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First normal bundle of ideal Lagrangian immersions in complex space forms

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Abstract

We give an elegant simple proof of a result concerning the first normal bundle of ideal Lagrangian immersions. This enables us to simplify and clarify several of the results in an earlier paper.

1. *δ-invariant and inequalities*

We follow the same notation and definitions of [**1**, **2**, **3**].

Let M be a Riemannian n-manifold. Denote by $K(\pi)$ the sectional curvature of M associated with a plane section $\pi \subset T_pM$, $p \in M$. For any orthonormal basis $e_1, \ldots, e_n$ of the tangent space T_pM, the scalar curvature τ at p is defined to be

$$\tau(p) = \sum_{i<j} K(e_i \wedge e_j). \tag{1·1}$$

Let L be a subspace of T_pM of dimension $r \geqslant 2$ and $\{e_1, \ldots, e_r\}$ an orthonormal basis of L. The scalar curvature $\tau(L)$ of the r-plane section L is defined by

$$\tau(L) = \sum_{\alpha<\beta} K(e_\alpha \wedge e_\beta), \quad 1 \leqslant \alpha, \beta \leqslant r. \tag{1·2}$$

For given integers $n \geqslant 2$ and $k \geqslant 0$, we denote by $\mathcal{S}(n, k)$ the finite set consisting of all k-tuples $(n_1, \ldots, n_k)$ of integers satisfying $2 \leqslant n_1 \leqslant \cdots \leqslant n_k < n$. Notice that $\mathcal{S}(n, 0) = \{\emptyset\}$. Denote by $\mathcal{S}(n)$ the union of $\mathcal{S}(n, k)$ with $k \geqslant 0$.

The invariants $\delta(n_1, \ldots, n_k)$, $(n_1, \ldots, n_k) \in \mathcal{S}(n)$, originally introduced in [**2**] are given by

$$\delta(n_1, \ldots, n_k)(p) = \tau(p) - \inf\{\tau(L_1) + \cdots + \tau(L_k)\}, \tag{1·3}$$

where $L_1, \ldots, L_k$ run over all k mutually orthogonal subspaces of T_pM such that $\dim L_j = n_j$, $j = 1, \ldots, k$.

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For each $(n_1,\dots,n_k)\in\mathcal{S}(n)$, let $c(n_1,\dots,n_k)$ and $b(n_1,\dots,n_k)$ denote the positive constants given by

$$c(n_1,\dots,n_k)=\frac{n^2(n+k-1-\sum n_j)}{2(n+k-\sum n_j)}, \tag{1·4}$$

$$b(n_1,\dots,n_k)=\frac{1}{2}\left(n(n-1)-\sum_{j=1}^{k}n_j(n_j-1)\right). \tag{1·5}$$

In [**2**, **3**] the author proved the following optimal general inequalities for arbitrary Lagrangian submanifolds in complex space forms.

THEOREM 1. *Let M be an n-dimensional Lagrangian submanifold of a complex space form $\tilde{M}^n(4c)$ of constant holomorphic sectional curvature $4c$. Then, for any k-tuple $(n_1,\dots,n_k)\in\mathcal{S}(n)$, we have*

$$\delta(n_1,\dots,n_k)\leqslant c(n_1,\dots,n_k)H^2+b(n_1,\dots,n_k)c. \tag{1·6}$$

The equality case of inequality (1·6) *holds identically if and only if we have*

(i) *M is minimal in $\tilde{M}^n(4c)$ and*

(ii) *There exists an orthonormal local frame field $e_1,\dots,e_{2n}$ such that the shape operators of M take the following forms*:

$$A_r=\begin{bmatrix} A_1^r & \dots & 0 & \\ \vdots & \ddots & \vdots & 0 \\ 0 & \dots & A_k^r & \\ & 0 & & 0 \end{bmatrix},\quad r=n+1,\dots,2n, \tag{1·7}$$

where each A_j^r is a symmetric $n_j\times n_j$ submatrix satisfying

$$\operatorname{trace}(A_1^r)=\cdots=\operatorname{trace}(A_k^r)=0. \tag{1·8}$$

For a Lagrangian submanifold M in a complex space form $M^n(4c)$ satisfying (1·7), we put $L_i=\operatorname{Span}\{e_\alpha:\alpha\in I_i\}$, with respect to the orthonormal frame field $\{e_1,\dots,e_n\}$ chosen above, where

$$I_i=\{n_1+\cdots+n_{i-1}+1,\dots,n_1+\cdots+n_i\},\ i=1,\dots,k.$$

We denote by $\mathcal{D}_i$ the distribution generated by L_i.

Recall that a Lagrangian immersion of a Riemannian manifold M into a complex space form $\tilde{M}^n(4c)$ is called *ideal* if it satisfies the equality case of (1·6) identically for some k-tuple $(n_1,\dots,n_k)\in\mathcal{S}(n)$.

2. *Lemma* 11 *of* [3]

The proof of theorems 2, 3 and 4 of [**3**] based on the following lemma 11 of [**3**].

LEMMA 11. *Let M^n be a Lagrangian submanifold in a complex space form with full first normal bundle which satisfies*

$$\delta(n_1,\dots,n_k)=c(n_1,\dots,n_k)H^2+b(n_1,\dots,n_k)c \tag{2·1}$$

for some $(n_1,\dots,n_k)\in\mathcal{S}(n)$. Then the corresponding distributions $\mathcal{D}_1,\dots,\mathcal{D}_k$ are integrable and their leaves are totally geodesic submanifolds of M^n.

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The proof of this lemma in [**3**] is very long, complicated, and at one point obscure. In this short note we give an elegant proof of this lemma. Thus, we get rid of the obscurity. Moreover, the proofs of theorems 2, 3 and 4 of [**3**] are then simplified dramatically.

3. *A simple proof of lemma* 11

We prove this lemma by induction. Since M^n is ideal Lagrangian, it satisfies (2·1) for some k-tuple $(n_1, \ldots, n_k)$. As in [**3**], let $\mathcal{D}_1, \ldots, \mathcal{D}_k$ be the associated distributions. Since the first normal bundle is full, lemma 4 of [**3**] implies that each $\mathcal{D}_i$ is integrable and its leaves are minimal in M^n. Lemma 7 of [**3**] implies that lemma 11 holds if $n_1, \ldots, n_k \leqslant 3$. Now, let us assume that lemma 11 also holds if $n_1, \ldots, n_k < m$ for a given integer $m > 3$. We shall prove that it also holds for $n_1, \ldots, n_k \leqslant m$.

Assume that $n_1 = m$. Let $\{e_1, \ldots, e_{n_1}\}$ be an orthonormal frame of $\mathcal{D}_1$. We put $h^\gamma_{\alpha\beta} := \langle h(e_\alpha, e_\beta), Je_\gamma\rangle$. Then we have $h^\gamma_{\alpha\beta} = h^\beta_{\alpha\gamma} = h^\alpha_{\beta\gamma}$. Since $h(\mathcal{D}_1, \mathcal{D}_1) = J\mathcal{D}_1$ by lemma 4 of [**3**], the equation of Codazzi yields

$$\sum_{\gamma\in I_1} h^\gamma_{\alpha\delta}\langle \nabla_{e_\beta} e_\gamma, Z\rangle = \sum_{\gamma\in I_1} h^\gamma_{\beta\delta}\langle \nabla_{e_\alpha} e_\gamma, Z\rangle \tag{3·1}$$

for $\alpha, \beta, \delta \in I_1$ and $Z \in \mathcal{D}_1^\perp := \mathcal{D}_2 \oplus \cdots \oplus \mathcal{D}_k$. If $\nabla_X Y \in \mathcal{D}_1$ for all $X, Y \in \mathcal{D}_1$, then $\mathcal{D}_1$ is autoparallel. In this case, leaves of $\mathcal{D}_1$ are totally geodesic. So, let us assume that this is not the case. Then there exists a unit vector field $Z \in \mathcal{D}_1^\perp$ such that $\langle \nabla_X Y, Z\rangle \neq 0$ for some vector fields $X, Y \in \mathcal{D}_1$. Let us put $\sigma_Z(X, Y) := \langle \nabla_X Y, Z\rangle$. Then σ_Z is a symmetric $(0, 2)$-tensor which is nothing but the second fundamental form of the leaves of $\mathcal{D}_1$ with respect to the normal direction Z.

Case (i). *σ_Z has only one eigenvalue for every $Z \in \mathcal{D}_1^\perp$.* In this case, leaves of $\mathcal{D}_1$ are totally umbilical in M. Combining this with lemma 4 of [**3**] implies that leaves of $\mathcal{D}_1$ are totally geodesic in M.

Case (ii). *σ_Z has at least two eigenvalues for some $Z \in \mathcal{D}_1^\perp$.* In this case, we choose an orthonormal basis $\{e_1, \ldots, e_{n_1}\}$ of $\mathcal{D}_1$ which diagonalizes σ_Z so that

$$\sigma_Z(e_\alpha, e_\beta) = \mu_\alpha \delta_{\alpha\beta}, \quad \alpha, \beta \in I_1. \tag{3·2}$$

From (3·1) and (3·2) we find $h^\delta_{\alpha\beta}(\mu_\alpha - \mu_\beta) = 0$ for $\alpha, \beta, \delta \in I_1$, which implies that

$$h^\delta_{\alpha\beta} = 0, \quad \forall \delta \in I_1, \ \text{ whenever } \ \mu_\alpha \neq \mu_\beta. \tag{3·3}$$

If the multiplicity of an eigenvalue, say μ_1, of σ_Z is one, then (3·3) gives

$$h^1_{\beta\delta} = 0, \quad \beta, \delta \in I_1; \ \beta \neq 1. \tag{3·4}$$

On the other hand, from theorem 1 of [**3**] we also have $\sum_{j=1}^{n_1} h^1_{jj} = 0$. Combining this with (3·3) and lemma 4 of [**3**] gives $e_1 \in \mathcal{N}_p^{rel}$ which contradicts to fullness of the first normal bundle, due to lemma 3 of [**3**]. Hence, the multiplicity of each eigenvalue is at least two. Let $\zeta_1, \ldots, \zeta_\ell$ be the distinct eigenvalues of σ_Z with multiplicities $m_1, \ldots, m_\ell$. Denote by $\mathcal{E}_1, \ldots, \mathcal{E}_\ell$ the corresponding eigenspace distributions. It follows from (3·3) and lemma 4 of [**3**] that $h(X, \mathcal{E}_j) \subset J(\mathcal{E}_j)$ for $j = 1, \ldots, \ell$, and $X \in TM^n$. Thus, M^n also satisfies (2·1) for the $(k + \ell - 1)$-tuple $(m_1, \ldots, m_\ell, n_2, \ldots, n_k)$ with $m_1, \ldots, m_\ell < m$ according to theorem 3·1 of [**2**]. Clearly, we may apply the same argument to each n_i with $n_i = m$ to conclude that M^n satisfies (2·1) for some q-tuple $(m_1, \ldots, m_q)$ with $m_1, \ldots, m_q < m$. Now, we may apply induction to conclude

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that $\mathcal{E}_1, \ldots, \mathcal{E}_\ell$ are autoparallel. Combining this with (3·2) we find $\sigma_Z = 0$ which is a contradiction. Thus, Case (ii) is impossible.

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