder-continuous for every $l, k \in \{1,2,\ldots,K\}$ then A2 is satisfied. In particular, if s satisfies A1 in addition, then A1 and A2 both hold.

To classify the support of σ we introduce the operator $B \equiv B_\zeta : \mathcal{B} \to \mathcal{B}$ given by

$$B_\zeta := D_{|a-\zeta|^2} - S\,, \tag{2.8}$$

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where $D_u\colon \mathcal{B} \to \mathcal{B}, x \mapsto ux$ denotes the multiplication operator on $\mathcal{B}$ with $u \in \mathcal{B}$. Since B maps real-valued functions to real-valued functions, we obtain a function $\beta\colon \mathbb{C} \to \mathbb{R}$ defined through

$$\beta(\zeta) := \inf_{x\in\mathcal{B}_+} \sup_{y\in\mathcal{B}_+} \frac{\langle x\,, B_\zeta\, y\rangle}{\langle x\,, y\rangle} \tag{2.9}$$

for $\zeta \in \mathbb{C}$, where the infimum and supremum are taken over $\mathcal{B}_+ := \{x \in \mathcal{B} : x > 0\}$ and $\langle u_1\,, u_2\rangle := \langle \overline{u}_1 u_2\rangle$ for all $u_1, u_2 \in \mathcal{B}$. The definition of β is motivated by the Birkhoff-Varga formula for the spectral radius of a matrix with positive entries [16]. In terms of β we define the set

$$\mathbb{S} := \{\zeta \in \mathbb{C}\colon \beta(\zeta) < 0\}\,, \tag{2.10}$$

whose closure coincides with $\operatorname{supp}\sigma$, as stated in the next theorem. We will see in Proposition 5.16 (i) that β is a continuous function and therefore $\mathbb{S}$ is an open set. Moreover, β is real analytic in a neighbourhood of $\partial\mathbb{S}$, see Corollary 5.20 below.

Theorem 2.2 (Properties of σ). *Let $a \in \mathcal{B} = L^\infty(\mathfrak{X}, \Sigma, \mu)$ and $\mathfrak{c}$ be a $\mathcal{B}$-valued circular element such that S and S^* from (2.2) satisfy (2.5) with s fulfilling A1 and A2.*

(i) *With respect to the Lebesgue measure, the Brown measure σ of $a + \mathfrak{c}$ has a bounded density on $\mathbb{C}$, which we also denote by σ, that is, $\sigma(\mathrm{d}\zeta) = \sigma(\zeta)\mathrm{d}^2\zeta$.*
(ii) *The density $\zeta \mapsto \sigma(\zeta)$ is strictly positive on $\mathbb{S}$ and admits a real analytic extension to an open neighbourhood of $\overline{\mathbb{S}}$.*
(iii) $\operatorname{supp}\sigma = \overline{\mathbb{S}}$ *and this set is bounded. Furthermore* $\operatorname{Spec}(D_a) \subset \mathbb{S}$.
(iv) $\partial\mathbb{S} = \{\zeta \in \mathbb{C}\colon \beta(\zeta) = 0\}$ *and it is a real analytic variety of (real) dimension at most* 1.
(v) *The unique continuous extension $\sigma\colon \overline{\mathbb{S}} \to [0,\infty)$ of the density $\sigma|_{\mathbb{S}}$ to $\overline{\mathbb{S}}$ satisfies $\sigma(\zeta) = g(\zeta)|\partial_\zeta\beta(\zeta)|^2$ for all $\zeta \in \partial\mathbb{S}$, where $g : \partial\mathbb{S} \to (0,\infty)$ is a strictly positive function that can be extended to a real analytic function on a neighbourhood of $\partial\mathbb{S}$.*

The proof of Theorem 2.2 is given at the end of Section 6 below.

Our next results describe in detail the edge behaviour of σ at points $\zeta_0 \in \partial\mathbb{S}$. Typically we observe a sharp drop of the density σ at the edge $\partial\mathbb{S}$. In fact, by Theorem 2.2 (v) this is the case if and only if $\partial_\zeta\beta(\zeta_0)$ does not vanish.

Definition 2.3 (Regular edge and singular points). Let $\zeta_0 \in \partial\mathbb{S}$ such that $\partial_\zeta\beta(\zeta_0) \neq 0$. Then we say that ζ_0 is a *regular edge point*. We denote the set of these points by Reg and its complement by $\mathrm{Sing} := \partial\mathbb{S} \setminus \mathrm{Reg}$, which is called the set of *singular points* for σ.

To give a short statement of the classification of the singular points for σ we introduce the following notion.

Definition 2.4 (Singularity types). Let $\alpha : U \to \mathbb{C}$ be a real analytic function on an open set $U \subset \mathbb{C}$ and $\zeta_0 \in U$. We say that α is of singularity type $p(x,y)$ at ζ_0 for a polynomial $p : \mathbb{R}^2 \to \mathbb{C}$ if there is a real analytic diffeomorphism $\Phi : V \to U_0$ from an open neighbourhood V of $0 \in \mathbb{C}$ to an open neighbourhood U_0 of $\zeta_0 = \Phi(0)$, such that

$$\alpha(\Phi(x + \mathrm{i}y)) = p(x,y)\,, \qquad x + \mathrm{i}y \in V\,.$$

In the sense above β is of singularity type x at any regular edge point. By Theorem 2.2 the edge $\partial\operatorname{supp}\sigma$ is locally analytically diffeomorphic to a line and σ has a jump discontinuity along this line.

Theorem 2.5 (Classification of edge and internal singularities). *Let $a \in \mathcal{B}$ and s satisfy A1 and A2. The set of singular points for σ admits a partition*

$$\mathrm{Sing} = \bigcup_{k=1}^{\infty} \left(\mathrm{Sing}_k^{\mathrm{int}} \cup \mathrm{Sing}_k^{\mathrm{edge}}\right) \cup \mathrm{Sing}_\infty^{\mathrm{int}}\,. \tag{2.11}$$

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Here $\mathrm{Sing} \setminus \mathrm{Sing}_\infty$ *is a finite set, while* Sing_∞ *is either empty or a finite disjoint union of closed real analytic paths without self-intersections. The different singularity types are characterised by the following properties:*

1) *Internal singularities of order k: Any* $\zeta \in \mathrm{Sing}_k^{\mathrm{int}}$ *is an isolated point of* $\mathbb{C} \setminus \mathbb{S}$ *and* σ *is of singularity type* $x^2 + y^{2k}$ *at* ζ.
2) *Edge singularities of order k: Any* $\zeta \in \mathrm{Sing}_k^{\mathrm{edge}}$ *lies in* $\partial\overline{\mathbb{S}}$ *and* β *is of singularity type* $y^{1+k} - x^2$ *at* ζ.
3) *Internal singularities of order* ∞*: Any* $\zeta \in \mathrm{Sing}_\infty^{\mathrm{int}}$ *lies in the interior of* $\overline{\mathbb{S}}$ *and* σ *is of singularity type* x^2 *at* ζ.

The proof of Theorem 2.5 is presented at the end of Section 6 below.

Remark 2.6. The internal singularities $\mathrm{Sing}_k^{\mathrm{int}} \cup \mathrm{Sing}_\infty^{\mathrm{int}}$ are classified by their local behaviour of the density σ. The edge singularities $\mathrm{Sing}_k^{\mathrm{edge}}$ are equivalently characterised by the local shape of the spectral edge $\partial \operatorname{supp} \sigma$, namely for $\zeta \in \mathrm{Sing}_k^{\mathrm{edge}}$ there is a local analytic diffeomorphism $\Psi\colon U \to V$ from an open neighbourhood V of ζ_0 to an open $U \subset \mathbb{C}$ with $\Psi(\zeta_0) = 0$ such that

$$\Psi(\operatorname{supp} \sigma \cap U) = \{x + \mathrm{i}y \in \mathbb{C} : x^2 - y^{1+k} \geq 0\} \cap V$$

Finally we show that all singularities allowed by the classification in Theorem 2.5 do occur in the simple case when $\mathfrak{c}$ is a standard circular element and $\mathfrak{X} = [0, 1]$ with the Lebesgue measure. A standard circular element satisfies $E[\mathfrak{c}^* b \mathfrak{c}] = E[\mathfrak{c} b \mathfrak{c}^*] = \langle b \rangle$ for all $b \in \mathcal{B}$.

Theorem 2.7 (Existence of all singularity types). *For each of the sets* $\mathrm{Sing}_k^{\mathrm{int}}$, $\mathrm{Sing}_k^{\mathrm{edge}}$ *with* $k \in \mathbb{N}$ *and* $\mathrm{Sing}_\infty^{\mathrm{int}}$, *there is a choice* $a \in L^\infty[0, 1]$ *such that for the Brown measure of* $a + \mathfrak{c}$, *with* $\mathfrak{c}$ *a standard circular element, this set is not empty. With the exception of* $\mathrm{Sing}_\infty^{\mathrm{int}}$, *the deformation a can be chosen to have finite image.*

The proof of Theorem 2.7 can be found just before Section 7.1.

Remark 2.8 (Circular element with general deformation). Although not directly covered by our results stated above, the classification of singularities from Theorem 2.5 also holds for the Brown measure of $a + \mathfrak{c}$, where a is a general operator in a tracial von Neumann algebra and $\mathfrak{c}$ is a standard circular element, which is $*$-free from a. We note that a can be non-normal in general. We define $\mathbb{S} = \{\zeta \in \mathbb{C} \colon f(\zeta) > 1\}$ with

$$f(\zeta) := \lim_{\eta \downarrow 0} \left\langle \frac{1}{(a - \zeta)(a - \zeta)^* + \eta^2} \right\rangle$$

for $\zeta \in \mathbb{C}$. Then the support of the Brown measure σ is $\overline{\mathbb{S}}$ [64] and the measure has a density on $\mathbb{C}$ [13], which is real analytic and strictly positive inside $\mathbb{S}$ [64]. Under the additional assumption

$$\operatorname{Spec}(a) \subset \mathbb{S}, \tag{2.12}$$

the dichotomy between regular edge points, at which the density has a jump discontinuity, and singular points of σ was established in [32]. The singular points $\zeta \in \mathrm{Sing}$ are called quadratic edges in [32] and are classified by satisfying $f(\zeta) = 1$ and $\partial_\zeta f(\zeta) = 0$, while regular edge points $\zeta \in \mathrm{Reg}$ satisfy $f(\zeta) = 1$ and $\partial_\zeta f(\zeta) \neq 0$. Here we provide a classification of the singular points $\zeta \in \mathrm{Sing}$ by showing that Sing is a disjoint union of the sets $\mathrm{Sing}_k^{\mathrm{int}}$, $\mathrm{Sing}_k^{\mathrm{edge}}$ with $k \in \mathbb{N}$ and $\mathrm{Sing}_\infty^{\mathrm{int}}$ as in (2.11), where these sets are defined as in Theorem 2.5 with $\beta := 1 - f$. We provide the proof for this classification in Section 8 below.

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2.1. Notations

We now introduce some notations used throughout. We write $[\![n]\!] := \{1, \dots, n\}$ for $n \in \mathbb{N}$. For $r > 0$, we denote by $\mathbb{D}_r := \{z \in \mathbb{C} \colon |z| < r\}$ the disk of radius r around the origin in $\mathbb{C}$ and by $\operatorname{dist}(x, A) := \inf\{|x - y| \colon y \in A\}$ the Euclidean distance of a point $x \in \mathbb{C}$ from a set $A \subset \mathbb{C}$.

We use the convention that c and C denote generic constants that may depend on the model parameters, but are otherwise uniform in all other parameters, for example, n, ζ, etc.. For two real scalars f and g we write $f \lesssim g$ and $g \gtrsim f$ if $f \le Cg$ for such a constant $C > 0$. In case $f \lesssim g$ and $f \gtrsim g$ both hold, we write $f \sim g$. If the constant C depends on a parameter δ that is not a model parameter, we write $\lesssim_\delta$, $\gtrsim_\delta$ and $\sim_\delta$, respectively. The notation for inequality up to constant is also used for self-adjoint matrices/operators f and g, where $f \le Cg$ is interpreted in the sense of quadratic forms. For complex f and $g \ge 0$ we write $f = O(g)$ in case $|f| \lesssim g$. Analogously $f = O_\delta(g)$ expresses the fact $|f| \lesssim_\delta g$.

In addition to L^∞, we introduce the usual L^p spaces on $(\mathfrak{X}, \mathcal{A}, \mu)$. We denote them by $L^p := L^p(\mathfrak{X}, \mathcal{A}, \mu)$ and the corresponding norms by $\|\cdot\|_p$. For functions $u_1, u_2 \in L^2$, we define their scalar product as

$$\langle u_1, u_2 \rangle := \langle \overline{u}_1 u_2 \rangle .$$

3. Examples

In this section, we present several examples that illustrate some of the different singularity types appearing in Theorem 2.5. Throughout this section, we choose $\mathfrak{X} = [0, 1]$, μ the Lebesgue-measure on $[0, 1]$ and s constant on $[0, 1]^2$. If $s \equiv t$ is constant on $[0, 1]^2$ then it is easy to see that $\beta(\zeta) = \frac{1}{t} - \int_0^1 \frac{\mathrm{d}x}{|a(x)-\zeta|^2}$ for all $\zeta \in \mathbb{C}$ and therefore, as has already been derived in [42, 57, 18],

$$\mathbb{S} = \left\{ \zeta \in \mathbb{C} \colon \int_0^1 \frac{\mathrm{d}x}{|a(x) - \zeta|^2} > \frac{1}{t} \right\}. \tag{3.1}$$

Example 3.1 (Simplest edge singularity of $\partial\mathbb{S}$). We choose $s \equiv 1$ and $a \colon [0, 1] \to \mathbb{C}$ with $a(x_1) = 1$ and $a(x_2) = -1$ for all $x_1 \in [0, 1/2)$ and $x_2 \in [1/2, 1]$. In that case, we obtain from (3.1) that

$$\mathbb{S} = \{\zeta \in \mathbb{C} \colon |1 - \zeta|^{-2} + |1 + \zeta|^{-2} > 1\}.$$

As we will see in Lemma 7.3 (a) below, this yields an edge singularity of order 1, that is, in $\mathrm{Sing}_1^{\mathrm{edge}}$ at zero. The boundary of $\mathbb{S}$ and the eigenvalues of a sampled corresponding random matrix model are depicted in Figure 1a. This example also appeared in [15, Example 5.2].

Example 3.2 (Simplest internal singularity of σ). With the choices $s \equiv 1/4$ and $a \colon [0, 1] \to \mathbb{C}$ with $a(x_1) = (1 + \mathrm{i})/\sqrt{2}$, $a(x_2) = (1 - \mathrm{i})/\sqrt{2}$, $a(x_3) = (-1 + \mathrm{i})/\sqrt{2}$ and $a(x_4) = -(1 + \mathrm{i})/\sqrt{2}$ for all $x_1 \in [0, 1/4)$, $x_2 \in [1/4, 1/2)$, $x_3 \in [1/2, 3/4)$ and $x_4 \in [3/4, 1]$, (3.1) yields

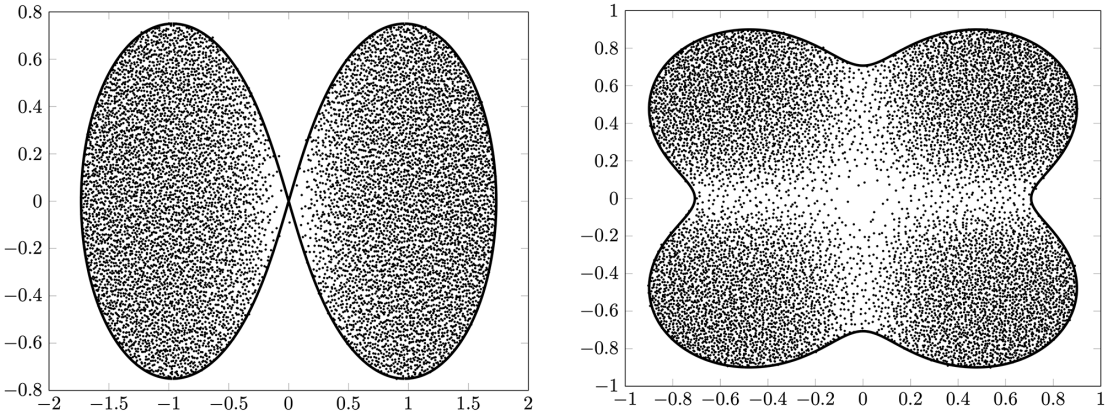
$$\mathbb{S} = \{\zeta \in \mathbb{C} \colon |\zeta - (1 + \mathrm{i})/\sqrt{2}|^{-2} + |\zeta - (1 - \mathrm{i})/\sqrt{2}|^{-2} + |\zeta + (1 - \mathrm{i})/\sqrt{2}|^{-2} + |\zeta + (1 + \mathrm{i})/\sqrt{2}|^{-2} > 1\}.$$

Lemma 7.3 (a) below will demonstrate that this gives rise to an internal singularity of order 1, that is, in $\mathrm{Sing}_2^{\mathrm{int}}$ at zero. Figure 1b shows the boundary of $\mathbb{S}$ and the eigenvalues of a sampled corresponding random matrix model. We note that a similar example was given in [32, Example 3.1(d)].

Example 3.3 (One-sided edge cusp of $\partial\mathbb{S}$). Let $s \equiv t := \frac{2}{3}(20 - 7\sqrt{7})$ be constant on $[0, 1]^2$, $\delta = (-17 + 7\sqrt{7})/8$ and $a \colon [0, 1] \to \mathbb{C}$ with $a(x_1) = 1$, $a(x_2) = -1$ and $a(x_3) = \mathrm{i}$ for all $x_1 \in [0, 1/(2 + \delta))$, $x_2 \in [1/(2 + \delta), 2/(2 + \delta))$ and $x_3 \in [2/(2 + \delta), 1]$. We note that (3.1) yields

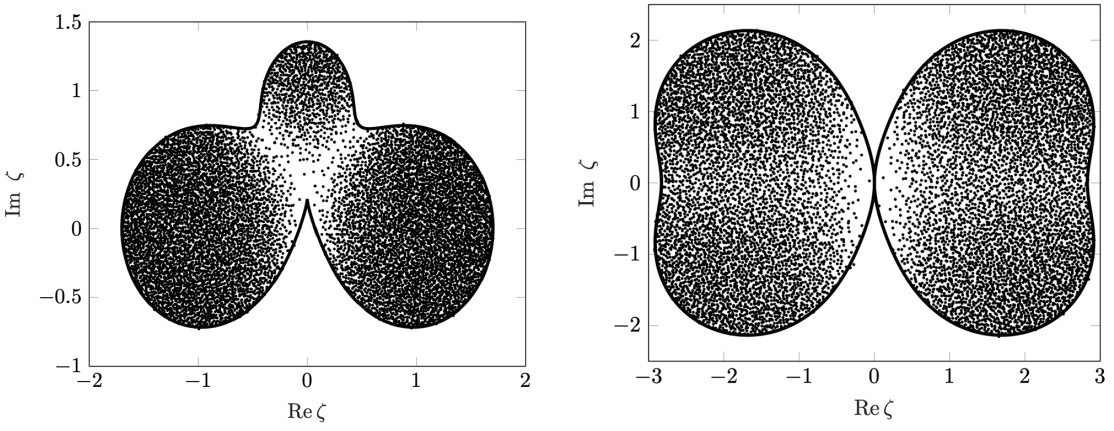
$$\mathbb{S} = \left\{ \zeta \in \mathbb{C} \colon \frac{1}{|1 - \zeta|^2} + \frac{1}{|1 + \zeta|^2} + \frac{\delta}{|\zeta - \mathrm{i}|^2} > \frac{2 + \delta}{t} \right\}.$$

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(a) Example 3.1 (b) Example 3.2

Figure 1. *The solid black lines in subfigures (1a) and (1b) show the boundary of $\mathbb{S}$ from Examples 3.1 and 3.2, respectively. The black dots are the eigenvalues of a sample of $A + X/\sqrt{n}$, where X is an $n \times n$ matrix with i.i.d. $N(0,1)$ standard real normal distributed entries, $n = 10000$ and $A = (\mathrm{diag}(a(i/n))\delta_{ij})_{i,j=1}^{n}$ is a diagonal matrix and a is chosen as in Examples 3.1 and 3.2, respectively.*



(a) Example 3.3 (b) Example 3.4

Figure 2. *The solid black lines in subfigures (2a) and (2b) show the boundary of $\mathbb{S}$ from Examples 3.3 and 3.4, respectively. The black dots are the eigenvalues of a sample of $A + X/\sqrt{n}$, where X is an $n \times n$ matrix with i.i.d. $N(0,1)$ standard real normal distributed entries, $n = 10000$ and $A = (\mathrm{diag}(a(i/n))\delta_{ij})_{i,j=1}^{n}$ is a diagonal matrix and a is chosen as in Examples 3.3 and 3.4, respectively.*

With $y_0 = (\sqrt{7} - 2)/3$, a simple calculation shows that β is of singularity type $x^2 + y^3$ at $\mathrm{i}y_0$ and, thus, $\mathrm{i}y_0$ is an edge singularity of order 2. For this example, a plot analogous to the previous examples is presented in Figure 2a.

Example 3.4 (Two-sided edge cusp). We set $s \equiv t := 4$ to be constant on $[0,1]^2$ and $a\colon [0,1] \to \mathbb{C}$, with $a(x_1) = \sqrt{3} + \mathrm{i}$, $a(x_2) = \sqrt{3} - \mathrm{i}$, $a(x_3) = -\sqrt{3} + \mathrm{i}$ and $a(x_4) = -\sqrt{3} - \mathrm{i}$ for all $x_1 \in [0, 1/4)$, $x_2 \in [1/4, 1/2)$, $x_3 \in [1/2, 3/4)$ and $x_4 \in [3/4, 1]$. We conclude from (3.1) that

$$\mathbb{S} = \left\{ \zeta \in \mathbb{C} \colon \frac{1}{|\sqrt{3} + \mathrm{i} - \zeta|^2} + \frac{1}{|\sqrt{3} - \mathrm{i} - \zeta|^2} + \frac{1}{|-\sqrt{3} + \mathrm{i} - \zeta|^2} + \frac{1}{|-\sqrt{3} - \mathrm{i} - \zeta|^2} > 1 \right\}.$$

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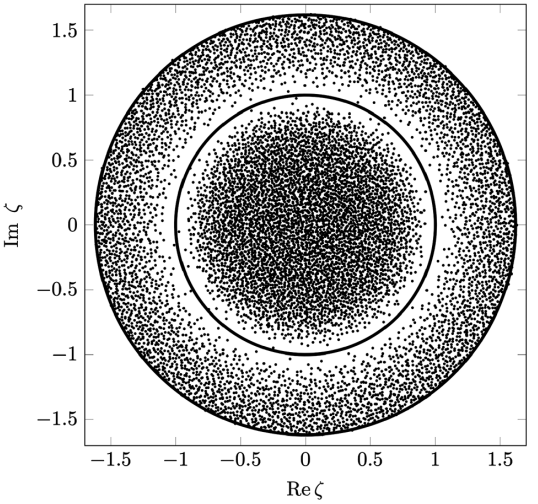


Figure 3. *The solid lines in this figure show the boundary of* $\mathbb{S}$ *from Example 3.5. The black dots show the eigenvalues of a sample of* $A + X/\sqrt{n}$*, where* X *is an* $n \times n$ *matrix with i.i.d.* $N(0,1)$ *standard real normal distributed entries,* $n = 10000$ *and* $A = (\mathrm{diag}(a(i/n))\delta_{ij})_{i,j=1}^{n}$ *is a diagonal matrix and* a *is chosen as in Example 3.5.*

Using Lemma 7.2 below, $-\beta$ and hence the boundary of $\mathbb{S}$ has a singularity of type $x^2 - y^4$ at zero, making this an edge singularity of order 2. Figure 2b shows a plot of this example analogous to the previous figures.

We refer to [7, Example 2.6 and Figure 1] for more examples in the spirit of the previous examples.

Example 3.5 (Internal singularity of type $\mathrm{Sing}_{\infty}^{\mathrm{int}}$). We choose $s \equiv 1$ on $[0,1]^2$ and

$$a\colon [0,1] \to \mathbb{C}, \qquad x \mapsto \begin{cases} \sqrt{2}\mathrm{e}^{4\pi \mathrm{i} x} & \text{if } x \in [0, 1/2], \\ 0 & \text{if } x \in (1/2, 1]. \end{cases}$$

As will be proved in Section 7.3 below, in this setup

$$\mathbb{S} = \left\{\zeta \in \mathbb{C}\colon 0 \le |\zeta|^2 < 1 \text{ or } 1 < |\zeta|^2 < (3+\sqrt{5})/2\right\} \tag{3.2}$$

and each point of $\partial\mathbb{D}_1$ belongs to $\mathrm{Sing}_{\infty}^{\mathrm{int}}$. Moreover, it is shown in Section 7.3 that, for each $\zeta \in \mathbb{C}$,

$$\sigma(\zeta) = \frac{1}{\pi}\left(1 - \frac{2}{2 + x + \frac{x}{(2x-1)^3}}\left(1 + \frac{1}{x^2}\right)\right)\mathbf{1}(\zeta \in \mathbb{S}), \tag{3.3}$$

where $x = x(|\zeta|^2) \in (0,\infty)$ is the unique positive solution of $\frac{1}{x} + \frac{1}{\sqrt{1+4x+x^2-8|\zeta|^2+3}} = 2$. The boundary of $\mathbb{S}$ and the sampled eigenvalues are depicted in Figure 3.

4. Brown measure, Hermitisation and Dyson equation

In this section, we derive the Dyson equation for the Cauchy transform of the Hermitisation of the deformed operator-valued circular element $a + \mathfrak{c}$. This Hermitisation itself is a family of operator-valued semicircular elements indexed by the spectral parameter $\zeta \in \mathbb{C}$ and the associated Dyson equation is our main tool for analysing the Brown measure of $a + \mathfrak{c}$.

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Throughout the following, let $(\mathcal{A}, E, \mathcal{B})$ be an operator-valued probability space with positive conditional expectation $E\colon \mathcal{A} \to \mathcal{B}$ such that $(\mathcal{A}, \langle E[\,\cdot\,]\rangle)$ is a tracial W^*-probability space. We assume that $\mathcal{B} \subset \mathcal{A}$ is a commutative von Neumann subalgebra with trace $\langle\cdot\rangle$ containing the unit of $\mathcal{A}$. We will also work on the operator-valued probability space $(\mathcal{A}^{2\times2}, \mathrm{id}\otimes E, \mathcal{B}^{2\times2})$, for which we employ the identifications $\mathcal{A}^{2\times2} = \mathbb{C}^{2\times2}\otimes\mathcal{A}$ and $\mathcal{B}^{2\times2} = \mathbb{C}^{2\times2}\otimes\mathcal{B}$.

For any $\mathfrak{c}\in\mathcal{A}$, $a\in\mathcal{B}$ and $\zeta\in\mathbb{C}$, we *hermitise* $\mathfrak{c}+a-\zeta$ and apply $\mathrm{id}\otimes E$ to the resolvent of the Hermitisation at $w\in\mathbb{C}$ with $\operatorname{Im} w>0$. We define

$$M(\zeta,w) = \mathrm{id}\otimes E\left[\begin{pmatrix} -w & \mathfrak{c}+a-\zeta \\ (\mathfrak{c}+a-\zeta)^* & -w\end{pmatrix}^{-1}\right]. \tag{4.1}$$

Here, w and ζ are interpreted as the constant functions on $\mathfrak{X}$ with the respective value. Note that the inverse is well defined as $\operatorname{Im} w>0$ and the operator whose inverse is taken in (4.1) is self-adjoint for $w=0$. Furthermore, the imaginary part $\operatorname{Im} M(\zeta,w) = \frac{1}{2\mathrm{i}}(M(\zeta,w)-M(\zeta,w)^*)$ is positive definite.

For the next lemma, we define $\Sigma\colon \mathcal{B}^{2\times2}\to\mathcal{B}^{2\times2}$ through

$$\Sigma\left[\begin{pmatrix} r_{11} & r_{12}\\ r_{21} & r_{22}\end{pmatrix}\right] = \begin{pmatrix} Sr_{22} & 0\\ 0 & S^*r_{11}\end{pmatrix} \tag{4.2}$$

for all $r_{11}, r_{12}, r_{21}, r_{22}\in\mathcal{B}$, where S and S^* are as in (2.2).

Lemma 4.1. *Let $\mathfrak{c}\in\mathcal{A}$ be a $\mathcal{B}$-valued circular element satisfying (2.1) and (2.2). Then $\begin{pmatrix}0 & \mathfrak{c}\\ \mathfrak{c}^* & 0\end{pmatrix}$ is an operator-valued semicircular element in $(\mathcal{A}^{2\times2}, \mathrm{id}\otimes E, \mathcal{B}^{2\times2})$ with covariance Σ from (4.2). Moreover, $M(\zeta,w)$ from (4.1) with such $\mathfrak{c}$ satisfies*

$$-M(\zeta,w)^{-1} = \begin{pmatrix} w & \zeta-a\\ \zeta-a & w\end{pmatrix} + \Sigma[M(\zeta,w)] \tag{4.3}$$

for any $a\in\mathcal{B}$, $w\in\mathbb{C}$ with $\operatorname{Im} w>0$ and $\zeta\in\mathbb{C}$.

The identity (4.3) is called *Dyson equation*. Under the constraint that $\operatorname{Im} M(\zeta,w)$ is positive definite, the Dyson equation has a unique solution [38, Theorem 2.1]. In particular, $M(\zeta,w)$ from (4.1) coincides with this solution. For constant function s the identity (4.3) has appeared in [18] and its analog for elliptic operators in [64].

Proof. An easy computation using the properties of the operator-valued circular element $\mathfrak{c}$, in particular, (2.1) and (2.2), yields the required $\mathcal{B}^{2\times2}$-valued free cumulants of $\mathfrak{s} = \begin{pmatrix}0 & \mathfrak{c}\\ \mathfrak{c}^* & 0\end{pmatrix}$ and shows that $\mathfrak{s}$ is an $\mathcal{B}^{2\times2}$-valued semicircular element, see, for example, [47, Definition 10 2) in Chapter 9].

Owing to (2.1) and (2.2), the covariance of $\mathfrak{s}$ satisfies $E[\mathfrak{s}b\mathfrak{s}] = \Sigma[b]$ for all $b\in\mathcal{B}^{2\times2}$ with Σ from (4.2). Since the R-transform of an operator-valued semicircular element coincides with its covariance, see, for example, [47, Theorem 11. 4) in Chapter 9], $M(\zeta,w)$ as defined in (4.1) satisfies (4.3) with Σ from (4.2) by [47, Theorem 11. 1) in Chapter 9]. □

We now express the Brown measure of $\mathfrak{c}+a$ through M from (4.1) or, equivalently, the solution to (4.3). To that end, it suffices to consider $w=\mathrm{i}\eta$ with $\eta\in(0,\infty)$ and we see that the Dyson equation can be simplified in this case. We consider two coupled equations for functions $v_1, v_2\in\mathcal{B}$ with $v_1>0$ and $v_2>0$, namely

$$\frac{1}{v_1} = \eta + Sv_2 + \frac{|\zeta-a|^2}{\eta+S^*v_1}, \tag{4.4a}$$

$$\frac{1}{v_2} = \eta + S^*v_1 + \frac{|\zeta-a|^2}{\eta+Sv_2}, \tag{4.4b}$$

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for all $\eta > 0$ and $\zeta \in \mathbb{C}$. We also call (4.4) *Dyson equation*. Given v_1 and v_2 from (4.4), we introduce y defined by

$$y := \frac{v_1(\bar{a} - \bar{\zeta})}{\eta + S^* v_1} = \frac{v_2(\bar{a} - \bar{\zeta})}{\eta + S v_2}, \tag{4.5}$$

where the second step follows from

$$v_2(\eta + S^* v_1) = v_1(\eta + S v_2). \tag{4.6}$$

Indeed, for the proof of the last identity, we multiply (4.4a) by $v_1 v_2(\eta + S^* v_1)$ and (4.4b) by $v_1 v_2(\eta + S v_2)$ and conclude (4.6) from the resulting relations.

The next lemma relates the solutions to (4.4) and (4.3) and implies existence and uniqueness to the one of (4.4).

Lemma 4.2. *For any $\zeta \in \mathbb{C}$ and $\eta > 0$, the following holds.*

(i) *Let $(v_1, v_2) \in \mathcal{B}^2$ be a solution of (4.4) with $v_1 > 0$ and $v_2 > 0$. Set*

$$M := \begin{pmatrix} \mathrm{i}v_1 & \overline{y} \\ y & \mathrm{i}v_2 \end{pmatrix} \in \mathcal{B}^{2\times 2} \tag{4.7}$$

with y as in (4.5). Then M solves (4.3) with $w = \mathrm{i}\eta$ and $\operatorname{Im} M$ is positive definite.

(ii) *Let $M = M(\zeta, \mathrm{i}\eta)$ be the solution of (4.3) such that $\operatorname{Im} M$ is positive definite. Then there are v_1, $v_2 \in \mathcal{B}$ such that $v_1 > 0$, $v_2 > 0$, M satisfies (4.7) with y as in (4.5) and (v_1, v_2) satisfy (4.4).*

In particular, there are unique v_1, $v_2 \in \mathcal{B}$ such that $v_1 > 0$ and $v_2 > 0$ and (4.4) holds.

Proof. The proofs of (i) and (ii) are easy computations. Therefore, the existence and uniqueness of the solution to (4.4) follows from the existence and uniqueness of the solution to (4.3), which is a special case of the general existence and uniqueness result [38, Theorem 2.1]. □

When evaluated on the imaginary axis at $w = \mathrm{i}\eta$ with $\eta > 0$, the representation (4.1) of the solution to the Dyson equation (4.3) with positive definite imaginary part implies the trivial bound

$$\|M(\zeta, \mathrm{i}\eta)\| \le \eta^{-1}. \tag{4.8}$$

Indeed, we note that the argument of $\mathrm{id} \otimes E$ in (4.1) with $w = \mathrm{i}\eta$ is bounded in norm by η^{-1} as the resolvent of a self-adjoint operator. Therefore, $\|M(\zeta, \mathrm{i}\eta)\| \le \eta^{-1}$ follows from $\|E\| = \|E(1)\| = 1$ due to the positivity of E and $E(1) = 1$. This completes the proof of (4.8).

Using the solution to (4.4) we characterise the Brown measure in the following proposition as the Laplacian on $\mathbb{C}$ of the function $-\frac{1}{2\pi} W$ defined through

$$W(\zeta) := \int_0^\infty \left(\langle v_1(\zeta, \eta) \rangle - \frac{1}{1+\eta} \right) \mathrm{d}\eta \tag{4.9}$$

for each $\zeta \in \mathbb{C}$, where v_1 is the first function in the solution pair (v_1, v_2) of the Dyson equation (4.4). If A1 holds then the integral in (4.9) exists in the Lebesgue sense, which follows from Lemma 6.1 below. The proposition below therefore connects the Brown measure of $a + \mathfrak{c}$ to v_1.

Proposition 4.3 (Characterisation of Brown measure). *Let σ be the Brown measure of the deformed $\mathcal{B}$-valued circular element $a + \mathfrak{c}$ with $a \in \mathcal{B}$, (2.1) and (2.2). Furthermore, let $v_1, v_2 > 0$ be the solutions of (4.4). If S and s satisfy (2.5) and A1, respectively, then the following holds.*

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(i) *The Brown measure σ is the unique probability measure on $\mathbb{C}$ such that*

$$\int_{\mathbb{C}} f(\zeta)\sigma(\mathrm{d}\zeta) = -\frac{1}{2\pi}\int_{\mathbb{C}} \Delta f(\zeta) W(\zeta)\mathrm{d}^2\zeta \tag{4.10}$$

for all $f \in C_0^2(\mathbb{C})$, where $\mathrm{d}^2\zeta$ denotes the Lebesgue measure on $\mathbb{C}$.

(ii) *The measure σ is also uniquely defined by the identity*

$$\int_{\mathbb{C}} f(\zeta)\sigma(\mathrm{d}\zeta) = \lim_{\eta\downarrow 0}\frac{1}{\pi}\int_{\mathbb{C}} \partial_\zeta f(\zeta)\left\langle \frac{v_1(\zeta,\eta)\,(a-\zeta)}{\eta + S^* v_1(\zeta,\eta)}\right\rangle \mathrm{d}^2\zeta \tag{4.11}$$

for all $f \in C_0^1(\mathbb{C})$.

The proof of Proposition 4.3 is presented at the end of Section 6.1 below. We conclude this section with a short overview of the contents of the remaining paper. The next section provides a detailed analysis of the solution (v_1, v_2) of the Dyson equation as well as a characterisation of $\mathbb{S}$ from (2.10). Section 6 is devoted to the translation of these insights to the analysis of the Brown measure. We establish the existence of all singularity types in Section 7.

5. The Dyson equation and its solution

In this section we analyse the solution v_1, v_2 to the Dyson equation (4.4). We begin by establishing bounds on the solution. Then we determine stability and regularity properties. Finally we use these insights to characterise the set $\mathbb{S}$ from (2.10) and expand v_1 and v_2 around edge points $\zeta_0 \in \partial\mathbb{S}$. We start with some useful identities and a priori estimates.

As S and S^* both act on $\mathcal{B} = L^\infty(\mathfrak{X}, \mu)$ as integral operators in the form (2.5), the identity (4.6) implies $\langle v_2 S^* v_1\rangle = \langle v_1 S v_2\rangle$ and thus

$$\langle v_1\rangle = \langle v_2\rangle\,. \tag{5.1}$$

We conclude from (4.4) and (4.5) that

$$v_1 \le \eta^{-1}, \qquad v_2 \le \eta^{-1}, \qquad |y| \le |a-\zeta|\eta^{-2} \tag{5.2}$$

for all $\zeta \in \mathbb{C}$ and $\eta > 0$. Furthermore, we have the identity

$$y = \frac{v_1(\bar a - \bar\zeta)}{\eta + S^* v_1} = \frac{1}{a-\zeta}(1 - v_1(\eta + S v_2)) = \frac{1}{a-\zeta} - \frac{v_1 v_2}{\bar y} \tag{5.3}$$

for any $\zeta \in \mathbb{C} \setminus \mathrm{Spec}(D_a)$. Here, $\mathrm{Spec}(D_a)$ denotes the spectrum of D_a considered as multiplication operator $\mathcal{B} \to \mathcal{B}$, which coincides with the essential range of a.

Throughout the remainder of this section, we assume that s satisfies A1. This implies

$$\mathbb{1}_{I_i} S w \sim \mathbb{1}_{I_i}\sum_{j=1}^K z_{ij}\langle w \mathbb{1}_{I_j}\rangle\,, \qquad \mathbb{1}_{I_i} S^* w \sim \mathbb{1}_{I_i}\sum_{j=1}^K z_{ji}\langle w \mathbb{1}_{I_j}\rangle \tag{5.4}$$

for all $w \in \mathcal{B}$ with $w \ge 0$ and all $i \in [\![K]\!]$.

5.1. Bounds on the solution

This subsection contains bounds on v_1 and v_2 under varying assumptions on s and a.

Lower bound on v_1 and v_2

We start with some preliminary bounds that require $a \in \mathcal{B}$ and the boundedness of the function s.

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Lemma 5.1 (Preliminary bounds on v_1 and v_2). *If for some constant $C > 0$, $s(x, y) \le C$ for all $x, y \in \mathfrak{X}$ then*

$$\frac{\eta}{\eta^2 + 1 + |\zeta|^2} \lesssim_C v_i \le \frac{1}{\eta} \tag{5.5}$$

uniformly for all $i = 1, 2$, $\zeta \in \mathbb{C}$ and $\eta > 0$.

Proof. The positivity of v_i and η in (4.4) imply $1/v_i \ge \eta$. Hence, the upper bounds in (5.5) follow.

As $s(x, y) \le C$ for all $x, y \in \mathfrak{X}$, we conclude from (4.4) and the upper bound $v_2 \le \eta^{-1}$ that

$$\frac{1}{v_1} \le \eta + C\eta^{-1} + \eta^{-1} 2(\|a\|_\infty^2 + |\zeta|^2),$$

which implies the missing lower bound in (5.5). □

We continue by showing that v_1 and v_2 are bounded from below by their average.

Lemma 5.2. *If s satisfies A1 then*

$$v_1 \gtrsim \langle v_1 \rangle, \qquad v_2 \gtrsim \langle v_2 \rangle \tag{5.6}$$

uniformly for all $\zeta \in \mathbb{C}$ and $\eta > 0$. As a consequence S^* and S act averaging on v_1 and v_2, respectively, that is,

$$S^* v_1 \sim \langle v_1 \rangle, \qquad S v_2 \sim \langle v_2 \rangle. \tag{5.7}$$

The following proof is based on identifying (v_1, v_2) as solution to a variational problem. In a similar context, such a strategy was also used in [2, Section 6.2 and Appendix A.4] and [4, Lemmas 3.11 and 3.17].

Proof. For the proof we rephrase (4.4) as a variational problem. This will help us to make use of the assumption A1. For $\eta > 0$, we define $J_\eta \colon \mathcal{B}_+ \times \mathcal{B}_+ \to \mathbb{R} \cup \{+\infty\}$, a functional which is minimised by (v_1, v_2), by

$$J_\eta(x_1, x_2) := \langle x_1 S x_2 \rangle + \eta \langle x_1 + x_2 \rangle - \big\langle (1 - u_\eta) \log x_1 \big\rangle - \big\langle (1 - u_\eta) \log x_2 \big\rangle,$$

where

$$u_\eta := \frac{|a - \zeta|^2 v_1}{\eta + S^* v_1} = \frac{|a - \zeta|^2 v_2}{\eta + S v_2}$$

by (4.6). Note that $1 - u_\eta = v_1(\eta + S v_2) \in \mathcal{B}_+$ by (4.4).

Next, we prove that $(x_1, x_2) = (v_1(\eta), v_2(\eta))$ is a minimiser for J_η. Indeed, the functional J_η satisfies the lower bounds

$$\begin{aligned} J_\eta(x_1, x_2) &\ge \langle \eta x_1 - \|u_\eta\|_\infty \mathbb{1}(x_1 \ge 1) \log x_1 \rangle + \langle \eta x_2 - \|u_\eta\|_\infty \mathbb{1}(x_2 \ge 1) \log x_2 \rangle \\ &\quad - (1 - \|u_\eta\|_\infty)(\langle \log x_1 \rangle + \langle \log x_2 \rangle) \\ &\ge \|u_\eta\|_\infty \beta_1 \left(\eta \frac{\alpha_1}{\beta_1} - \log \frac{\alpha_1}{\beta_1} \right) + (1 - \|u_\eta\|_\infty)(\eta \langle x_1 \rangle - \langle \log x_1 \rangle) \\ &\quad + \|u_\eta\|_\infty \beta_2 \left(\eta \frac{\alpha_2}{\beta_2} - \log \frac{\alpha_2}{\beta_2} \right) + (1 - \|u_\eta\|_\infty)(\eta \langle x_2 \rangle - \langle \log x_2 \rangle), \end{aligned}$$

with $\alpha_i := \langle x_i \mathbb{1}(x_i \ge 1) \rangle$ and $\beta_i := \langle \mathbb{1}(x_i \ge 1) \rangle$. Here, in the second step, we applied Jensen's inequality with the expectation $f \mapsto \beta_i^{-1} \langle \mathbb{1}(x_i \ge 1) f \rangle$, that is, $\langle \mathbb{1}(x_i \ge 1) \log x_i \rangle \le \beta_i \log \alpha_i / \beta_i$. Since $\langle \log x_i \rangle \le \log \langle x_i \rangle$ by Jensen's inequality, the functional J_η is bounded from below for any

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$\eta > 0$. Therefore, $\iota_\eta = \inf_{(x_1,x_2)\in\mathcal{B}_+\times\mathcal{B}_+} J_\eta(x_1,x_2) > -\infty$. Moreover, (v_1, v_2) satisfies the corresponding Euler-Lagrange equations and J_η is strictly convex due to $1 - u_\eta = v_1(\eta + Sv_2)$ and (5.5). Hence, $J_\eta(v_1, v_2) = \iota_\eta$.

In the regime $\eta \gg 1$, that is, there is a constant $\eta_0 > 0$ depending only on the model parameters such that for all $\eta \geq \eta_0$, we deduce from (4.4) and the bound $v_i \leq 1/\eta$ in Lemma 5.1 that

$$v_i \sim \frac{\eta}{|\zeta|^2 + \eta^2},$$

which implies $v_i \sim \langle v_i \rangle$. Thus, we restrict to $\eta \lesssim 1$. In case $\eta \lesssim 1$ and $|\zeta| \gg 1$ we get

$$v_i \sim \frac{\eta}{|\zeta|^2},$$

and therefore $v_i \sim \langle v_i \rangle$. Hence, we restrict to $\eta \lesssim 1$ and $|\zeta| \lesssim 1$ in the following. As (v_1, v_2) is a minimiser of J_η, the value $J_\eta(v_1, v_2)$ is bounded by

$$J_\eta(v_1, v_2) \leq J_\eta(1, 1) = 2\eta + \langle S1 \rangle .$$

With the notations $c_i := \mu(I_i)$, $\langle x \rangle_i := \frac{1}{c_i}\langle x \mathbb{1}_{I_i} \rangle$ and $z_{ij} := \mathbb{1}(s|_{I_i\times I_j} \neq 0)$, we see by Jensen's inequality that

$$\begin{aligned}
1 &\gtrsim J_\eta(v_1, v_2) \\
&= \langle v_1 S v_2 \rangle - \sum_j c_j \big\langle (1-u)\log v_1 \big\rangle_j - \sum_j c_j \big\langle (1-u)\log v_2 \big\rangle_j + \eta\langle v_1 + v_2 \rangle \\
&\geq c \sum_{i,j} z_{ij} \langle v_1 \rangle_i \langle v_2 \rangle_j - \sum_j c_j \langle 1-u \rangle_j \log \frac{\langle (1-u)v_1 \rangle_j \langle (1-u)v_2 \rangle_j}{\langle 1-u \rangle_j^2} \\
&\geq c \sum_{i,j} z_{ij} \langle v_1 \rangle_i \langle v_2 \rangle_j - \sum_j c_j \langle 1-u \rangle_j \log \big(\langle v_1 \rangle_j \langle v_2 \rangle_j \big) + 2 \sum_j c_j \langle 1-u \rangle_j \log \langle 1-u \rangle_j \\
&\geq \sum_j \varphi_j(\chi_j) - \frac{2}{\mathrm{e}} \sum_j c_j ,
\end{aligned}$$

where in the last step we used $z_{jj} = 1$ for all $j \in [\![K]\!]$ and defined

$$\chi_j := \langle v_1 \rangle_j \langle v_2 \rangle_j , \qquad \varphi_j(\chi) := c\chi - c_j \mathbb{1}(\chi \leq 1)(1 - \langle u \rangle_j) \log \chi - \mathbb{1}(\chi > 1) \log \chi$$

for all $j \in [\![K]\!]$ and $\chi \in (0, \infty)$. For all $j \in [\![K]\!]$, we infer $\chi_j \lesssim 1$ and

$$\chi_j \geq \exp\left(-\frac{C}{1 - \langle u \rangle_j} \right) = \exp\left(-\frac{C}{\langle v_1(\eta + Sv_2) \rangle_j} \right) \tag{5.8}$$

for some positive constant $C \sim 1$. We rewrite the Dyson equation (4.4) in the form

$$v_1 = \frac{\eta + S^* v_1}{|a - \zeta|^2 + (\eta + S^* v_1)(\eta + Sv_2)} , \qquad v_2 = \frac{\eta + Sv_2}{|a - \zeta|^2 + (\eta + S^* v_1)(\eta + Sv_2)} . \tag{5.9}$$

Now we set $x := (\eta + S^* v_1)(\eta + Sv_2)$. In particular, we have

$$\mathbb{1}_{I_i} x \sim \mathbb{1}_{I_i} \left(\eta^2 + \sum_{j,k} z_{ij} z_{ki} \langle v_1 \rangle_k \langle v_2 \rangle_j \right) \sim \mathbb{1}_{I_i} \langle x \rangle_i \tag{5.10}$$

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due to (5.4). Suppose now that $\langle x\rangle_i \gg 1$ for an $i \in [\![K]\!]$. Then by taking the $\langle\cdot\rangle_i$-average of (5.9), multiplying the two equations and using (5.4) and (5.10) as well as $\langle x\rangle_i \gg 1$ we get

$$\chi_i \sim \frac{1}{\langle x\rangle_i} \ll 1\,.$$

On the other hand by multiplying the first equation in (5.9) with $\eta + Sv_2$ we find

$$\langle v_1(\eta + Sv_2)\rangle_i \sim 1$$

as $\langle x\rangle_i \gg 1$. This contradicts (5.8) and $\chi_i \ll 1$ and we conclude $\langle x\rangle_i \lesssim 1$ for all $i \in [\![K]\!]$. With (5.9) and (5.10), this implies

$$v_1 \gtrsim S^* v_1\,, \qquad v_2 \gtrsim Sv_2\,.$$

Iterating these inequalities yields $v_1 \gtrsim \langle v_1\rangle$ and $v_2 \gtrsim \langle v_2\rangle$, that is, (5.6), since $(z_{ij})_{i,j}$ is primitive by A1. Since $z_{ii} = 1$ for all $i \in [\![K]\!]$, we have $S1 \gtrsim 1$ and $S^*1 \gtrsim 1$. This implies $Sv_2 \gtrsim \langle v_2\rangle$ and $S^*v_1 \gtrsim \langle v_1\rangle$, respectively. The upper bounds on Sv_2 and S^*v_1 in (5.7) follow from the upper bound on s in A1. □

Bound in L^2-norm:
Now we show a bound with respect to the norm on L^2.

Lemma 5.3. *If s satisfies A1 then*

$$\langle v_1^2\rangle + \langle v_2^2\rangle + \langle |y|^2\rangle \lesssim 1 \tag{5.11}$$

uniformly for all $\zeta \in \mathbb{C}$ and $\eta > 0$.

Proof. We multiply the first relation in (4.4a) by v_1^2 and estimate $v_1 \geq v_1^2 Sv_2 \gtrsim v_1^2\langle v_2\rangle = v_1^2\langle v_1\rangle$ due to (5.7) and (5.1). Averaging this estimate and using $\langle v_1\rangle > 0$ yields $\langle v_1^2\rangle \lesssim 1$. The bound $\langle v_2^2\rangle \lesssim 1$ is proved analogously. From (4.5), we conclude

$$|y|^2 = \frac{v_1^2|a-\zeta|^2}{(\eta + S^*v_1)^2} \leq \frac{v_1^2|a-\zeta|^2}{(\eta + S^*v_1)^2} + \frac{v_1^2(\eta + Sv_2)}{\eta + S^*v_1} = \frac{v_1}{\eta + S^*v_1}, \tag{5.12}$$

where we used (4.4) in the last step. Hence, (5.7) implies

$$\langle |y|^2\rangle \lesssim \frac{\langle v_1\rangle}{\eta + \langle v_1\rangle} \leq 1.$$

□

Corollary 5.4. *Let $a \in \mathcal{B}$ and s satisfy A1. Then*

$$\frac{\eta + \langle v_i\rangle}{1 + \eta^2 + |\zeta|^2} \lesssim v_i \lesssim \frac{1}{\eta + \langle v_i\rangle}\,, \qquad i = 1, 2\,.$$

Proof. We only consider the case $i = 1$. The case $i = 2$ follows analogously. For the lower bound, we start from (4.4) and get

$$v_1 = \frac{\eta + S^*v_1}{(\eta + S^*v_1)(\eta + Sv_2) + |\zeta - a|^2} \gtrsim \frac{\eta + \langle v_1\rangle}{\eta^2 + \langle v_1\rangle^2 + |\zeta|^2 + \|a\|_\infty^2} \gtrsim \frac{\eta + \langle v_1\rangle}{1 + \eta^2 + |\zeta|^2}\,,$$

using $\langle v_1\rangle = \langle v_2\rangle$ by (5.1), (5.11) and (5.7). For the upper bound, (4.4), $\langle v_1\rangle = \langle v_2\rangle$ and (5.7) imply

$$v_1 \leq \frac{1}{\eta + Sv_2} \sim \frac{1}{\eta + \langle v_1\rangle}\,.$$

□

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Bound in supremum norm:
Under stronger assumptions on s and a, we can also get a bound on v_i in the L^∞-norm. Let $\Gamma_{a,s}$ be as defined in (2.7).

Lemma 5.5. *Assuming the upper bound* $\max\{\|v_1\|_2, \|v_2\|_2\} \leq \Lambda$ *for some* $\Lambda \geq 1$ *and* $4\Lambda^2 < \lim_{\tau\to\infty} \Gamma_{a,s}(\tau)$*, then we have the bound*

$$\max\{\|v_1\|_\infty, \|v_2\|_\infty\} \leq \frac{\Gamma_{a,s}^{-1}(4\Lambda^2)}{\Lambda}.$$

We remark that if s satisfies A1, then by Lemma 5.3 a $\Lambda \geq 1$ with $\Lambda \sim 1$ exists such that $\max\{\|v_1\|_2, \|v_2\|_2\} \leq \Lambda$ holds uniformly for all $\eta > 0$ and $\zeta \in \mathbb{C}$.

Proof. We specialise the proof of [6, Lemma 9.1] to our setting. □

Corollary 5.6. *Let* $a \in \mathcal{B}$*. If* s *and* a *satisfy A1 and A2 then*

$$\|v_1\|_\infty + \|v_2\|_\infty \lesssim 1$$

uniformly for all $\zeta \in \mathbb{C}$ *and* $\eta > 0$*.*

Scaling relations
Lemma 5.7. *Let* $a \in \mathcal{B}$ *and* s *satisfy A1 and A2. Then the following holds.*

(i) *Uniformly for all* $\zeta \in \mathbb{C}$ *and* $\eta > 0$*, we have*

$$v_1 \sim \langle v_1\rangle = \langle v_2\rangle \sim v_2, \qquad |y| \lesssim 1.$$

(ii) *For any sufficiently small positive constant* $c \sim 1$ *the inequalities* $\eta + \langle v_1(\zeta,\eta)\rangle \leq c$ *and* $|\zeta| \leq 1/c$ *imply* $|\zeta - a| \sim 1$ *and* $|y| \sim 1$*.*

Before going into the proof of Lemma 5.7, we remark that if $a \in \mathcal{B}$ and s satisfies A1 then

$$\|v_1(\zeta,\eta) - (1+\eta)^{-1}\| \lesssim (1+|\zeta|)\eta^{-2} \tag{5.13}$$

uniformly for $\eta > 0$ and $\zeta \in \mathbb{C}$. Indeed, for $\eta \in (0,1]$, (5.13) is a trivial consequence of (5.2). For $\eta \geq 1$, (5.13) follows by inverting (4.4a), subtracting $(1+\eta)^{-1}$ on both sides and estimating the right hand side using (5.7) and Lemma 5.3.

Proof. We first prove that $v_1 \sim \langle v_1\rangle = \langle v_2\rangle \sim v_2$. As s satisfies A1, equation (4.4), $v_1, v_2 > 0$ and (5.7) imply

$$v_1 \sim v_2\,. \tag{5.14}$$

Hence, it suffices to show $v_1 \sim \langle v_1\rangle$ due to (5.1).

From (5.13), we conclude that $v_1 \sim (1+\eta)^{-1}$ and $\langle v_1\rangle \sim (1+\eta)^{-1}$ uniformly for $\eta \gtrsim 1$ and $|\zeta| \lesssim 1$. This proves Lemma 5.7 in that regime. If, on the other hand, $|\zeta| \geq \|a\|_\infty + 1$ then $|\zeta - a| \sim |\zeta|$. Hence, for such ζ, we conclude from (4.4a) and (5.7) that

$$\frac{1}{v_1} \sim \eta + \langle v_2\rangle + \frac{|\zeta|^2}{\eta + \langle v_1\rangle}.$$

As the right hand side is a constant function on $\mathfrak{X}$, we obtain $v_1 \sim \langle v_1\rangle$ if $|\zeta| \geq \|a\|_\infty + 1$.

Hence, it remains to consider $|\zeta| \lesssim 1$ and $\eta \lesssim 1$. In particular, $|\zeta - a| \lesssim 1$ as $a \in \mathcal{B}$. Thus, (4.4a), (5.7) and (5.1) imply

$$\eta + \langle v_1\rangle \sim v_1((\eta + \langle v_1\rangle)^2 + |\zeta - a|^2). \tag{5.15}$$

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Together with Lemma 5.3, this yields $\eta + \langle v_1\rangle \lesssim v_1$. We conclude $v_1 \gtrsim \langle v_1\rangle$ and $v_1 \gtrsim \eta$ as well as $\langle v_1\rangle \gtrsim \eta$. If $|\zeta - a| \geq c$ for any $c \sim 1$ then $v_1 \lesssim \eta + \langle v_1\rangle \sim \langle v_1\rangle$ by (5.15). Therefore we conclude $v_1 \sim \langle v_1\rangle$ if $|\zeta - a| \geq c$. What remains is the case $|\zeta - a| \leq c$ and $\eta \leq c$ for some constant $c \sim 1$. As $v_1 \lesssim 1$ by A2 and Lemma 5.5, we conclude from (5.15) that $1 \gtrsim \eta + \langle v_1\rangle$ or $1 \gtrsim \frac{|\zeta-a|^2}{\eta+\langle v_1\rangle}$. In the second case, (5.15) implies $\langle v_1\rangle \lesssim \eta + \langle v_1\rangle \sim v_1(|\zeta - a|^4 + |\zeta - a|^2)$. Using $|\zeta - a| \leq c$, choosing $c \sim 1$ sufficiently small and averaging $\langle v_1\rangle \lesssim v_1(|\zeta - a|^4 + |\zeta - a|^2)$ yield a contradiction as $\langle v_1\rangle > 0$. Hence, $\langle v_1\rangle \gtrsim 1$ and, thus, $\langle v_1\rangle \sim 1$ by Lemma 5.3 as well as $1 \gtrsim v_1$ by (5.15) as $\eta \leq c$ for some small enough $c \sim 1$. Since $v_1 \lesssim 1$ by Lemma 5.5, this completes the proof of $v_1 \sim \langle v_1\rangle$ uniformly for $\zeta \in \mathbb{C}$ and $\eta > 0$. This completes the proof of (i).

For the proof of (ii), from $v_1 \sim \langle v_1\rangle$, (5.7) and (5.12), we conclude $|y| \lesssim 1$ uniformly for $\zeta \in \mathbb{C}$ and $\eta > 0$. Owing to (5.15) and $v_1 \sim \langle v_1\rangle$, we have $\eta + \langle v_1\rangle \sim \langle v_1\rangle(\eta + \langle v_1\rangle)^2 + \langle v_1\rangle|\zeta - a|^2$. As $\eta + \langle v_1\rangle \leq c$, by choosing $c \sim 1$ sufficiently small, we can incorporate $\langle v_1\rangle(\eta + \langle v_1\rangle)^2$ into the left-hand side and obtain $\langle v_1\rangle \lesssim |\zeta - a|^2\langle v_1\rangle$. Hence, $1 \lesssim |\zeta - a|$ as $\langle v_1\rangle > 0$ for $\eta > 0$. The bound $|y| \sim 1$ follows from (5.3). This proves the additional statement and completes the proof of Lemma 5.7. □

5.2. A relation between the derivatives of M

In this subsection, we restrict the Dyson equation (4.3) to the imaginary axis $w = \mathrm{i}\eta$ for $\eta > 0$, that is, we consider the solution $M = M(\zeta, \mathrm{i}\eta)$ of

$$-M^{-1} = \begin{pmatrix} \mathrm{i}\eta & \zeta - a \\ \overline{\zeta - a} & \mathrm{i}\eta \end{pmatrix} + \Sigma[M]. \tag{5.16}$$

We take derivatives of M with respect to η, ζ and $\bar{\zeta}$. We establish a useful relation between these derivatives in the next lemma. Here, we use the notation

$$\langle R\rangle := \frac{1}{2}(\langle r_{11}\rangle + \langle r_{22}\rangle), \qquad R = \begin{pmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{pmatrix} \in \mathcal{B}^{2\times 2}.$$

Lemma 5.8. *In the setup of Lemma 4.1 we have*

$$\langle \partial_\eta M_{21}(\zeta, \mathrm{i}\eta)\rangle = 2\mathrm{i}\langle \partial_\zeta M(\zeta, \mathrm{i}\eta)\rangle, \qquad \langle \partial_\eta M_{12}(\zeta, \mathrm{i}\eta)\rangle = 2\mathrm{i}\langle \partial_{\bar\zeta} M(\zeta, \mathrm{i}\eta)\rangle \tag{5.17}$$

for every $\zeta \in \mathbb{C}$ and $\eta > 0$, where we decomposed

$$M(\zeta, \mathrm{i}\eta) = \begin{pmatrix} M_{11}(\zeta, \mathrm{i}\eta) & M_{12}(\zeta, \mathrm{i}\eta) \\ M_{21}(\zeta, \mathrm{i}\eta) & M_{22}(\zeta, \mathrm{i}\eta) \end{pmatrix}.$$

Here, the derivatives of M are taken entrywise. Before proving Lemma 5.8, we note that the Schur complement formula implies

$$\begin{pmatrix} -w & \mathfrak{d} \\ \mathfrak{d}^* & -w \end{pmatrix}^{-1} = \begin{pmatrix} w(\mathfrak{d}\mathfrak{d}^* - w^2)^{-1} & \mathfrak{d}(\mathfrak{d}^*\mathfrak{d} - w^2)^{-1} \\ (\mathfrak{d}^*\mathfrak{d} - w^2)^{-1}\mathfrak{d}^* & w(\mathfrak{d}^*\mathfrak{d} - w^2)^{-1} \end{pmatrix} \tag{5.18}$$

for any $\mathfrak{d} \in \mathcal{A}$ and any $w \in \mathbb{C}$ with $\operatorname{Im} w > 0$.

Proof of Lemma 5.8. Throughout the following, we set $|\mathfrak{d}| = \sqrt{\mathfrak{d}^*\mathfrak{d}}$ for any $\mathfrak{d} \in \mathcal{A}$. Let $\eta \in (0, \infty)$ and $\zeta \in \mathbb{C}$. Owing to (2.4) and the invertibility of $|a + \mathfrak{c} - \zeta|^2 + \eta^2$, we have

$$\log D(|a + \mathfrak{c} - \zeta|^2 + \eta^2) = \langle E[\log(|a + \mathfrak{c} - \zeta|^2 + \eta^2)]\rangle.$$

In particular, the right-hand side is infinitely often differentiable with respect to ζ, $\overline{\zeta}$ and η if $\eta > 0$.

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Differentiating with respect to ζ, $\overline{\zeta}$ and η yield

$$\partial_\zeta \log D(|a+\mathfrak{c}-\zeta|^2+\eta^2) = -\langle E[(|a+\mathfrak{c}-\zeta|^2+\eta^2)^{-1}(a+\mathfrak{c}-\zeta)^*]\rangle = -\langle M_{21}(\zeta,\eta)\rangle, \tag{5.19a}$$

$$\partial_{\bar\zeta} \log D(|a+\mathfrak{c}-\zeta|^2+\eta^2) = -\langle E[(|a+\mathfrak{c}-\zeta|^2+\eta^2)^{-1}(a+\mathfrak{c}-\zeta)]\rangle = -\langle M_{12}(\zeta,\eta)\rangle, \tag{5.19b}$$

$$\partial_\eta \log D(|a+\mathfrak{c}-\zeta|^2+\eta^2) = 2\eta\langle E[(|a+\mathfrak{c}-\zeta|^2+\eta^2)^{-1}]\rangle = -2\mathrm{i}\langle M_{22}(\zeta,\eta)\rangle. \tag{5.19c}$$

Here, we used in the third steps that, for all $\eta > 0$ and $\zeta \in \mathbb{C}$, (4.1) and (5.18) imply

$$M(\zeta,\mathrm{i}\eta) = \begin{pmatrix} M_{11}(\zeta,\mathrm{i}\eta) & M_{12}(\zeta,\mathrm{i}\eta) \\ M_{21}(\zeta,\mathrm{i}\eta) & M_{22}(\zeta,\mathrm{i}\eta) \end{pmatrix} = \begin{pmatrix} \mathrm{i}\eta E[(\mathfrak{d}\mathfrak{d}^*+\eta^2)^{-1}] & E[\mathfrak{d}(\mathfrak{d}^*\mathfrak{d}+\eta^2)^{-1}] \\ E[(\mathfrak{d}^*\mathfrak{d}+\eta^2)^{-1}\mathfrak{d}^*] & \mathrm{i}\eta E[(\mathfrak{d}^*\mathfrak{d}+\eta^2)^{-1}] \end{pmatrix}$$

with $\mathfrak{d} = a+\mathfrak{c}-\zeta$. As $\langle M_{22}\rangle = \langle v_2\rangle = \langle v_1\rangle = \langle M_{11}\rangle$ by (5.1), Lemma 5.8 follows from (5.19) due to the exchangeability of the derivatives. □

Let $\mathcal{L}$ be the stability operator of (5.16), defined as

$$\mathcal{L}\colon \mathcal{B}^{2\times 2} \to \mathcal{B}^{2\times 2}, \quad R \mapsto \mathcal{L}[R] := M^{-1}RM^{-1} - \Sigma[R]. \tag{5.20}$$

This operator is invertible for any $\zeta \in \mathbb{C}$ and $\eta > 0$ due to Lemma B.1 below. Therefore, the implicit function theorem applied to (5.16) and simple computations show that

$$\partial_\zeta M = \mathcal{L}^{-1}[E_{12}], \qquad \partial_{\bar\zeta} M = \mathcal{L}^{-1}[E_{21}], \qquad \partial_\eta M = \mathrm{i}\mathcal{L}^{-1}[E_+] \tag{5.21}$$

for all $\eta > 0$ and $\zeta \in \mathbb{C}$, where we used the notations E_{12}, E_{21} and E_+ for the elements of $\mathcal{B}^{2\times 2}$ defined through

$$E_{12} := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad E_{21} := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad E_+ := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{5.22}$$

5.3. Stability of Dyson equation and analyticity of its solution

In this section we show how the solution v_1, v_2 of (4.4) can be extended to $\eta = 0$. The characterisation of the Brown measure in Proposition 4.3 in combination with Lemma 4.2 shows how the Brown measure σ relates to the solution of (4.4) at the origin, that is, at $\eta = 0$. Now we introduce a deterministic analog of the ε-pseudospectrum. For this purpose we note that the map $w \mapsto \langle M(\zeta,w)\rangle$ is the Stieltjes transform of a probability measure on $\mathbb{R}$. Indeed, [6, Proposition 2.1 and Definition 2.2] show that

$$\langle M(\zeta,w)\rangle = \int_{\mathbb{R}} \frac{\rho_\zeta(\mathrm{d}\tau)}{\tau - w} \tag{5.23}$$

holds for all $w \in \mathbb{C}$ with $\operatorname{Im} w > 0$ and a unique probability measure ρ_ζ whose support satisfies $\operatorname{supp}\rho_\zeta \subset \big(\operatorname{Spec}(D_{|a-\zeta|}) \cup \operatorname{Spec}(-D_{|a-\zeta|})\big) + [-2\|S\|_\infty^{1/2}, 2\|S\|_\infty^{1/2}]$.

Definition 5.9. Let ρ_ζ be the unique probability measure on $\mathbb{R}$ for which (5.23) holds. Through ρ_ζ we define

$$\mathbb{S}_\varepsilon := \{\zeta \in \mathbb{C} : \operatorname{dist}(0, \operatorname{supp}\rho_\zeta) \le \varepsilon\} \tag{5.24}$$

for any $\varepsilon \ge 0$.

Remark 5.10. The sets $\mathbb{S}_\varepsilon$ defined in (5.24) are monotonously nondecreasing in $\varepsilon \ge 0$, that is, $\mathbb{S}_{\varepsilon_1} \subset \mathbb{S}_{\varepsilon_2}$ if $\varepsilon_1 \le \varepsilon_2$. Moreover, they are bounded, in fact, $\mathbb{S}_\varepsilon \subset \{\zeta \in \mathbb{C} : |\zeta| \le \varepsilon + \|a\|_\infty + 2(\|S\|_\infty)^{1/2}\}$ for all $\varepsilon \ge 0$ as a consequence of [6, Proposition 2.1]. Here, $\|S\|_\infty$ denotes the operator norm of S viewed as an operator from $\mathcal{B}$ to $\mathcal{B}$.

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If we stay away from $\mathbb{S}_\varepsilon$, then the solution is extended to $\eta = 0$ by setting $v_i = 0$ by the following lemma.

Lemma 5.11. *Let $\varepsilon > 0$. Let $\zeta \in (\mathbb{C} \setminus \mathbb{S}_\varepsilon) \cap \mathbb{D}_{1/\varepsilon}$. Then $v_i(\zeta, \eta) \sim_\varepsilon \eta$ for all $\eta \in (0, 1]$ and $i = 1, 2$. In particular, v_i is continuously extended to $\zeta \in \mathbb{C} \setminus \mathbb{S}_0$ and $\eta = 0$ by setting $v_i(\zeta, 0) := 0$.*

Proof. From (4.4a), we conclude $v_1(|\zeta - a|^2 + (\eta + Sv_2)(\eta + S^*v_1)) = \eta + S^*v_1 \geq \eta$. Thus, $|\zeta| \leq \varepsilon^{-1}$, $a \in \mathcal{B}, \eta \leq 1$, (5.7) and Lemma 5.3 imply $v_1 \gtrsim_\varepsilon \eta$ for all $\eta \in (0, 1]$. Similarly, $v_2 \gtrsim_\varepsilon \eta$ for all $\eta \in (0, 1]$. On the other hand, as $\zeta \in \mathbb{C} \setminus \mathbb{S}_\varepsilon$, the statement (v) of [6, Lemma D.1] holds for $\tau = 0$. Hence, [6, Lemma D.1 (i)] implies $\max\{v_1(\zeta, \eta), v_2(\zeta, \eta)\} \leq \|\mathrm{Im}\, M(\zeta, \mathrm{i}\eta))\| \lesssim \eta$ for all $\eta \in (0, c]$ for some sufficiently small $c \sim_\varepsilon 1$. If $\eta \in (c, 1]$ then the upper bound in Corollary 5.4 yields $v_i \lesssim \eta^{-1} \sim \eta$ for all $\eta \in (c, 1]$. This completes the proof. □

The next proposition states that if $\langle v_1 \rangle = \langle v_2 \rangle$ remains bounded away from zero as $\eta \downarrow 0$, then the solution has an analytic extension to $\eta = 0$.

Proposition 5.12 (Analyticity in the bulk). *Let s satisfy A1 and $\zeta \in \mathbb{C}$ with $\limsup_{\eta \downarrow 0} \langle v_1(\zeta, \eta) \rangle > 0$. Then $v_1, v_2 : \mathbb{C} \times (0, \infty) \to (0, \infty)$ has an extension to a neighbourhood of $(\zeta, 0)$ in $\mathbb{C} \times \mathbb{R}$ which is real analytic in all variables.*

To prove this proposition, we show that the Dyson equation (4.4) is stable even for $\eta = 0$. However, the equation does not have a unique solution on $\mathcal{B}_+^2$ for $\eta = 0$ without the additional constraint $\langle v_1 \rangle = \langle v_2 \rangle$. Therefore, we have to reformulate the equation to incorporate this constraint. Proposition 5.12 is proved at the end of this subsection.

We recall that $\mathcal{B}_+ := \{w \in \mathcal{B} : w > 0\}$ and set

$$e_- = (1, -1) \in \mathcal{B}^2 := \mathcal{B} \oplus \mathcal{B}, \qquad e_-^\perp := \{h = (h_1, h_2) \in \mathcal{B}^2 : \langle h_1 \rangle = \langle h_2 \rangle\}.$$

For $\eta > 0$ and $\zeta \in \mathbb{C}$, we define $J \equiv J_{\zeta,\eta} \colon e_-^\perp \cap \mathcal{B}_+^2 \to e_-^\perp$, $(w_1, w_2) \mapsto (J_1(w_1, w_2), J_2(w_1, w_2))$ through

$$J_1(w_1, w_2) := (\eta + Sw_2)\left(w_1 - \frac{\eta + S^*w_1}{(\eta + S^*w_1)(\eta + Sw_2) + |a - \zeta|^2}\right),$$
$$J_2(w_1, w_2) := (\eta + S^*w_1)\left(w_2 - \frac{\eta + Sw_2}{(\eta + S^*w_1)(\eta + Sw_2) + |a - \zeta|^2}\right).$$

Then (4.4) takes the form $J(v) = 0$ with $v = (v_1, v_2) \in \mathcal{B}_+^2$.

On $\mathcal{B}^2$, we introduce a tracial state and a scalar product defined through

$$\left\langle \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right\rangle := \frac{1}{2}\big(\langle x_1 \rangle + \langle x_2 \rangle\big), \qquad \left\langle \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \right\rangle := \frac{1}{2}\big(\langle \overline{x}_1 y_1 \rangle + \langle \overline{x}_2 y_2 \rangle\big) \tag{5.25}$$

for $x_1, x_2, y_1, y_2 \in \mathcal{B}$. For $x \in \mathcal{B}^2$, we write $\|x\|_2 := \sqrt{\langle x, x \rangle}$. We also interpret $\mathcal{B}^2$ as an algebra equipped with the componentwise multiplication $(x_1 x_2)(y_1, y_2) := (x_1 y_1, x_2 y_2)$.

For the rest of this section we will assume that s satisfies A1. Until the proof of Proposition 5.12, we fix $\zeta \in \mathbb{C}$ such that $\limsup_{\eta \downarrow 0} \langle v_1(\zeta, \eta) \rangle \geq \delta$ for some $\delta > 0$. Under these conditions, J remains well defined on $e_-^\perp \cap \mathcal{B}_+^2$ even for $\eta = 0$ and we set $J_0 := J_{\zeta,\eta=0}$. We now pick candidates for $v_1(\zeta, 0)$ and $v_2(\zeta, 0)$ by choosing weakly convergent subsequences in the limit $\eta \downarrow 0$. By Lemma 5.3, there are $v_0 \in (L^2)^2 := L^2 \oplus L^2$ and a monotonically decreasing sequence $\eta_n \downarrow 0$ in $(0, 1]$ such that $v^{(n)} = v(\zeta, \eta_n)$ is weakly convergent to v_0 in $(L^2)^2$, that is, for any $h \in (L^2)^2$, $\langle h, v^{(n)} - v_0 \rangle \to 0$ in the limit $n \to \infty$. We recall that $L^2 = L^2(\mathfrak{X}, \mathcal{A}, \mu)$.

Lemma 5.13. *Let $v_0 = \lim_{n \to \infty} v^{(n)}$ be a weak limit as above. Then $v_0 \in \mathcal{B}_+^2 \cap e_-^\perp$ and $\delta \lesssim v_0 \lesssim \frac{1}{\delta}$. Furthermore, v_0 satisfies (4.4) for $\eta = 0$, that is, $J_0(v_0) = 0$.*

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For the following arguments, we introduce the operators S_o and S_d on $\mathcal{B}^2$ defined through

$$S_o := \begin{pmatrix} 0 & S \\ S^* & 0 \end{pmatrix}, \quad S_d := \begin{pmatrix} S^* & 0 \\ 0 & S \end{pmatrix}. \tag{5.26}$$

Owing to the upper bound in A1, S_o and S_d can be extended naturally to operators on $(L^2)^2$.

Proof. Since $v^{(n)} \rightharpoonup v_0$ weakly and $\langle e_- v^{(n)} \rangle = 0$, we conclude $v_0 \perp e_-$. Furthermore, for any $h \in \mathcal{B}^2_+$ we get $\langle h v_0 \rangle = \lim_{n\to\infty} \langle h v^{(n)} \rangle \gtrsim \delta \langle h \rangle$ because of Corollary 5.4 and $\limsup_{n\to\infty} \langle v^{(n)} \rangle \geq \delta$. From this we conclude $v_0 \gtrsim \delta$. Similarly, Corollary 5.4 implies $v_0 \lesssim \frac{1}{\delta}$ and thus $v_0 \in \mathcal{B}^2_+$.

The natural extensions of S and S^* to operators on L^2 are Hilbert-Schmidt operators because $s \in L^2(\mathfrak{X} \times \mathfrak{X}, \mu \otimes \mu)$ due to the upper bound in A1. In particular, S and S^* are compact operators on L^2 and, thus, $S_o v^{(n)} \to S_o v_0$ and $S_d v^{(n)} \to S_d v_0$ in $(L^2)^2$. The bounds $\delta \lesssim v^{(n)} \lesssim \delta^{-1}$ then imply that $J_{\zeta,\eta_n}(v^{(n)}) \rightharpoonup J_0(v_0)$ weakly in $(L^2)^2$. Consequently, $J_0(v_0) = 0$. □

For the formulation of the next lemma, we note that $\|T\|_\infty$ denotes the operator norm of an operator $T\colon \mathcal{B}^2 \to \mathcal{B}^2$ and, analogously, $\|T\|_2$ is the operator norm if $T\colon (L^2)^2 \to (L^2)^2$.

Lemma 5.14. *Let v_0 be a weak limit of a sequence $v^{(n)} = v(\zeta, \eta_n)$ as above. Then*

$$\|(\nabla J_0|_{w=v_0})^{-1}\|_2 + \|(\nabla J_0|_{w=v_0})^{-1}\|_\infty \lesssim_\delta 1.$$

Proof of Lemma 5.14. Within this proof we will make use of the auxiliary Lemma 5.15 below whose proof relies on ideas from [5]. Therefore we introduce notations that match the ones from [5], namely

$$\tau := (|\zeta - a|^2, |\zeta - a|^2) \tag{5.27}$$

and recall the definitions of S_o and S_d from (5.26). Using the notations (5.26) and (5.27), we write J in the form

$$J(w) = (\eta + S_o w)\left(w - \frac{1}{\eta + S_o w + \frac{\tau}{\eta + S_d w}}\right).$$

Now we take the directional derivative $\nabla_h J$ of J in the direction $h \in \mathcal{B}^2$ with $h \perp e_-$, that is, $\langle h e_- \rangle = 0$, and evaluate at the solution $v = (v_1, v_2)$. Thus, we find

$$\nabla_h J|_{w=v} = (\eta + S_o v)\left(h + v^2 S_o h - \frac{v^2 \tau}{(\eta + S_d v)^2} S_d h\right) = (\eta + S_o v)\mathscr{L} h\,, \tag{5.28}$$

where we used $J(v) = 0$ and introduced the linear operator $\mathscr{L} \equiv \mathscr{L}_{\zeta,\eta}(v)\colon \mathcal{B}^2 \to \mathcal{B}^2$ as

$$\mathscr{L} h := h + v^2 S_o h - \frac{v^2 \tau}{(\eta + S_d v)^2} S_d h \tag{5.29}$$

to guarantee the last equality. Here, $v^2 = (v_1^2, v_2^2)$ since the algebra $\mathcal{B}^2$ is naturally equipped with entrywise multiplication. We now restrict our analysis to $\eta = 0$ and use the following lemma that provides a resolvent estimate for $\mathscr{L}_0 = \mathscr{L}_{\zeta,0}(v_0)$, the operator evaluated on the weak limit v_0.

Lemma 5.15. *There is $\varepsilon_* \sim_\delta 1$ such that for any $\varepsilon \in (0, \varepsilon_*)$ we have the bound*

$$\sup\left\{\|(\mathscr{L}_0 - \omega)^{-1}\|_\# : \omega \notin \mathbb{D}_\varepsilon \cup ((1 + \mathbb{D}_{1+\varepsilon}) \setminus \mathbb{D}_{2\varepsilon})\right\} \lesssim_{\delta,\varepsilon} 1 \tag{5.30}$$

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for $\# = 2, \infty$. *Here,* $\mathbb{D}_\varepsilon$ *contains the single isolated eigenvalue* 0 *of* $\mathscr{L}_0$ *with corresponding right and left eigenvectors* $v_- := e_- v_0$ *and* $S_o v_-$, *that is,*

$$\mathbb{D}_\varepsilon \cap \mathrm{Spec}(\mathscr{L}_0) = \{0\}\,, \qquad \ker \mathscr{L}_0^2 = \mathrm{Span}(v_-)\,, \qquad \mathscr{L}_0 v_- = 0\,, \qquad \mathscr{L}_0^* S_o v_- = 0\,.$$

Here, $\mathscr{L}_0^*$ *is the adjoint of* $\mathscr{L}$ *with respect to the* L^2*-scalar product introduced in (5.25).*

The proof of Lemma 5.15 follows a strategy similar to the one used to prove stability of the Dyson equation in [5], where the case $a = 0$ was treated. For completeness we present the proof, adjusted to our setup with nontrivial a, in Appendix B below. Using Lemma 5.15 we now show that

$$\|\mathscr{L}_0^{-1}|_{(S_o v_-)^\perp}\|_\# \lesssim_\delta 1\,, \qquad \# = 2, \infty\,, \tag{5.31}$$

from which the claim of Lemma 5.14 immediately follows due to (5.28), A1 and $v_0 \gtrsim \delta$. To see (5.31), we apply Lemma B.4 to $A := C\mathscr{L}_0$ for some appropriately large positive constant $C \sim_\delta 1$. We now check the assumptions of the lemma. Note that $\mathscr{L}_0$ maps $e_-^\perp$ to $(S_o v_-)^\perp$. By Lemma 5.15 the right and left eigenvectors of $\mathscr{L}_0$ corresponding to the eigenvalue 0 are v_- and $S_o v_-$, respectively. Moreover, $\langle v_-, e_-\rangle \gtrsim_\delta 1$ as $v_0 \gtrsim \delta$, $|\langle e_-, w\rangle| \le \|w\|_\#$ and that $\|\mathscr{L}_0 w\|_\# \gtrsim_\delta \|w\|_\#$ for any $w \perp S_o v_-$ due to Lemma 5.15. By Lemma B.4 with the choices $\alpha := 0$, $x := v_-$ and $y := \frac{1}{2} e_-$ we get $\|\mathscr{L}_0 w\|_\# \gtrsim_\delta \|w\|_\#$ for any $w \perp e_-$. Thus, (5.31) is shown. □

Now we use the stability at $\eta = 0$ to finish the proof of the main result of this subsection.

Proof of Proposition 5.12. Let $\zeta_0 \in \mathbb{C}$ be such that $\limsup_{\eta \downarrow 0} \langle v_1(\zeta_0, \eta)\rangle > 0$. Let $\eta_n \downarrow 0$ such that $v^{(n)} = v(\zeta_0, \eta_n)$ is weakly convergent in $(L^2)^2$. This is possible, because the family $v(\zeta_0, \eta)$ with $\eta \in (0, 1]$ is bounded in $(L^2)^2$ due to Lemma 5.3. By Lemma 5.13 the weak limit $v_0 = \lim_{n\to\infty} v^{(n)}$ satisfies the Dyson equation, $J_{\zeta,0}(v_0) = 0$, and by Lemma 5.14 the Dyson equation is stable at $v = v_0$ and $\eta = 0$. By the implicit function theorem we find a real analytic function $\widetilde{v}$, defined on a neighbourhood U of $(\zeta_0, 0)$ in $\mathbb{C} \times \mathbb{R}$, such that $\widetilde{v}(\zeta, \eta)$ solves (4.4) and $\widetilde{v}(\zeta_0, 0) = v_0$. Since $v_0 \gtrsim \delta$ according to Lemma 5.13, $\widetilde{v}(\zeta, \eta) > 0$ on U if the neighbourhood U is chosen sufficiently small. By uniqueness of the solution to the Dyson equation we conclude $\widetilde{v}(\zeta, \eta) = v(\zeta, \eta)$ for all $(\zeta, \eta) \in U$. □

5.4. *Characterisation of* $\mathbb{S}$

Throughout this section we assume that $a \in \mathcal{B}$ and s satisfies A1 and A2. To generalize (2.8), (2.9) and (2.10) to the setup introduced in Section 5, we define an operator $B_\zeta : \mathcal{B} \to \mathcal{B}$, a function $\beta : \mathbb{C} \to \mathbb{R}$ and a subset $\mathbb{S} \subset \mathbb{C}$ through

$$\beta(\zeta) := \inf_{x \in \mathcal{B}_+} \sup_{y \in \mathcal{B}_+} \frac{\langle x, B_\zeta\, y\rangle}{\langle x, y\rangle}\,, \qquad B_\zeta := D_{|a-\zeta|^2} - S\,, \qquad \mathbb{S} := \{\zeta \in \mathbb{C} : \beta(\zeta) < 0\}\,. \tag{5.32}$$

We also set

$$\rho_\zeta(0) := \frac{1}{\pi} \lim_{\eta \downarrow 0} \langle v_1(\zeta, \eta)\rangle\,. \tag{5.33}$$

This limit exists, because either $\limsup_{\eta\to 0}\langle v_1(\zeta,\eta)\rangle > 0$, in which case v_1 can be analytically extended to $\eta = 0$ by Proposition 5.12, or $\limsup_{\eta\to 0}\langle v_1(\zeta,\eta)\rangle = 0$ in which case the limit equals zero as well. The definition in (5.33) is motivated by the fact that the measure ρ_ζ from Definition 5.9 can be shown to have a density on $\mathbb{R}$ and the value of this density at zero would be given by the right hand side in (5.33).

In the following we will denote by $\lambda_{\mathrm{PF}}(T)$ the spectral radius of a compact and positivity preserving operator T. By the Krein-Rutman theorem $\lambda_{\mathrm{PF}}(T)$ is an eigenvalue of T with a positive eigenvector. We also refer to this eigenvalue as the Perron-Frobenius eigenvalue of T. In particular the operators S

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and S^* are compact as mentioned in the proof of Lemma 5.13 and therefore so are $D_x S D_y$ and $D_x S^* D_y$ for $x, y \in \mathcal{B}$. We use this fact in the statement of the following proposition.

Proposition 5.16. *Let $a \in \mathcal{B}$ and s satisfy A1 and A2. The following relations between β, $\mathbb{S}$, $\mathbb{S}_\varepsilon$ and ρ_ζ apply.*

(i) *The function $\mathbb{C} \ni \zeta \mapsto \beta(\zeta)$ is continuous and satisfies $\lim_{\zeta\to\infty} \beta(\zeta) = +\infty$. In particular, $\mathbb{S}$ is bounded.*

(ii) *The spectrum of D_a lies inside $\mathbb{S}$, that is,*

$$\operatorname{Spec}(D_a) \subset \mathbb{S}\,. \tag{5.34}$$

(iii) *The sign of β satisfies*

$$\operatorname{sign}\beta(\zeta) = \operatorname{sign}\Big(1 - \lambda_{\mathrm{PF}}\Big(S D_{|a-\zeta|}^{-2}\Big)\Big)\,, \qquad \zeta \in \mathbb{C}\,. \tag{5.35}$$

(iv) *For any $\zeta \in \mathbb{C}$ with $\beta(\zeta) > 0$ the operator B_ζ is invertible. Furthermore, all such ζ are characterised by*

$$\{\zeta \in \mathbb{C} : \beta(\zeta) > 0\} = \{\zeta \in \mathbb{C} : \operatorname{dist}(0, \operatorname{supp}\rho_\zeta) > 0\} = \mathbb{C} \setminus \mathbb{S}_0\,. \tag{5.36}$$

(v) *The set $\mathbb{S}$ is characterised by having a positive singular value density at the origin, that is,*

$$\mathbb{S} = \{\zeta \in \mathbb{C} : \rho_\zeta(0) > 0\}\,. \tag{5.37}$$

Before proving Proposition 5.16, we state a corollary, which follows directly from Proposition 5.12 and Proposition 5.16 (v).

Corollary 5.17 (Existence and uniqueness for (1.1)). *Let $a \in \mathcal{B}$ and s satisfy A1 and A2. Then for each $\zeta \in \mathbb{S}$, the relations (1.1) have a unique solution $(v_1, v_2) \in \mathcal{B}_+ \times \mathcal{B}_+$.*

Proof. Proof of (i): The continuity of $\zeta \mapsto \beta(\zeta) = \beta$ with $B = B_\zeta$ is a consequence of the bound

$$\left| \inf_{x\in\mathcal{B}_+} \sup_{y\in\mathcal{B}_+} \frac{\langle x, (B + D_w)y\rangle}{\langle x, y\rangle} - \beta \right| \le \sup_{x\in\mathcal{B}_+} \sup_{y\in\mathcal{B}_+} \frac{\langle x, D_{|w|}y\rangle}{\langle x, y\rangle} \le \|w\|_\infty$$

for any real valued $w \in \mathcal{B}$ since $B - B_{\zeta+\omega} = D_w$ with $w = a - \zeta|^2 - |a - \zeta + \omega|^2$ and $\|w\|_\infty \to 0$ for $\omega \in \mathbb{C}$ with $|\omega| \to 0$. This bound follows from

$$\frac{\langle x, (B + D_w)y\rangle}{\langle x, y\rangle} \le \frac{\langle x, By\rangle}{\langle x, y\rangle} + \sup_{\widetilde{x}\in\mathcal{B}_+} \sup_{\widetilde{y}\in\mathcal{B}_+} \frac{\langle \widetilde{x}, D_{|w|}\widetilde{y}\rangle}{\langle \widetilde{x}, \widetilde{y}\rangle} \le \frac{\langle x, By\rangle}{\langle x, y\rangle} + \|w\|_\infty$$

for every $x, y \in \mathcal{B}_+$, as well as

$$\frac{\langle x, (B + D_w)y\rangle}{\langle x, y\rangle} \ge \frac{\langle x, By\rangle}{\langle x, y\rangle} - \sup_{\widetilde{x}\in\mathcal{B}_+} \sup_{\widetilde{y}\in\mathcal{B}_+} \frac{\langle \widetilde{x}, D_{|w|}\widetilde{y}\rangle}{\langle \widetilde{x}, \widetilde{y}\rangle} \ge \frac{\langle x, By\rangle}{\langle x, y\rangle} - \|w\|_\infty,$$

and then taking the supremum over $y \in \mathcal{B}_+$ and the infimum over $x \in \mathcal{B}_+$ in both inequalities. The statement $\beta(\zeta) \to +\infty$ as $\zeta \to \infty$ is obvious.

Before we start with the proof of other individual statements of the proposition, we show that $\mathbb{S}$ can be classified in terms of the Perron-Frobenius eigenvalue of $S D_{|a-\zeta|}^{-2}$ in the sense that

$$\mathbb{S} = \{\zeta \in \mathbb{C} : \lambda(\zeta) > 1\}\,, \tag{5.38}$$

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where we introduced $\lambda : \mathbb{C} \to [0, \infty]$ as the limit of a strictly increasing sequence via

$$\lambda(\zeta) := \lim_{\varepsilon \downarrow 0} \lambda_\varepsilon(\zeta)\,, \qquad \lambda_\varepsilon(\zeta) := \lambda_{\mathrm{PF}}\big(S(\varepsilon + D_{|a-\zeta|^2})^{-1}\big).$$

We recall that $\lambda_{\mathrm{PF}}(T)$ denotes the Perron-Frobenius eigenvalue of T and refer to the comments before the statement of the proposition for its definition.

To show (5.38) let $\varepsilon \in (0,1)$, $D := D_{|\zeta-a|^2}$, $\lambda_\varepsilon = \lambda_\varepsilon(\zeta)$ and $C > 0$ such that $1 + |\zeta - a|^2 \le C$. For $\zeta \in \mathbb{C}$ with $\beta(\zeta) \ge 0$ we get

$$\beta + \varepsilon \le C \inf_{x \in \mathcal{B}_+} \sup_{y \in \mathcal{B}_+} \frac{\langle x\,, (\varepsilon + B)y\rangle}{\langle x\,, (\varepsilon + D)y\rangle} = C\left(1 - \sup_{x \in \mathcal{B}_+} \inf_{y \in \mathcal{B}_+} \frac{\langle x\,, S(\varepsilon + D)^{-1}y\rangle}{\langle x\,, y\rangle}\right) = C\big(1 - \lambda_\varepsilon\big),$$

where we used $\varepsilon + |\zeta - a|^2 \le C$ in the first inequality, and conclude

$$\beta \le C(1 - \lambda) \qquad \text{if } \beta \ge 0\,. \tag{5.39}$$

For $\zeta \in \mathbb{C}$ with $\beta(\zeta) < 0$ we use

$$-\beta - \varepsilon = \sup_{x \in \mathcal{B}_+} \inf_{y \in \mathcal{B}_+} \frac{-\langle x\,, (\varepsilon + B)y\rangle}{\langle x\,, y\rangle}$$

for sufficiently small $\varepsilon > 0$ and find analogously that

$$\beta \ge C\big(1 - \lambda\big) \qquad \text{if } \beta < 0\,. \tag{5.40}$$

From (5.39) and (5.40) we conclude (5.38).

We also show that

$$\mathrm{Spec}(D_a) \subset \{\zeta \in \mathbb{C} : \beta(\zeta) \le 0\}. \tag{5.41}$$

We will improve this to (5.34) below. Let $\zeta \in \mathrm{Spec}(D_a)$. Then $\operatorname{ess\,inf}|\zeta - a| = 0$. Thus, for any $\varepsilon > 0$ we find $x \in \mathcal{B} \setminus \{0\}$ with $x \ge 0$ such that $|\zeta - a|^2 x \le \varepsilon x$. In the definition of β from (5.32) we can take the supremum over all $x \in \overline{\mathcal{B}}_+$, Thus, we get

$$\beta \le \varepsilon - \inf_{y \in \mathcal{B}_+} \frac{\langle x\,, Sy\rangle}{\langle x\,, y\rangle} \le \varepsilon\,.$$

Since $\varepsilon > 0$ was arbitrarily small, we conclude $\beta \le 0$.

Proof of (iv): Let $\zeta \in \mathbb{C}$ such that $\beta(\zeta) = \beta > 0$. Then (5.39) implies $\lambda_{\mathrm{PF}}(SD^{-1}) < 1$ with $D = D_{|\zeta-a|^2}$. Here, D is invertible because $\operatorname{ess\,inf}|\zeta-a| > 0$ due to (5.41). In particular, $B = (1 - SD^{-1})D$ is invertible.

Now we show $\mathrm{dist}(0, \operatorname{supp}\rho_\zeta) > 0$ to see one inclusion in the characterisation (5.36). The Dyson equation in the matrix representation, (4.3) is solved by

$$M_0 := \begin{pmatrix} 0 & (\overline{a-\zeta})^{-1} \\ (a-\zeta)^{-1} & 0 \end{pmatrix} \tag{5.42}$$

at $w = 0$. Furthermore, the associated stability operator (cf. (5.20))

$$\mathcal{L}_0 \colon \mathcal{B}^{2\times 2} \to \mathcal{B}^{2\times 2}, \quad R \mapsto M_0^{-1} R M_0^{-1} - \Sigma R = \begin{pmatrix} |a-\zeta|^2 r_{22} - S r_{22} & (a-\zeta)^2 r_{21} \\ (\overline{a-\zeta})^2 r_{12} & |a-\zeta|^2 r_{11} - S^* r_{11} \end{pmatrix} \tag{5.43}$$

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is invertible because B_ζ is invertible and ess inf$|a-\zeta| > 0$. Therefore (4.3) can be uniquely solved for sufficiently small w as an analytic function $w \mapsto M(\zeta, w)$ with $M(\zeta, 0) = M_0$ and we get

$$\mathcal{L}_0[\partial_w M(\zeta, w)|_{w=0}] = 1 \quad \text{and} \quad \partial_\eta M(\zeta, \mathrm{i}\eta)|_{\eta=0} = \mathrm{i}\mathcal{L}_0^{-1}[1]$$

In particular, $\operatorname{Im} M(\zeta, \mathrm{i}\eta)$ is positive definite for sufficiently small $\eta > 0$ because $\mathcal{L}_0^{-1}$ is positivity preserving, as can be seen from a Neumann series expansion using that $\lambda_{\mathrm{PF}}(SD^{-1}) < 1$. Therefore $M(\zeta, \mathrm{i}\eta)$ is the unique solution of (4.3) with $m_{11}(\zeta, \mathrm{i}\eta) = \mathrm{i}v_1(\zeta, \eta)$ and $m_{22}(\zeta, \mathrm{i}\eta) = \mathrm{i}v_2(\zeta, \eta)$. Since $m_{11}|_{\eta=0} = m_{22}|_{\eta=0} = 0$ we conclude that $\rho_\zeta(0) = \langle v_1(\zeta, 0)\rangle = 0$. The invertibility of $\mathcal{L}_0$ also implies analyticity of $M(\zeta, w)$ in w in a small neighbourhood of zero. Thus, $\rho_\zeta([-\varepsilon, \varepsilon]) = 0$ for $\varepsilon > 0$ sufficiently small and $\operatorname{dist}(0, \operatorname{supp}\rho_\zeta) > 0$.

To see the other inclusion in (5.36), let $\zeta \in \mathbb{C}$ be such that $\operatorname{dist}(0, \operatorname{supp}\rho_\zeta) \geq \delta$ for some $\delta > 0$. From [6, Lemma D.1 (iv)] we know that $M = M(\zeta, \mathrm{i}\eta)$ is locally a real analytic function of η with an expansion $M = M_0 + \mathrm{i}\eta M_1 + O(\eta^2)$, where $M_0 = M_0^*$. Taking the imaginary part of (4.3) at $w = \mathrm{i}\eta$, dividing both sides by η shows that

$$(M^*)^{-1} K_\eta M^{-1} = 1 + \Sigma[K_\eta]\,, \tag{5.44}$$

where $K_\eta = \frac{1}{\eta} \operatorname{Im} M = \operatorname{Re} M_1 + O(\eta)$. In particular, $K_0 := \lim_{\eta \downarrow 0} K_\eta$ exists and since $\operatorname{Im} M(\zeta, \mathrm{i}\eta) \sim_\delta \eta$ by Lemma 5.11 we get $K_0 \sim_\delta 1$. Evaluating (5.44) at $\eta = 0$ yields

$$M_0(\Sigma K_0) M_0 = K_0 - 1\,. \tag{5.45}$$

Taking the scalar product of (5.45) with the left Perron-Frobenius eigenvector of $R \mapsto M_0(\Sigma R)M_0$ and using $K_0 \sim_\delta 1$ we see that $\lambda_{\mathrm{PF}}(R \mapsto M_0(\Sigma R)M_0) < 1$. This is equivalent to $\lambda := \lambda_{\mathrm{PF}}(SD^{-1}) < 1$ with $D := D_{|a-\zeta|^2}$. Now let $u \in \mathcal{B}_+$ be the Perron-Frobenius eigenvector of SD^{-1}. Since $\varepsilon := \text{ess inf}|a-\zeta| > 0$ we get with $y_0 := D^{-1}u$ that

$$(D - S)y_0 = (1 - \lambda)Dy_0 > 0\,.$$

Thus,

$$\beta = \inf_{x>0} \sup_{y>0} \frac{\langle x, By\rangle}{\langle x, y\rangle} \geq \inf_{x>0} \frac{\langle x, By_0\rangle}{\langle x, y_0\rangle} \geq (1 - \lambda)\varepsilon^2 > 0\,.$$

This finishes the proof of (5.36), that is, of (iv).

Proof of (iii): We have now collected enough information to improve (5.38) to (5.35). Indeed, by (5.39) and (5.40) it remains to show that $\lambda < 1$ implies $\beta > 0$. Due to (5.38) we already know $\beta \geq 0$ in case $\lambda < 1$. Now let $\beta = 0$ and $\lambda \leq 1$. Then we show that $\lambda = 1$. Indeed by the characterisation (5.36) we have $0 \in \operatorname{supp}\rho_\zeta$. Now we consider the identity

$$B_\zeta v_2 = \eta - v_2(\eta + S^* v_1)(\eta + Sv_2)$$

which follows from (4.4b). For some $\varepsilon > 0$ we add εv_2 to both sides and apply the inverse of $\varepsilon + D$ with $D = D_{|a-\zeta|^2}$. Then we take the scalar product with the right Perron-Frobenius eigenvector $x_\varepsilon \in \mathcal{B}_+$ of $S^*(\varepsilon + D)^{-1}$ corresponding to its Perron-Frobenius eigenvalue $\lambda_\varepsilon > 0$. Note that the Perron-Frobenius eigenvalues of $S^*(\varepsilon + D)^{-1}$, $(\varepsilon + D)^{-1}S$ and $S(\varepsilon + D)^{-1}$ all coincide. Thus we get

$$(1 - \lambda_\varepsilon)\langle x_\varepsilon v_2\rangle = \eta\langle(\varepsilon + D)^{-1}x_\varepsilon\rangle - \langle x_\varepsilon(\varepsilon + D)^{-1}(v_2(\eta + S^* v_1)(\eta + Sv_2) - \varepsilon v_2)\rangle\,. \tag{5.46}$$

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From [6, Corollary D.2] and $\langle v_i\rangle \sim v_i$ by Lemma 5.7 (i) we see that $\eta/\langle v_i\rangle \to 0$ for $\eta \downarrow 0$. Thus, dividing (5.46) by $\langle v_2\rangle$, taking the limit $\eta \downarrow 0$ and using (5.7) reveals

$$(1-\lambda_\varepsilon)\langle x_\varepsilon\rangle \sim (1-\lambda_\varepsilon)\langle x_\varepsilon k\rangle = \varepsilon\langle x_\varepsilon(\varepsilon+D)^{-1}k\rangle \sim \varepsilon\langle x_\varepsilon(\varepsilon+D)^{-1}S1\rangle = \varepsilon\langle S^*(\varepsilon+D)^{-1}x_\varepsilon\rangle = \varepsilon\lambda_\varepsilon\langle x_\varepsilon\rangle\,,$$

where $k := \limsup_{\eta\downarrow 0}\frac{v_2}{\langle v_2\rangle} \sim 1$. Letting $\varepsilon \downarrow 0$ shows $\lambda = 1$. Thus, (5.35) is proven.

Proof of (v): By (5.36) we know that $\rho_\zeta(0) > 0$ implies $\beta(\zeta) \le 0$. Thus, it suffices to show that for $\zeta \in \mathbb{C}$ with $\beta(\zeta) \le 0$ we get $\beta(\zeta) = 0$ if and only if $\rho_\zeta(0) = 0$. Now let $\beta = \beta(\zeta) \le 0$. As above, we consider the identity (5.46). First, suppose $\rho_\zeta(0) > 0$, that is, we can analytically extend v to $\eta = 0$ by Proposition 5.12 and have $v|_{\eta=0} > 0$. Then in the limit $\eta \downarrow 0$ we find

$$(\lambda_\varepsilon - 1)\langle x_\varepsilon v_2\rangle = \langle x_\varepsilon(\varepsilon+D)^{-1}(v_2(S^*v_1Sv_2 - \varepsilon))\rangle$$

Using $v_2 \sim \langle v_2\rangle \sim \rho_\zeta(0)$ for small enough $\varepsilon > 0$ the right hand side satisfies

$$\langle x_\varepsilon(\varepsilon+D)^{-1}(v_2(S^*v_1Sv_2-\varepsilon))\rangle \sim \rho_\zeta(0)^3\langle x_\varepsilon(\varepsilon+D)^{-1}S1\rangle \sim \lambda_\varepsilon\rho_\zeta(0)^3\langle x_\varepsilon\rangle.$$

Since $\langle x_\varepsilon v_2\rangle \sim \rho_\zeta(0)\langle x_\varepsilon\rangle$ we infer $\lambda_\varepsilon - 1 \sim \lambda_\varepsilon\rho_\zeta(0)^2$. Thus, $\lambda > 1$ and by (5.35) therefore $\beta < 0$.

Conversely, let $v|_{\eta=0} = 0$. Then we know from Lemma 5.7 (ii) that $\delta := \operatorname{ess\,inf}|a-\zeta| > 0$. Since $\beta(\zeta) \le 0$ the characterisation (5.36) implies $0 \in \operatorname{supp}\rho_\zeta$ and by (5.35) we have $\lambda \ge 1$. Since $\eta/\langle v\rangle \to 0$ for $\eta \downarrow 0$ by [6, Corollary D.2] we get, dividing (5.46) by $\langle v_2\rangle$ and taking the limit $\eta \downarrow 0$, the scaling behaviour

$$1 - \lambda_\varepsilon \sim \varepsilon\lambda_\varepsilon\,.$$

This implies $\lambda_\varepsilon \le 1$, thus $\lambda = 1$, and completes the proof of (v).

Proof of (ii): Let $\zeta \in \operatorname{Spec}(D_a)$. By (5.41) we know $\beta(\zeta) \le 0$. Suppose $\beta(\zeta) = 0$. Then (5.37) would imply $\rho_\zeta(0) = 0$. However, this contradicts $\zeta \in \operatorname{Spec}(D_a)$ because of Lemma 5.7 (ii) and the definition of $\rho_\zeta(0)$ in (5.33). This finishes the proof of the proposition. □

5.5. *Edge expansion of* v_1 *and* v_2

In this section we expand the solution (v_1, v_2) of (4.4) around any $\zeta_0 \in \mathbb{C}$ with $\beta(\zeta_0) = 0$. We will see later in Lemma 6.6 below that the set of these points coincides with $\partial\mathbb{S}$, which will turn out to be the regular edge and singular points of the Brown measure σ. Therefore we consider in this section a fixed $\zeta_0 \in \mathbb{C}$ with $\beta(\zeta_0) = 0$. The expansion of v_1, v_2 around ζ_0 is based on analytic perturbation theory for β. Throughout this section we will always assume $|\zeta - \zeta_0| + \eta \le c$ for some sufficiently small positive constant $c \sim 1$, that is, we assume that (ζ, η) lies within a small neighbourhood of $(\zeta_0, 0)$. We assume A1 and A2 throughout the remainder of this section.

To shorten notation, we denote $v_i = v_i(\zeta, \eta)$. The identities

$$B_\zeta v_2 = \eta - v_2(\eta + S^*v_1)(\eta + Sv_2), \qquad B_\zeta^* v_1 = \eta - v_1(\eta + S^*v_1)(\eta + Sv_2), \tag{5.47}$$

which follow from (4.4a) and (4.4b), respectively, are used to expand v_1 and v_2 in a neighbourhood of ζ_0. We will see in Corollary 5.20 below that the function $\zeta \mapsto \beta(\zeta)$, used to define $\mathbb{S}$ in (5.32), coincides locally around ζ_0 with the isolated nondegenerate eigenvalue of B_ζ closest to zero. We denote by $b = b_\zeta \in \mathcal{B}_+$ and $\ell = \ell_\zeta \in \mathcal{B}_+$ the right and left eigenvectors of $B = B_\zeta$, corresponding to the eigenvalue $\beta = \beta(\zeta)$ with normalisation $\langle b\rangle = \langle \ell\rangle = 1$, that is,

$$Bb = \beta b, \qquad B^*\ell = \beta\ell\,. \tag{5.48}$$

The existence and uniqueness of b and ℓ is a consequence of analytic perturbation theory and Lemma 5.19 below. This lemma also implies that $\zeta \mapsto b_\zeta$ and $\zeta \mapsto \ell_\zeta$ are real analytic functions. The main result of this section is the following proposition.

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Proposition 5.18. *Let s and a satisfy A1 and A2. Furthermore, let $\zeta_0 \in \mathbb{C}$ such that $\beta(\zeta_0) = 0$. Then there is an open neighbourhood $U \subset \mathbb{C} \times \mathbb{R}^2$ of $(\zeta_0, 0, 0)$, an open neighbourhood $V \subset \mathbb{C} \times \mathbb{R}$ of $(\zeta_0, 0)$ and real analytic functions $\widetilde{w}_1, \widetilde{w}_2 \colon U \to \mathcal{B}$ such that*

$$v_1(\zeta,\eta) = \vartheta(\zeta,\eta)\ell_\zeta + \widetilde{w}_i(\zeta,\eta,\vartheta(\zeta,\eta))\,, \qquad v_2(\zeta,\eta) = \vartheta(\zeta,\eta)b_\zeta + \widetilde{w}_i(\zeta,\eta,\vartheta(\zeta,\eta))$$

for $(\zeta,\eta) \in V$ and $\eta > 0$. Furthermore, $\vartheta = \vartheta(\zeta,\eta)$ satisfies

$$\vartheta^3\big\langle \ell b(S^*\ell)(Sb)\big\rangle + \beta\vartheta\langle \ell b\rangle - \eta = g(\zeta,\eta,\vartheta)\,, \tag{5.49}$$

for all $(\zeta,\eta) \in V$ where $\ell = \ell_\zeta$, $b = b_\zeta, \beta = \beta(\zeta)$ and $g \colon U \to \mathbb{R}$ is a real analytic function, such that

$$g(\zeta,\eta,x) = O(|\eta x|^2 + |x|^5)\,, \qquad (\zeta,\eta,x) \in U\,.$$

The proof of Proposition 5.18 is the content of the remainder of this section and will be summarised at its end. We remark that as a solution to the cubic equation (5.49) the quantity ϑ and with it v_1, v_2 are not analytic at $\zeta = \zeta_0$ and $\eta = 0$.

The following lemma collects spectral properties of B_{ζ_0}. These properties yield corresponding properties of B_ζ for sufficiently small $|\zeta - \zeta_0|$, using analytic perturbation theory. We will use this idea throughout the remainder of this section after the statement of Lemma 5.19.

Lemma 5.19 (Properties of B). *Let $\zeta_0 \in \mathbb{C}$ with $\beta(\zeta_0) = 0$ and $B_0 := B_{\zeta_0}$. Then there is a constant $\varepsilon > 0$ with $\varepsilon \sim 1$ such that*

$$\sup\Big\{\|(B_0-\omega)^{-1}\|_\# + \|(B_0^*-\omega)^{-1}\|_\# : \omega \in \mathbb{D}_{2\varepsilon} \setminus \mathbb{D}_\varepsilon\Big\} \lesssim 1 \tag{5.50}$$

for $\# = 2, \infty$. Here $\mathbb{D}_\varepsilon$ contains a single isolated nondegenerate eigenvalue 0 of B_0, that is,

$$\mathbb{D}_\varepsilon \cap \operatorname{Spec}(B_0) = \{0\}\,, \qquad \dim\ker B_0^2 = 1\,. \tag{5.51}$$

Moreover, the right and left eigenvectors, $b_0 \in \mathcal{B}_+$ and $\ell_0 \in \mathcal{B}_+$, corresponding to this eigenvalue with normalisation $\langle b_0\rangle = \langle \ell_0\rangle = 1$ satisfy the bounds $\ell_0 \sim b_0 \sim 1$. Furthermore, if

$$P_0 := \frac{\langle \ell_0\,\cdot\,\rangle}{\langle \ell_0 b_0\rangle} b_0\,, \qquad Q_0 := 1 - P_0$$

denote the associated spectral projections then

$$\|B_0^{-1}Q_0\|_\# + \|(B_0^*)^{-1}Q_0^*\|_\# \lesssim 1\,. \tag{5.52}$$

Proof. Here we present the proofs of the bounds (5.50) and (5.52) for B_0. The corresponding bounds for B_0^* follow analogously. From Proposition 5.16 (ii) and since $\mathbb{S}$ is bounded we know that $|a - \zeta_0| \sim 1$. Thus, b_0 is the right eigenvector of $D^{-1}S$ with eigenvalue 1 and ℓ_0 is the right eigenvector of $D^{-1}S^*$ with eigenvalue 1, where $D := D_{|a-\zeta_0|^2}$. In particular, $b_0, \ell_0 \in \mathcal{B}_+$ by the Krein-Rutman theorem and the geometric multiplicity of the eigenvalue 0 of B_0 is 1. Furthermore, the nondegeneracy of the eigenvalue 0 is a consequence of $b_0, \ell_0 \in \mathcal{B}_+$. Indeed, suppose we had $\dim\ker B^2 > 1$. Then there would be a generalised eigenvector x with $Bx = b_0$ and $\langle \ell_0 b_0\rangle = \langle \ell_0 B_0 x\rangle = 0$ which contradicts $\ell_0 > 0$ and $b_0 > 0$. This proves (5.51), which together with (5.50) implies (5.52). The relation $b_0 \sim \ell_0 \sim 1$ is a direct consequence of $|a - \zeta_0| \sim 1$ and Assumption A1.

We are left with proving (5.50). Instead of controlling the resolvent of B_0, it suffices to bound the inverse of $1 - SD^{-1} - \omega D^{-1}$ because

$$\frac{1}{B_0 - \omega} = \frac{1}{D}\left(\frac{1}{1 - SD^{-1} - \omega D^{-1}}\widetilde{Q}_\omega + \frac{1}{1 - SD^{-1} - \omega D^{-1}}\widetilde{P}_\omega\right), \tag{5.53}$$

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where $\widetilde{P}_\omega$ and $\widetilde{Q}_\omega := 1-\widetilde{P}_\omega$ are the analytic spectral projections associated with $SD^{-1}-\omega D^{-1}$ such that

$$\widetilde{P}_0 = \frac{\langle \ell_0 \cdot \rangle}{\langle \ell_0 D b_0\rangle} D b_0 \,.$$

Analytic perturbation theory can be applied to SD^{-1} because of Lemma C.2, which shows that the resolvent of the operator SD^{-1} is bounded in annulus around its isolated eigenvalue 1. Consequently, the first summand in (5.53) is bounded for sufficiently small $|\omega|$. The second summand admits the expansion

$$\frac{1}{1-SD^{-1}-\omega D^{-1}}\widetilde{P}_\omega = \frac{1}{\widetilde{\beta}(\omega)}\widetilde{P}_\omega\,, \qquad \widetilde{\beta}(\omega) = -\omega\,\frac{\langle \ell_0 b_0\rangle}{\langle \ell_0 D b_0\rangle} + O(|\omega|^2)$$

by standard analytic perturbation formulas, see, for example, [6, Lemma C.1]. Therefore the second summand is bounded for $\omega \in \mathbb{C}\setminus \mathbb{D}_\varepsilon$ for sufficiently small ε. □

Corollary 5.20. *Let $\zeta_0 \in \mathbb{C}$ with $\beta(\zeta_0)=0$. Then $0 \in \operatorname{Spec}(B_{\zeta_0})$, ess inf$|a-\zeta_0| > 0$ and $\lambda_{\mathrm{PF}}(SD^{-1}_{|a-\zeta_0|^2}) = 1$. Furthermore, there is $\varepsilon > 0$ such that $\beta(\zeta)$ is an isolated nondegenerate eigenvalue of B_ζ for all $\zeta \in \zeta_0 + \mathbb{D}_\varepsilon$. In particular $\zeta_0 + \mathbb{D}_\varepsilon \ni \zeta \mapsto \beta(\zeta)$ is real analytic and has the expansion*

$$\begin{aligned}\beta(\zeta) = -2\mathrm{Re}\left[\frac{\langle \ell_0 b_0 (a-\zeta_0)\rangle}{\langle \ell_0 b_0\rangle}\overline{(\zeta-\zeta_0)}\right] + \left(1 - 2\,\mathrm{Re}\left[\frac{\langle \ell_0 \overline{(a-\zeta_0)} B_0^{-1} Q_0[b_0(a-\zeta_0)]\rangle}{\langle \ell_0 b_0\rangle}\right]\right)|\zeta-\zeta_0|^2 \\ - 2\,\mathrm{Re}\left[\frac{\langle \ell_0 (a-\zeta_0) B_0^{-1} Q_0[b_0(a-\zeta_0)]\rangle}{\langle \ell_0 b_0\rangle}\overline{(\zeta-\zeta_0)}^2\right] + O(|\zeta-\zeta_0|^3)\,,\end{aligned} \tag{5.54}$$

which implies the formulas

$$\partial_\zeta \beta(\zeta_0) = -\frac{\langle \ell_0 b_0 \overline{(a-\zeta_0)}\rangle}{\langle \ell_0 b_0\rangle}\,, \qquad \partial_\zeta \partial_{\overline{\zeta}} \beta(\zeta_0) = 1 - 2\,\mathrm{Re}\left[\frac{\langle \ell_0 \overline{(a-\zeta_0)} B_0^{-1} Q_0[b_0(a-\zeta_0)]\rangle}{\langle \ell_0 b_0\rangle}\right] \tag{5.55}$$

for the derivatives of β at $\zeta=\zeta_0$.

Proof. Let $\zeta_0 \in \mathbb{C}$ be such that $\beta(\zeta_0)=0$. By Lemma 5.19 we have $0 \in \operatorname{Spec}(B_{\zeta_0})$ and by Proposition 5.16 (ii) we get ess inf$|a-\zeta_0| > 0$. The fact that $\lambda_{\mathrm{PF}}(SD^{-1}_{|\zeta_0-a|^2}) = 1$ was shown in (5.35).

Now we show that $\beta(\zeta)$ is an eigenvalue of B_ζ for sufficiently small $|\zeta-\zeta_0|$. Using analytic perturbation theory, let $b(\zeta)$ and $\ell(\zeta)$ be the right and left eigenvectors of B_ζ corresponding to the isolated nondegenerate eigenvalue $\widetilde{\beta}(\zeta)$ with $\widetilde{\beta}(\zeta_0)=0$ that depends real analytically on ζ. As $\widetilde{\beta}(\zeta_0)$ is a real isolated eigenvalue and B_{ζ_0} as well as $B_\zeta - B_{\zeta_0}$ are invariant under complex conjugation, $\widetilde{\beta}(\zeta)$, $b(\zeta)$ and $\ell(\zeta)$ are also real. Since $\ell(\zeta_0) \sim b(\zeta_0) \sim 1$ we have $b(\zeta)$, $\ell(\zeta) \in \mathcal{B}_+$ for sufficiently small $|\zeta-\zeta_0|$. Therefore

$$\widetilde{\beta}(\zeta) = \inf_{x>0} \frac{\langle x, B_\zeta b(\zeta)\rangle}{\langle x, b(\zeta)\rangle} \le \beta(\zeta) \le \sup_{y>0} \frac{\langle \ell(\zeta), B_\zeta y\rangle}{\langle \ell(\zeta), y\rangle} = \widetilde{\beta}(\zeta)\,,$$

which proves $\widetilde{\beta} = \beta$.

The expansion (5.54) is now a direct consequence of analytic perturbation theory, as we see for example, by using [6, Lemma C.1] with $B = B_0 + E$ and $E = D_{|a-\zeta|^2} - D_{|a-\zeta_0|^2} = D_{|\zeta-\zeta_0|^2 - 2\mathrm{Re}\,(\overline{(a-\zeta_0)}(\zeta-\zeta_0))}$. □

Due to analytic perturbation theory with ζ in a small neighbourhood of ζ_0 and by Lemma 5.19 we have $b \sim \ell \sim 1$. We split v_1 and v_2 according to the spectral decompositions of B^* and B, namely

$$v_1 = \vartheta_1 \ell + \widetilde{v}_1, \qquad v_2 = \vartheta_2 b + \widetilde{v}_2 \tag{5.56}$$

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with the contributions $\vartheta_i = \vartheta_i(\zeta, \eta)$ to the eigendirections ℓ and b of B^* and B as well as their complements $\widetilde{v}_i = \widetilde{v}_i(\zeta, \eta)$ given as

$$\vartheta_1 := \frac{\langle b v_1 \rangle}{\langle \ell b \rangle}, \qquad \vartheta_2 := \frac{\langle \ell v_2 \rangle}{\langle \ell b \rangle}, \qquad \widetilde{v}_1 := Q^* v_1, \qquad \widetilde{v}_2 := Q v_2, \qquad Q := 1 - \frac{\langle \ell \cdot \rangle}{\langle \ell b \rangle} b. \tag{5.57}$$

To quantify the error terms we introduce

$$\alpha := \|v_1\|_\infty + \|v_2\|_\infty .$$

Projecting the identities (5.47) with Q and Q^*, respectively, leads to

$$B_\zeta \widetilde{v}_2 = O(\eta + \alpha^3), \qquad B_\zeta^* \widetilde{v}_1 = O(\eta + \alpha^3). \tag{5.58}$$

Using $\|B^{-1} Q\|_\infty \lesssim 1$, a consequence of (5.52) and analytic perturbation theory, we find

$$\|\widetilde{v}_1\|_\infty + \|\widetilde{v}_2\|_\infty = O(\eta + \alpha^3). \tag{5.59}$$

Because of $\langle v_1 \rangle = \langle v_2 \rangle$, that is, by (5.1), (5.56) and the normalisation $\langle b \rangle = \langle \ell \rangle = 1$, (5.59) implies

$$\vartheta_1 = \vartheta_2 + O(\eta + \alpha^3). \tag{5.60}$$

Inserting the decomposition (5.56) into (5.47) and using (5.59), as well as (5.60), leads to

$$\begin{aligned} \beta \vartheta_2 b + B\widetilde{v}_2 &= \eta - \vartheta^3 b (Sb)(S^*\ell) + O(\eta\alpha^2 + \alpha^5), \\ \beta \vartheta_1 \ell + B^*\widetilde{v}_1 &= \eta - \vartheta^3 \ell (Sb)(S^*\ell) + O(\eta\alpha^2 + \alpha^5), \end{aligned}$$

where we set $\vartheta := \frac{1}{2}(\vartheta_1 + \vartheta_2)$. Now we average the first equation against ℓ and the second equation against b, use $\langle b \rangle = \langle \ell \rangle = 1$ and then take the arithmetic mean of the resulting equations to find

$$\vartheta^3 \big\langle \ell b (S^*\ell)(Sb) \big\rangle + \beta \vartheta \langle \ell b \rangle - \eta = O(\eta\alpha^2 + \alpha^5). \tag{5.61}$$

From this approximate cubic equation we conclude the scaling behaviours

$$\alpha \sim \vartheta \sim \sqrt{\max\{0, -\beta\}} + \frac{\eta}{\eta^{2/3} + |\beta|}, \qquad \|\widetilde{v}_1\|_\infty + \|\widetilde{v}_2\|_\infty \sim \eta + \max\{0, -\beta\}^{3/2}, \tag{5.62}$$

in the regime of sufficiently small α, where we used $\vartheta \geq 0$ and $\vartheta > 0$ for $\eta > 0$ to choose the correct branch of the solution. The corresponding argument is summarised in Lemma C.1 in the appendix.

To apply this lemma we absorb the $O(\alpha^5)$-term on the right hand side of (5.61) into the cubic term in ϑ on the left hand side, that is, we write $O(\alpha^5) = \gamma \vartheta^3$ for some $\gamma = O(\alpha^2)$, which we absorb into the coefficient of the ϑ^3-term. Such rewriting is possible since $\alpha = O(\vartheta)$ in the regime where α is sufficiently small. This holds because $\vartheta \gtrsim \eta$ and $\alpha = O(\vartheta + \eta + \alpha^3)$ by (5.56), (5.60) and (5.59). Now we see that α is indeed small for (ζ, η) in a neighbourhood of $(\zeta_0, 0)$. Due to the characterisation of $\mathbb{S}$ in (5.37) we have $\lim_{\eta \downarrow 0} \alpha|_{\zeta=\zeta_0} = 0$. With $\beta(\zeta_0) = 0$ and because α is a continuous function of η when $\eta > 0$, the scaling (5.62) implies $\alpha|_{\zeta=\zeta_0} \sim \eta^{1/3}$. Since $\zeta \mapsto \beta(\zeta)$ is continuous by Proposition 5.16 (i) and α is a continuous function of ζ for any $\eta > 0$ the behaviour (5.62) holds as long as $\eta + |\zeta - \zeta_0|$ is sufficiently small.

We now summarise our insights by finishing the proof of Proposition 5.18.

Proof of Proposition 5.18. By following the computation leading to (5.58) we easily see that the right hand side of these equations are real analytic functions of ϑ, η, ζ and $\widetilde{v}_i$. By the implicit function theorem and the invertibility of B on the range of Q the $\widetilde{v}_i$ are real analytic functions of ϑ, η and ζ. Similarly, the

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right hand side of (5.61) is a real analytic function of ϑ, η and ζ. Together, we have proved Proposition 5.18 with

$$\widetilde{v}_i(\zeta,\eta) = \widetilde{w}_i(\vartheta(\zeta,\eta),\eta,\zeta)\,. \qquad \square$$

6. Properties of the Brown measure σ

In this section, we show our main results about the existence and properties of the Brown measure σ, Proposition 4.3, Theorem 2.2 and Theorem 2.5. We start by proving that the Brown measure has an explicit construction as a distributional derivative of the function L defined in (4.9) through the solution of the Dyson equation as stated in Proposition 4.3. The proof of Proposition 4.3 is presented in Subsection 6.1, and the proofs of Theorem 2.2 and Theorem 2.5 are given at the end of this section.

6.1. Characterisation of the Brown measure σ

Here, we present the proof of Proposition 4.3. The main idea of this proof is to show that $-L$ from (4.9) is subharmonic and, therefore, the distribution $-\frac{1}{2\pi}\Delta L$ is induced by a measure. Before we present this proof, we establish a few necessary ingredients. The next lemma, in particular, implies that L is well-defined.

Lemma 6.1 (Integrating $\langle v_1\rangle$ with respect to η). *Let $a \in \mathcal{B}$ and s satisfy A1. Then, uniformly for $\zeta \in \mathbb{C}$ and $\eta > 0$, we have*

$$0 \le \langle v_1(\zeta,\eta)\rangle \lesssim \frac{1}{1+\eta}. \tag{6.1}$$

Furthermore, uniformly for any $T > 0$ and $\zeta \in \mathbb{C}$, we have

$$\int_0^T \left|\langle v_1(\zeta,\eta)\rangle - \frac{1}{1+\eta}\right| \mathrm{d}\eta \lesssim \min\left\{T, \sqrt{1+|\zeta|}\right\}, \qquad \int_T^\infty \left|\langle v_1(\zeta,\eta)\rangle - \frac{1}{1+\eta}\right| \mathrm{d}\eta \lesssim \frac{1+|\zeta|}{T}. \tag{6.2}$$

Proof. From (5.2) and Lemma 5.3, we immediately conclude (6.1). The bounds in (6.2) follow directly from (6.1) and (5.13). $\square$

Proof of Proposition 4.3. We first note that each of the identities (4.10) and (4.11) uniquely characterises a probability measure on $\mathbb{C}$. We now show that (4.10) and (4.11) characterise the same probability measure on $\mathbb{C}$. Let $f \in C_0^2(\mathbb{C})$. By dominated convergence and (6.2), we obtain

$$\begin{aligned}\int_{\mathbb{C}} \Delta f(\zeta) W(\zeta)\mathrm{d}^2\zeta &= \lim_{\varepsilon\downarrow 0} \int_{\mathbb{C}} \Delta f(\zeta) \int_\varepsilon^\infty \left(\langle v_1(\zeta,\eta)\rangle - \frac{1}{1+\eta}\right)\mathrm{d}\eta\mathrm{d}^2\zeta \\ &= 2\lim_{\varepsilon\downarrow 0} \int_{\mathbb{C}} \partial_\zeta f(\zeta) \int_\varepsilon^\infty \partial_\eta \langle \overline{y(\zeta,\eta)}\rangle \mathrm{d}\eta\mathrm{d}^2\zeta \\ &= -2\lim_{\varepsilon\downarrow 0} \int_{\mathbb{C}} \partial_\zeta f(\zeta) \langle \overline{y(\zeta,\varepsilon)}\rangle \mathrm{d}^2\zeta.\end{aligned}$$

Here, in the second step, we integrated by parts, exchanged differentiation and integration and used $\partial_{\bar\zeta}\langle v_1\rangle = -\partial_\eta\langle\overline{y}\rangle/2$ due to (5.17), (5.1) and (4.7). To see that the ∂_ζ-derivative and the integral in the second step can be exchanged we use the bound $|\partial_{\bar\zeta}\langle v_1\rangle| \lesssim \eta^{-2}$ that follows from (5.21) and Lemma B.1. The third step is a consequence of $\lim_{\eta\to\infty}\langle y(\zeta,\eta)\rangle = 0$ for all $\zeta \in \mathbb{C}$, which follows from the definition of y in (4.5) and the upper bound in (5.5). Owing to (4.5), this shows that the right-hand sides of (4.10) and (4.11) coincide.

What remains is to prove that (4.11) characterises the Brown measure of $a + \mathfrak{c}$. From (2.3), we see that the Brown measure $\sigma = \sigma_{a+\mathfrak{c}}$ of $a + \mathfrak{c}$ coincides with $\frac{1}{2\pi}\Delta \log D(a + \mathfrak{c} - \zeta)$, where Δ denotes the

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distributional Laplacian with respect to ζ. On the other hand, the properties of the Fuglede-Kadison determinant from (2.4) imply

$$D(a+\mathfrak{c}-\zeta) = D(|a+\mathfrak{c}-\zeta|) = \lim_{\eta\downarrow 0}(D(|a+\mathfrak{c}-\zeta|^2+\eta^2))^{1/2}.$$

Hence, integration by parts, (5.19a) and (4.7) yield

$$\int_{\mathbb{C}} \Delta f(\zeta)\sigma(\mathrm{d}^2\zeta) = \lim_{\eta\downarrow 0}\frac{1}{4\pi}\int_{\mathbb{C}} \Delta f(\zeta)\log D(|a+\mathfrak{c}-\zeta|^2+\eta^2)\mathrm{d}^2\zeta = \lim_{\eta\downarrow 0}\frac{1}{\pi}\int_{\mathbb{C}}(\partial_{\bar\zeta} f(\zeta))\langle y(\zeta,\eta)\rangle \mathrm{d}^2\zeta.$$

Note that the Fuglede-Kadison determinant here is monotone in η, justifying the exchange of limit and integration. Thus, owing to (4.5), (4.11) characterises the Brown measure $\sigma_{a+\mathfrak{c}}$, which completes the proof of Proposition 4.3. □

6.2. *Strict positivity in the bulk*

In this subsection we show that the Brown measure has strictly positive density in the bulk, that is, inside $\mathbb{S}$ as defined in (5.32).

Proposition 6.2. *Let $a \in \mathcal{B}$ and s satisfy A1. Then there is $C \sim 1$ such that $-\partial_{\bar\zeta}\langle y(\zeta,\eta)\rangle \in (0, C]$ for all $\zeta \in \mathbb{C}$ and $\eta > 0$.*

Differentiating the last identity in (5.19a) and a straightforward computation using (4.7) and $Y(Y^*Y+\eta^2)^{-1}Y^* = 1-\eta^2(YY^*+\eta^2)^{-1}$ with $Y = a+\mathfrak{c}-\zeta$ yield

$$\begin{aligned}-\partial_{\bar\zeta}\langle y(\zeta,\eta)\rangle &= \eta^2\langle E[(|a+\mathfrak{c}-\zeta|^2+\eta^2)^{-1}(|a+\mathfrak{c}-\zeta|_*^2+\eta^2)^{-1}]\rangle \\ &= \eta^2\langle E[(|a+\mathfrak{c}-\zeta|^2+\eta^2)^{-1/2}(|a+\mathfrak{c}-\zeta|_*^2+\eta^2)^{-1}(|a+\mathfrak{c}-\zeta|^2+\eta^2)^{-1/2}]\rangle,\end{aligned}$$

where we used the abbreviations $|Y|^2 = Y^*Y$ and $|Y|_*^2 = YY^*$. This implies

$$0 < -\partial_{\bar\zeta}\langle y(\zeta,\eta)\rangle \le \eta^{-2}. \tag{6.3}$$

The lower bound $-\partial_{\bar\zeta}\langle y(\zeta,\eta)\rangle \ge 0$ is equivalent to $\zeta \mapsto \log D(|a+\mathfrak{c}-\zeta|^2+\eta^2)$ being subharmonic, which was observed in [37, Lemma 2.8].

Proof of Proposition 6.2. The main work in this proof is to represent $-\partial_{\bar\zeta}\langle y(\zeta,\eta)\rangle$ as the quadratic form of a self-adjoint operator, which we show to be positive and bounded. For $\eta > 0$ and $\zeta \in \mathbb{C}$, we start from the second identity in (5.21) and compute

$$\partial_{\bar\zeta}\langle E_{21}, M\rangle = \left\langle E_{21}\,,\, \mathcal{L}^{-1}E_{21}\right\rangle = \left\langle \mathcal{C}_M^* E_{21}\,,\, E_{21}\right\rangle + \left\langle \mathcal{C}_M^* E_{21}\,,\, (1-\Sigma\mathcal{C}_M)^{-1}\Sigma\mathcal{C}_M E_{21}\right\rangle,$$

where $\mathcal{C}_M R := MRM$ and $\mathcal{C}_M^* = \mathcal{C}_{M^*}$. Note that $1-\Sigma\mathcal{C}_M$ is invertible since $1-\Sigma\mathcal{C}_M = \mathcal{L}\mathcal{C}_M$, $\mathcal{C}_M$ is invertible and $\mathcal{L}$ is invertible by Lemma B.1. With

$$\Sigma\mathcal{C}_M E_{21} = \begin{pmatrix} \mathrm{i}S(v_2\overline{y}) & 0 \\ 0 & \mathrm{i}S^*(v_1\overline{y})\end{pmatrix}, \qquad \mathcal{C}_{M^*}E_{21} = \begin{pmatrix} -\mathrm{i}v_1\overline{y} & \overline{y}^2 \\ -v_1v_2 & -\mathrm{i}v_2\overline{y}\end{pmatrix},$$

and the action of $\Sigma\mathcal{C}_M$ on diagonal matrices in $\mathcal{B}^{2\times 2}$ given by

$$\Sigma\mathcal{C}_M\begin{pmatrix} r_1 & 0 \\ 0 & r_2\end{pmatrix} = \begin{pmatrix} S(|y|^2r_1 - v_2^2r_2) & 0 \\ 0 & S^*(-v_1^2r_1 + |y|^2r_2)\end{pmatrix}$$

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this simplifies to

$$-\partial_{\overline{\zeta}}\langle E_{21}, M\rangle = \frac{1}{2}\langle v_1 v_2\rangle + \left\langle \begin{pmatrix} v_1\overline{y} \\ v_2\overline{y}\end{pmatrix}, (1-Y)^{-1}\begin{pmatrix} S(v_2\overline{y}) \\ S^*(v_1\overline{y})\end{pmatrix}\right\rangle, \tag{6.4}$$

where the scalar product on $\mathcal{B}^2$ is the one from (5.25) and $Y\colon \mathcal{B}^2 \to \mathcal{B}^2$ is defined as

$$Y\begin{pmatrix} r_1 \\ r_2\end{pmatrix} = \begin{pmatrix} S(|y|^2 r_1 - v_2^2 r_2) \\ S^*(-v_1^2 r_1 + |y|^2 r_2)\end{pmatrix} = \begin{pmatrix} SD^2_{|y|} & -SD^2_{v_2} \\ -S^* D^2_{v_1} & S^* D^2_{|y|}\end{pmatrix}\begin{pmatrix} r_1 \\ r_2\end{pmatrix}.$$

Now we introduce a symmetrisation of Y. For this purpose we define $\widehat{v} \in \mathcal{B}$ via

$$\widehat{v} := \sqrt{v_1(\eta + Sv_2)} = \sqrt{v_2(\eta + S^* v_1)}\,, \tag{6.5}$$

where the second identity is due to (4.6), and $V, F, T \in \mathcal{B}^{2\times 2}$ as

$$T := \begin{pmatrix} -\hat{v}^2 & |a-\zeta|^2 \frac{v_1 v_2}{\hat{v}^2} \\ |a-\zeta|^2 \frac{v_1 v_2}{\hat{v}^2} & -\hat{v}^2\end{pmatrix}, \quad V := \begin{pmatrix} \frac{\hat{v}}{v_1} & 0 \\ 0 & \frac{\hat{v}}{v_2}\end{pmatrix}, \quad F := V^{-1} S_o V^{-1} \tag{6.6}$$

analogous to [5, (3.27)]. Note that $|a-\zeta|^2 \frac{v_1 v_2}{\hat{v}^2} = \frac{|y|^2}{v_1 v_2}$ by the definition of y in (4.5). Then $VFTV^{-1} = Y$ and represented in terms of F and T the formula (6.4) reads

$$-2\partial_{\overline{\zeta}}\langle E_{21}, M\rangle = \left\langle \begin{pmatrix} \widehat{v}\overline{y} \\ \widehat{v}\overline{y}\end{pmatrix}, \left(\frac{1}{X} + \frac{2}{1-FT}F\right)\begin{pmatrix} \widehat{v}\overline{y} \\ \widehat{v}\overline{y}\end{pmatrix}\right\rangle, \tag{6.7}$$

where we introduced

$$X := \begin{pmatrix} D\left(\frac{\hat{v}^2}{v_1 v_2}|y|^2\right) & 0 \\ 0 & D\left(\frac{\hat{v}^2}{v_1 v_2}|y|^2\right)\end{pmatrix}. \tag{6.8}$$

In particular, (6.4) is the quadratic form of a self-adjoint operator, evaluated on a vector in the subspace of $\mathcal{B}^2$ with identical entries in the first and second component. With the orthogonal projection onto this subspace represented by

$$E := \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1\end{pmatrix} \in \mathcal{B}^{2\times 2}\,,$$

we have $TE = ET$, $XE = EX$ and

$$T = -1 + 2XE\,. \tag{6.9}$$

The representation (6.9) of T holds because of

$$1 = \widehat{v}^2 + \frac{\widehat{v}^2}{v_1 v_2}|y|^2\,, \tag{6.10}$$

which follows directly from (4.4), (4.5) and the definition of $\widehat{v}$ in (6.5).

Inserting (6.9) into (6.7), we see that proving positivity of the right hand side of (6.7) reduces to proving that the operator

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$$
\begin{aligned}
E\left(\frac{1}{X}+\frac{2}{1+F-2FXE}F\right)E &= E\frac{1}{\sqrt{X}}\left(1+\frac{2}{1+\widetilde{F}\frac{1}{X}-2\widetilde{F}E}\widetilde{F}E\right)\frac{1}{\sqrt{X}}E \\
&= E\frac{1}{\sqrt{X}}\left(1+\frac{1}{1-\frac{2}{1+\widetilde{F}\frac{1}{X}}\widetilde{F}E}\frac{2}{1+\widetilde{F}\frac{1}{X}}\widetilde{F}E\right)\frac{1}{\sqrt{X}}E \\
&= E\frac{1}{\sqrt{X}}\left(1-\frac{2}{1+\widetilde{F}\frac{1}{X}}\widetilde{F}E\right)^{-1}\frac{1}{\sqrt{X}}E = E\frac{1}{\sqrt{X}}\frac{1}{1-K}\frac{1}{\sqrt{X}}E
\end{aligned}
\tag{6.11}
$$

on $\mathcal{B}^2$ is positive definite on the image of E, where we introduced

$$
\widetilde{F} := \sqrt{X}F\sqrt{X} \qquad \text{and} \qquad K := E\frac{2}{1+\widetilde{F}\frac{1}{X}}\widetilde{F}E = E\sqrt{X}\frac{2F}{1+F}\sqrt{X}E \tag{6.12}
$$

in the calculation and used $E^2 = E$ as well as $EX = XE$. Combining (6.7), (6.9), (6.8) and (6.11) yields

$$
-2\partial_{\bar{\zeta}}\langle E_{21}, M\rangle = \left\langle \begin{pmatrix} \overline{e}_y\sqrt{v_1v_2} \\ \overline{e}_y\sqrt{v_1v_2} \end{pmatrix}, \frac{1}{1-K}\begin{pmatrix} \overline{e}_y\sqrt{v_1v_2} \\ \overline{e}_y\sqrt{v_1v_2} \end{pmatrix}\right\rangle, \tag{6.13}
$$

where $e_y := \frac{y}{|y|} \in \mathcal{B}$. We now split the self-adjoint operator $F = F_+ - F_-$ into its positive and negative parts and estimate

$$
\sqrt{X}\left(\frac{2F_+}{1+F_+}-\frac{2F_-}{1-F_-}\right)\sqrt{X} \le \sqrt{X}\left(\frac{2F_+}{1+F_+}\right)\sqrt{X} \le 1 - D_{\widehat{v}}^2, \tag{6.14}
$$

where we used $\|F\| < 1$ for the first inequality. To see this, from the definitions of F in (6.6) and $\widehat{v}$ in (6.5) we read off

$$
F\begin{pmatrix} \widehat{v} \\ \widehat{v} \end{pmatrix} = \begin{pmatrix} \widehat{v} \\ \widehat{v} \end{pmatrix} - \eta\begin{pmatrix} v_1/\hat{v} \\ v_2/\hat{v} \end{pmatrix}.
$$

Since F is a positivity preserving compact operator, there is a positive eigenvector $f = (f_1, f_2)$ with $Ff = \|F\|f$. Taking the scalar product with f yields

$$
\|F\| = 1 - \eta\frac{\langle f_1v_1/\widehat{v}\rangle + \langle f_2v_2/\widehat{v}\rangle}{\langle f_1\widehat{v}\rangle + \langle f_2\widehat{v}\rangle} < 1\,.
$$

The final inequality in (6.14) follows from $\frac{2F_+}{1+F_+} < 1$ because $0 \le F_+ \le \|F\| < 1$ and (6.10). Hence, $1 - K > 0$, which implies $-\partial_{\bar{\zeta}}\langle y(\zeta,\eta)\rangle > 0$. Moreover, we plug (6.14) into (6.13) and use (6.5) as well as (5.7) to obtain

$$
-2\partial_{\bar{\zeta}}\langle E_{21}, M\rangle \le \left\langle \frac{v_2}{\eta + Sv_2}\right\rangle \lesssim 1,
$$

which proves the upper bound on $-\partial_{\bar{\zeta}}\langle y(\zeta,\eta)\rangle$. This completes the proof of Proposition 6.2. □

The next corollary follows from Proposition 4.3 and the upper bound in Proposition 6.2. We recall the definition of $\mathbb{S}_\varepsilon$ from (5.24).

Corollary 6.3. *Let $a \in \mathcal{B}$, s satisfy A1 and σ be the measure from Proposition 4.3. Then* $\operatorname{supp}\sigma \subset \mathbb{S}_0$ *and the measure σ satisfies the identity*

$$
\sigma(\zeta) = -\lim_{\eta\downarrow 0}\frac{1}{\pi}\partial_{\bar{\zeta}}\langle y(\zeta,\eta)\rangle \tag{6.15}
$$

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in the sense of distributions on $\mathbb{C}$*, where* y *is the* $(2,1)$ *component of* M *from (4.7). Moreover,* σ *is absolutely continuous with respect to the Lebesgue measure on* $\mathbb{C}$ *and its density is bounded.*

Owing to (6.13) and (6.15), we see that the Brown measure admits the representation

$$\pi\sigma = \lim_{\eta\downarrow 0} \left\langle \begin{pmatrix} \overline{e}_y\sqrt{v_1v_2} \\ e_y\sqrt{v_1v_2} \end{pmatrix}, \frac{1}{1-K}\begin{pmatrix} \overline{e}_y\sqrt{v_1v_2} \\ e_y\sqrt{v_1v_2} \end{pmatrix} \right\rangle \tag{6.16}$$

in a distributional sense, where K is defined in (6.12).

Proposition 6.4 (Strict positivity of Brown measure on $\mathbb{S}$). *Let* $a \in \mathcal{B}$ *and* s *satisfy A1 and A2. Then the density of the Brown measure* σ *(cf. Corollary 6.3) is strictly positive and real analytic on* $\mathbb{S}$*.*

For the proof of Proposition 6.4 we compute the Brown measure through the formula (6.15), that is, the distributional identity $\pi\sigma = -2\lim_{\eta\downarrow 0}\overline{\partial}_\zeta\langle E_{21}, M\rangle$. First we will see that the right hand side in (6.15) is nonnegative and is in fact positive when evaluated at $\eta > 0$, that is, we prove Proposition 6.2. After that we will see that under assumption A2 the right hand side can be continuously extended to $\eta = 0$ away from $\partial\mathbb{S}$ and remains a bounded function of ζ, that is, σ has a density.

Proof of Proposition 6.4. For the proof of analyticity of σ, we recall the definition of y from (4.5). We conclude from Proposition 5.16 (v), (5.33) and Proposition 5.12 that $\mathbb{S} \to \mathbb{C}$, $\zeta \mapsto y(\zeta, 0)$ is real analytic. Therefore, (6.15) implies that σ is real analytic on $\mathbb{S}$.

To prove a lower bound on σ, we use (6.16) and see that $1 - K$ remains bounded on the image of E as $\eta \downarrow 0$. Indeed by the identity in (6.14) the only contribution to K that may potentially be unbounded is the one associated with F_-. However, $EF_-E|_{\eta=0} \le 1 - \varepsilon$ for some $\varepsilon > 0$ because of the spectral gap of F above -1 in Lemma B.2 and the fact that $(\widehat{v}, -\widehat{v})$, the eigenvector corresponding to eigenvalue -1, is mapped to zero by E. □

6.3. Edge behaviour of the Brown measure

Here we show that σ can be continuously extended to the boundary of $\mathbb{S}$ and compute its boundary values. Throughout this subsection we assume A1 and A2.

Proposition 6.5 (Boundary values of σ). *There exists a real analytic extension of* $\sigma|_{\mathbb{S}}$ *to a neighbourhood of* $\overline{\mathbb{S}}$*. The extension satisfies*

$$\sigma(\zeta_0) = \frac{1}{\pi}\frac{|\langle (a-\zeta_0)\ell_0 b_0\rangle|^2}{\langle |a-\zeta_0|^4\ell_0^2 b_0^2\rangle} = \frac{\langle \ell_0 b_0\rangle^2 |\partial_\zeta \beta(\zeta_0)|^2}{\pi\langle |a-\zeta_0|^4\ell_0^2 b_0^2\rangle} \tag{6.17}$$

for any $\zeta_0 \in \partial\mathbb{S}$*, where* $\ell_0 := \ell|_{\zeta=\zeta_0}$ *and* $b_0 := b|_{\zeta=\zeta_0}$*. Furthermore, for* $\zeta_0 \in \mathrm{Sing}$ *a singular boundary point the extension satisfies* $\partial_\zeta\sigma(\zeta_0) = 0$ *and*

$$\Delta\sigma(\zeta_0) = \frac{32|\langle \ell_0(a-\zeta_0)B_0^{-1}[b_0(a-\zeta_0)]\rangle|^2 + \langle \ell_0 b_0\rangle^2(\Delta\beta(\zeta_0))^2}{2\pi\langle |a-\zeta_0|^4\ell_0^2 b_0^2\rangle}. \tag{6.18}$$

Proof. We use the identity (6.15) to compute σ at some ζ in a sufficiently small neighbourhood of $\zeta_0 \in \partial\mathbb{S}$ in terms of y. We expand y in terms of v_1, v_2 with the help of (5.3) and expand v_1, v_2 in terms of β. In this proof we set $\eta = 0$ and by shifting the spectrum we assume without loss of generality that $\zeta_0 = 0$. In this case, instead of the decomposition (5.56), it is more convenient to write

$$v_1 = \vartheta_1(\ell - \theta w_1), \qquad v_2 = \vartheta_2(b - \theta w_2) \tag{6.19}$$

where we introduced $\theta := \vartheta_1\vartheta_2$ and where due to (5.59) the vectors w_1 and w_2 remain bounded, that is, $\|w_1\|_\infty + \|w_2\|_\infty \lesssim 1$. We already know $\alpha \sim \vartheta_1 \sim \vartheta_2$ due to (5.62) and (5.60) and that ϑ_i are real

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analytic functions of ζ for $\zeta \in \mathbb{S}$ by Proposition 5.18 even for $\eta = 0$. Furthermore, $\vartheta = 0$ at $\eta = 0$ for $\zeta \in \mathbb{C} \setminus \overline{\mathbb{S}}$. For $\zeta \in \overline{\mathbb{S}}$ and $\eta = 0$, the first equation in (5.47) becomes

$$\beta \langle \ell b \rangle + \theta \langle \ell (b - \theta w_2) S^*(\ell - \theta w_1) S(b - \theta w_2) \rangle = 0, \tag{6.20}$$

after projecting onto the left eigenvector $\ell = \ell_\zeta$ of $B = B_\zeta$ and

$$\begin{aligned} w_1 &= (B^{-1})^* Q^* [(\ell - \theta w_1) S^*(\ell - \theta w_1) S(b - \theta w_2)], \\ w_2 &= B^{-1} Q [(b - \theta w_2) S^*(\ell - \theta w_1) S(b - \theta w_2)], \end{aligned} \tag{6.21}$$

after projecting (5.47) onto the complement of the left and right eigendirections, where we recall the projection Q from (5.57). Owing to the implicit function theorem, we see from the structure of the equations (6.20) and (6.21) that β, w_1 and w_2 are locally analytic function of θ for fixed B, and therefore fixed b, ℓ, Q. Furthermore, as $\|w_1\|_\infty + \|w_2\|_\infty \lesssim 1$, (6.20) shows

$$\beta \langle \ell b \rangle + \theta \langle \ell b (S^* \ell)(Sb) \rangle = O(\theta^2) . \tag{6.22}$$

Combining (5.3) with (6.15) we get the formula

$$\pi \sigma = \left\langle \frac{1}{a - \zeta} \partial_{\overline{\zeta}} (v_1 S v_2) \right\rangle . \tag{6.23}$$

Since $v_1 S v_2 = \theta \ell S b + O(\theta^2)$ is an analytic function of θ for fixed B, by the relation (6.22) between θ and β, as well as the fact that β is locally real analytic around $\zeta_0 = 0$ due to Corollary 5.20, we see that σ is the $\overline{\zeta}$-derivative of an analytic function of β for fixed B. Since B also analytically depends on ζ, we conclude that the right hand side of (6.23) can be analytically extended to ζ in a neighbourhood of 0. We denote this extension again by σ. Furthermore, we have

$$\partial_{\overline{\zeta}} (v_1 S v_2) = (\ell S b) \partial_{\overline{\zeta}} \theta + \theta \partial_{\overline{\zeta}} (\ell S b) + O(|\partial_{\overline{\zeta}} \theta| \theta + \theta^2) . \tag{6.24}$$

Evaluating at $\zeta = 0$ yields

$$\pi \sigma|_{\zeta=0} = \left\langle \frac{1}{a} (\ell_0 S b_0) \partial_{\overline{\zeta}} \theta|_{\zeta=0} \right\rangle = -\langle \overline{a} \ell_0 b_0 \rangle \frac{\langle \ell_0 b_0 \rangle}{\langle \ell_0^2 b_0^2 |a|^4 \rangle} \partial_{\overline{\zeta}} \beta|_{\zeta=0}$$

where we used (6.22) and $B_0^* \ell_0 = 0 = B_0 b_0$. With (5.55) the claim (6.17) follows.

Now we assume $\zeta_0 = 0 \in \mathrm{Sing}$ is a singular boundary point. Since β is locally analytic in a neighbourhood of 0 and $\partial_{\overline{\zeta}} \beta(0) = 0$, we have $\beta = O(|\zeta|^2)$ and $\partial_{\overline{\zeta}} \beta = O(|\zeta|)$. Thus, using (6.24), (6.22) and $Bb = \beta b$ we get

$$\partial_{\overline{\zeta}} (v_1 S v_2) = |\zeta - a|^2 \ell b \, \partial_{\overline{\zeta}} \theta + \theta \partial_{\overline{\zeta}} (\ell S b) + O(|\zeta|^3) . \tag{6.25}$$

Since $\partial_{\overline{\zeta}} B = D_{\zeta - a}$, $\beta(\zeta) = 0$, $\partial_\zeta \beta(\zeta) = 0$ and $\ell, b \in \mathcal{B}_+$, we have

$$\partial_{\overline{\zeta}} b|_{\zeta=0} = \overline{\partial_\zeta b}|_{\zeta=0} = B_0^{-1} [a b_0] , \qquad \partial_{\overline{\zeta}} \ell|_{\zeta=0} = \overline{\partial_\zeta \ell}|_{\zeta=0} = (B_0^*)^{-1} [a \ell_0] , \tag{6.26}$$

where all quantities with index 0 are evaluated at $\zeta = \zeta_0 = 0$ and we used $\langle a \ell_0 b_0 \rangle = 0$ because $\partial_\zeta \beta(0) = 0$ (cf. (5.55)) to guarantee the inverses of B_0 and B_0^* can be applied. Furthermore,

$$\begin{aligned} \left\langle \overline{(a - \zeta)} \ell b \right\rangle &= -\overline{\zeta} \langle \ell_0 b_0 \rangle + 2 \langle \overline{a} \, \mathrm{Re} [\zeta \partial_\zeta (\ell b)|_{\zeta=0}] \rangle + O(|\zeta|^2) \\ &= -\overline{\zeta} \langle \ell_0 b_0 \rangle + 2 \langle \overline{a} \, \mathrm{Re} [\zeta \ell_0 B_0^{-1} [\overline{a} b_0] + \zeta b_0 (B_0^*)^{-1} [\overline{a} \ell_0]] \rangle + O(|\zeta|^2) \end{aligned} \tag{6.27}$$

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where we used $\langle a\ell_0 b_0\rangle = 0$ again. Inserting (6.25) into (6.23) and using (6.26), (6.22) and (6.27) we find

$$\begin{aligned}\pi\sigma &= \Big\langle \overline{(a-\zeta)}\ell b\Big\rangle \partial_{\overline{\zeta}}\theta + \left\langle \frac{1}{a-\zeta}\partial_{\overline{\zeta}}(\ell S b)\right\rangle \theta + O(|\zeta|^3)\\ &= \frac{\langle \ell_0 b_0\rangle}{\langle \ell_0^2 b_0^2 |a|^4\rangle}\Big(\big(\overline{\zeta}\langle \ell_0 b_0\rangle - 2\langle \overline{a}\,\mathrm{Re}[\zeta\ell_0\, B_0^{-1}[\overline{a}b_0] + \zeta b_0 (B_0^*)^{-1}[\overline{a}\ell_0]]\rangle\big)\partial_{\overline{\zeta}}\beta \\ &\quad - \left\langle \frac{1}{a}\big((Sb_0)(B_0^*)^{-1}[a\ell_0] + \ell_0 S B_0^{-1}[ab_0]\big)\right\rangle \beta\Big) + O(|\zeta|^3)\end{aligned} \tag{6.28}$$

In particular, $\partial_\zeta \sigma|_{\zeta=0} = 0$. By (5.54) the derivative of β satisfies

$$\partial_{\overline{\zeta}}\beta = \left(1 - 2\,\mathrm{Re}\left[\frac{\langle \ell_0 \overline{a} B_0^{-1}[b_0 a]\rangle}{\langle \ell_0\, b_0\rangle}\right]\right)\zeta - 2\frac{\langle \ell_0 a B_0^{-1}[b_0 a]\rangle}{\langle \ell_0\, b_0\rangle}\overline{\zeta} + O(|\zeta|^2)\,.$$

Using this, $B_0 b_0 = 0$ and the expansion of β from (5.54) in (6.28), we conclude

$$\frac{\pi\langle \ell_0^2 b_0^2 |a|^4\rangle}{8\langle \ell_0 b_0\rangle^2}\Delta\sigma|_{\zeta=0} = 2|R[a,a]|^2 + (1 - R[a,\overline{a}] - R[\overline{a},a])^2 = 2|R[a,a]|^2 + (\partial_\zeta\partial_{\overline{\zeta}}\beta(0))^2,$$

where we introduced

$$R[x,y] := \frac{\langle \ell_0 x B_0^{-1}[b_0 y]\rangle}{\langle \ell_0\, b_0\rangle}\,, \qquad x, y \in \{a, \overline{a}\}\,.$$

With (5.55) this proves (6.18). □

Lemma 6.6. *Let $\zeta \in \mathbb{C}$ such that $\beta(\zeta) = 0$ and $\partial_\zeta\beta(\zeta) = 0$. Then $\Delta\beta(\zeta) < 0$. In particular,*

$$\partial\mathbb{S} = \{\zeta \in \mathbb{C} : \beta(\zeta) = 0\}. \tag{6.29}$$

Proof. Let $\zeta \in \mathbb{C}$ with $\beta(\zeta) = 0$ and $\partial_\zeta\beta(\zeta) = 0$. From (5.55) we read off

$$\langle \ell\, b(a-\zeta)\rangle = 0\,, \qquad \partial_\zeta\partial_{\overline{\zeta}}\beta = 1 - 2\,\mathrm{Re}\left[\frac{\langle \ell\overline{(a-\zeta)}B^{-1}b(a-\zeta)\rangle}{\langle \ell\, b\rangle}\right], \tag{6.30}$$

where we evaluated the expressions in (5.55) at $\zeta_0 = \zeta$ and omitted the projection Q_0 in the formula for $\partial_\zeta\partial_{\overline{\zeta}}\beta$ since $\langle \ell\, b(a-\zeta)\rangle = 0$ implies $Q_0[b(a-\zeta)] = b(a-\zeta)$.

We write $\partial_\zeta\partial_{\overline{\zeta}}\beta$ in terms of

$$K := D\left(\frac{\sqrt{\ell}}{|a-\zeta|\sqrt{b}}\right)SD\left(\frac{\sqrt{b}}{|a-\zeta|\sqrt{\ell}}\right), \qquad x := \sqrt{\ell b}\,\frac{a-\zeta}{|a-\zeta|}$$

and arrive at

$$\partial_\zeta\partial_{\overline{\zeta}}\beta = 1 - 2\,\mathrm{Re}\left[\frac{\langle \overline{x}(1-K)^{-1}x\rangle}{\langle |x|^2\rangle}\right] = -\frac{1}{\langle |x|^2\rangle}\left\langle \frac{1}{1-K}x\,,\,(1-K^*K)\frac{1}{1-K}x\right\rangle.$$

Here we used that $(1-K)^{-1}x$ is well-defined since $x \perp k$ due to (6.30), where $k := |a-\zeta|\sqrt{\ell b}$ is the right and left Perron-Frobenius eigenvector of K, that is, $(1-K)k = 0 = (1-K^*)k$ due to (5.48) with $\beta = 0$. Furthermore, $(1-K^*K)k = 0$ implies that k is the Perron-Frobenius eigenvector of K^*K and thus $1-K^*K$ is strictly positive definite on $k^\perp$, implying $\Delta\beta < 0$. Since β is real analytic in a neighbourhood of ζ with $\beta(\zeta) = \partial_\zeta\beta(\zeta) = 0$ according to Corollary 5.20 and such ζ cannot be a local minimum of β due to $\Delta\beta(\zeta) < 0$ we infer (6.29). □

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As a consequence of Lemma 6.6, the definition of $\mathbb{S}$ in (5.32) and Proposition 5.16 (iv) yield

$$\mathbb{S}_0 = \overline{\mathbb{S}}. \tag{6.31}$$

We now have all ingredients to prove Theorems 2.2 and 2.5.

Proof of Theorem 2.2. Items (i) and (ii) are proved in Propositions 6.4 and 6.5. For the proof of (iii), we conclude $\operatorname{supp}\sigma \subset \mathbb{S}_0 = \overline{\mathbb{S}}$ from Corollary 6.3 and (6.31). Moreover, $\mathbb{S} \subset \operatorname{supp}\sigma$ follows from (ii), which completes the proof of (iii) due to Proposition 5.16 (ii). Note that $\partial\mathbb{S}$ is a real analytic variety due to (6.29) and Corollary 5.20. The dimension of $\partial\mathbb{S}$ is at most one as $\Delta\beta(\zeta) \neq 0$ if $\partial_\zeta\beta(\zeta) = 0$ by Lemma 6.6. This shows (iv). Part (v) follows from Proposition 6.5 and the fact that ℓ_0 and b_0 and, thus, g are real analytic by Corollary 5.20. □

Proof of Theorem 2.5. Let $\zeta \in \operatorname{Sing}$ be a singular point of σ, that is, $\beta(\zeta) = 0$ and $\partial_\zeta\beta(\zeta) = 0$. By Lemma 6.6 we have $\Delta\beta(\zeta) < 0$ and by (6.18) also $\Delta\sigma(\zeta) > 0$. Now we apply the following lemma about critical points of real analytic functions in two dimensions with positive Laplacian, which simplifies to the well-known *Morse lemma* (see, for example, [46, Lemma 2.2]) if the critical point is nondegenerate.

Lemma 6.7 (Higher order Morse lemma). *Let $f : U \to \mathbb{R}$ be a real analytic function on an open set $U \subset \mathbb{C}$ and $\zeta_0 \in U$. Suppose that $\partial_\zeta f(\zeta_0) = 0$ and $\Delta f(\zeta_0) > 0$. Then there is a real analytic diffeomorphism $\Phi : V \to U_0$ from an open neighbourhood $V \subset \mathbb{C}$ of zero to an open neighbourhood $U_0 \subset U$ of $\zeta_0 = \Phi(0)$, a natural number $K \geq 2$ and $\tau \in \{0, \pm 1\}$, such that*

$$f \circ \Phi(z) = f(\zeta_0) + (\operatorname{Re} z)^2 + \tau(\operatorname{Im} z)^K, \qquad \text{for all } z \in V.$$

The proof of Lemma 6.7 is given in Appendix A. Since $-\beta$ satisfies the assumptions of Lemma 6.7 at ζ and $\beta(\zeta) = 0$ we conclude that $-\beta$ is of singularity type $x^2 + \tau y^K$ at ζ, where we used the terminology from Definition 2.4. If $\tau = -1$ or $\tau = 1$ and odd K, then $\zeta \in \partial\overline{\mathbb{S}} = \partial \operatorname{supp}\sigma$. In this case $\zeta \in \operatorname{Sing}^{\text{edge}}_{K-1}$.

Otherwise, that is, if $\tau = 0$ or $\tau = 1$ and even K, ζ lies in the interior of $\overline{\mathbb{S}}$. In this case σ satisfies the conditions of Lemma 6.7 at ζ with $\sigma(\zeta) = 0$, as $\partial_\zeta\sigma(\zeta) = 0$ because of Proposition 6.5. Thus, σ is either of singularity type $x^2 + y^{2n}$ for some $n \in \mathbb{N}$ or of singularity type x^2 at ζ. We note that the singularity types $x^2 - y^n$ and $x^2 + y^{2n+1}$ are not possible since σ is a real analytic function in a neighbourhood of ζ by Proposition 6.5, which is nonnegative at ζ as ζ lies in the interior of $\overline{\mathbb{S}}$. In case σ is of singularity type $x^2 + y^{2n}$, we have $\zeta \in \operatorname{Sing}^{\text{int}}_{2n}$ and ζ is an isolated point of $\mathbb{C} \setminus \mathbb{S}$. If σ is of singularity type x^2, then $\zeta \in \operatorname{Sing}^{\text{int}}_{\infty}$. In this case the fact that σ is a restriction of a real analytic function on an open neighbourhood of $\overline{\mathbb{S}}$ by Theorem 2.2 (ii) implies that the connected component of $\operatorname{Sing}^{\text{int}}_{\infty}$ containing ζ is a closed analytic path without self-intersections. □

7. Existence of singularity types

In this section we provide examples that show that all possible singularity types that are allowed by Theorem 2.5 appear in the Brown measures of suitably chosen deformations of standard circular elements, that is, we prove Theorem 2.7. The proof of the theorem is summarised before Subsection 7.1. We work with the case $\mathcal{B} = L^\infty[0,1]$ and such that $\mathfrak{c}$ is a standard circular element, that is, $E[\mathfrak{c}^* x \mathfrak{c}] = E[\mathfrak{c} x \mathfrak{c}^*] = \langle x \rangle$ for $x \in \mathcal{B}$. We fix $a \in \mathcal{B}$. Hence, a and $\mathfrak{c}$ are ordinary $*$-free in $(\mathcal{A}, \langle\cdot\rangle)$ and a is automatically normal. In this setup $B_\zeta x = |a - \zeta|^2 x - \langle x \rangle$ for all $x \in \mathcal{B}$ and the Dyson equation (4.4) simplifies to

$$\kappa = \eta + \left\langle \frac{\kappa}{\kappa^2 + |\zeta - a|^2} \right\rangle, \qquad v_1 = v_2 = \frac{\kappa}{\kappa^2 + |\zeta - a|^2}, \tag{7.1}$$

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where $\kappa = \kappa(\zeta, \eta) = \eta + \langle v_1(\zeta, \eta)\rangle \geq 0$. To apply our theory, we will choose $a\colon [0,1] \to \mathbb{C}$ such that

$$\operatorname{ess\,inf}_{x\in[0,1]} \int_{[0,1]} \frac{\mathrm{d}y}{|a(x) - a(y)|^2} = \infty\,.$$

For $\zeta \in \mathbb{S}$ we have $\kappa_0 = \kappa(\zeta, 0) > 0$ and get

$$\beta(\zeta) = \inf_{x\in\mathcal{B}_+} \sup_{y\in\mathcal{B}_+} \frac{\langle x\,, (B_\zeta + \kappa_0^2)y\rangle}{\langle x\,, y\rangle} - \kappa_0^2 \geq -\kappa_0^2$$

for β from (2.9), by choosing for y the function $b := (|a-\zeta|^2 + \kappa_0^2)^{-1}$, for which $(B_\zeta + \kappa_0^2)b = 0$ due to (7.1) and $\eta = 0$. Similarly, we get $\beta(\zeta) \leq -\kappa_0^2$ by choosing $x = b$. Thus, $\beta(\zeta) = -\kappa(\zeta, 0)^2$ for $\zeta \in \mathbb{S}$ and

$$\left\langle \frac{1}{|a-\zeta|^2 - \beta(\zeta)} \right\rangle = 1\,. \tag{7.2}$$

In particular,

$$\left\langle \frac{1}{(|a-\zeta|^2 - \beta)^2} \right\rangle \partial_{\overline{\zeta}}\beta = \left\langle \frac{\zeta - a}{(|a-\zeta|^2 - \beta)^2} \right\rangle. \tag{7.3}$$

Thus, we express the Brown measure σ on $\mathbb{S}$ in terms of β, using (6.23), (7.2) and (7.3), through

$$\begin{aligned} \pi\sigma &= -\left\langle \frac{1}{a-\zeta}\partial_{\overline{\zeta}} \frac{\beta}{|a-\zeta|^2 - \beta} \right\rangle \\ &= \left\langle \frac{-\beta}{(|a-\zeta|^2 - \beta)^2} \right\rangle - \left\langle \frac{1}{a-\zeta}\left(\frac{1}{|a-\zeta|^2 - \beta} + \frac{\beta}{(|a-\zeta|^2 - \beta)^2} \right) \right\rangle \partial_{\overline{\zeta}}\beta \\ &= \left\langle \frac{1}{(|a-\zeta|^2 - \beta)^2} \right\rangle (-\beta + |\partial_{\overline{\zeta}}\beta|^2)\,. \end{aligned} \tag{7.4}$$

The previous formulas have been derived earlier, see, for example, [13, 18, 42, 64]. From (7.2) and (2.10), we see that $\mathbb{S}$ is given by the equation $\mathbb{S} = \{\zeta \in \mathbb{C} : f(\zeta) > 1\}$, where

$$f(\zeta) := \int \frac{\nu(\mathrm{d}\omega)}{|\zeta - \omega|^2} = \left\langle \frac{1}{|\zeta - a|^2} \right\rangle \tag{7.5}$$

and ν is the spectral measure of a, that is, ν is the unique measure on $\mathbb{C}$ for which $\int \zeta^k \overline{\zeta}^l \nu(\mathrm{d}\zeta) = \langle a^k \overline{a}^l\rangle$. We interpret $f(\zeta) = \infty$ for $\zeta \in \operatorname{supp}\nu$. We note that

$$\Delta f(\zeta) = \left\langle \frac{4}{|a-\zeta|^4} \right\rangle > 0$$

for all $\zeta \notin \operatorname{supp}\nu$.

In the following we consider deformations a such that ν is symmetric with respect to the imaginary axis. In this case f, β and σ inherit this symmetry from ν. In particular, $f(\zeta) = f(-\overline{\zeta})$ for all $\zeta \in \mathbb{C}$. We will make use of the following corollary of Lemma 6.7, which is proven at the end of Appendix A.

Corollary 7.1 (Higher order Morse lemma with symmetry). *Let $f : U \to \mathbb{R}$ be a real analytic function on an open set $U \subset \mathbb{C}$ with $0 \in U$. Suppose that $\partial_\zeta f(0) = 0$ and $\Delta f(0) > 0$. Assume furthermore, that f is symmetric with respect to the imaginary axis, that is, $f(\zeta) = f(-\overline{\zeta})$ for all $\zeta \in U$ such that $-\overline{\zeta} \in U$. If $\partial^2_{\mathrm{Re}\,\zeta} f(0) > 0$, then the diffeomorphism Φ from Lemma 6.7 with $\Phi(0) = 0$ can be chosen*

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such that $D\Phi(0)$ is positive definite and diagonal. Furthermore, the exponent $K \geq 2$ and the sign τ in the statement of Lemma 6.7 are

$$K := 1 + \sup\{k \in \mathbb{N} : \partial^l_{\operatorname{Im}\zeta} f(0) = 0 \quad \textit{for all } l \leq k\}\,, \qquad \tau = \begin{cases} \operatorname{sign} \partial^K_{\operatorname{Im}\zeta} f(0) & \textit{if } K < \infty\,, \\ 0 & \textit{if } K = \infty\,. \end{cases}$$

The following lemma connects the singularity types of f, β and σ in this case. It shows that in order to prove Theorem 2.7, it suffices to show that ν from (7.5) can be chosen such that $f - 1$ has a given singularity type of the form $x^2 + \tau y^K$ with $\tau \in \{0, \pm 1\}$ and $K \in \mathbb{N}$ with $K \geq 2$ at 0.

Lemma 7.2. *Let f be defined as in (7.5), assume that $f(\zeta) = f(-\overline{\zeta})$ for all $\zeta \in \mathbb{C}$, that $\partial^2_{\operatorname{Re}\zeta} f(0) > 0$ and that $f(0) = 1$. Then $0 \in \partial\mathbb{S}$ and the singularity types of $f - 1$ and $-\beta$ agree at 0. If $0 \in \operatorname{Sing}$ and $-\beta$ is of singularity type $x^2 + y^{2n}$ at 0 for some $n \in \mathbb{N}$, then σ is also of singularity type $x^2 + y^{2n}$.*

Proof. For $f(0) = 1$ we have $0 \in \partial\mathbb{S}$ by definition of $\mathbb{S}$. Then (7.3) implies

$$\left\langle \frac{1}{|a - \zeta|^4} \right\rangle \partial_\zeta \beta = -\partial_\zeta f\,. \tag{7.6}$$

Evaluating at $\zeta = 0$ shows that $\partial_\zeta \beta = 0$ if and only if $\partial_\zeta f = 0$, proving that the singularity types of $f - 1$ and $-\beta$ agree for a regular edge point $0 \in \operatorname{Reg}$. We now denote $x = \operatorname{Re}\zeta$ and $y = \operatorname{Im}\zeta$ and assume $0 \in \operatorname{Sing}$. Then f and $-\beta$ satisfy the assumptions in Corollary 7.1. By applying the derivative ∂_y repeatedly to (7.3) and evaluating at $\zeta = 0$, we see that the exponent K and the sign τ from Corollary 7.1 for f and $-\beta$ agree.

Finally, we show that the singularity types of σ and $-\beta$ at 0 agree. From (7.4) we see that

$$\partial_\zeta \sigma(0) = 0\,, \qquad \partial_w^2 \sigma(0) = \frac{1}{\pi}\left(-\partial_w^2 \beta(0) + \frac{1}{4}(\partial_w^2 \beta(0))^2\right), \tag{7.7}$$

where $w = x, y$ since β is an even function of x and $f(0) = 1$. In particular $\Delta\sigma(0) > 0$ and $\partial_x^2 \sigma(0) > 0$ because $-\partial_x^2 \beta(0) = \partial_x^2 f(0) > 0$ by (7.3). We conclude that σ also satisfies the assumptions of Corollary 7.1. Suppose $-\beta$ has singularity type $x^2 + y^2$ at 0, then $\partial_y^2 \sigma(0) > 0$ by (7.7). Now suppose $-\beta$ has singularity type $x^2 + y^{2n}$ at 0 for some $n \geq 2$. By repeated application of the derivative ∂_y on (7.4) and evaluating at $\zeta = 0$ we see that

$$\partial_y^k \sigma(0) = -\frac{1}{\pi} \partial_y^k \beta(0)\,, \qquad k = 3, \ldots, 2n\,,$$

which implies that σ is of singularity type $x^2 + y^{2n}$ at 0. □

To construct singularity types of f at 0 of the form $x^2 \pm y^K$ with a finite $K \geq 2$, we will choose ν with finite support, that is, $\operatorname{supp} \nu = \{a_1, \ldots, a_N\}$ with distinct nonzero $a_i \in \mathbb{C}$ and $\nu_i = \nu(\{a_i\})$. In this case

$$f(\zeta) = \sum_{i=1}^{N} \frac{\nu_i}{|\zeta - a_i|^2}\,. \tag{7.8}$$

To guarantee symmetry of f with respect to the imaginary axis we write f as a sum of functions of the form

$$g_a(\zeta) := \frac{1}{2}\left(\frac{1}{|\zeta + 1/a|^2} + \frac{1}{|\zeta - 1/\overline{a}|^2}\right), \tag{7.9}$$

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with $a \in \mathbb{C}_> := \{z \in \mathbb{C} : \operatorname{Re} z > 0, \operatorname{Im} z > 0\}$ a complex number in the first quadrant. For even K we can even restrict to f being a sum of functions of the form

$$f_a(\zeta) := \frac{1}{2}\big(g_a(\zeta) + g_{\overline{a}}(\zeta)\big) \tag{7.10}$$

with $a \in \mathbb{C}_>$. With these definitions any convex combination of functions of the form g_a and f_a are of the form (7.8).

In the following subsections we construct an example for each singularity type that is allowed by Theorem 2.5. The result of these constructions is summarised in the following lemma.

Lemma 7.3. *The following examples exist.*

(a) *Quadratic singularity types: With the choice* $\nu = \frac{1}{2}(\delta_1 + \delta_{-1})$ *the function* $f - 1$*, where* f *is from (7.5), has singularity type* $x^2 - y^2$ *at* 0*. The function* $f - 1$ *with* $f(\zeta) := f_a(\zeta)$ *and* $a := \frac{1}{\sqrt{2}}(1 + \mathrm{i})$ *has singularity type* $x^2 + y^2$ *at* 0*.*

(b) *Singularity types of even y-power: Let* $n \in \mathbb{N}$ *with* $n \geq 2$ *and* $\tau \in \{\pm 1\}$*. There are* $c_1, \dots, c_n > 0$ *with* $\sum_{i=1}^n c_i = 1$ *and distinct* $a_1, \dots, a_n \in \mathbb{C}_>$*, such that the function* $f - 1$ *with*

$$f(\zeta) := \sum_{i=1}^{n} c_i f_{a_i}(\zeta)$$

has singularity type $x^2 + \tau y^{2n}$ *at* 0*.*

(c) *Singularity types of odd y-power: Let* $n \in \mathbb{N}$*. There are* $c_1, \dots, c_{2n+2} > 0$ *with* $\sum_{i=1}^{2n+2} c_i = 1$ *and distinct* $a_1, \dots, a_{2n+2} \in \mathbb{C}_>$*, such that the function* $f - 1$ *with*

$$f(\zeta) := \sum_{i=1}^{2n+2} c_i g_{a_i}(\zeta)$$

has singularity type $x^2 + y^{2n+1}$ *at* 0*.*

(d) *Singularity types of infinite order: There is a choice* $r > 0$ *and* $\alpha > 0$ *such that for* $\nu = \frac{1}{2}\nu_{\mathrm{c}} + \frac{1}{2}\delta_{\mathrm{i}\alpha}$*, where* ν_{c} *is the uniform distribution on the circle* $\mathrm{i}\alpha + r S^1$*, and* f *from (7.5) the function* $f - 1$ *has singularity type* x^2 *at* 0*.*

Lemma 7.3 (a) is easily verified by a simple calculation. The other parts of Lemma 7.3 are proven in each of the following subsections, that is, we prove Part (b) in Subsection 7.1, Part (c) in Subsection 7.2 and Part (d) in Subsection 7.3. With these ingredients we now prove our main result about the existence of all singularity types.

Proof of Theorem 2.7. Since all examples from Lemma 7.3 have ν which is symmetric with respect to the imaginary axis, we can apply Lemma 7.2 in each of these examples. To see the existence of a singularity of the type $\mathrm{Sing}_1^{\mathrm{int}}$, we take the example with $f(\zeta) := f_a(\zeta)$ and $a := \frac{1}{\sqrt{2}}(1+\mathrm{i})$ from Lemma 7.3 (a). By Lemma 7.2 the singularity type of $f - 1$ coincides with the singularity type of $-\beta$ and also with the singularity type of σ, showing $0 \in \mathrm{Sing}_1^{\mathrm{int}}$. To see an example with $0 \in \mathrm{Sing}_k^{\mathrm{int}}$ and order $k \geq 2$, we consider Lemma 7.3 (b) with $\tau = +1$. Again by Lemma 7.2 the singularity type of $f - 1$ coincides with the singularity type of σ.

Now we show that $\mathrm{Sing}_k^{\mathrm{edge}}$ with order $k \in \mathbb{N}$ is not empty for some example. For the case $k = 1$ the example is taken from Lemma 7.3 (a) with the choice $\nu = \frac{1}{2}(\delta_1 + \delta_{-1})$, for odd $k > 1$ from Lemma 7.3 (b) with $\tau = -1$ and for even $k > 1$ from Lemma 7.3 (c). In each case the singularity types of $f - 1$ and $-\beta$ at 0 coincide due to Lemma 7.2.

Finally $0 \in \mathrm{Sing}_\infty^{\mathrm{int}}$ for the example from Lemma 7.3 (d) since here the singularity types of $f - 1$ and σ coincide by Lemma 7.2. □

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7.1. Singularity types of even y-power

Here we prove Lemma 7.3 (b). Since $f_{ca}(0) = c^2 f_a(0)$ for f_a defined as in (7.10) and $c > 0$, it suffices to show that $f - f(0)$ has singularity type $x^2 + \tau y^{2n}$ at 0, that is, we can drop the condition $f(0) = 1$. Furthermore, we can consider f of the form

$$f(\zeta) := \sum_{i=1}^{n} c_i \frac{\operatorname{Re} a_i}{|a_i|^2} f_{a_i}(\zeta)\,,$$

with $a_i \in \mathbb{C}_>$, $c_i > 0$ and show that this f has singularity type $x^2 + \tau y^{2n}$ at 0. This is sufficient since such f can be transformed into a convex combination of the f_{a_i} by dividing by $\sum_{i=1}^{n} c_i \frac{\operatorname{Re} a_i}{|a_i|^2}$ and without changing the singularity type. Properties of the function f_a are summarised in Lemma C.3. The symmetries of f_a (see (C.2)) in particular imply that its Hessian at $\zeta = 0$ is diagonal. Furthermore, f satisfies $\partial_y^{2k+1} f(0) = 0$ for all $k \in \mathbb{N}_0$ due to (C.1), where $\zeta = x + \mathrm{i}y$. It remains to show that a_i and c_i can be chosen such that $\partial_y^{2k} f(0) = 0$ for all $k = 1, \dots, n-1$ and $\operatorname{sign} \partial_y^{2n} f(0) = \tau$ for a given $\tau \in \{\pm 1\}$. Together with the symmetry of $f(\zeta) = f(-\overline{\zeta})$ around the imaginary axis and since $\Delta f(0) > 0$ this is enough to guarantee the local behaviour because of Corollary 7.1. By the formula (C.1) for $\partial_y^{2k} f_a(0)$ the choices a_i and c_i defining f have to be made such that

$$\operatorname{sign}\left((-1)^n \operatorname{Re} \sum_{i=1}^{n} c_i a_i^{2n+1}\right) = \tau\,, \qquad \operatorname{Re} \sum_{i=1}^{n} c_i a_i^{2k+1} = 0\,, \qquad k = 1, \dots, n-1\,.$$

That such a choice exists is ensured by the following lemma, where we replaced τ by $(-1)^n \tau$ to avoid tracking the additional sign.

Lemma 7.4 (Discrete truncated moment problem for even case). *For each $n \in \mathbb{N}$ and $\tau \in \{\pm 1\}$ there are positive numbers $c_1, \dots, c_n > 0$ and distinct complex numbers $z_1, \dots, z_n \in \mathbb{C}_>$ such that*

$$\operatorname{Re} \sum_{i=1}^{n} c_i z_i^{2n+1} = \tau\,, \qquad \operatorname{Re} \sum_{i=1}^{n} c_i z_i^{2k+1} = 0\,, \qquad \text{for all } \; k = 1, \dots, n-1\,.$$

Proof. We perform the proof by induction over n. For $n = 1$ we choose $z_1 \in \mathbb{C}$ with $\operatorname{Im} z_1 > 0, \operatorname{Re} z_1 > 0$ and $\operatorname{Re} z_1^3 = \tau$. Then we set $c_1 := 1$. Now suppose for $n \in \mathbb{N}$ with $n \geq 2$ we have positive numbers $c_1, \dots, c_n \in (0, \infty)$ and distinct complex numbers $\widehat{z}_1, \dots, \widehat{z}_{n-1} \in \mathbb{C}_>$ such that

$$\operatorname{Re} \sum_{i=1}^{n-1} c_i \widehat{z}_i^{2n-1} = \pm 1\,, \qquad \operatorname{Re} \sum_{i=1}^{n-1} c_i \widehat{z}_i^{2k+1} = 0\,, \qquad k = 1, \dots, n-2\,. \tag{7.11}$$

The construction works for either choice of sign in the induction hypothesis. We will cover both choices simultaneously. Since we can always normalise the coefficients, it suffices to show that there are $c_n > 0$ and $z_i \in \mathbb{C}_>$ in the first quadrant such that

$$\operatorname{sign} \operatorname{Re} \sum_{i=1}^{n} c_i z_i^{2n+1} = \tau\,, \qquad \operatorname{Re} \sum_{i=1}^{n} c_i z_i^{2k+1} = 0\,, \qquad k = 1, \dots, n-1\,. \tag{7.12}$$

To see that this is possible we pick $\omega \in \mathbb{C}$ with $|\omega| = 1, \operatorname{Re} \omega > 0$ and $\operatorname{Im} \omega > 0$ such that $\operatorname{sign} \operatorname{Re} \omega^{2n-1} = \mp 1$ and $\operatorname{sign} \operatorname{Re} \omega^{2n+1} = \tau$. Then we set

$$z_n(r) := \frac{\omega}{r}\,, \qquad c_n(r) := \frac{r^{2n-1}}{|\operatorname{Re} \omega^{2n-1}|}$$

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for $r > 0$, as well as

$$F_k(w) := \sum_{i=1}^{n-1} c_i w_i^{2k+1}, \qquad G_k(w,r) := F_k(w) + r^{2n-2k-2} \frac{\omega^{2k+1}}{|\operatorname{Re} \omega^{2n-1}|}$$

for $w = (w_1, \dots, w_{n-1})$, $k \in [\![n-1]\!]$ and $r \geq 0$. In particular, $G_k(w,r) := F_k(w) + c_n(r) z_n(r)^{2k+1}$ for $r > 0$. For small enough $r > 0$ we now solve the system

$$G_k(w,r) = G_k(\widehat{z}, 0), \qquad k \in [\![n-1]\!]$$

for $w = z(r)$. Since $\partial_{w_i} G_k(w) = \partial_{w_i} F_k(w)$ this system is locally uniquely solvable by the implicit function theorem due to

$$\det(\partial_{w_i} F_k(w))_{i,k=1}^{n-1} = \det((2k+1) c_i w_i^{2k})_{i,k=1}^{n-1} = \prod_{k=1}^{n-1} (2k+1) \prod_{i=1}^{n-1} c_i w_i^2 \prod_{i<j} (w_j^2 - w_i^2),$$

which does not vanish for $w = \widehat{z}$, because by assumption the $\widehat{z}_i$ are distinct. With $z_i = z_i(r)$ the system of equations (7.12) is now satisfied. Indeed, $\operatorname{Re} G_k(w,r) = \operatorname{Re} F_k(\widehat{z}, 0) = 0$ for $k \in [\![n-2]\!]$ by (7.11), as well as $\operatorname{Re} G_{n-1}(w,r) = \operatorname{Re} G_{n-1}(\widehat{z}, 0) = 0$ by (7.11) and the choice of the sign of $\operatorname{Re} \omega^{2n-1}$. Finally

$$\operatorname{sign} \operatorname{Re} \sum_{i=1}^{n} c_i z_i(r)^{2n+1} = \operatorname{sign} \operatorname{Re} \left(O(1) + \frac{1}{r^2} \frac{\operatorname{Re} \omega^{2n+1}}{|\operatorname{Re} \omega^{2n-1}|} \right) = \tau$$

by the choice of the sign of $\operatorname{Re} \omega^{2n+1}$ and for sufficiently small $r > 0$. For small $r > 0$ the $z_i(r)$ with $i \in [\![n]\!]$ are also distinct. □

7.2. Singularity types of odd y-power

Now we prove prove Lemma 7.3 (c). As in the treatment of singularity types of even order, it suffices to show that $f - f(0)$ with f of the form

$$f(\zeta) := \sum_{i=1}^{2n+2} c_i \frac{\operatorname{Re} a_i}{|a_i|^2} g_{a_i}(\zeta) + \frac{1}{|\zeta + \mathrm{i}|^2} \tag{7.13}$$

where $a_1, \dots, a_{2n+2} \in \mathbb{C}_>$ and $c_1, \dots, c_{2n+2} > 0$ has singularity type $x^2 - y^{2n+1}$ at 0. The function g_a is defined in (7.9) and its properties are summarised in Lemma C.3.

To construct such singularity type we choose f such that it satisfies

$$\partial_y^{2n+1} f(0) < 0, \qquad \partial_y^k f(0) = 0, \qquad k = 1, \dots, 2n \tag{7.14}$$

and apply Corollary 7.1. The symmetry $f(\zeta) = f(-\overline{\zeta})$ follows from our ansatz (7.13) and $\Delta f(0) > 0$ because $\Delta g_a(0) = 4|a|^4$ by Lemma C.3 and $\Delta |z + \mathrm{i}|^{-2}|_{z=0} = 4$. With

$$\partial_y^{2k-1} \frac{1}{|z+\mathrm{i}|^2}\bigg|_{z=0} = -(2k)!, \qquad \partial_y^{2k} \frac{1}{|z+\mathrm{i}|^2}\bigg|_{z=0} = (2k+1)!,$$

as well as the formulas for the derivatives of g_a from (C.3) the conditions (7.14) on our ansatz (7.13) translate to

$$(-1)^n \operatorname{Im} \sum_{i=1}^{n} c_i a_i^{2(n+1)} < 2(n+1), \quad (-1)^{k+1} \operatorname{Im} \sum_{i=1}^{n} c_i a_i^{2k} = 2k, \quad (-1)^{k+1} \operatorname{Re} \sum_{i=1}^{n} c_i a_i^{2k+1} = 2k+1$$

for all $k \in [\![n]\!]$. Such a choice of c_i and a_i exists due to the following lemma.

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Lemma 7.5 (Discrete truncated moment problem for odd case). *For each $n \in \mathbb{N}$ there are positive numbers $c_1, \dots, c_{2n+2} > 0$ and distinct complex numbers $z_1, \dots, z_{2n+2} \in \mathbb{C}_>$ such that*

$$(-1)^{k+1} \operatorname{Im} \sum_{i=1}^{2n+2} c_i z_i^{2k} = 2k\,, \qquad (-1)^{k+1} \operatorname{Re} \sum_{i=1}^{2n+2} c_i z_i^{2k+1} = 2k+1\,, \qquad k = 1, \dots, n\,, \tag{7.15}$$

as well as

$$(-1)^{n} \operatorname{Im} \sum_{i=1}^{2n+2} c_i z_i^{2(n+1)} < 2(n+1)\,. \tag{7.16}$$

Proof. We perform an induction over n. For the case $n = 1$ we first find $z \in \mathbb{C}_>$ and $c > 0$ such that

$$c \operatorname{Im} z^2 = 2\,, \qquad c \operatorname{Re} z^3 = 3\,, \qquad -c \operatorname{Im} z^4 < 4\,.$$

The choice $z = \frac{3}{4}\sqrt{2}\mathrm{e}^{\mathrm{i}\pi/12}$ and $c = \frac{32}{9}$ satisfies these conditions. Now we set $c_1 := c$ and fix three additional complex numbers $z_2, z_3, z_4 \in \mathbb{C}_>$ in the first quadrant that are all distinct and distinct from z_1. We also set $c_i := \delta$ for $i = 2, 3, 4$, where $\delta > 0$ is chosen sufficiently small. Then the relations

$$\operatorname{Im} \sum_{i=1}^{4} c_i z_i^2 = 2\,, \qquad \operatorname{Re} \sum_{i=1}^{4} c_i z_i^3 = 3\,, \qquad -\operatorname{Im} \sum_{i=1}^{4} c_i z_i^4 < 4$$

with $z_1 = z_1(\delta)$ are satisfied for $\delta = 0$ with $z_1(0) = z$ and by the implicit function theorem can also be satisfied for small $\delta > 0$. In fact the two identities have the form $c \operatorname{Im} z_1^2 = 2 + O(\delta)$ and $c \operatorname{Re} z_1^3 = 3 + O(\delta)$ and the map $z \mapsto (\operatorname{Im} z^2, \operatorname{Re} z^3) = (2xy, x^3 - xy^2)$ with $z = x + \mathrm{i}y$ has the nonvanishing Jacobian

$$\det \begin{pmatrix} 2y & 2x \\ 3x^2 - 3y^2 & -6xy \end{pmatrix} = -6x(x^2 + y^2)\,.$$

For the induction step we assume that $c_1, \dots, c_{2n} > 0$ and $\widehat{z}_1, \dots, \widehat{z}_{2n} \in \mathbb{C}_>$ are given such that

$$(-1)^{k+1} \operatorname{Im} \sum_{i=1}^{2n} c_i \widehat{z}_i^{2k} = 2k\,, \quad (-1)^{k+1} \operatorname{Re} \sum_{i=1}^{2n} c_i \widehat{z}_i^{2k+1} = 2k+1\,, \quad (-1)^{n-1} \operatorname{Im} \sum_{i=1}^{2n} c_i \widehat{z}_i^{2n} < 2n \tag{7.17}$$

for all $k = 2, \dots, n-1$. With the short hand notation $w = (w_1, \dots, w_{2n})$ we define

$$F_k(w) := \sum_{i=1}^{2n} c_i w_i^k\,, \qquad k = 2, \dots, 2n+2\,.$$

We now construct an additional $c_{2n+1}(r) = \alpha(r) r^{2n} > 0$ and $z_{2n+1}(r) = \frac{1}{r}\mathrm{e}^{\mathrm{i}\varphi(r)}$ with functions $\varphi = \varphi(r) \in (0, \pi/2)$ and $\alpha = \alpha(r) > 0$ for $r > 0$ that we choose appropriately such that

$$(-1)^{n+1} \big(\operatorname{Im} F_{2n}(\widehat{z}) + \alpha \sin(2n\varphi) \big) = 2n\,, \quad (-1)^{n+1} \Big(\operatorname{Re} F_{2n+1}(\widehat{z}) + \frac{\alpha}{r} \cos((2n+1)\varphi) \Big) = 2n+1\,, \tag{7.18}$$

as well as

$$(-1)^{n} \Big(\operatorname{Im} F_{2n+2}(\widehat{z}) + \frac{\alpha}{r^2} \sin((2n+2)\varphi) \Big) < 2n+1 \tag{7.19}$$

are satisfied for sufficiently small $r > 0$. To do so we define $\varphi_* := \frac{\pi}{2} - \frac{1}{2n+1}\pi$. Then $\cos((2n+1)\varphi_*) = 0$ and $\sin(2n\varphi_*) = \sin((2n+2)\varphi_*)$ because $2n\varphi_* + (2n+2)\varphi_* = \pi(2n-1)$ is an odd multiple of π.

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Furthermore, we have the inequality $(-1)^{n+1}\sin(2n\varphi_*) = \sin\left(\frac{2n}{2n+1}\pi\right) > 0$. Now let

$$\sigma := \operatorname{sign}\left((-1)^{n+1}\operatorname{Re} F_{2n+1}(\widehat{z}) - (2n+1)\right).$$

We set $\varphi(r) := \varphi_* + \sigma\beta(r)r$ for some appropriately chosen $\beta(r) > 0$ that is bounded and bounded away from zero. In particular, we have the expansion $(-1)^{n+1}\cos((2n+1)\varphi(r)) = -\sigma(2n+1)r + O(r^2)$. The identities (7.18) become

$$\alpha(r) = \frac{2n - (-1)^{n-1}\operatorname{Im} F_{2n}(\widehat{z})}{(-1)^{n+1}\sin(2n\varphi(r))} \tag{7.20}$$

and

$$(-1)^{n+1}\operatorname{Re} F_{2n+1}(\widehat{z}) - (2n+1) - \sigma(2n+1)\beta(r)\alpha(r) = \begin{cases} 0 & \text{if } \sigma = 0\,, \\ O(r) & \text{if } \sigma \neq 0\,. \end{cases} \tag{7.21}$$

For $\sigma = 0$ there is no need to choose β since $\varphi(r) = \varphi_*$ and we set $\alpha(r) := \alpha_* > 0$, satisfying

$$\alpha_* = \frac{2n - (-1)^{n-1}\operatorname{Im} F_{2n}(\widehat{z})}{(-1)^{n+1}\sin(2n\varphi_*)}\,, \tag{7.22}$$

in this case. That α_* is positive follows from the choice of φ_* and (7.17). For $\sigma \neq 0$ we solve the system (7.20) and (7.21). By the definition of σ, φ_* and (7.17) there is a solution $\alpha(0) = \alpha_* > 0$ and $\beta(0) = \beta_* > 0$ for $r = 0$. In fact, α_* is determined by (7.22) and β_* then by evaluating (7.21) on $r = 0$. By the implicit function theorem this solution can be extended to sufficiently small $r > 0$. Thus, (7.18) is satisfied for our choice of $\alpha(r)$ and $\varphi(r)$. Since $\alpha(0) = \alpha_* > 0$ and $(-1)^n\sin((2n+2)\varphi(0)) = (-1)^n\sin(2n\varphi_*) < 0$, by the choice of φ_*, the inequality (7.19) is also satisfied for sufficiently small $r > 0$.

Now we define

$$G_k(w, r) := F_k(w) + c_{2n+1}(r)z_{2n+1}(r)^k\,, \qquad k = 2, \dots, 2n+2$$

for sufficiently small $r > 0$. By the choices of the functions φ and α above, we can analytically extend $G_k(w, r)$ to a neighbourhood of $r = 0$ for $k = 2, \dots, 2n+1$. By construction we have

$$(-1)^{n+1}\operatorname{Im} G_{2n}(\widehat{z}, r) = 2n\,, \qquad (-1)^{n+1}\operatorname{Re} G_{2n+1}(\widehat{z}, r) = 2n+1\,, \qquad (-1)^n \operatorname{Im} G_{2n+2}(\widehat{z}, r) < 2n+1$$

and $G_{k+1}(w, 0) = F_{k+1}(w)$ for $k \in [\![2n-2]\!]$. Now we solve the system of equations $G_{k+1}(z(r), r) = F_{k+1}(\widehat{z})$ for $k \in [\![2n-2]\!]$ together with $G_{2n}(z(r), r) = G_{2n}(\widehat{z}, 0)$ and $G_{2n+1}(z(r), r) = G_{2n+1}(\widehat{z}, 0)$ for a solution $z = (z_1, \dots, z_{2n})$ with $z(0) = \widehat{z}$. This is possible in a neighbourhood of $r = 0$ by the implicit function theorem since

$$\det(\partial_{w_i} F_{k+1}(w))_{i,k=1}^{2n} = \det((k+1)c_i w_i^k)_{i,k=1}^{2n} = \prod_{k=1}^{2n}(k+1)\prod_{i=1}^{2n} c_i w_i \prod_{i<j}(w_j - w_i)\,, \tag{7.23}$$

which does not vanish when evaluated at $w = \widehat{z}$.

Altogether we have found distinct $z_1, \dots, z_{2n+1} \in \mathbb{C}_>$ and $c_{2n+1} > 0$ such that (7.15) and (7.16) are satisfied with $c_{2n+2} = 0$. Finally we add $z_{2n+2} \in \mathbb{C}_>$ distinct from all $z_1, \dots, z_{2n+1}$. We set $z_* := (z_1, \dots, z_{2n})$ and $c_{2n+2} = \delta$ and find a choice $z(\delta) \in \mathbb{C}_>^{2n}$ for $\delta \geq 0$ with $z(0) = z_*$ such that this choice also satisfies (7.15) and (7.16). The choice of c_{2n+1} and z_{2n+1} remains independent of δ. That such choice exists follows again from (7.23) and the implicit function theorem. □

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7.3. Singularity types of infinite order

In this subsection we prove Lemma 7.3 (d). As before it suffices to show that $f - f(0)$ is of singularity type x^2, that is, we drop the condition $f(0) = 1$. Furthermore, by shifting ν we can place the singularity at $\mathrm{i} \in \mathrm{Sing}_\infty^{\mathrm{int}}$. We follow Example 3.5 and choose a as in that example. A short computation using (7.2) reveals that $\mathbb{S}$ satisfies (3.2). Thus, since $\partial\mathbb{D}_1 \cap \mathbb{S} = \emptyset$ while points on both sides of $\partial\mathbb{D}_1$ belong to $\mathbb{S}$, we conclude from the analyticity of β that $\partial_\zeta \beta(\zeta) = \partial_{\bar\zeta}\beta(\zeta) = 0$ for all $\zeta \in \partial\mathbb{D}_1$. Hence, $\partial\mathbb{D}_1 \subset \mathrm{Sing}_\infty^{\mathrm{int}}$. Furthermore, using (4.4) and (6.15), we find that σ is given by (3.3) in this example. We refer to Example 3.5 for more explanations and to Figure 3 for a plot of the boundary of $\mathbb{S}$ and some sampled eigenvalues.

8. Circular element with general deformation

In this section, we consider the Brown measure σ of $a + \mathfrak{c}$, where $\mathfrak{c}$ is a standard circular element and a is a general operator in a tracial von Neumann algebra that is $*$-free from $\mathfrak{c}$. We prove the classification of singular points stated in Remark 2.8. Here we use the notation $|x|$ and $|x|_*$ for the positive semidefinite self-adjoint elements defined by $|x|^2 = xx^*$ and $|x|_*^2 = x^*x$, where x is a general operator. The Brown measure in this setup has a density which satisfies the identity

$$\pi\sigma(\zeta) = \left\langle \frac{1}{|a-\zeta|^2+\kappa^2} \frac{\kappa^2}{|a-\zeta|_*^2+\kappa^2} \right\rangle + \left\langle \frac{1}{(|a-\zeta|^2+\kappa^2)^2} \right\rangle^{-1} \left| \left\langle \frac{1}{(|a-\zeta|^2+\kappa^2)^2}(a-\zeta) \right\rangle \right|^2, \tag{8.1}$$

for all $\zeta \in \mathbb{S}$, which was derived earlier, see, for example, [42, equation (16)], [18, Section 4.4], [13, Theorem 7.10] and [32, equation (4.25)]. Here $\kappa = \kappa(\zeta) \geq 0$ is determined by the well-known equation

$$\left\langle \frac{1}{|a-\zeta|^2+\kappa^2} \right\rangle = 1 \tag{8.2}$$

for $\zeta \in \mathbb{S}$, see, for example, [42, equation (17)], [64, Lemma 3.6] and [32, equation (4.19)]. We note that $|\zeta - a|^2 \geq \|(a-\zeta)^{-1}\|^{-2}$ in the sense of quadratic forms for all $\zeta \in \mathbb{C} \setminus \mathrm{Spec}(a)$.

The boundary of $\mathrm{supp}\,\sigma$ is contained in $\partial\mathbb{S} = \{\zeta \in \mathbb{C} \colon f(\zeta) = 1\}$ with $f(\zeta) = \langle |a-\zeta|^{-2}\rangle$, that is, in this setup $f(\zeta) - 1$ plays the same role as $-\beta(\zeta)$ in our setup above. Since $\Delta f(\zeta) > 0$ for any $\zeta \notin \mathrm{Spec}(a)$, the classification of edge points $\zeta_0 \in \partial\overline{\mathbb{S}} \cap \mathrm{Sing} = \bigcup_{n=2}^\infty \mathrm{Sing}_n^{\mathrm{edge}}$ from Remark 2.8 follows from Lemma 6.7. In particular, for such ζ_0 the function $f - 1$ is of singularity type $x^2 - y^n$ at ζ_0.

To see that a point $\zeta_0 \in \partial\mathbb{S}$ in the interior of $\overline{\mathbb{S}}$ satisfies $\zeta_0 \in \mathrm{Sing}_k^{\mathrm{int}} \cup \mathrm{Sing}_\infty^{\mathrm{int}}$, that is, σ is of singularity type $x^2 + y^{2k}$ or x^2 at ζ_0, we realise that $\Delta\sigma(\zeta_0) > 0$ at such ζ_0 and use Lemma 6.7 again. Indeed, as $\zeta \mapsto \kappa(\zeta)$ is real analytic on $\mathbb{S}$ by [64, Lemma 3.6], differentiating (8.2) shows

$$\left\langle \frac{1}{(|a-\zeta|^2+\kappa^2)^2} \right\rangle \partial_\zeta \kappa^2 = \left\langle \frac{1}{(|a-\zeta|^2+\kappa^2)^2}(a-\zeta)^* \right\rangle \tag{8.3}$$

for $\zeta \in \mathbb{S}$.

By the implicit function theorem, we conclude from (8.2) and (2.12) that $\zeta \mapsto \kappa(\zeta)^2$ can be extended to a real analytic function in a neighbourhood of $\overline{\mathbb{S}}$. The nonnegativity of κ^2 on $\overline{\mathbb{S}}$ and $\kappa(\zeta_0) = 0$ imply that κ^2 has a local minimum at ζ_0 and, thus, $\partial_\zeta \kappa^2(\zeta_0) = 0$. Hence, evaluating (8.3) at $\zeta_0 \in \partial\mathbb{S} \setminus \partial\overline{\mathbb{S}}$ yields that the right hand side of (8.3) vanishes. Therefore, differentiating (8.3) with respect to $\bar\zeta$ shows

$$\left\langle \frac{1}{|a-\zeta_0|^4} \right\rangle \partial_\zeta \partial_{\bar\zeta} \kappa^2(\zeta_0) = \left\langle \frac{1}{|a-\zeta_0|^2} \frac{1}{|a-\zeta_0|_*^2} \right\rangle$$

and, thus, $\Delta\kappa^2(\zeta_0) > 0$. Moreover, differentiating (8.1) with respect to ζ and $\bar\zeta$ and using $\partial_\zeta \kappa^2(\zeta_0) = 0 = \kappa^2(\zeta_0)$, as well as (8.3) shows

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$$\pi\partial_\zeta\partial_{\bar\zeta}\sigma(\zeta_0)=\left\langle\frac{1}{|a-\zeta_0|^2}\frac{1}{|a-\zeta_0|_*^2}\right\rangle\partial_\zeta\partial_{\bar\zeta}\kappa^2(\zeta_0)+\left\langle\frac{1}{|a-\zeta|^4}\right\rangle^{-1}\left|\partial_\zeta\left\langle\frac{1}{(|a-\zeta|^2+\kappa^2)^2}(a-\zeta)\right\rangle\right|^2_{\zeta=\zeta_0}.$$

Hence, $\Delta\sigma(\zeta_0)=4\partial_\zeta\partial_{\bar\zeta}\sigma(\zeta_0)>0$ as $\Delta\kappa^2(\zeta_0)>0$.

A. Proof of higher order Morse lemma

Proof of Lemma 6.7. Without loss of generality we assume $\zeta_0=0$ and $f(0)=0$. We denote $x=\operatorname{Re}\zeta$ and $y=\operatorname{Im}\zeta$. Furthermore, by rotating the coordinates (x,y) so that they coincide with the eigendirections of the Hessian $\operatorname{Hess}(f)$ of f, we may assume $f(x,y)=\alpha x^2+\beta y^2+O(|x|^3+|y|^3)$, where 2α and 2β are the eigenvalues of $\operatorname{Hess}(f)$. Here we used $f(0)=0$, $\partial f(0)=0$. Since $\Delta f(0)>0$, we may assume $\alpha>0$ by potentially exchanging the roles of x and y. By rescaling we can also assume $\alpha=1$. In case $\beta\neq 0$ we use the Morse lemma (see, for example, [46, Lemma 2.2]) to bring $\widetilde f:=f\circ\Phi$ into the form $\widetilde f=x^2\pm y^2$.

For the remainder of the proof, we assume $\beta=0$ and tacitly shrink the neighbourhood of 0 whenever necessary. In the case when $\beta=0$ we see that

$$f(x,y)=x^2(1+xg(x,y)+yh(y))+xy^2j(y)+y^3k(y)\,, \tag{A.1}$$

where g,h,j,k are real analytic functions. Now we define a local real analytic diffeomorphism through the ansatz $\Phi_1(x,y):=(x+y^2\psi_1(y),y)$, where the real analytic function ψ_1 will be chosen appropriately below. Then $f_1:=f\circ\Phi_1$ has the form

$$f_1(x,y)=x^2(1+xg_1(x,y)+yh_1(y))+xy^2j_1(y)+y^3k_1(y)$$

with

$$\begin{aligned}
g_1(x,y)&=g(x+y^2\psi_1(y),y)+3y^2\psi_1(y)(\partial_1 g)(y^2\psi_1(y),y)+3y^2(\psi_1(y)x+y^2\psi_1(y)^2)(\partial_1^2 g)(y^2\psi_1(y),y)/2\\
&\quad+(3y^2\psi_1(y)x^2+3y^4\psi_1(y)^2x+y^6\psi_1(y)^3)\widetilde g(x,y),\\
h_1(y)&=h(y)+3y\psi_1(y)g(y^2\psi_1(y),y)+3y^3\psi_1(y)^2(\partial_1 g)(y^2\psi_1(y),y)+y^5\psi_1(y)^3(\partial_1^2 g)(y^2\psi_1(y),y)/2,\\
j_1(y)&=j(y)+\psi_1(y)(2+2yh(y)+3y^2g(y^2\psi_1(y),y)\psi_1(y)+y^4\psi_1(y)^2\partial_1 g(y^2\psi_1(y),y))\,,\\
k_1(y)&=k(y)+y\psi_1(y)(j(y)+\psi_1(y)(1+yh(y)+y^2g(y^2\psi_1(y),y)\psi_1(y)))\,,
\end{aligned}$$

where ∂_1 denotes the partial derivative with respect to the first argument and $\widetilde g$ is defined through the expansion

$$g(x+y^2\psi_1(y),y)=g(y^2\psi_1(y),y)+x(\partial_1 g)(y^2\psi_1(y),y)+x^2(\partial_1^2 g)(y^2\psi_1(y),y)/2+x^3\widetilde g(x,y).$$

By the implicit function theorem, we find a real analytic function $\psi_1(y)$ such that $j_1(y)=0$ for all small enough y. In particular, $\psi_1(y)=(-\frac12+O(y))j(y)$.

Hence, f_1 simplifies to

$$f_1(x,y)=x^2(1+xg_1(x,y)+yh_1(y))+y^3k_1(y)\,,$$

where $k_1(y)=\tau y^{K-3}\psi_2(y)$ for some $\tau\in\{0,\pm1\}$, $K\geq 3$ and a real analytic function ψ_2 with $\psi_2(0)>0$. We define the local real analytic diffeomorphism $\Phi_2(x,y):=(x,y\psi_2(y)^{1/K})$. Then $f_2:=f\circ\Phi_2^{-1}$ is of the form

$$f_2(x,y)=x^2(1+xg_2(x,y)+yh_2(y))+\tau y^K$$

for some real analytic functions g_2 and h_2. Finally, we choose the local real analytic diffeomorphism $\Phi_3(x,y):=(x(1+xg_2(x,y)+yh_2(y))^{1/2},y)$ such that $\widetilde f:=f_2\circ\Phi_3^{-1}$ is of the form claimed in the lemma. The diffeomorphism from the statement of the lemma is $\Phi:=\Phi_1\circ\Phi_2^{-1}\circ\Phi_3^{-1}$. □

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Proof of Corollary 7.1. As in the proof of Lemma 6.7 the case $\beta \neq 0$ and $\alpha \neq 0$ is clear, where 2α and 2β are the two eigenvalues of $\mathrm{Hess}(f)$. Thus, by rescaling it suffices to consider $\beta = 0$ and $\alpha = 1$. As f is symmetric with respect to the imaginary axis, we have the representation

$$f(x, y) = f(0) + x^2 p(x^2, y) + y^3 k(y)\,,$$

for real analytic functions p and k with $p(0,0) = 1$. In particular, $j(y) = 0$ in the representation (A.1). Hence, from the proof of Lemma 6.7 we know that $\psi_1 \equiv 0$ and $k(y) = k_1(y) = \tau y^{K-3}\psi_2(y)$ with some real analytic ψ_2 such that $\psi_2(0) > 0$. Choosing the diffeomorphism $\Phi^{-1}(x, y) := (x\sqrt{p(x^2, y)}, y\psi_2(y)^{1/K})$ shows the claim. □

B. Stability of Dyson equation

Let M be the solution of (5.16). The stability operator of (5.16) is given by (5.20).

Lemma B.1. *Let $a \in \mathcal{B}$, $s\colon \mathfrak{X} \times \mathfrak{X} \to [0, \infty)$ a bounded measurable function and $\mathcal{L}$ defined as in (5.20). Then $\mathcal{L}$ is invertible for any $\eta > 0$ and satisfies*

$$\|\mathcal{L}^{-1}\|_\infty \leq \frac{1}{\eta^2}\,. \tag{B.1}$$

Proof. We write $\mathcal{L}$ as a directional derivative of $M := M_R(\zeta, \mathrm{i}\eta)$, where

$$M_R(\zeta, \mathrm{i}\eta) := E\left[\left(\begin{pmatrix} -\mathrm{i}\eta & Y \\ Y^* & -\mathrm{i}\eta \end{pmatrix} - R\right)^{-1}\right], \qquad Y := a + \mathfrak{c} - \zeta\,.$$

We note that for $\eta > 0$ the map $R \mapsto M_R(\zeta, \mathrm{i}\eta)$ is differentiable in a neighbourhood of $R = 0$. Let $\widetilde{R} \in \mathcal{B}^{2\times 2}$ be arbitrary. Taking the directional derivative $\nabla_{\widetilde{R}}$ with respect to R in the direction $\widetilde{R}$ yields

$$\nabla_{\widetilde{R}} M = E\left[\begin{pmatrix} -\mathrm{i}\eta & Y \\ Y^* & -\mathrm{i}\eta \end{pmatrix}^{-1} \widetilde{R} \begin{pmatrix} -\mathrm{i}\eta & Y \\ Y^* & -\mathrm{i}\eta \end{pmatrix}^{-1}\right]. \tag{B.2}$$

On the other hand M satisfies a modified version of the Dyson equation (4.3), namely

$$-\frac{1}{M} = R + \begin{pmatrix} \mathrm{i}\eta & \zeta - a \\ \zeta - a & \mathrm{i}\eta \end{pmatrix} + \Sigma[M]\,.$$

Taking the directional derivative $\nabla_{\widetilde{R}}$ of this equation shows $\mathcal{L}\nabla_{\widetilde{R}} M = \widetilde{R}$. Since $\widetilde{R}$ was arbitrary, we conclude that $\mathcal{L}$ is invertible and $\mathcal{L}^{-1}\widetilde{R} = \nabla_{\widetilde{R}} M$. The bound (B.1) now follows from the representation (B.2). □

Proof of Lemma 5.15. We recall that $\zeta \in \mathbb{C}$ is fixed such that $\limsup_{\eta \downarrow 0} \langle v_1(\zeta, \eta)\rangle \geq \delta$ for some $\delta > 0$ and that $v^{(n)} = v(\zeta, \eta_n) \to v_0$ weakly in $(L^2)^2$, where $v_0 \sim_\delta 1$. First we use the identities

$$\mathscr{L} v_- = -\eta \frac{\tau v^2}{(\eta + S_d v)^2}\,, \qquad \mathscr{L}^*(e_-(\eta + S_o v)) = \eta e_-$$

with $v_- = v e_-$ and $v = v^{(n)}$. In the limit $\eta \downarrow 0$ we see $\mathscr{L}_0 v_- = 0$ and $\mathscr{L}_0^* S_o v_- = 0$. Here we used that $v = v_0$ satisfies the Dyson equation, (4.4) at $\eta = 0$. For the rest of this proof we drop the 0-index from our notation. We introduce $T, V, F : \mathcal{B}^2 \to \mathcal{B}^2$ as in (6.6), evaluated at $v = v_0$ and $\eta = 0$. In terms of T, F and V we obtain

$$\mathscr{L} = V^{-1}(1 - TF)V\,. \tag{B.3}$$

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We consider the natural extensions $F\colon (L^2)^2 \to (L^2)^2$ and $T\colon (L^2)^2 \to (L^2)^2$. These operators are self-adjoint and we use their spectral properties presented in Lemma B.2 below. The proof of this lemma is postponed to after the proof of Lemma 5.15.

Lemma B.2 (Spectral properties of F and T). *The Hermitian operator $F\colon (L^2)^2 \to (L^2)^2$ satisfies the following properties:*

(i) *F has nondegenerate isolated eigenvalues at ± 1 and a spectral gap $\varepsilon \sim_\delta 1$, that is,*

$$\operatorname{Spec}(F) \subset \{-1\} \cup [-1+\varepsilon, 1-\varepsilon] \cup \{1\}\,. \tag{B.4}$$

(ii) *The eigenvectors corresponding to the eigenvalues ± 1 are*

$$FVv = Vv\,, \qquad FVv_- = -Vv_-\,, \tag{B.5}$$

where $v = (v_1, v_2)$ and $v_- = (v_1, -v_2)$.

The Hermitian operator $T\colon (L^2)^2 \to (L^2)^2$ satisfies the following properties:

(iii) *The spectrum is bounded away from 1 by a gap of size $\varepsilon \sim_\delta 1$, that is,*

$$\operatorname{Spec}(T) \subset [-1, 1-\varepsilon]\,.$$

(iv) *For $x \in \{(y,-y) : y \in L^2\}$ we have $Tx = -x$.*

We now prove Lemma 5.15 by tacitly using the properties of F and T from Lemma B.2. Since $f_- := Vv_- \in \{(y,-y) : y \in \mathcal{B}\}$ we see that F and T both leave the subspace $(f_-)^\perp$ invariant. Now, using (B.3), we rewrite the resolvent of $\mathscr{L}$ as

$$\frac{1}{\mathscr{L}-\omega} = V^{-1}\frac{1}{1-TF-\omega}V\,.$$

The operator V and its inverse satisfy the bound $\|V\|_\# + \|V^{-1}\|_\# \lesssim_\delta 1$ for $\# = 2, \infty$, where we used $\delta \lesssim v \lesssim \frac{1}{\delta}$ from Lemma 5.13. Thus, it suffices to show (5.30) with $\mathscr{L}$ replaced by $1-TF$. Furthermore, we can restrict to the case $\# = 2$, since [9, Lemma 4.5] is applicable because

$$\|TF\|_\infty + \|TF\|_{\infty\to 2}\|TF\|_{2\to\infty} \lesssim_\delta 1\,.$$

Here, $\|TF\|_{\infty\to 2}$ and $\|TF\|_{2\to\infty}$ denote the operator norms of TF viewed as operator $\mathcal{B}^2 \to (L^2)^2$ and $(L^2)^2 \to \mathcal{B}^2$, respectively.

Since $\|TF\|_2 \le 1$ we have the bound $\|(1-TF-\omega)^{-1}\|_2 \lesssim_{\varepsilon,\delta} 1$ for any $\omega \notin 1+\mathbb{D}_{1+\varepsilon}$ and any $\varepsilon > 0$. Now we can decompose

$$\frac{1}{1-TF-\omega} = (1-P)\frac{1}{1-TF-\omega}(1-P) - \frac{1}{\omega}P\,, \tag{B.6}$$

where P is the orthogonal projection onto the span of f_-. Provided $\varepsilon > 0$ is chosen sufficiently small, the first summand in (B.6) is bounded for $\omega \in \mathbb{D}_{2\varepsilon}$. Indeed $1-TF$ has a bounded inverse on $f_-^\perp$ by applying Lemma B.3 with $H = (L^2)^2$. Finally, the second summand in (B.6) is bounded on $\mathbb{C}\setminus\mathbb{D}_\varepsilon$. Altogether the bound (5.30) is proven. The nondegeneracy of the eigenvector v_- of $\mathscr{L}$ also follows from the decomposition (B.6). □

Proof of Lemma B.2. The identities (B.5) are verified by using the definitions of V and F from (6.6). That the eigenvalues ± 1 are nondegenerate and the existence of the spectral gap in (B.4) are seen by studying

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$$F^2 = \begin{pmatrix} RR^* & 0 \\ 0 & R^*R \end{pmatrix}, \qquad R := D_{\frac{v_1}{\widehat{v}}} S D_{\frac{v_2}{\widehat{v}}} .$$

As we now show, both RR^* and R^*R have spectrum contained in $[0, 1-\varepsilon] \cup \{1\}$ for some $\varepsilon \sim_\delta 1$, where 1 is a nondegenerate eigenvalue. We show this for RR^*. The argument for R^*R is analogous. The operator $RR^* : \mathcal{B} \to \mathcal{B}$ has a kernel representation

$$(RR^*u)(x) = \int_{\mathfrak{X}} t_1(x,y) u(y) \mu(\mathrm{d}y)$$

with a symmetric non-negative kernel

$$t_1(x,y) = \frac{v_1(x)}{\widehat{v}(x)} \int_{\mathfrak{X}} s(x,q) \frac{v_2(q)^2}{\widehat{v}(q)^2} s(y,q) \mu(\mathrm{d}q) \frac{v_1(y)}{\widehat{v}(y)}$$

because of (2.5). Since $v = v_0 \sim_\delta 1$ in our regime due to Lemma 5.13, assumption A1 implies

$$t_1(x,y) \sim_\delta \sum_{i,j,l=1}^{K} z_{il} z_{jl} \mathbb{1}_{I_i \times I_j}(x,y) . \tag{B.7}$$

Now let t_k be the kernel of $(RR^*)^k$ for any $k \in \mathbb{N}$. Since the matrix $Z = (z_{ij})_{i,j=1}^K$ is primitive with positive diagonal there is a $k \in \mathbb{N}$ such that $t_k(x,y) \sim_\delta 1$ for μ-almost all $x, y \in \mathfrak{X}$. Indeed, by (B.7) we have

$$t_k(x,y) \sim_\delta \sum_{i,j}^{K} ((ZZ^*)^k)_{ij} \mathbb{1}_{I_i \times I_j}(x,y) ,$$

where the entries of $(ZZ^*)^k$ are all positive for large enough $k \in \mathbb{N}$ due to the primitivity of Z and its positive diagonal. By [2, Lemma 5.7] with the necessary eigenfunction from (B.5), we conclude that $(RR^*)^k$ has a spectral gap $\sim_\delta 1$ and therefore so does RR^*. A similar argument to the one presented here has been used in [4, Lemma 3.3].

Now we show the spectral properties of T. The proof follows the strategy from [5, Lemma 3.6]. Since the argument is short we present it here for the convenience of the reader. The property (iv) is obvious from the definition of T in (6.6). It remains to see that $\mathrm{Spec}(T) \subset [-1, 1-\varepsilon]$. Since restricted to vectors $(y,-y)$, the operator T is -1 it suffices to consider the restriction of T to vectors (y,y). Here we find $T(y,y) = (D_p y, D_p y)$ with $p := |a-\zeta|^2 \frac{v_1 v_2}{\widehat{v}^2} - \widehat{v}^2$. Recalling the definition of $\widehat{v}$ in (6.5) we see that

$$2\frac{|a-\zeta|^2 v_2}{S v_2} - 1 = p = 1 - 2 v_2 S^* v_1,$$

where we used the Dyson equation (4.4) at $\eta = 0$. Since $v_i \sim_\delta 1$ this shows the claim about the spectrum of T. □

Lemma B.3 (Contraction-Inversion Lemma). *Let $\varepsilon > 0$ and T, F be two bounded self-adjoint operators on a Hilbert space H such that $\|T\| \le 1$ and $\|F\| \le 1$. Suppose that there are normalized vectors $f_\pm \in H$ satisfying*

$$F f_+ = \|F\| f_+, \quad F f_- = -\|F\| f_-, \quad \|Fx\| \le (1-\varepsilon)\|x\| \tag{B.8}$$

for all $x \in H$ such that $x \perp \mathrm{span}\{f_+, f_-\}$.

Furthermore, assume that

$$\langle f_+, T f_+ \rangle \le 1 - \varepsilon, \quad T f_- = -f_-. \tag{B.9}$$

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Then $1-TF$ maps $f_-^\perp$ into itself and its restriction to $f_-^\perp$ is invertible such that there is a constant $c>0$, depending only on ε, with

$$\|(1-TF)x\| \geq c\|x\|. \tag{B.10}$$

for all $x \perp f_-$.

Proof. First we see that TF maps $f_-^\perp$ into itself. Indeed, let $h \in H$ with $\|h\| = 1$ and $h \perp f_-$. Then $\langle f_-, TFh\rangle = -\langle f_-, Fh\rangle = 0$ because of (B.9). Now we write $h = \beta f_+ + \gamma x$ for some normalized $x \perp \operatorname{span}\{f_+, f_-\}$ and $\beta, \gamma \in \mathbb{C}$ with $|\beta|^2 + |\gamma|^2 = 1$. From here on we follow the proof of [5, Lemma 3.7] with the notational identifications $\mathbf{A} \leftrightarrow T$ and $\mathbf{B} \leftrightarrow F$. □

Lemma B.4 (Twist lemma). *Let H be a Hilbert space with a scalar product $\langle\cdot\,,\cdot\rangle$ and equipped with a norm $\|\cdot\|_\#$ (not necessarily induced by the scalar product), $\varepsilon \in (0,1)$ and A a bounded linear operator on $(H, \langle\cdot\,,\cdot\rangle)$ such that $\overline{\mathbb{D}}_\varepsilon \cap \operatorname{Spec} A = \{\alpha\}$. We assume that α is a nondegenerate eigenvalue of A and $Ax = \alpha x$ for some $x \in H$ with $\|x\|_\# = 1$. Let*

$$P := -\frac{1}{2\pi\mathrm{i}} \oint_{\partial\mathbb{D}_\varepsilon} \frac{\mathrm{d}\zeta}{A-\zeta} = \langle p\,,\cdot\rangle x\,, \tag{B.11}$$

with some $p \in H$ be the corresponding spectral projection and $y \in H$ a vector such that

$$|\langle x\,, y\rangle| \geq 2\varepsilon\,, \qquad |\langle y\,, w\rangle| \leq \|w\|_\#\,, \qquad \forall\, w \in H\,. \tag{B.12}$$

Suppose that A has a bounded inverse on the range of $1-P$ with respect to the norm $\|\cdot\|_\#$, that is,

$$\|Aw\|_\# \geq \|w\|_\#\,, \qquad \forall\, w \perp p\,. \tag{B.13}$$

Then A has a bounded inverse when restricted to $y^\perp$, namely

$$\|Aw\|_\# \geq \frac{\varepsilon}{3}\|w\|_\#\,, \qquad \forall\, w \perp y\,. \tag{B.14}$$

Proof. The proof follows the one of [9, Lemma 4.6] line by line, replacing $\mathbb{C}^d$ by H. □

C. Auxiliary results

Lemma C.1. *Let $y>0$ be the unique solution to the equation $y^3 + \beta y = x$ for $x>0$ and $\beta \in \mathbb{R}$. Then*

$$y \sim \sqrt{\max\{0,-\beta\}} + \frac{x}{x^{2/3}+|\beta|}\,.$$

Proof. First we consider the case $\beta \geq 0$. Then clearly

$$y \sim \frac{x}{x^{2/3}+|\beta|}\,.$$

Now let $\beta<0$, then we must have $y = \sqrt{-\beta}(1+\varepsilon)$ for some $\varepsilon>0$ since $y^2+\beta y>0$. For this ε we get the equation $\varepsilon^3 + 3\varepsilon^2 + 2\varepsilon = x|\beta|^{-3/2}$. Thus, we have the scaling

$$\varepsilon \sim \min\left\{\frac{x}{|\beta|^{3/2}}, \frac{x^{1/3}}{|\beta|^{1/2}}\right\}.$$

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Therefore we conclude

$$y = \sqrt{|\beta|} + \min\left\{\frac{x}{|\beta|}, x^{1/3}\right\} \sim \sqrt{|\beta|} + \frac{x}{x^{2/3} + |\beta|},$$

which is the claim of the lemma. □

Lemma C.2. *Let $S\colon \mathcal{B} \to \mathcal{B}$ be an integral operator as in (2.5) with a kernel $s : \mathfrak{X}^2 \to (0, \infty)$ that satisfies the bounds $\varepsilon \le s(x, y) \le \frac{1}{\varepsilon}$ for all $x, y \in \mathfrak{X}$ and some constant $\varepsilon > 0$ and such that the spectral radius of S is normalised to $\varrho(S) = 1$. Then there are constants $\delta, C > 0$, depending only on ε, such that*

$$\sup\left\{\|(S - z)^{-1}\|_2 \colon z \notin D_{1-\delta}(0) \cup D_\delta(1)\right\} \le C$$

and $\operatorname{Spec}(S) \cap D_\delta(1) = \{1\}$ *is nondegenerate.*

Proof. We follow line by line the arguments from the proof of [33, Lemma A.1], where the finite dimensional case with $|\mathfrak{X}| < \infty$ and μ the counting measure is carried out. □

Lemma C.3 (Properties of f_a and g_a). *Let $a \in \mathbb{C}_>$, as well as $f_a\colon \mathbb{C} \setminus \{1/a, 1/\overline{a}, -1/a, -1/\overline{a}\} \to \mathbb{R}$ and $g_a\colon \mathbb{C} \setminus \{-1/a, 1/\overline{a}\} \to \mathbb{R}$ be defined as in (7.10) and (7.9), respectively. Then f_a and g_a are real analytic. Moreover, the Laplacian and derivatives of f_a along the imaginary axis are*

$$\Delta f_a(0) = 4|a|^4, \qquad \partial_y^{2k+1} f_a(0) = 0, \qquad \partial_y^{2k} f_a(0) = (-1)^k (2k)! |a|^2 \frac{\operatorname{Re} a^{2k+1}}{\operatorname{Re} a} \tag{C.1}$$

for all $k \in \mathbb{N}_0$, where $\zeta = x + \mathrm{i}y$. Furthermore, f_a has the symmetries

$$f_a(\zeta) = f_a(-\zeta) = f_a(\overline{\zeta}) = f_a(-\overline{\zeta}). \tag{C.2}$$

The Laplacian and derivatives of g_a along the imaginary axis are

$$\Delta g_a(0) = 4|a|^4,\ \partial_y^{2k-1} g_a(0) = (-1)^{k+1}(2k-1)!|a|^2 \frac{\operatorname{Im} a^{2k}}{\operatorname{Re} a},\ \partial_y^{2k} g_a(0) = (-1)^k (2k)! |a|^2 \frac{\operatorname{Re} a^{2k+1}}{\operatorname{Re} a} \tag{C.3}$$

and g_a has the symmetry $g_a(\zeta) = g_a(-\overline{\zeta})$.

Proof. To verify the formulas for the derivatives of f_a and g_a we first compute

$$\partial_\zeta^k \frac{1}{|\zeta|^2} = \frac{(-1)^k k!}{\overline{\zeta}\zeta^{k+1}}.$$

From this we conclude

$$\begin{aligned}\partial_y^k \frac{1}{|\zeta|^2} &= \left(\mathrm{i}(\partial_\zeta - \partial_{\overline{\zeta}})\right)^k \frac{1}{|\zeta|^2} = \mathrm{i}^k \sum_{l=0}^{k} \binom{k}{l} (-1)^l \partial_\zeta^{k-l} \partial_{\overline{\zeta}}^l \frac{1}{|\zeta|^2} \\ &= (-\mathrm{i})^k k! \sum_{l=0}^{k} \frac{(-1)^l}{\overline{\zeta}^{l+1}\zeta^{k-l+1}} = \frac{(-\mathrm{i})^k k!}{|\zeta|^2\zeta^k} \frac{1 - (-\frac{\zeta}{\overline{\zeta}})^{k+1}}{1 + \frac{\zeta}{\overline{\zeta}}} = \frac{(-\mathrm{i})^k k!}{|\zeta|^{2k+2}} \frac{\overline{\zeta}^{k+1} + (-1)^k \zeta^{k+1}}{\zeta + \overline{\zeta}},\end{aligned}$$

where $\zeta = x + \mathrm{i}y$. For the cases of odd and even numbers of derivatives we get

$$\partial_y^{2k+1} \frac{1}{|\zeta|^2} = (-1)^{k+1} \frac{(2k+1)!}{|\zeta|^{4k+4}} \frac{\operatorname{Im} \zeta^{2k+2}}{\operatorname{Re} \zeta}, \qquad \partial_y^{2k} \frac{1}{|\zeta|^2} = \frac{(-1)^k (2k)!}{|\zeta|^{4k+2}} \frac{\operatorname{Re} \zeta^{2k+1}}{\operatorname{Re} \zeta}$$

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for $k \in \mathbb{N}_0$. Now we use the definition of f_a in (7.10) and of g_a in (7.9), take the derivative and evaluate at $\zeta = 0$ to get (C.1) and (C.3), respectively. □

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