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Volume lemmas and large deviations for partially hyperbolic endomorphisms

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Abstract. We consider partially hyperbolic attractors for non-singular endomorphisms admitting an invariant stable bundle and a positively invariant cone field with non-uniform cone expansion at a positive Lebesgue measure set of points. We prove volume lemmas for both Lebesgue measure on the topological basin of the attractor and the SRB measure supported on the attractor. As a consequence, under a mild assumption we prove exponential large-deviation bounds for the convergence of Birkhoff averages associated to continuous observables with respect to the SRB measure.

Key words: volume lemmas, partially hyperbolic endomorphism, SRB measure, large deviations

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1. *Introduction*

One of the main goals of ergodic theory is to describe the statistical properties of dynamical systems using invariant measures. In particular, the thermodynamic formalism aims to construct and describe the statistical properties of invariant measures that are physically relevant, meaning equilibrium states with respect to some potential. Among these, one should mention the Sinai–Ruelle–Bowen (SRB) measures, the construction and statistical properties of which rapidly became a topic of interest among the mathematics and physics communities. Unfortunately, apart from the uniformly hyperbolic setting, where the

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existence of finite Markov partitions allows us to semiconjugate the dynamics to subshifts of finite type (see, for example, [**3**]) there is no systematic approach to the construction of SRB measures. We refer the reader to [**28**] for an accessible paper on SRB measures.

With respect to that, there have been important contributions to the study of non-singular endomorphisms, that is, local diffeomorphisms. In the mid-1970s, Przytycki [**15**] extended the notion of uniform hyperbolicity to the context of endomorphisms and studied Anosov endomorphisms. Here, due to the non-invertibility of the dynamics, the existence of an invariant unstable subbundle for uniformly hyperbolic basic pieces needs to be replaced by the existence of positively invariant cone fields on which vectors are uniformly expanded by all positive iterates (we refer the reader to §3.2 for more details). In [**17**], Qian and Zhu constructed SRB measures for Axiom A attractors of endomorphisms, obtaining these as equilibrium states for the geometric potential defined using inverse limits. A characterization of SRB measures for uniformly hyperbolic endomorphisms can also be given in terms of dimensional characteristics of the stable manifold (cf. [**22**]). Mihailescu [**12**] also contributed to the thermodynamic formalism of hyperbolic basic pieces for endomorphisms, introducing the concept of inverse SRB measures. The entropy production of this class of inverse SRB measures was later described by Mihailescu and Urbanski [**14**]. Among the difficulties that arise when dealing with non-invertible hyperbolic dynamics one should mention that unstable manifolds may have a complicated geometric structure (cf. [**16**, **22**]).

The ergodic theory for endomorphisms beyond the scope of uniform hyperbolicity is quite incomplete. In the context of surface endomorphisms, a major contribution is due to Tsujii [**21**], who proved that for $r \geq 19$, C^r-generic partially hyperbolic endomorphisms with an unstable cone field admit finitely many SRB measures whose ergodic basins of attraction cover Lebesgue almost every point in the manifold (the regularity can be dropped to $r \geq 2$ on the space of non-singular endomorphisms). While it remains unknown if these SRB measures are (generically) hyperbolic, an extension to higher dimension should present significant difficulties as discussed by the author (cf. [**21**, p. 43]). In [**7**] the first and third authors constructed SRB measures for robust classes of non-singular C^2-partially hyperbolic endomorphisms, proving their statistical stability and the continuous dependence of their entropies with respect to the dynamics.

Our purpose here is to contribute to the ergodic theory of partially hyperbolic attractors for endomorphisms. Given the non-invertible structure of the attractor, even in the case of hyperbolic endomorphisms unstable manifolds have self-intersections (hence are badly understood in the ambient space) and stable manifolds may fail to have good geometric properties when lifted to the natural extension (see, for example, [**13**] for a detailed survey on geometric and ergodic aspects of hyperbolic sets for endomorphisms). In the non-hyperbolic context the situation becomes even more intricate. Here we consider attractors for $C^{1+\alpha}$ ($\alpha > 0$) non-singular endomorphisms on compact Riemannian manifolds that admit uniform contraction along a well-defined invariant stable subbundle and have a positively invariant cone field. Under a mild non-uniform cone-expansion assumption, these endomorphisms admit finitely many hyperbolic and physical SRB measures, these are statistically stable and their metric entropies vary continuously with respect to the underlying dynamics, and the SRB measure is unique if the attractor is transitive (here

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we mean SRB measures as measures which are absolutely continuous disintegrations with respect to Lebesgue measure along Pesin unstable manifolds) [**7**]. In this context, unstable subbundles depend on the pre-orbits and are almost everywhere defined via Oseledets's theorem, hence the unstable Jacobian is defined almost everywhere (with respect to the SRB measure) and could fail to be Hölder continuous on the attractor. Our main contribution here is to overcome such lack of regularity of the unstable Jacobian. For this, we prove a bounded distortion property and obtain two volume lemmas at instants of hyperbolicity (c-cone-hyperbolic times defined in [**7**]). The (invariant) SRB measure is supported on the attractor, and the volume lemma for the SRB measure lies on the regularity of the unstable Jacobian along pre-images of unstable disks. The volume lemma for Lebesgue measure on the topological basin of the attractor is even more subtle. Since local unstable manifolds may intersect we make use of some 'fake foliations' tangent to the cone field and, in order to use a Fubini-like argument, we need to estimate the largest possible distortion using *any* subspaces in the cone field (cf. Proposition 4.4 for the precise statement) and its dependence on a small open neighborhood. These volume lemmas are the main results of the paper. As a consequence we derive large-deviation upper bounds for the velocity of convergence of Birkhoff averages associated to continuous observables using subadditive observables associated to the invariant cone field.

This paper is organized as follows. In §2 we describe the class of partially hyperbolic attractors considered here and state our results on large deviations. We recall some preliminary results in §3. In §4 we prove a first volume lemma, for the Lebesgue measure on the basin on the attractor. The volume lemma for the SRB measure is proven in §5. In §6 we prove the large-deviation bounds for the volume and SRB measure, and in §7 we give some examples.

2. *Statement of the main result*

2.1. *Setting.* Throughout, let M be a compact connected Riemannian d^*-dimensional manifold. Assume that $f : M \to M$ is a $C^{1+\alpha}$, $\alpha > 0$, non-singular endomorphism and $\Lambda \subset M$ a compact positively f-invariant subset. Let $U \supset \Lambda$ be an open set such that $f(\overline{U}) \subset U$ and assume

$$\Lambda = \Lambda_f := \bigcap_{n\geq 0} f^n(\overline{U}),$$

where $\overline{U}$ denotes the closure of the set U. We say that Λ is a *partially hyperbolic attractor* if there are a continuous splitting of the tangent bundle $T_U M = E^s \oplus F$ (where F is not necessarily Df-invariant) and constants $c > 0$ and $0 < \lambda < 1$ satisfying:

(H1) $Df(x) \cdot E^s_x = E^s_{f(x)}$ for every $x \in U$;

(H2) $\|Df^n(x)|_{E^s_x}\| \leq \lambda^n$ for every $x \in U$ and $n \in \mathbb{N}$;

(H3) there is a cone field $U \ni x \mapsto C(x)$ of constant dimension $\dim F$ so that $Df(x)(C(x)) \subseteq C(f(x))$ for every $x \in U$, and there is a positive Lebesgue measure set $H \subset U$ so that

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log \|(Df(f^j(x))|_{C(f^j(x))})^{-1}\| \leq -2c < 0, \tag{2.1}$$

for every $x \in H$ (the expression $\|(Df(f^j(x))|_{C(f^j(x))})^{-1}\|$ is made precise in §3);

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(H4) $\|Df(x)\,v\|\,\|Df(x)^{-1}w\| \le \lambda\|v\|\,\|w\|$ for every $v \in E^s_x$, all $w \in Df(x)(C(x))$ and $x \in U$.

Throughout the paper the constant $c > 0$ will be fixed as above. We say that Λ is an *attractor*, E^s is a *uniformly contracting bundle* and C is a *non-uniformly expanding cone field*(or C exhibits non-uniform expansion). We will denote $d := \dim(F) = \dim(C)$.

THEOREM 2.1. [**7**, Theorem A] *Let f be a $C^{1+\alpha}$ non-singular endomorphism, Λ be a partially hyperbolic attractor and U be the basin of attraction of Λ. Then there are finitely many SRB measures for f whose basins of attraction cover H, Lebesgue mod zero. Moreover, these measures are hyperbolic, and if* $\mathrm{Leb}(U \setminus H) = 0$ *then there are finitely many SRB measures for f in U. Furthermore, if $f\,|_\Lambda$ is transitive then there is a unique SRB measure for f in U.*

Many large-deviation estimates for non-uniformly hyperbolic dynamics rely on the differentiability of the pressure function on regular (e.g. Hölder continuous) potentials, which is still an open question in our context. The level-1 large-deviation estimates that we shall obtain here stand for *continuous* observables and rely on the following volume lemmas, which are the main results in this paper and of independent interest.

FIRST VOLUME LEMMA. *There exists $\delta > 0$ so that for every $0 < \varepsilon < \delta$ there exists $K(\varepsilon) > 0$ so that if n is a c-cone-hyperbolic time for x and F_1 is* any *subspace of dimension d contained in $C(x)$ then*

$$K(\varepsilon)^{-1} e^{-\log|\det Df^n(x)|_{F_1}|} \le \mathrm{Leb}(B(x,n,\varepsilon)) \le K(\varepsilon) e^{-\log|\det Df^n(x)|_{F_1}|}.$$

SECOND VOLUME LEMMA. *Assume that the density of the SRB measure μ is bounded away from zero and infinity. There exists $\delta > 0$ so that for every $0 < \varepsilon < \delta$ there exists $K(\varepsilon) > 0$ so that if n is a c-cone-hyperbolic time for x and F_1 is* any *subspace of dimension d contained in $C(x)$ then*

$$K(\varepsilon)^{-1} e^{-\log|\det Df^n(x)|_{F_1}|} \le \mu(\Lambda \cap B(x,n,\varepsilon)) \le K(\varepsilon) e^{-\log|\det Df^n(x)|_{F_1}|}.$$

Some comments are in order. First, the precise statements of the volume lemmas appear as Propositions 4.5 and 5.2, respectively. Moreover, since invariance of subbundles in the cone field is obtained when the entire pre-orbit is fixed, the observable $\log|\det Df^n(x)\,|_{F_1}|$ above leads to a non-stationary Birkhoff sum associated to the family of potentials $\{\log|\det Df\,|_{Df^j(F_1)}|\}_{0\le j\le n-1}$. We overcome this fact by proving that the asymptotic behavior of this family of potentials coincides with the sub-additive behavior of the maximal volume computed using subspaces in the cone field. More precisely, the sequence of functions $\Gamma_n(x) = \max_{F_1 \subset C(x)} |\det Df^n(x)|_{F_1}|$ is sub-multiplicative and, by Kingman's subadditive ergodic theorem, if ν is an invariant probability measure then

$$\Gamma_\nu(x) = \lim_{n\to\infty} \frac{1}{n} \log\,\Gamma_n(x) \quad \text{exists for } \nu\text{-almost every } x \tag{2.2}$$

and

$$\int \Gamma_\nu(x)\,d\nu = \inf_{n\ge 1} \frac{1}{n} \int \log\,\Gamma_n(x)\,d\nu(x),$$

and these are related with the volume expansion of any subspace of fixed dimension contained in the cone field (cf. Corollary 4.2). Moreover, the latter are almost everywhere

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constant provided that ν is ergodic. The following result asserts that under a mild assumption on the sequence of cone-hyperbolic times (see Definition 4.1 for the definition of non-lacunarity and the first cone-hyperbolic time map n_1) the SRB measure is a weak Gibbs measure (see Proposition 6.1 for the precise statement) and satisfies large-deviation estimates for continuous observables.

THEOREM A. *Let f be a $C^{1+\alpha}$ non-singular endomorphism, Λ be a transitive partially hyperbolic attractor, U be the basin of attraction of Λ, and μ be the unique SRB measure for $f|_\Lambda$. Assume that the density of the SRB measure is bounded away from zero and infinity and that the sequence of cone-hyperbolic times is non-lacunar μ-almost everywhere. Then (i) μ is a weak Gibbs measure, and (ii) there exists $C > 0$ so that for every continuous observable ϕ and every closed interval $F \subset \mathbb{R}$,*

$$\limsup_{n\to\infty} \frac{1}{n} \log \mu\left(x \in \Lambda : \frac{1}{n}\sum_{j=0}^{n-1} \phi(f^j(x)) \in F\right) \le \max\left\{\inf_{\beta>0} \mathcal{E}_\mu(\beta), -\inf_{c\in F} I(c) + \beta\right\}, \tag{2.3}$$

where $\mathcal{E}_\mu(\beta) = \limsup_{n\to\infty} (1/n) \log \mu(x : n_1(x) > \beta n/(2 \log C))$ and

$$I(c) = \sup\left\{\int \Gamma_\nu(x)\, d\nu(x) - h_\nu(f) : \nu \in \mathcal{M}_f, \int \phi\, d\nu = c\right\} \ge 0.$$

The large-deviation bound (2.3) estimates the exponential decay rate of the set of points whose Birkhoff averages do not belong to some closed interval J using two constants. First, note that the number $\mathcal{E}_\mu(\beta)/\beta$ is roughly the exponential decay rate of the tail of the first cone-hyperbolic time map (cf. Remark 6.2). Second, the number $I(c)$ can be viewed as a kind of 'distance' from a set of measures to the equilibrium, computed among measures with the prescribed average equal to c (in the closed interval F).

The proof of Theorem A explores the fact that under the previous assumptions the SRB measure is a weak Gibbs measure (cf. Proposition 6.1). Moreover, our results also apply for attractors of partially hyperbolic diffeomorphisms, hence providing an alternative large-deviation upper bound for [**2**, Theorem B]. For an account of large deviations for non-uniformly hyperbolic diffeomorphisms and endomorphisms we refer the reader to [**2**, **6**, **8**, **10**, **11**, **18**] and references therein.

Remark 2.1. If we assume that the sequence of cone-hyperbolic times is non-lacunar for Leb-almost every point then we also obtain the upper bound

$$\limsup_{n\to\infty} \frac{1}{n} \log \mathrm{Leb}\left(x \in U : \frac{1}{n}\sum_{j=0}^{n-1} \phi(f^j(x)) \in F\right) \le \max\left\{\inf_{\beta>0} \mathcal{E}_\mu(\beta), -\inf_{c\in F} I(c)\right\} \tag{2.4}$$

for every continuous observable ϕ and every $c \in \mathbb{R}$. However, while in the case of the SRB measure the non-lacunarity of the sequences of cone-hyperbolic times follows from integrability of the first cone-hyperbolic time map, this is far from immediate in the case of non-invariant probability measures (see Remark 4.1 for a more detailed discussion).

In view of Theorem A we can now provide a criterion for exponential large deviations. Note that the first cone-hyperbolic time map n_1 is well defined μ-almost everywhere.

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We say the first cone-hyperbolic time map n_1 has *exponential tail* (with respect to μ) if there exists $\eta > 0$ so that $\mu(n_1 > n) \leq e^{-\eta n}$ for all $n \gg 1$. The latter condition is satisfied by the SRB measures in the case of Anosov endomorphisms (in which case $n_1(x) = 1$ for every $x \in \Lambda$) and certain open sets of non-hyperbolic endomorphisms derived from Anosov (see §7).

COROLLARY A. *In the context of Theorem A, if the first cone-hyperbolic time map n_1 has exponential tail with respect to μ then for every $\delta > 0$ it follows that*

$$\limsup_{n\to\infty} \frac{1}{n} \log \mu\left(x \in \Lambda : \left|\frac{1}{n}\sum_{j=0}^{n-1} \phi(f^j(x)) - \int \phi \, d\mu\right| \geq \delta\right) < 0.$$

3. *Preliminaries*

The present section is devoted to a preliminary discussion on the notion of uniform and non-uniform hyperbolicity for non-singular endomorphisms. Given the vector spaces V and W and a cone $E \subset V$, let the vector space $\mathcal{L}_E(V, W) := \{T|_E : T \in \mathcal{L}(V, W)\}$ be endowed with the norm $\|T|_E\| = \sup_{0\neq v\in E} (\|T \cdot v\|/\|v\|)$. Given a non-singular endomorphism $f : M \to M$, a Df-invariant and convex cone field C, and $x \in M$, denote by $(Df(x)|_{C(x)})^{-1}$ the map

$$Df(x)^{-1}|_{Df(x)(C(x))} : Df(x)(C(x)) \to C(x) \subset T_x M.$$

If $TM = E \oplus F$ is a continuous splitting of the tangent bundle, the *cone field of width $a >$ 0 centered at F* is the continuous map $x \mapsto C(x)$ that assigns to each $x \in M$ the cone of width $a > 0$ centered at F_x, where $C := \{v = v_1 \oplus v_2 \in E \oplus F : \|v_1\| \leq a\|v_2\|\}$. Finally, a submanifold $D \subset M$ is *tangent to the cone field* $x \mapsto C(x)$ if $\dim(D) = \dim(C)$ and $T_x D \subset C(x)$, for every $x \in D$.

Let $\Lambda \subset M$ be a compact positively invariant subset of M and $\Lambda \ni x \mapsto C(x)$ be a Df-invariant cone field on Λ.

Definition 3.1. Let M be a compact Riemannian manifold, $f : M \to M$ be a non-singular endomorphism and $c > 0$. We say that $n \in \mathbb{N}$ is a *c-cone-hyperbolic time* for $x \in M$ (with respect to C) if $Df(f^j(x))(C(f^j(x))) \subset C(f^{j+1}(x))$ for every $0 \leq j \leq n-1$ and

$$\prod_{j=n-k}^{n-1} \|(Df(f^j(x))|_{C(f^j(x))})^{-1}\| \leq e^{-ck} \quad \text{for every } 1 \leq k \leq n. \tag{3.1}$$

It is easy to see that if $n < m$ are c-cone hyperbolic times then $m - n$ is a c-cone hyperbolic time for $f^n(x)$. Moreover, if n is a c-cone-hyperbolic time for $x \in M$ then

$$\|Df^{n-j}(f^j(x)) \cdot v\| \geq e^{c(n-j)}\|v\|, \tag{3.2}$$

for every $v \in C(f^j(x))$ and $0 \leq j \leq n-1$ (cf. [**7**]).

3.1. *Natural extension.*

The natural extension of M by f is the set

$$M^f := \{\hat{x} = (x_{-j})_{j\in\mathbb{N}} : x_{-j} \in M \text{ and } f(x_{-j}) = x_{-j+1}, \text{ for every } j \in \mathbb{N}\}.$$

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We can induce a metric on M^f by $\hat{d}(\hat{x}, \hat{y}) := \sum_{j\in\mathbb{N}} 2^{-j} d(x_{-j}, y_{-j})$. The space M^f endowed with the metric $\hat{d}$ is a compact metric space and the topology induced by $\hat{d}$ is equivalent to the topology induced by $M^{\mathbb{N}}$ endowed with Tychonov's product topology. The projection $\pi : M^f \to M$ given by $\pi(\hat{x}) := x_0$ is a continuous map. The lift of f is the map $\hat{f} : M^f \to M^f$ given by $\hat{f}(\hat{x}) := (\ldots, x_{-n}, \ldots, x_{-1}, x_0, f(x_0))$, and it is clear that $f \circ \pi = \pi \circ \hat{f}$. Finally, for each $\hat{x} \in M^f$ we take $T_{\hat{x}} M^f := T_{\pi(\hat{x})} M$ and set

$$\begin{array}{rccc} D\hat{f}(\hat{x}): & T_{\hat{x}} M^f & \to & T_{\hat{f}(\hat{x})} M^f \\ & v & \mapsto & Df(\pi(\hat{x}))v. \end{array}$$

If $\Lambda \subset M$ is a compact and positively invariant, that is, $f(\Lambda) \subset \Lambda$, the natural extension of Λ by f is the set of pre-orbits that lie on Λ, that is, $\Lambda^f := \{\hat{x} = (x_{-j})_{j\in\mathbb{N}} : x_{-j} \in \Lambda$ and $f(x_{-j}) = x_{-j+1}$ for every $j \in \mathbb{N}\}$. The projection π induces a continuous bijection between $\hat{f}$-invariant probability measures and f-invariant probability measures, assigning every $\hat{\mu}$ in M^f to the pushforward $\pi_*\hat{\mu} = \mu$ (see, for example, [**16**] for more details).

3.2. *Non-uniform hyperbolicity for endomorphisms.*

3.2.1. *Lyapunov exponents and non-uniform hyperbolicity.*

We need the following proposition.

PROPOSITION 3.1. [**16**, Proposition I.3.5] *Suppose that M is a compact Riemannian manifold. Let $f : M \to M$ be a C^1 map preserving a probability measure μ. There is a Borel set $\hat{\Delta} \subset M^f$ with $\hat{f}(\hat{\Delta}) = \hat{\Delta}$ and $\hat{\mu}(\hat{\Delta}) = 1$ satisfying that for every $\hat{x} \in \hat{\Delta}$ there is a splitting $T_{\hat{x}} M^f = E_1(\hat{x}) \oplus \cdots \oplus E_{r(\hat{x})}(\hat{x})$, numbers $+\infty > \lambda_1(\hat{x}) > \lambda_2(\hat{x}) > \cdots > \lambda_{r(\hat{x})}(\hat{x}) > -\infty$ (Lyapunov exponents), and multiplicities $m_i(\hat{x})$ for $1 \le i \le r(\hat{x})$ such that:*

(1) *$D\hat{f}(\hat{f}^n(\hat{x})) : T_{\hat{f}^n(\hat{x})} M^f \to T_{\hat{f}^{n+1}(\hat{x})} M^f$ is a linear isomorphism for every $n \in \mathbb{Z}$;*
(2) *the functions $r : \hat{\Delta} \to \mathbb{N}$, $m_i : \hat{\Delta} \to \mathbb{N}$ and $\lambda_i : \hat{\Delta} \to \mathbb{R}$ are $\hat{f}$-invariant;*
(3) *$\dim(E_i(\hat{x})) = m_i(\hat{x})$ for every $1 \le i \le r(\hat{x}) \le \dim M$;*
(4) *the splitting is $D\hat{f}$-invariant, that is, $D\hat{f}(\hat{x}) \cdot E_i(\hat{x}) = E_i(\hat{f}(\hat{x}))$, for every $1 \le i \le r(\hat{x})$;*
(5) *$\lim_{n\to\pm\infty} (1/n) \log \|D\hat{f}^n(\hat{x}) \cdot u\| = \lambda_i(\hat{x})$ for every $u \in E_i(\hat{x}) \backslash \{0\}$ and for every $1 \le i \le r(\hat{x})$;*
(6) *if $\rho_1(\hat{x}) \ge \rho_2(\hat{x}) \ge \cdots \ge \rho_d(\hat{x})$ represent the numbers $\lambda_i(\hat{x})$ repeated $m_i(\hat{x})$ times for each $1 \le i \le r(\hat{x})$ and $\{u_1, u_2, \ldots, u_d\}$ is a basis for $T_{\hat{x}} M^f$ satisfying $\lim_{n\to\pm\infty} (1/n) \log \|D\hat{f}^n(\hat{x}) \cdot u_i\| = \rho_i(\hat{x})$ for every $1 \le i \le d$, then for any subsets $P, Q \subset \{1, 2, \ldots, d\}$ with $P \cap Q = \emptyset$ we have $\lim_{n\to\infty} (1/n) \log \angle(D\hat{f}^n(\hat{x}) \cdot E_P, D\hat{f}^n(\hat{x}) \cdot E_Q) = 0$ where E_P and E_Q are, respectively, the subspaces generated by $\{u_i\}_{i\in P}$ and $\{u_i\}_{i\in Q}$.*

Moreover, if $\hat{\mu}$ is ergodic then the previous functions are constant $\hat{\mu}$-almost everywhere.

Given an f-invariant probability measure μ, we say that μ is *hyperbolic* if is has no zero Lyapunov exponents. Associated to this formulation of the concept of Lyapunov exponents we have the existence of unstable manifolds (stable manifolds, see [**16**, Proposition V.4.5]) for almost every point with respect to some invariant measure with some positive Lyapunov exponents. Consider the subspaces $E^u_{\hat{x}} := \bigoplus_{\lambda_i(\hat{x})>0} E_i(\hat{x})$ and $E^{cs}_{\hat{x}} := \bigoplus_{\lambda_i(\hat{x})\le 0} E_i(\hat{x})$.

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PROPOSITION 3.2. [**16**, Proposition V.4.4] *There is a countable number of compact subsets* $(\hat{\Delta}_i)_{i\in\mathbb{N}}$, *of* M^f *with* $\bigcup_{i\in\mathbb{N}} \hat{\Delta}_i \subset \hat{\Delta}$ *and* $\hat{\mu}(\hat{\Delta}\setminus\bigcup_{i\in\mathbb{N}} \hat{\Delta}_i) = 0$ *such that the following statements hold.*

(1) *For each* $\hat{\Delta}_i$ *there is* $k_i \in \mathbb{N}$ *satisfying* $\dim(E^u(\hat{x})) = k_i$ *for every* $\hat{x} \in \hat{\Delta}_i$, *and the subspaces* $E^u_{\hat{x}}$ *and* $E^{cs}_{\hat{x}}$ *depend continuously on* $\hat{x} \in \hat{\Delta}_i$.

(2) *For any* $\hat{\Delta}_i$ *there is a family of* C^1 *embedded* k_i*-dimensional disks* $\{W^u_{\mathrm{loc}}(\hat{x})\}_{\hat{x}\in\hat{\Delta}_i}$ *in* M *and real numbers* λ_i, $\varepsilon \ll \lambda_i$, $r_i < 1$, γ_i, α_i *and* β_i *such that the following properties hold for each* $\hat{x} \in \hat{\Delta}_i$.

   (a) *There is a* C^1 *map* $h_{\hat{x}} : O_{\hat{x}} \to E^{cs}_{\hat{x}}$, *where* $O_{\hat{x}}$ *is an open set of* $E^u_{\hat{x}}$ *which contains* $\{v \in E^u_{\hat{x}} : \|v\| < \alpha_i\}$ *satisfying: (i)* $h_{\hat{x}}(0) = 0$ *and* $Dh_{\hat{x}}(0) = 0$; *(ii)* $\mathrm{Lip}(h_{\hat{x}}) \le \beta_i$ *and* $\mathrm{Lip}(Dh_{\hat{x}}(\cdot)) \le \beta_i$; *and (iii)* $W^u_{\mathrm{loc}}(\hat{x}) = \exp_{x_0}(\mathrm{graph}(h_{\hat{x}}))$.

   (b) *For any* $y_0 \in W^u_{\mathrm{loc}}(\hat{x})$ *there is a unique* $\hat{y} \in M^f$ *such that* $\pi(\hat{y}) = y_0$ *and*

$$\mathrm{dist}(x_{-n}, y_{-n}) \le \min\{r_i e^{-\varepsilon_i n}, \gamma_i e^{-\lambda_i n}\,\mathrm{dist}(x_0, y_0)\} \quad \textit{for every } n \in \mathbb{N}.$$

   (c) *If*

$$\hat{W}^u_{\mathrm{loc}}(\hat{x}) := \{\hat{y} \in M^f : y_{-n} \in W^u_{\mathrm{loc}}(\hat{f}^{-n}(\hat{x})),\ \textit{for every } n \in \mathbb{N}\}$$

   then $\pi : \hat{W}^u_{\mathrm{loc}}(\hat{x}) \to W^u_{\mathrm{loc}}(\hat{x})$ *is bijective and* $\hat{f}^{-n}(\hat{W}^u_{\mathrm{loc}}(\hat{x})) \subset \hat{W}^u_{\mathrm{loc}}(\hat{f}^{-n}(\hat{x}))$.

   (d) *For any* $\hat{y}, \hat{z} \in \hat{W}^u_{\mathrm{loc}}(\hat{x})$, *the inequality*

$$\mathrm{dist}^u_{\hat{f}^{-n}(\hat{x})}(y_{-n}, z_{-n}) \le \gamma_i e^{-\lambda_i n}\,\mathrm{dist}^u_{\hat{x}}(y_0, z_0)$$

   holds for every $n \in \mathbb{N}$, *where* $\mathrm{dist}^u_{\hat{x}}$ *is the distance along* $W^u_{\mathrm{loc}}(\hat{x})$.

3.3. *SRB measures.* The notion of SRB measure for endomorphisms depends intrinsically on the existence of unstable manifolds, whose geometry and construction are more complicated than for diffeomorphisms as unstable manifolds may have self-intersections (hence do not generate an invariant foliation).

Let $(X, \mathcal{A}, \mu)$ be a probability space and $\mathcal{P}$ be a partition of X. We say that $\mathcal{P}$ is a measurable partition if there is a sequence of countable partitions of X, $(\mathcal{P}_j)_{j\in\mathbb{N}}$, such that $\mathcal{P} = \bigvee_{j=0}^{\infty} \mathcal{P}_j \mod 0$. Consider the projection $p : X \to \mathcal{P}$ that associates to each $x \in X$ the atom $\mathcal{P}(x)$ that contains x. We say that a subset $\mathcal{Q} \subset \mathcal{P}$ is measurable if and only if $p^{-1}(\mathcal{Q})$ is measurable. It is easy to see that the family of all measurable sets of $\mathcal{P}$ is a σ-algebra of $\mathcal{P}$. If μ is an f-invariant probability measure with at least one positive Lyapunov exponent at μ-almost every point, a measurable partition $\mathcal{P}$ of M^f is *subordinated to unstable manifolds* if for $\hat{\mu}$-almost every $\hat{x} \in M^f$ (1) $\pi|_{\mathcal{P}(\hat{x})} : \mathcal{P}(\hat{x}) \to \pi(\mathcal{P}(\hat{x}))$ is bijective, and (2) there is a submanifold $W_{\hat{x}}$ with dimension $k(\hat{x})$ in M such that $W_{\hat{x}} \subset W^u(\hat{x})$, $\pi(\mathcal{P}(\hat{x})) \subset W_{\hat{x}}$ and $\pi(\mathcal{P}(\hat{x}))$ contains an open neighborhood of x_0 in $W_{\hat{x}}$, where $\mathcal{P}(\hat{x})$ denotes the element of the partition $\mathcal{P}$ that contains $\hat{x}$ and $k(\hat{x})$ denotes the number of positive Lyapunov exponents at $\hat{x}$. Moreover, taking $\tilde{\mu} := p_*\mu$, a *disintegration of* μ *with respect to* $\mathcal{P}$ is a family $(\mu_P)_{P\in\mathcal{P}}$ of probability measures on X such that:

(1) $\mu_P(P) = 1$ for $\tilde{\mu}$-almost every $P \in \mathcal{P}$;

(2) for every measurable set $E \subset X$ the map $\mathcal{P} \ni P \mapsto \mu_P(E)$ is measurable;

(3) $\mu(E) = \int \mu_P(E)\, d\tilde{\mu}(P)$ for every measurable set $E \subset X$.

We are now in a position to define the SRB property for invariant measures.

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Definition 3.2. We say that an f-invariant and ergodic probability measure μ is an *SRB measure* if it has at least one positive Lyapunov exponent almost everywhere and, for every partition $\mathcal{P}$ subordinated to unstable manifolds, we have $\pi_*\hat{\mu}_{\mathcal{P}(\hat{x})} \ll \mathrm{Leb}_{W_{\hat{x}}}$ for $\hat{\mu}$-almost every $\hat{x} \in M^f$, where $(\hat{\mu}_{\mathcal{P}(\hat{x})})_{\hat{x}\in M^f}$ is a disintegration of $\hat{\mu}$ with respect to $\mathcal{P}$.

The construction of the SRB measures for these partially hyperbolic attractors follows from careful control of densities of the pushforward of the Lebesgue measure on disks tangent to the cone field [**7**]. By (H3), there exists a disk D that is tangent to the cone field C and so that (2.1) holds for a positive Lebesgue measure set in D and SRB measures arise from the ergodic components of the accumulation points of the Cesàro averages

$$\mu_n = \frac{1}{n}\sum_{j=0}^{n-1} f_*^j \,\mathrm{Leb}_D\,. \tag{3.3}$$

In what follows we say that a set $A = \{a_n : n \in \mathbb{N}\} \subset \mathbb{N}$ has positive frequency in the integers if $\liminf_{n\to\infty} (1/n)\#(A \cap \{1, 2, \ldots, n\}) > 0$. The non-uniform expansion on the cone field ensures that for every $x \in H \cap D$ there are infinitely many values of n (with positive frequency in the integers, as a consequence of Pliss's lemma) so that

$$\|Df^n(x)\cdot v\| \geq e^{cn}\|v\| \quad \text{for } \textit{all } v \in C(x). \tag{3.4}$$

In (3.4), the uniform expansion of all vectors in the cone field is crucial because unstable directions (to be defined *a posteriori* at almost every point by means of Oseledets's theorem) will be contained in the cone field C but cannot be determined *a priori*, and to prove the SRB measures are hyperbolic measures. Moreover, it guarantees that we can choose a suitable family $\mathcal{D}_n$ of disks contained in D that are expanded by f^n and such that the family $\{f^n(D_n) : D_n \in \mathcal{D}_n\}$ is composed of disks of uniform size (cf. [**7**, Propositions 5.2 and 5.4]). If ν is an accumulation point of the measures

$$\nu_n := \frac{1}{n}\sum_{j=0}^{n-1} f_*^j \,\mathrm{Leb}_{\mathcal{D}_j} \tag{3.5}$$

then its support is contained in a union of disks obtained as accumulations of disks $f^n(D_n)$, for $D_n \in \mathcal{D}_n$, as $n \to \infty$ (the limit is guaranteed by Arzelà–Ascoli theorem, cf. [**7**, Proposition 5.5]). Due to possible self-intersections of disks Δ obtained as accumulations of the family of disks $f^n(\mathcal{D}_n)$ tangent to C, a selection procedure on the space of pre-orbits (which also involves the lifting of a reference measure to the natural extension) becomes necessary. Such a construction involves a careful selection of pre-orbits so that accumulation disks are contained in unstable manifolds parametrized by elements of the natural extension. In consequence, any invariant measure supported in these disks will be hyperbolic since it will have only positive Lyapunov exponents in the direction complementary to the stable bundle. The proof of the SRB property involves a careful and technical choice of partitions adapted to unstable disks on the natural extension that resemble an inverse Markov tower.

4. *A first volume lemma*

4.1. *c-Cone-hyperbolic times and geometry of disks tangent to C.*

In this subsection we collect some results from [**7**], which were inspired by [**1**]. Consider D a disk in M which is tangent to the cone field C and assume that D satisfies $\mathrm{Leb}_D(H) > 0$, where

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H is the subset given by hypothesis (H3). For $\varepsilon > 0$ and $x \in M$ denote by $T_xM(\varepsilon) := \{v \in T_xM : \|v\| \leq \varepsilon\}$ the ball of radius ε-centered on $0 \in T_xM$. Fix $\delta_0 > 0$ such that the exponential map restricted to $T_xM(\delta_0)$ is a diffeomorphism onto its image for every $x \in M$, and set $B(x, \varepsilon) := \exp_x(T_xM(\varepsilon))$ for every $0 < \varepsilon \leq \delta_0$. As D is tangent to the cone field there exists $0 < \delta \leq \delta_0$ such that if $y \in B(x, \delta) \cap D$ then there exists a unique linear map $A_x(y) : T_xD \to E^s_x$ whose graph is parallel to T_yD. Moreover, by compactness of M, δ can be taken independent of x. Thus one can express the Hölder variation of the tangent bundle in these local coordinates as follows: given $C > 0$ and $0 < \alpha < 1$, we say that the tangent bundle TD of D is (C, α)-*Hölder* if $\|A_x(y)\| \leq C \cdot \text{dist}_D(x, y)^\alpha$, for every $y \in B(x, \delta_0) \cap D$ and $x \in D$. Moreover, the α-Hölder constant of the tangent bundle TD is given by the number $\kappa(TD, \alpha) := \inf\{C > 0 : TD \text{ is } (C, \alpha)\text{-Hölder}\}$. The Df-invariance of the cone field (assumption (H3)) implies that disks tangent to the cone field are preserved under positive iteration by the dynamics. Furthermore, while assumption (H4) guarantees domination and there exists a control of the distortion along these disks (cf. [**7**, Proposition 5.1]), one can expect a volume lemma for the Lebesgue measure at cone-hyperbolic times. This will appear in §4.2.

PROPOSITION 4.1. [**7**, Proposition 5.1] *There are $\alpha, \beta \in (0, 1)$ and $C_0 > 0$ such that if N and $f(N)$ are submanifolds of M tangent to the cone field C then $\kappa(Df(N), \alpha) \leq \beta \cdot \kappa(TN, \alpha) + C_0$. In consequence, there is $L_1 > 0$ (depending only on f) such that if $f^j(N)$ is a submanifold of M tangent to C for all $0 \leq j \leq k$ then the map*

$$\begin{array}{rccc} J_k : & f^k(N) & \to & \mathbb{R} \\ & x & \mapsto & \log|\det(Df(x)|_{T_x f^k(N)})|, \end{array}$$

is (L_1, α)-Hölder continuous.

Hypothesis (H3) implies that there exist a disk $D \subset U$ tangent to C and a positive Lebesgue measure subset $H \subset D$ satisfying

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log \|(Df(f^j(x))|_{C(f^j(x))})^{-1}\| \leq -2c < 0 \tag{4.1}$$

for every $x \in H$. Pliss's lemma (see, for example, [**9**, Lemma 11.8]) ensures that there exists $\theta > 0$ (depending only on f and c) such that the following holds: for every $x \in H$ and $n \geq 1$ large there exists a sequence $1 \leq n_1(x) < \cdots < n_l(x) \leq n$ of c-cone-hyperbolic times for x with $l \geq \theta n$.

Definition 4.1. Given $x \in \Lambda$ with infinitely many cone-hyperbolic times, let $n_i = n_i(x)$ denote the *ith cone hyperbolic time* for x. We say that the sequence $(n_i)_{i\geq 1}$ is *non-lacunar* if $n_{i+1}/n_i \to 1$ as $i \to \infty$.

Remark 4.1. We observe that the non-lacunarity of a given sequence of positive integers requires the gaps between consecutive elements to be small in comparison with the growth of the elements in the sequence.

Remark 4.2. A simple computation shows that $n_1(f^{n_i(x)}(x)) = n_{i+1}(x) - n_i(x)$ and that $\text{Leb}(x \in U : n_{i+1}(x) - n_i(x) > \varepsilon n_i(x))$ coincides with $\text{Leb}(x \in U : n_1(f^{n_i(x)}(x)) > \varepsilon n_i(x))$, for every $\varepsilon > 0$. It is well known that the integrability of the first cone-hyperbolic time map with respect to an invariant probability η ensures that the sequence $(n_i(x))_{i\geq 1}$ is

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non-lacunary for η-almost every x (see, for example, [**24**, Lemma 3.7]). In the case of the (not necessarily invariant) Lebesgue measure, the non-lacunarity of the sequence of cone-hyperbolic times follows as a direct consequence of the Borel–Cantelli lemma whenever the condition $\sum_{k\geq 1} \mathrm{Leb}(x \in U : n_1(f^k(x)) > \varepsilon k) < \infty$ holds. Araújo and Pacífico [**2**] proved large-deviation upper bounds for the Lebesgue measure using an alternative method of covering at instants of hyperbolicity.

Given a submanifold $D \subset M$, we denote by dist_D the Riemannian distance on D. Then $\mathrm{dist}_M \leq \mathrm{dist}_D$.

PROPOSITION 4.2. [**7**, Proposition 5.2] *Let D be a C^1 disk embedded in U that is tangent to C. There is $\delta > 0$ such that if n is a c-cone-hyperbolic time for $x \in D$ and $d(x, \partial D) > \delta$ then there is an open neighborhood $D(x, n, \delta)$ of x in D diffeomorphic to a C^1-disk $\Delta(f^n(x), \delta)$ by f^n. Moreover,*

$$\mathrm{dist}_{f^{n-k}(D(x,n,\delta))}(f^{n-k}(x),\ f^{n-k}(y)) \leq e^{-(c/2)k}\, \mathrm{dist}_{\Delta(f^n(x),\delta)}(f^n(x),\ f^n(y))$$

for every $y \in D(x, n, \delta)$ and $1 \leq k \leq n$.

We observe the constant $\delta > 0$ in Proposition 4.2 is independent of $x \in M$. The disks $D(x, n, \delta) \subset D$ are called *hyperbolic pre-disks* and their images $\Delta(f^n(x), \delta)$ are called *n-hyperbolic disks*. We need the following bounded distortion property (cf. [**7**, Proposition 5.2]) for the Jacobian of f along disks tangent to the invariant cone field.

PROPOSITION 4.3. *Suppose D is a C^1 disk embedded in U, $x \in D$ and let n be a c-cone hyperbolic time for x. There is $C_1 > 0$ such that*

$$C_1^{-1} \leq \frac{|\det Df^n(y)|_{T_y D(x,n,\delta)}|}{|\det Df^n(z)|_{T_z D(x,n,\delta)}|} \leq C_1$$

for every $y, z \in D(x, n, \delta)$, where $D(x, n, \delta)$ is the hyperbolic pre-disk at x.

4.2. *The first volume lemma.* In this subsection we restate and prove a volume lemma for the Lebesgue measure of dynamic balls on the ambient space. The volume lemma follows from a Fubini-like argument together with two bounded distortion lemmas, one obtained as consequence of the backward contraction property for disks tangent to the cone field at c-cone-hyperbolic times, and the other that requires the control of distortion for all subspaces of maximal dimension contained in the cone field.

Recall that the cone field C has constant dimension d. Given $x \in M$ and two subspaces F_1, F_2 of dimension d contained in the cone $C(x)$, there exist unique linear maps $L_{F_2} : F_1 \to E^s_x$ and $L_{F_1} : F_2 \to E^s_x$ such that

$$F_2 = \mathrm{graph}(L_{F_2}) := \{v + L_{F_2}(v) \in F_1 \oplus E^s_x : v \in F_1\} \tag{4.2}$$

and $F_1 = \mathrm{graph}(L_{F_1})$. We measure the distance between two subspaces F_1 and F_2 on $T_x M$ by

$$\theta_x(F_1, F_2) := \max\{\|L_{F_1}\|, \|L_{F_2}\|\}.$$

Note that $\theta_x(F_1, F_2) = 0$ if and only if F_1 and F_2 coincide, and this is roughly a projective metric.

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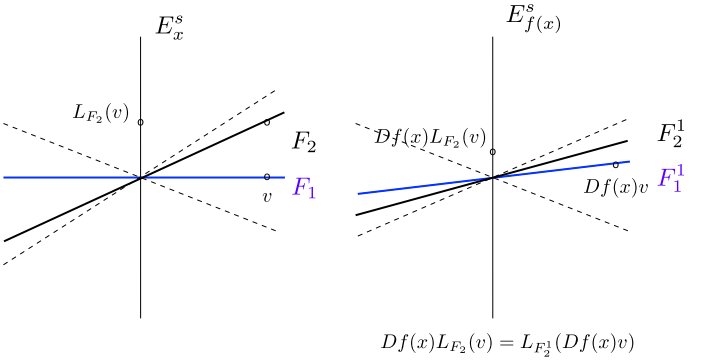


FIGURE 1. Iteration of subspaces that are graphs.

Remark 4.3. Since f is a local diffeomorphism, given $x \in M$, two subspaces $F_1, F_2 \subset T_xM$ of dimension d and the linear map $L_{F_2} : F_1 \to E^s_x$ such that $F_2 = \text{graph}(L_{F_2})$, the subspaces $Df(x)F_1$ and $Df(x)F_2$ also have dimension d. Moreover, $Df(x)F_2$ is obtained as the graph of a unique linear map $L^1_{F_2} : Df(x)F_1 \to E^s_{f(x)}$. In addition, by (4.2) and the Df-invariance of the stable subbundle, we get

$$Df(x)[v + L_{F_2}(v)] = Df(x)v + Df(x)L_{F_2}(v) \in [Df(x)F_1 \oplus E^s_{f(x)}] \cap \text{graph}(L^1_{F_2}),$$

and, consequently, $Df(x)L_{F_2}(v) = L^1_{F_2}(Df(x)v)$ for each $v \in F_2$ (see Figure 1). By induction, we get that

$$Df^i(x)L_{F_2}(v) = L^i_{F_2}(Df^i(x)v) \quad \text{for every } v \in F_2 \text{ and every } i \geq 1, \tag{4.3}$$

where $F^i_2 = Df^i(x)F_2 = \text{graph}(L^i_{F_2})$, $L^i_{F_2} : F^i_1 \to E^s_{f^i(x)}$, and $F^i_1 := Df^i(x)F_1$.

LEMMA 4.1. *There exists $\kappa \in (0, 1)$ such that for every $n \geq 1$,*

$$\theta_{f^i(x)}(Df^i(x)F_1, Df^i(x)F_2) \leq \kappa^i \cdot \theta_x(F_1, F_2) \quad \textit{for every } 1 \leq i \leq n.$$

Proof. Given $n \geq 1$, $v \in F_1$ and $v_i = Df^i(x)v$ for every $0 \leq i \leq n$. By the invariance condition (4.3) and the domination condition (H4),

$$\frac{\|L_{F^i_2} v_i\|}{\|v_i\|} = \frac{\|Df^i(x)L_{F_2}v\|}{\|Df^i(x)v\|} \leq \lambda^i \frac{\|L_{F_2}v\|}{\|v\|} \leq \lambda^i \theta_x(F_1, F_2)$$

for every $0 \leq i \leq n$. Since estimates for $\|L_{F^i_1}\|$ are analogous, the lemma follows by taking $\kappa = \lambda$. □

PROPOSITION 4.4. *If $n \geq 1$, $d > 1$, $F_1, F_2 \subset T_xM$ are subspaces of dimension d contained in $C(x)$ and $\theta_x(F_1, F_2) < 1/(d^2 - 1)$ then there exists $K > 0$ such that*

$$\left| \frac{|\det(Df^n(x)|_{F_2})|}{|\det(Df^n(x)|_{F_1})|} - 1 \right| \leq K \frac{\theta_x(F_1, F_2)}{1 - (d^2 - 1)\theta_x(F_1, F_2)}.$$

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Proof. Assume $d > 1$. Given a orthonormal basis $e_1, \dots, e_d$ of F_1, we have that $\|e_1 \wedge \cdots \wedge e_d\| = 1$ and

$$|\det(Df^n(x)|_{F_1})| = \|Df^n(x)e_1 \wedge \cdots \wedge Df^n(x)e_d\|. \tag{4.4}$$

We proceed to estimate $|\det(Df^n(x)|_{F_2})|$. Note that for any pair of orthonormal vectors e_s, e_r on F_1 we have

$$\begin{aligned}
&| \, \|Df^n(x)(e_r + L_{F_2}e_r) \wedge Df^n(x)(e_s + L_{F_2}e_s)\| - \|Df^n(x)e_r \wedge Df^n(x)e_s\| \, | \\
&\quad \le \|Df^n(x)e_r \wedge Df^n(x)L_{F_2}e_s + Df^n(x)L_{F_2}e_r \\
&\qquad \wedge Df^n(x)e_s + Df^n(x)L_{F_2}e_r \wedge Df^n(x)L_{F_2}e_s\| \\
&\quad = \|Df^n(x)e_r \wedge L^n_{F_2}Df^n(x)e_s + L^n_{F_2}Df^n(x)e_r \\
&\qquad \wedge Df^n(x)e_s + L^n_{F_2}Df^n(x)e_r \wedge L^n_{F_2}Df^n(x)e_s\| \\
&\quad \le 3\|L^n_{F_2}\| \, \|Df^n(x)e_r \wedge Df^n(x)e_s\|.
\end{aligned}$$

Using the previous argument recursively, we get

$$\begin{aligned}
&| \, \|Df^n(x)(e_1 + L_{F_2}e_1) \wedge \cdots \wedge Df^n(x)(e_d + L_{F_2}e_d)\| \\
&\qquad - \|Df^n(x)e_1 \wedge \cdots \wedge Df^n(x)e_d\| \, | \\
&\quad \le (d^2 - 1)\|L^n_{F_2}\| \, \|Df^n(x)e_1 \wedge \cdots \wedge Df^n(x)e_d\|,
\end{aligned}$$

which implies that

$$\begin{aligned}
&\|Df^n(x)(e_1 + L_{F_2}e_1) \wedge \cdots \wedge Df^n(x)(e_d + L_{F_2}e_d)\| \\
&\quad \le [1 + (d^2 - 1)\|L^n_{F_2}\|] \, \|Df^n(x)e_1 \wedge \cdots \wedge Df^n(x)e_d\|
\end{aligned} \tag{4.5}$$

and

$$\begin{aligned}
&\|Df^n(x)(e_1 + L_{F_2}e_1) \wedge \cdots \wedge Df^n(x)(e_d + L_{F_2}e_d)\| \\
&\quad \ge [1 - (d^2 - 1)\|L^n_{F_2}\|] \, \|Df^n(x)e_1 \wedge \cdots \wedge Df^n(x)e_d\|.
\end{aligned} \tag{4.6}$$

Proceeding as above, we also obtain that

$$| \, \|(e_1 + L_{F_2}e_1) \wedge \cdots \wedge (e_d + L_{F_2}e_d)\| - \|e_1 \wedge \cdots \wedge e_d\| \, | \le (d^2 - 1)\theta_x(F_1, F_2)$$

or, equivalently,

$$\begin{aligned}
1 - (d^2 - 1)\theta_x(F_1, F_2) &\le \|(e_1 + L_{F_2}e_1) \wedge \cdots \wedge (e_d + L_{F_2}e_d)\| \\
&\le 1 + (d^2 - 1)\theta_x(F_1, F_2).
\end{aligned} \tag{4.7}$$

By Lemma 4.1, together with (4.4), (4.5) and (4.7), we conclude that

$$\begin{aligned}
|\det(Df^n(x)|_{F_2})| &= \frac{\|Df^n(x)(e_1 + L_{F_2}e_1) \wedge \cdots \wedge Df^n(x)(e_d + L_{F_2}e_d)\|}{\|(e_1 + L_{F_2}e_1) \wedge \cdots \wedge (e_d + L_{F_2}e_d)\|} \\
&\le \frac{1 + (d^2 - 1)\|L^n_{F_2}\|}{1 - (d^2 - 1)\theta_x(F_1, F_2)} \cdot \|Df^n(x)e_1 \wedge \cdots \wedge Df^n(x)e_d\| \\
&\le \frac{1 + (d^2 - 1)\theta_{f^n(x)}(F_1^n, F_2^n)}{1 - (d^2 - 1)\theta_x(F_1, F_2)} \cdot \|Df^n(x)e_1 \wedge \cdots \wedge Df^n(x)e_d\|,
\end{aligned}$$

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and so

$$\begin{aligned}\frac{|\det(Df^n(x)|_{F_2})|}{|\det(Df^n(x)|_{F_1})|} &\le \frac{1+(d^2-1)\theta_{f^n(x)}(F_1^n, F_2^n)}{1-(d^2-1)\theta_x(F_1, F_2)}\\ &= 1+\frac{(d^2-1)[\theta_x(F_1, F_2)+\theta_{f^n(x)}(F_1^n, F_2^n)]}{1-(d^2-1)\theta_x(F_1, F_2)}\\ &\le 1+\frac{(d^2-1)[\theta_x(F_1, F_2)+\theta_{f^n(x)}(F_1^n, F_2^n)]}{1-(d^2-1)\theta_x(F_1, F_2)}\\ &\le 1+\frac{(d^2-1)(1+\kappa^n)}{1-(d^2-1)\theta_x(F_1, F_2)}\theta_x(F_1, F_2)\\ &\le 1+2(d^2-1)\frac{\theta_x(F_1, F_2)}{1-(d^2-1)\theta_x(F_1, F_2)}\end{aligned}$$

for every $n \ge 1$. Analogously, using (4.4), (4.6) and (4.7),

$$\begin{aligned}\frac{|\det(Df^n(x)|_{F_2})|}{|\det(Df^n(x)|_{F_1})|} &\ge \frac{1-(d^2-1)\theta_{f^n(x)}(F_1^n, F_2^n)}{1+(d^2-1)\theta_x(F_1, F_2)}\\ &= 1-\frac{(d^2-1)\theta_{f^n(x)}(F_1^n, F_2^n)+(d^2-1)\theta_x(F_1, F_2)}{1+(d^2-1)\theta_x(F_1, F_2)}\\ &\ge 1-\frac{(d^2-1)(1+\kappa^n)\theta_x(F_1, F_2)}{1+(d^2-1)\theta_x(F_1, F_2)} \ge 1-\frac{2(d^2-1)\theta_x(F_1, F_2)}{1+(d^2-1)\theta_x(F_1, F_2)}.\end{aligned}$$

Taking $K = 2(d^2-1)$, the lemma follows. □

Remark 4.4. If $d = 1$ then a simplification of the previous argument yields that if $e_1 \in F_1$ and $\|e_1\| = 1$ then

$$\begin{aligned}\frac{|\det(Df^n(x)|_{F_2})|}{|\det(Df^n(x)|_{F_1})|} &= \frac{\|Df^n(x)(e_1+L_{F_2}e_1)\|}{\|e_1+L_{F_2}e_1\|} \le \frac{1+\|Df^n(x)L_{F_2}e_1\|}{1-\theta_x(F_1, F_2)}\\ &\le \frac{1+\theta_{f^n(x)}(F_1^n, F_2^n)}{1-\theta_x(F_1, F_2)}\end{aligned}$$

and the lower bound follows analogously.

Using Lemma 4.1 and Proposition 4.4, we deduce the following bounded distortion property.

COROLLARY 4.1. *There exists $K_0 > 0$ so that for every $n \ge 1$,*

$$\frac{1}{K_0} \le \frac{|\det(Df^n(x)|_{F_2})|}{|\det(Df^n(x)|_{F_1})|} \le K_0 \tag{4.8}$$

for any *subspaces F_1, F_2 of dimension d contained in the cone $C(x)$.*

Proof. Assumptions (H1)–(H4) ensure that the angle between vectors in E^s and vectors contained in the cone field $C(\cdot)$ is uniformly bounded away from zero on the attractor Λ. Hence $\theta_x(F_1, F_2)$ is uniformly bounded, for every $x \in \Lambda$ and any pair of subspaces F_1, F_2 contained in $C(x)$. This, together with Proposition 4.4 and Remark 4.4, implies that (4.8) holds with $K_0 = 1 + 2K \max_{x\in\Lambda} \max_{F_1,F_2\subset C(x)} \theta_x(F_1, F_2)$ for any subspaces F_1, F_2 of dimension d contained in the cone $C(x)$ so that $\theta_x(F_1, F_2) < \min\{1/2(d^2-1), 1/2\}$.

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Now, let F_1, F_2 be arbitrary subspaces of dimension d contained in $C(x)$. Recall that F_2 is the graph of a linear map $L_{F_2} : F_1 \to E^s_x$. The Grassmannian $\mathrm{Grass}_d(\mathbb{R}^{\dim M})$, that is, the set of all d-dimensional subspaces in $\mathbb{R}^{\dim M}$, is compact (see, for example, [**19**]). Therefore, there exists $N = N(d) \geq 1$ so that $\mathrm{Grass}_d(\mathbb{R}^{\dim M})$ admits a finite cover by N balls of radius $1/2(d^2-1)$. Using that M is compact and that the fiber bundle TM is locally isomorphic to $U \times \mathbb{R}^{\dim M}$ ($U \subset M$), we have the following consequence: there exist $1 \leq r \leq N$ and a family $(E_i)_{1\leq i\leq r}$ of subspaces of dimension d contained in the cone such that $E_1 = F_1$, $E_r = F_2$ and $\theta_x(E_i, E_{i+1}) < 1/2(d^2-1)$ for every $1 \leq i \leq r-1$. In particular, (4.8) holds with $K_0 = (1 + 2K \max_{x\in\Lambda} \max_{F_1,F_2\subset C(x)} \theta_x(F_1, F_2))^N$. This proves the corollary. □

We will need the following result.

COROLLARY 4.2. *Let ν be an f-invariant probability measure supported on Λ. Then, for ν-almost every x and for* any *subspace $F_1 \subset C(x)$ of dimension d,*

$$\Gamma_\nu(x) = \lim_{n\to\infty} \frac{1}{n} \log |\det Df^n(x)|_{F_1}| = \sum_{\lambda_i(\nu,x)>0} \lambda_i(\nu, x),$$

where $\lambda_1(\nu, x) \leq \lambda_2(\nu, x) \leq \cdots \leq \lambda_d(\nu, x)$ are the ν-almost everywhere defined Lyapunov exponents of f with respect to ν along the cone direction.

Proof. Let ν be f-invariant probability measure supported on Λ and let $\hat{\nu}$ be the unique $\hat{f}$-invariant probability measure such that $\pi_*\hat{\nu} = \nu$. By Oseledets's theorem there exists an almost everywhere defined $D\hat{f}$-invariant splitting $T_{\pi(\hat{x})}M = E_1(\hat{x}) \oplus \cdots \oplus E_{r(\hat{x})}(\hat{x})$ so that $\lim_{n\to\pm\infty} (1/n) \log \|D\hat{f}^n(\hat{x})u\| = \lambda_i(\hat{x})$ for every $u \in E_i(\hat{x})\backslash\{0\}$ and for every $1 \leq i \leq r(\hat{x})$ (cf. Proposition 3.1). However, by partial hyperbolicity (recall assumptions (H1) and (H3)), E^s is a Df-invariant subbundle and $C(\cdot)$ is a Df-invariant cone field. This fact, together with the latter, ensures that for $\hat{\nu}$-almost every $\hat{x}$ there exists $1 < s(\hat{x}) < r(\hat{x})$ so that

$$E^s_{\pi(\hat{x})} = E_1(\hat{x}) \oplus \cdots \oplus E_{s(\hat{x})}(\hat{x}) \quad \text{and} \quad T_{\pi(\hat{x})}M = E^s_{\pi(\hat{x})} \oplus E_{s(\hat{x})+1}(\hat{x}) \oplus \cdots \oplus E_{r(\hat{x})}(\hat{x}),$$

and the $D\hat{f}$-invariant subspace $\hat{F}_{\pi(\hat{x})} := E_{s(\hat{x})+1}(\hat{x}) \oplus \cdots \oplus E_{r(\hat{x})}(\hat{x})$ is contained in the cone $C(\pi(\hat{x}))$.

Now fix an arbitrary subspace $F_1 \subset C(x)$ of dimension d. By definition of Γ_n (recall (2.2)) together with Corollary 4.1 it follows that

$$|\det Df^n(x)|_{F_1}| \leq \Gamma_n(x) := \max_{\tilde{F}\subset C(x)} |\det Df^n(x)|_{\tilde{F}}| \leq K_0 |\det Df^n(x)|_{F_1}|. \tag{4.9}$$

Moreover, since $(\Gamma_n)_n$ is sub-multiplicative it follows from Kingman's subadditive ergodic theorem that the limit $\lim_{n\to\infty} (1/n) \log \Gamma_n(x)$ does exist for ν-almost every x. This, together with (4.9), ensures that

$$\begin{aligned} \limsup_{n\to\infty} \frac{1}{n} \log \det |Df^n(x)|_{F_1}| &\leq \lim_{n\to\infty} \frac{1}{n} \log\ \Gamma_n(x) = \Gamma_\nu(x) \\ &\leq \liminf_{n\to\infty} \frac{1}{n} \log \det |Df^n(x)|_{F_1}| \end{aligned}$$

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for ν-Oseledets theorem x (the same property holds taking lim inf instead of lim sup). Then $\Gamma_\nu(x) = \lim_{n\to\infty} (1/n) \log |\det Df^n(x)|_{F_1}|$, as claimed. Finally, the second equality in the corollary follows from the previous argument and the fact that if $x = \pi(\hat{x})$ then K_0^{-1} $|\det Df^n(x)|_{F_1}| \le |\det Df^n(x)|_{\hat{F}_x}| \le K_0 |\det Df^n(x)|_{F_1}|$. This proves the corollary. □

Remark 4.5. Given $c > 0$ and $\delta = \delta(c) > 0$ as in Proposition 4.2, there exists $\delta_1 = \delta_1(c) \in (0, \delta)$ so that if n is a c-cone-hyperbolic time for x then n is a $c/2$-cone-hyperbolic time for every $y \in B(x, n, \delta_1)$ (see, for example, [**7**, Lemma 5.3 and Proposition 5.7]).

Let Leb denote the Lebesgue measure on the compact Riemannian manifold M. The next proposition ensures, loosely speaking, that the Lebesgue measure satisfies a volume lemma (at cone-hyperbolic times) expressed in terms of the Jacobian along of any subspace of full dimension in the cone field.

PROPOSITION 4.5. *Let $\delta = \delta(c/2) > 0$ be given by Proposition 4.2. Given $0 < \varepsilon < \delta$, there exists $K(\varepsilon) > 0$ so that if n is a c-cone-hyperbolic time for x and F_1 is* any *subspace of dimension d contained in $C(x)$ then*

$$K(\varepsilon)^{-1} e^{-\log|\det Df^n(x)|_{F_1}|} \le \mathrm{Leb}(B(x, n, \varepsilon)) \le K(\varepsilon) e^{-\log|\det Df^n(x)|_{F_1}|}. \tag{4.10}$$

Proof. Given $c > 0$, let $0 < \delta_1 < \delta = \delta(c/2)$ be given by Proposition 4.2 and Remark 4.5. Assume that n is a c-cone-hyperbolic time for x and take $0 < \varepsilon \ll \delta_1$. By Remark 4.5, n is a $c/2$-cone hyperbolic time for every $y \in B(x, n, \delta_1)$ and, consequently, if D is a C^1-disk tangent to $C(x)$ passing through y and $d(y, \partial D) > \varepsilon$ then there is an open neighborhood $D(y, n, \varepsilon)$ of y in D diffeomorphic to a C^1-disk $\Delta(f^n(y), \varepsilon)$ by f^n and such that

$$\mathrm{dist}_{f^{n-k}(D(y,n,\varepsilon))}(f^{n-k}(z),\ f^{n-k}(y)) \le e^{-(c/2)k}\ \mathrm{dist}_{\Delta(f^n(y),\varepsilon)}(f^n(z),\ f^n(y)),$$

for every $z \in D(y, n, \varepsilon)$ and $1 \le k \le n$. Since the curvature of disks tangent to $C(x)$ is bounded, we may reduce δ_1 as necessary to get that $\mathrm{dist}_M \le \mathrm{dist}_{\tilde{D}} \le 2\,\mathrm{dist}_M$ for every C^1-disk $\tilde{D}$ of diameter δ_1 tangent to the cone field.

Let D be a C^1-disk of diameter δ_1 tangent to $C(x)$ passing through x and let $D(x, n, \varepsilon)$ be as above. Then it is not hard to check that

$$D \cap B\left(x, n, \frac{\varepsilon}{2}\right) \subset D(x, n, \varepsilon) \subset D \cap B(x, n, \varepsilon).$$

Moreover, as the local stable manifolds are exponentially contracted by iteration of the dynamics, we have that $W^s_{\mathrm{loc}}(x) \cap B(x, \varepsilon) = W^s_{\mathrm{loc}}(x) \cap B(x, n, \varepsilon)$. Set $W := W^s_{\mathrm{loc}}(x) \cap B(x, \varepsilon)$, consider $\hat{W} = \exp_x^{-1}(W)$, $\hat{D}_x := \exp_x^{-1}(D)$, and for every $y \in \hat{W}$ let $\hat{D}_y$ be the C^1-submanifold passing through y and obtained by parallel transport from $\hat{D}_x$ (see Figure 2). In particular, $(\exp_x(\hat{D}_y))_{y\in\hat{W}}$ is a foliation by C^1-submanifolds $(\gamma_y)_{y\in W}$ tangent to the cone field C that cover $B(x, n, \varepsilon)$. By the Fubini–Tonelli theorem, we have that $\mathrm{Leb}(B(x, n, \varepsilon)) = \int_W \mathrm{Leb}_{\gamma_y}(B(x, n, \varepsilon) \cap \gamma_y)\, dy$. Using change of variables and Proposition 4.3, we conclude that, for every $y \in W$,

$$\begin{aligned}\mathrm{Leb}(f^n(B(x, n, \varepsilon) \cap \gamma_y)) &= \int_{B(x,n,\varepsilon)\cap\gamma_y} |\det Df^n(z)\,|_{T_z\gamma_y}\,|\, d\,\mathrm{Leb}_{\gamma_y}(z) \\ &\le C_1\,\mathrm{Leb}_{\gamma_y}(B(x, n, \varepsilon) \cap \gamma_y) |\det Df^n(y)\,|_{T_y\gamma_y}\,|\end{aligned}$$

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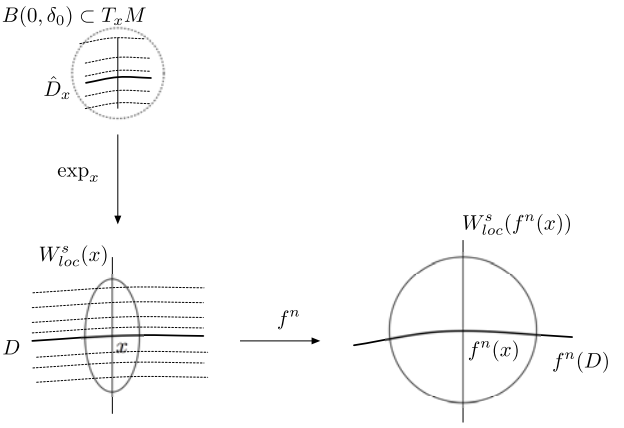


FIGURE 2. Construction of a foliation of the dynamic ball by disks tangent to the cone field.

(the lower bound is analogous). Since n is a c-cone-hyperbolic time, $f^n(B(x, n, \varepsilon) \cap \gamma_y)$ contains a disk of uniform radius ε in γ_y, and the left-hand side above is bounded away from zero by some constant $C_2(\varepsilon) > 0$. By Corollary 4.1,

$$\begin{aligned} \mathrm{Leb}_{\gamma_y}(B(x, n, \varepsilon) \cap \gamma_y) &\geq C_1^{-1} C_2(\varepsilon) |\det Df^n(y)\,|_{T_y\gamma_y}\,|^{-1} \\ &\geq K_0^{-1} C_1^{-1} C_2(\varepsilon) |\det Df^n(y)\,|_{F_1}\,|^{-1} \end{aligned} \tag{4.11}$$

for all $y \in W$ (by an abuse of notation F_1 stands for the subspace of T_yM obtained from $F_1 \subset T_xM$ by parallel transport). Since $\mathrm{Leb}(f^n(B(x, n, \varepsilon) \cap \gamma_y))$ is bounded above by a uniform constant $C_3(\varepsilon) > 0$, a similar reasoning implies that

$$\mathrm{Leb}_{\gamma_y}(B(x, n, \varepsilon) \cap \gamma_y) \leq K_0 C_1 C_3(\varepsilon) |\det Df^n(y)\,|_{F_1}\,|^{-1}. \tag{4.12}$$

Now, since the distance between any pair of points in W is exponentially contracted by forward iteration and $W \ni y \mapsto |\det Df(y)\,|_{F_1}\,|$ is Hölder continuous, there exists $K_1 > 0$ so that $K_1^{-1} |\det Df^n(y)\,|_{F_1}\,| \leq |\det Df^n(x)\,|_{F_1}\,| \leq K_1 |\det Df^n(y)\,|_{F_1}\,|$ for any $x, y \in W$ and $n \geq 1$. Therefore

$$\mathrm{Leb}(B(x, n, \varepsilon)) = \int \mathrm{Leb}_{\gamma_y}(B(x, n, \varepsilon) \cap \gamma_y)\, dy \geq K(\varepsilon)^{-1} e^{-\log|\det Df^n(x)|_{F_1}|},$$

for every $n \geq 1$, where $K(\varepsilon) = K_0 K_1 C_1 \max\{C_2(\varepsilon)^{-1}, C_3(\varepsilon)\}$. Since the proof of the upper bound is entirely analogous using (4.12) instead of (4.11), this completes the proof of the proposition. □

5. *A second volume lemma*

5.1. *The construction of the SRB measures.* As defined in §3.3, SRB measures are absolutely continuous with respect to unstable manifolds. In our context, there are finitely many SRB measures the union of whose basins of attraction has full Lebesgue measure in a neighborhood of the partially hyperbolic attractor. Moreover, if the local diffeomorphism is transitive on the attractor then the SRB measure is unique. See [**7**, Theorem A] for more details.

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We need to recall some of the arguments in the construction of the SRB measures, referring the reader to [7] for full details and proofs. Fix $\delta > 0$ given by Proposition 4.2 and let $H = H(c) \subset U$ be the set of points with infinitely many c-cone hyperbolic times and such that $\mathrm{Leb}_D(H) > 0$, guaranteed by hypothesis (H3). Reducing δ, if necessary, we can assume that $B := B(D, c, \delta) := \{x \in D \cap H : \mathrm{dist}_D(x, \partial D) \geq \delta\}$ satisfies $\mathrm{Leb}_D(B) > 0$. For each $n \in \mathbb{N}$ define $H_n := \{x \in B : n \text{ is a } c\text{-cone hyperbolic time for } x\}$. By [7, §5], there exists $\tau > 0$ and for each $n \in \mathbb{N}$ there is a finite set $H_n^* \subset H_n$ such that the hyperbolic pre-disks $(D(x, n, \delta))_{x \in H_n^*}$ are contained in D, pairwise disjoint, and their union $\mathscr{D}_n := \bigcup_{x \in H_n^*} D(x, n, \delta)$ satisfies $\mathrm{Leb}_D(\mathscr{D}_n \cap H_n) \geq \tau\, \mathrm{Leb}_D(H_n)$. The sequence of probability measures $(\nu_n)_n$ given by (3.5) have non-trivial accumulation points ν in the weak* topology. Each measure ν_n (recall (3.5)) is supported on the union of disks $\bigcup_{j=0}^{n-1} \Delta_j$, with $\Delta_j := \bigcup_{x \in H_j^*} \Delta(f^j x, \delta)$, and the support of any accumulation point ν of $(\nu_n)_{n \in \mathbb{N}}$ is contained in a union of unstable manifolds of uniform size.

PROPOSITION 5.1. [7] *Given $y \in \mathrm{supp}(\nu)$, there exists $\hat{x} = (x_{-n})_{n \in \mathbb{N}} \in M^f$ such that y belongs to a disk $\Delta(\hat{x})$ of radius $\delta > 0$ accumulated by n_j-hyperbolic disks $\Delta(f^{n_j}(x), \delta)$ with $j \to \infty$, and satisfying that:*

(1) *the inverse branch $f_{x_{-n}}^{-n}$ is well defined in $\Delta(\hat{x})$ for every $n \geq 0$;*
(2) *if for each $y \in \Delta(\hat{x})$ we take $y_{-n} := f_{x_{-n}}^{-n}(y)$ then $\mathrm{dist}_M(y_{-n}, x_{-n}) \leq e^{-(c/2)\cdot n}\delta$ for every $n \geq 0$.*

By Proposition 5.1, given $y \in \mathrm{supp}(\nu)$, there is a disk $\Delta(\hat{x})$ which is contractive along the pre-orbit $\hat{x} \in M^f$. We proceed to prove $\hat{\nu}$ is supported on a union of unstable manifolds. For that, if $\Delta(\hat{x})$ ($\hat{x} \in \hat{H}_\infty$) is a disk given by Proposition 5.1 then there exists a sequence of points $(z_{j_k})_{k \in \mathbb{N}}$ satisfying $z_{j_k} \in H_{j_k}^*$ for every $k \in \mathbb{N}$ such that $\lim_{k \to \infty} \Delta(f^{j_k}(z_{j_k}), \delta) = \Delta(\hat{x})$. Since $\lim_{k \to \infty} f^{j_k - n}(z_{j_k}) = x_{-n}$ for every $n \in \mathbb{N}$ (where the limit is taken over $j_k \geq n$) then it is natural to define

$$\hat{\Delta}(\hat{x}) := \Big\{\hat{y} \in M^f : \hat{y} = \lim_{s \to \infty} \hat{y}_{j_{k_s}},\ \text{for some subsequence } (\hat{y}_{j_{k_s}})_{s \in \mathbb{N}} \\ \text{where } \hat{y}_{j_{k_s}} \in \hat{f}^{j_{k_s}}(\pi^{-1}(D(z_{j_{k_s}}, j_{k_s}, \delta))) \text{ and } (z_{j_{k_s}})_{s \in \mathbb{N}} \subset (z_{j_k})_{k \in \mathbb{N}}\Big\}. \quad (5.1)$$

Roughly, the set $\hat{\Delta}(\hat{x})$ is formed by all points obtained as accumulation points of a sequence $(\hat{y}_{j_k})_{k \in \mathbb{N}}$ where $\hat{y}_{j_k} \in \hat{f}^{j_k}(\pi^{-1}(D(z_{j_k}, j_k, \delta)))$ for every $k \geq 1$. The support of the lift $\hat{\nu}$ of ν to M^f is contained in the union of this family of sets. More precisely, we have the following result.

LEMMA 5.1. *Given $\hat{y} \in \mathrm{supp}(\hat{\nu})$, there exist $\hat{x} \in M^f$ and $\hat{\Delta}(\hat{x}) \subset M^f$ (given by (5.1)) such that $\hat{y} \in \hat{\Delta}(\hat{x})$ and $\pi|_{\hat{\Delta}(\hat{x})} : \hat{\Delta}(\hat{x}) \to \Delta(\hat{x})$ is a continuous bijection.*

Let $\hat{H}_\infty$ be the set of points $\hat{x} \in M^f$ given by Lemma 5.1. Then $\mathrm{supp}(\hat{\nu}) \subset \bigcup_{\hat{x} \in \hat{H}_\infty} \hat{\Delta}(\hat{x})$. This is a key step in the proof that ν is a f-invariant hyperbolic measure. Moreover, we have the following theorem.

THEOREM 5.1. [7, Theorem 5.1] *There exists $C_3 > 1$ so that the following holds: given $(\hat{x}, l)$, there exists a family of conditional measures $(\hat{\nu}_{\hat{\Delta}})_{\hat{\Delta} \in \hat{\mathcal{K}}_\infty(\hat{x}, l)}$ of $\hat{\nu}|_{\hat{K}_\infty(\hat{x}, l)}$ such that*

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$\pi_*\hat{\nu}_{\hat{\Delta}} \ll \mathrm{Leb}_\Delta$, *where* $\Delta = \pi(\hat{\Delta})$, *and* $(1/C_3)\,\mathrm{Leb}_\Delta(B) \le \pi_*\hat{\nu}_{\hat{\Delta}}(B) \le C_3\,\mathrm{Leb}_\Delta(B)$ *for every measurable subset* $B \subset \Delta$ *and for almost every* $\hat{\Delta} \in \hat{\mathcal{K}}_\infty(\hat{x}, l)$.

The absolute continuity of the Lebesgue measure on the hyperbolic disks given by the above theorem implies that the SRB measures are obtained as ergodic components of the limit measure $\mu = \lim_{k\to\infty}(1/n_k)\sum_{j=0}^{n_k-1} f_*^j\,\mathrm{Leb}_D$ (cf. [**7**, Proposition 5.8]). Moreover, a Hopf-like argument ensures that the basin of attraction of these ergodic components contains open sets (mod 0), foliated by local stable leaves, whose Lebesgue measure is bounded from below by a uniform constant. This is then used to conclude that there are finitely many SRB measures for the non-uniform hyperbolic endomorphism f (and uniqueness under the transitivity assumption). Finally, both the SRB measures and their entropies vary continuously with the endomorphism (where continuity of the measures is taken with respect to weak* topology) (cf. [**7**, Theorem B]).

5.2. *The second volume lemma.* In this subsection we restate and prove a volume lemma for the SRB measure μ on the attractor Λ. This will be crucial to proving, in §6.1, that the SRB measure satisfies a non-uniform version of the Gibbs property. Recall that a SRB measure for f is an invariant measure satisfying $\pi_*\hat{\mu}_{\mathcal{P}(\hat{x})} \ll \mathrm{Leb}_{\pi(\mathcal{P}(\hat{x}))}$, for $\hat{\mu}$ almost every $\hat{x} \in \Lambda^f$, where $\hat{\mu}$ is the lift of μ to Λ^f, for any partition $\mathcal{P}$ subordinated to unstable manifolds. Since $\pi(\mathcal{P}(\hat{x})) \subset W^u_{\mathrm{loc}}(\hat{x})$, $\mathrm{Leb}_{\pi(\mathcal{P}(\hat{x}))}$ denotes the Lebesgue measure inherited by the Riemannian metric on the unstable manifold associated to $\hat{x}$.

Let us fix a partition $\mathcal{P}$ subordinated to unstable manifolds. Since the natural projection π is a homeomorphism with its image when restricted to an atom of $\mathcal{P}$, we can define a measure $\hat{m}_{\hat{\gamma}} = (\pi|_{\hat{\gamma}})^*\,\mathrm{Leb}_{\hat{\gamma}}$ for all $\hat{\gamma} \in \mathcal{P}$. So if μ is a SRB measure then $\hat{\mu}_{\hat{\gamma}} \ll \hat{m}_{\hat{\gamma}}$. By the Radon–Nikodym theorem, there is a measurable function $\rho_{\hat{\gamma}} : \hat{\gamma} \to \mathbb{R}$, almost everywhere positive and satisfying

$$\hat{\mu}_{\hat{\gamma}}(\hat{B}) = \int_{\hat{B}} \rho_{\hat{\gamma}}(\hat{x})\, d\hat{m}_{\hat{\gamma}}(\hat{x}). \tag{5.2}$$

The following lemma shows that the function $\rho_{\hat{\gamma}}$ is uniformly bounded away from zero and infinity in local unstable manifolds.

LEMMA 5.2. *Let* $\mathcal{P}$ *be a partition of* Λ^f *subordinated to unstable manifolds. There exists* $C > 0$ *such that*

$$e^{-C\,\mathrm{dist}_\gamma(x_0,y_0)^\alpha} \le \frac{\rho_{\hat{\gamma}}(\hat{x})}{\rho_{\hat{\gamma}}(\hat{y})} \le e^{C\,\mathrm{dist}_\gamma(x_0,y_0)^\alpha},$$

for all $\hat{x}, \hat{y} \in \hat{\gamma}$, *for all* $\hat{\gamma} \in \mathcal{P}$, *where* $\gamma = \pi(\hat{\gamma})$, $x_0 = \pi(\hat{x})$ *and* $y_0 = \pi(\hat{y})$.

Proof. Let $\hat{\gamma}$ be an atom of $\mathcal{P}$, let $\gamma = \pi(\hat{\gamma})$ and let $n \in \mathbb{N}$ be fixed. Since the lifts of unstable manifolds to Λ^f are invariant by $\hat{f}^{-n}$ and the map $\pi|_{\hat{f}^{-n}(\hat{\gamma})}$ is injective, we conclude that $\hat{f}^{-n}(\mathcal{P})$ is also a partition subordinated to unstable manifolds. Denote $\hat{\gamma}_{-n} := \hat{f}^{-n}(\hat{\gamma})$ and let ρ_n be the density of $\hat{\mu}_{\hat{\gamma}_{-n}}$ with respect to $\hat{m}_\gamma$. By (5.2) and change of variables,

$$\hat{\mu}_{\hat{\gamma}_{-n}}(\hat{B}) = \int_{\hat{f}^n(\hat{B})} \rho_n \circ \hat{f}^{-n}(\hat{x}) |\det(D\hat{f}^{-n}|_{T_{\pi(\hat{x})}\gamma})|\, d\hat{m}_\gamma(\hat{x}).$$

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Here $D\hat{f}^{-n}(\hat{x}) = (Df^n(x_{-n}))^{-1}$. As the disintegrations are unique (in a $\hat{\mu}$-full measure subset) then $\hat{f}^n_*\hat{\mu}_{\hat{\gamma}_{-n}} = \hat{\mu}_{\hat{\gamma}}$, and so $\rho(\hat{x}) = \rho_n(\hat{f}^{-n}(\hat{x})) \cdot |\det(D\hat{f}^{-n}(\hat{x})|_{T_{\pi(\hat{x})}\gamma})|$. Therefore, given $\hat{x}$, $\hat{y} \in \hat{\gamma}$, we get

$$\frac{\rho(\hat{x})}{\rho(\hat{y})} = \frac{\rho_n(\hat{f}^{-n}(\hat{x}))}{\rho_n(\hat{f}^{-n}(\hat{y}))} \cdot \frac{|\det(D\hat{f}^{-n}(\hat{x})|_{T_{\pi(\hat{x})}\gamma})|}{|\det(D\hat{f}^{-n}(\hat{y})|_{T_{\pi(\hat{y})}\gamma})|}. \tag{5.3}$$

Since γ is contained in a local unstable manifold and the Jacobian of disks tangent to the cone field is Hölder continuous (recall Proposition 4.1), we conclude that the second quotient on the right-hand side above is bounded above by $e^{C\mathrm{dist}_\gamma(x_0,y_0)^\alpha}$, where $C = L_1 \cdot e^{c\alpha/2}/(e^{c\alpha/2} - 1)$.

Consider the map $\rho : \Lambda^f \to \mathbb{R}$ given by $\rho(\hat{x}) = \rho_{\hat{\gamma}_{\hat{x}}}(\hat{x})$, where $\hat{\gamma}_{\hat{x}}$ is the atom of $\mathcal{P}$ that contains $\hat{x}$. By Lusin's theorem, for every $\varepsilon > 0$ there is a compact subset $\hat{\Omega} \subset \Lambda^f$ such that $\rho\,|_{\hat{\Omega}}$ is a (uniformly) continuous function and $\hat{\mu}(\Lambda^f \setminus \hat{\Omega}) < \varepsilon$. In particular, given $k \in \mathbb{N}$ there exist $\delta_k > 0$ such that

$$\left|\frac{\rho(\hat{z})}{\rho(\hat{w})} - 1\right| < \frac{1}{k},$$

for all $\hat{z}$, $\hat{w} \in \hat{\Omega}$ with $\hat{d}(\hat{z}, \hat{w}) < \delta_k$. We will use the previous estimate for suitable returns of $\hat{x}$ and $\hat{y}$. By Birkhoff's ergodic theorem the frequency of visits of $\hat{x}$ and $\hat{y}$ to $\hat{\Omega}$ tends to $\hat{\mu}(\hat{\Omega}) \geq 1 - \epsilon$. In particular, there are infinitely many simultaneous returns of $\hat{x}$, $\hat{y} \in \hat{\gamma}$ to $\hat{\Omega}$. Hence there exists $n_k \in \mathbb{N}$ satisfying $\hat{f}^{-n_k}(\hat{x})$, $\hat{f}^{-n_k}(\hat{y}) \in \hat{\Omega}$, and $\hat{d}(\hat{x}, \hat{y}) < \delta_k$. Therefore,

$$\left|\frac{\rho(\hat{f}^{-n_k}(\hat{x}))}{\rho(\hat{f}^{-n_k}(\hat{y}))} - 1\right| < \frac{1}{k} \quad \text{or, equivalently,} \quad \left|\frac{\rho_{n_k}(\hat{f}^{-n_k}(\hat{x}))}{\rho_{n_k}(\hat{f}^{-n_k}(\hat{y}))} - 1\right| < \frac{1}{k}.$$

In consequence, we find a sequence $n_k \to \infty$ satisfying

$$\lim_{k\to\infty} \frac{\rho_{n_k}(\hat{f}^{-n_k}(\hat{x}))}{\rho_{n_k}(\hat{f}^{-n_k}(\hat{y}))} = 1.$$

Then expression (5.3) also can be bounded by $e^{C\mathrm{dist}_\gamma(x_0,y_0)^\alpha}$. The other inequality in the lemma follows by inversion of the roles of $\hat{x}$ and $\hat{y}$ above. □

PROPOSITION 5.2. *Assume that the density $\rho : \Lambda^f \to \mathbb{R}$ satisfies $1/K^* \leq \rho_{\hat{\gamma}}(\hat{y}) \leq K^*$ for all $\hat{y} \in \hat{B}$. Let $\delta > 0$ be given by the definition of c-cone-hyperbolic times. Given $0 < \varepsilon < \delta$, there exists $K_2 = K_2(\varepsilon) > 0$ so that if n is a c-cone-hyperbolic time for x and F is any subspace of dimension d contained in $C(x)$ then*

$$K_2(\varepsilon)^{-1} e^{-\log|\det Df^n(x)|_F|} \leq \mu(\Lambda \cap B(x, n, \varepsilon)) \leq K_2(\varepsilon) e^{-\log|\det Df^n(x)|_F|}. \tag{5.4}$$

Proof. Let n be a c-cone-hyperbolic time for x and $\varepsilon > 0$ be as above. If $\hat{\mu}$ is the lift of the SRB measure μ to Λ^f then $\mu = \pi_*\hat{\mu}$. Fix a partition $\mathcal{P}$ of Λ^f subordinated to the unstable manifolds. Then

$$\begin{aligned}\mu(\Lambda \cap B(x, n, \varepsilon)) &= \pi_*\hat{\mu}(B(x, n, \varepsilon)) = \hat{\mu}(\pi^{-1}(B(x, n, \varepsilon))) \\ &= \int_{\mathcal{P}} \hat{\mu}_{\hat{\gamma}}(\pi^{-1}(B(x, n, \varepsilon)))\, d\hat{\mu}(\hat{\gamma}).\end{aligned}$$

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Denote $\hat{B} := \pi^{-1}(B(x, n, \varepsilon))$. Since $\pi(\hat{\gamma})$ is contained in some local unstable manifold, we have that $\pi(\hat{\gamma})$ is tangent to the cone field C. In particular, $\mathrm{dist}_\gamma(x_0, y_0) \le 2\,\mathrm{dist}_M(x_0, y_0) < 4\varepsilon$ for all $\hat{x}, \hat{y} \in \hat{\gamma} \cap \hat{B}$. Using Lemma 5.2 and the fact that the density of μ is bounded away from zero and infinity, for every $\hat{\gamma} \in \mathcal{P}$ there exists $\hat{y}_{\hat{\gamma}} \in \hat{\gamma}$ so that

$$\hat{\mu}_{\hat{\gamma}}(\hat{B}) = \int_{\hat{B}} \rho_{\hat{\gamma}}(\hat{x})\, d\hat{m}_{\hat{\gamma}}(\hat{x}) \le e^{C(4\varepsilon)^\alpha} \rho_{\hat{\gamma}}(\hat{y}_{\hat{\gamma}}) \hat{m}_{\hat{\gamma}}(\hat{B}) \le K^* e^{C(4\varepsilon)^\alpha} \hat{m}_{\hat{\gamma}}(\hat{B})$$

and, consequently, $\mu(\Lambda \cap B(x, n, \varepsilon)) \le e^{C(4\varepsilon)^\alpha} K^* \int_{\mathcal{P}} \hat{m}_{\hat{\gamma}}(\hat{B})\, d\hat{\mu}(\hat{\gamma})$, for every sufficiently small ε. Inequality (4.12) now ensures that

$$\hat{m}_{\hat{\gamma}}(\hat{B}) = \mathrm{Leb}_{\pi(\hat{\gamma})}(B(x, n, \varepsilon)) \le K(\varepsilon) |\det Df^n(x)\,|_F\,|^{-1}$$

where F is the tangent space to some unstable disk that contains x, for all $\hat{\gamma}$ intersecting $\hat{B}$. Therefore,

$$\mu(\Lambda \cap B(x, n, \varepsilon)) \le K^* e^{C(4\varepsilon)^\alpha} K(\varepsilon) e^{-\log|\det Df^n(x)|_F|}.$$

The inverse inequality is obtained in the same way, using the inverse inequalities from Lemma 5.2 and Proposition 4.5. So, taking $K_2(\varepsilon) = K^* e^{C(4\varepsilon)^\alpha} K(\varepsilon)$, we complete the proof of the proposition. □

6. *Large deviations: Proof of Theorem A*

6.1. *The SRB measure is a weak Gibbs measure.*

In this subsection we prove that under the mild assumptions in Theorem A the SRB measure is a weak Gibbs measure, that is, it satisfies the conclusion of Proposition 6.1 below. Let H be the set of points with infinitely many c-cone hyperbolic times. It is a standard argument to prove that the volume lemma together with the non-lacunarity of the sequences of cone-hyperbolic times ensures the weak Gibbs measure.

PROPOSITION 6.1. *Let μ the SRB measure given by Theorem A. Assume that the sequence of cone-hyperbolic times is non-lacunar for μ-almost every x. Then, for μ-almost every $x \in H$, there exists $\delta > 0$ so that, given $0 < \varepsilon < \delta$, there exists a sequence $(K_n(x, \varepsilon))_n$ of positive integers such that $\lim_{\varepsilon \to 0} \limsup_{n \to \infty} (1/n) \log K_n(x, \varepsilon) = 0$ for μ-almost every x and, for* any *subspace F_1 of dimension d contained in $C(x)$,*

$$K_n(x, \varepsilon)^{-1} e^{-\log|\det Df^n(x)|_{F_1}|} \le \mu(\Lambda \cap B(x, n, \varepsilon)) \le K_n(x, \varepsilon) e^{-\log|\det Df^n(x)|_{F_1}|}$$

for every $n \ge 1$.

Proof. We will prove the upper bound in the proposition (the lower bound is analogous). Let $\delta > 0$ be given by the second volume lemma. Then, for every $0 < \varepsilon < \delta$, there exists $K_2(\varepsilon) > 1$ so that if n is a c-cone-hyperbolic time for x and F_1 is any subspace of dimension d contained in $C(x)$ then

$$K_2(\varepsilon)^{-1} e^{-\log|\det Df^n(x)|_{F_1}|} \le \mu(\Lambda \cap B(x, n, \varepsilon)) \le K_2(\varepsilon) e^{-\log|\det Df^n(x)|_{F_1}|}.$$

Given an arbitrary $n \ge 1$, write $n_i(x) \le n < n_{i+1}(x)$, where $n_i(x)$ and $n_{i+1}(x)$ are consecutive c-cone hyperbolic times for x. Then, taking $n_i = n_i(x)$, it follows that

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$$\begin{aligned}\mu(B(x,n,\varepsilon)) \le \mu(B(x,n_i,\varepsilon)) &\le K_2(\varepsilon)e^{-\log|\det Df^{n_i}(x)|_{F_1}|}\\ &= K_2(\varepsilon)e^{-\log|\det Df^{n}(x)|_{F_1}|+\log|\det Df^{n-n_i}(f^{n_i}(x))|_{F_1}|}\\ &= K_n(x,\varepsilon)e^{-\log|\det Df^{n}(x)|_{F_1}|},\end{aligned}$$

where

$$K_n(x,\varepsilon) := K_2(\varepsilon)\Big(d^{-1}\min_{y\in M}\|Df(y)^{-1}\|\Big)^{n-n_i} \tag{6.1}$$

(depending only on the center x). Now, since c-cone-hyperbolic times concatenate, note that

$$\begin{aligned}\frac{1}{n}\log K_n(x,\varepsilon) &\le \frac{1}{n}\log K_2(\varepsilon) + \frac{n-n_i(x)}{n}\log\Big(d^{-1}\min_{y\in M}\|Df(y)^{-1}\|\Big)\\ &\le \frac{1}{n}\log K_2(\varepsilon) + \frac{n_{i+1}(x)-n_i(x)}{n}\log\Big(d^{-1}\min_{y\in M}\|Df(y)^{-1}\|\Big)\end{aligned}$$

which, by assumption, tends to zero as $n\to\infty$. This finishes the proof. □

Remark 6.1. In the case where Λ is a hyperbolic set for the endomorphism f, it admits a Markov partition. If this is the case, since the SRB measure μ satisfies a weak Gibbs property, several consequences can be obtained from [**26**]: exponential large-deviation bounds and almost sure estimates for the error term in the Shannon–McMillan–Breiman convergence to entropy, and a topological characterization of large-deviation bounds for Gibbs measures, and it is deduced that their local entropy is zero (see [**26**] for the precise statements).

6.2. *Large deviations for the SRB measure.* Since the SRB measure is a weak Gibbs measure (Proposition 6.1), we adapt the methods used in [**23**, **25**, **27**] to obtain large-deviation estimates for weak Gibbs measures. The main difference here lies in the non-additive characterization of volume expansion along the invariant cone field which in turn requires some ingredients from the variational principle for sub-additive sequences of potentials.

We proceed with the proof of Theorem A. Let $\beta>0$, $0<\varepsilon<\delta_0/2$, and $n\ge 1$ be fixed and let $F\subset\mathbb{R}$ be a closed interval. Given $n\ge 1$, set

$$B_n = \left\{x\in M : \frac{1}{n}\sum_{j=0}^{n-1}\phi(f^j(x))\in F\right\}$$

and write

$$B_n \subset \{x\in\Lambda:\, K_n(x,2\varepsilon) > e^{\beta n}\}\cup(B_n\cap\{x\in\Lambda:\, K_n(x,2\varepsilon)\le e^{\beta n}\}) \tag{6.2}$$

where $K_n(x,2\varepsilon)$ is given by (6.1). Set $C = d^{-1}\min_{y\in M}\|Df(y)^{-1}\|$. Observe that if $n\ge 1$ is large (depending only on ε) then

$$\begin{aligned}\{x\in\Lambda:\, K_n(x,2\varepsilon) > e^{\beta n}\} &\subset \left\{x\in\Lambda:\,(n-n_i(x))\log C > \beta n - \log K_2(2\varepsilon) > \frac{\beta}{2}n\right\}\\ &\subset \left\{x\in\Lambda:\, n_1(f^{n_i(x)}(x)) > \frac{\beta n}{2\log C}\right\}\\ &\subset \bigcup_{k=1}^{n}\left\{x\in\Lambda:\, n_1(f^{k}(x)) > \frac{\beta n}{2\log C}\right\}\end{aligned}$$

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where $n_i(x) \le n < n_{i+1}(x)$ are consecutive cone-hyperbolic times for x. Since μ is f-invariant,

$$\mu(\{x \in \Lambda : K_n(x, \varepsilon) > e^{\beta n}\}) \le n\mu\left(\left\{x \in \Lambda : n_1(x) > \frac{\beta n}{2 \log C}\right\}\right). \tag{6.3}$$

On the other hand, if $E_n \subset B_n \cap \{x \in \Lambda : K_n(x, 2\varepsilon) \le e^{\beta n}\}$ is a maximal (n, ε)-separated set, $B_n \cap \Delta_n$ is contained in the union of the dynamical balls $B(x, n, 2\varepsilon)$ centered at points of E_n, and, consequently,

$$\mu(B_n) < n\mu\left(\left\{x \in \Lambda : n_1(x) > \frac{\beta n}{2 \log C}\right\}\right) + e^{\beta n} \sum_{x \in E_n} e^{-J_n(x)} \tag{6.4}$$

for every large n, where $J_n(x) = \log |\det Df^n(x) \mid_{F_1}|$ and F_1 is any subspace of dimension d in $C(x)$. Now, consider the probability measures σ_n and η_n given by

$$\sigma_n = \frac{1}{Z_n} \sum_{x \in E_n} e^{-J_n(x)} \delta_x \quad \text{and} \quad \eta_n = \frac{1}{n} \sum_{j=0}^{n-1} f_*^j \sigma_n,$$

where $Z_n = \sum_{x \in E_n} e^{-J_n(x)}$, and let η be a weak* accumulation point of the sequence $(\eta_n)_n$. It is not hard to check that η is an f-invariant probability measure. Assume $\mathcal{P}$ is a partition of M with diameter smaller than ε and $\eta(\partial \mathcal{P}) = 0$. Each element of $\mathcal{P}^{(n)}$ contains a unique point of E_n. As in the proof of the variational principle for subadditive potentials (cf. [**4**, pp. 648–650]),

$$H_{\sigma_n}(\mathcal{P}^{(n)}) - \int J_n(x)\, d\sigma_n = \log\left(\sum_{x \in E_n} e^{-J_n(x)}\right),$$

which guarantees that

$$\limsup_{n \to \infty} \frac{1}{n} \log Z_n \le h_\eta(f) - \lim_{n \to \infty} \frac{1}{n} \int J_n(x)\, d\eta(x) = h_\eta(f) - \int \Gamma_\eta(x)\, d\eta. \tag{6.5}$$

We used the definition of Γ_η in (2.2) and Corollary 4.1 to guarantee that the limit in the right-hand side above exists. Since $(1/n) \sum_{j=0}^{n-1} \phi(f^j(x)) \in F$ for every $x \in E_n$, it follows that $\int \phi\, d\eta_n \in F$ for every $n \ge 1$ and $\int \phi\, d\eta \in F$ by weak* convergence. Finally, equations (6.1), (6.4) and (6.5) imply that, for every $\beta > 0$,

$$\limsup_{n \to \infty} \frac{1}{n} \log \mu(B_n) \le \max\left\{\mathcal{E}_\mu(\beta), -\inf_{c \in F} I(c) + \beta\right\} \tag{6.6}$$

where

$$\mathcal{E}_\mu(\beta) = \limsup_{n \to \infty} \frac{1}{n} \log \mu\left(x : n_1(x) > \frac{\beta n}{2 \log C}\right)$$

and

$$I(c) = \sup\left\{h_\eta(f) - \int \Gamma_\eta(x)\, d\eta : \eta \in \mathcal{M}_1(f), \int \phi\, d\eta \in F\right\}.$$

Since (6.6) holds and $\beta > 0$ is arbitrary, taking the infimum over all possible $\beta > 0$ in the second alternative in (6.6), we deduce that, for every $\beta > 0$,

$$\limsup_{n \to \infty} \frac{1}{n} \log \mu(B_n) \le \inf_{\beta > 0}\left[\max\{\mathcal{E}_\mu(\beta), -\inf_{c \in F} I(c) + \beta\}\right],$$

which proves the second item in Theorem A. This completes the proof of the theorem.

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Remark 6.2. Note that

$$\frac{\mathcal{E}_\mu(\beta)}{\beta} = \limsup_{n\to\infty} \frac{1}{\beta n} \log \mu\left(x : n_1(x) > \frac{\beta n}{2\log C}\right) \leq \frac{1}{2\log C} \limsup_{n\to\infty} \frac{1}{n} \log \mu(x : n_1(x) > n),$$

where the right-hand side is a constant $\mathcal{E} \leq 0$ which is independent of β. In consequence, $\mathcal{E}_\mu(\beta) \leq \beta\mathcal{E} \leq 0$ for every $\beta > 0$. Hence, the inequality is effective for large deviations when there exists $\beta > 0$ so that $\mathcal{E}_\mu(\beta) < 0$ or, in other words, when the first cone-hyperbolic time n_1 has exponential tail.

6.3. *Proof of Corollary A.* Assume that we are in the context of Theorem A and that n_1 has exponential tails. On the one hand, there exists $\mathcal{E} > 0$ so that $\mathcal{E}_\mu(\beta) \leq -\beta\mathcal{E}$ for all $\beta > 0$. On the other hand, it follows from the proof of Theorem A that for every $\beta > 0$ there exists a probability measure η such that $\int \phi\, d\eta \notin (\int \phi\, d\mu - \delta, \int \phi\, d\mu + \delta)$ and

$$\limsup_{n\to\infty} \frac{1}{n} \log \mu(B_n) \leq \max\left\{\mathcal{E}_\mu(\beta), h_\eta(f) - \int \Gamma_\eta(x)\, d\eta + \beta\right\} \tag{6.7}$$

for every $\beta > 0$ (recall equations (6.4)–(6.6)). By the Ruelle inequality it follows that

$$h_\eta(f) - \int \Gamma_\eta(x)\, d\eta = h_\eta(f) - \int \sum_{\lambda_i(\eta,x)>0} \lambda_i(\eta, x)\, d\eta \leq 0.$$

By Pesin's entropy formula [**16**, Theorem VII.1.1], the unique SRB measure μ for f on the attractor Λ is the unique invariant measure that satisfies the equality $h_\mu(f) = \int \sum_{\lambda_i(\mu,x)>0} \lambda_i(\mu, x)\, d\mu$. In consequence, there exists $\beta > 0$ so that the right-hand side of (6.7) is strictly negative. This proves the corollary.

7. *Some examples*

In this section we give some applications of our main results in the context of Anosov and partially hyperbolic endomorphisms.

7.1. *Anosov endomorphisms.* Let $f : M \to M$ be an Anosov endomorphism, that is, f be a local diffeomorphism satisfying (H1)–(H4) on M with condition (2.1) replaced by $\|(Df(x))|_{C(x)}^{-1}\| \leq e^{-2c} < 1$ for all $x \in M$. Then for all $\hat{x} \in M^f$ there exist a splitting $T_{\hat{x}}M^f = E^s_{x_0} \oplus E^u_{\hat{x}}$ and constants $C > 0$ and $\lambda \in (0, 1)$ satisfying that: (1) $Df(x_0) \cdot E^s_{x_0} = E^s_{f(x_0)}$ and $Df(x_0) \cdot E^u_{\hat{x}} = E^u_{\hat{f}(\hat{x})}$; (2) $\|Df^n(x_0) \cdot v\| \leq C \cdot \lambda^n \|v\|$ for all $v \in E^s_{x_0}$ and $n \in \mathbb{N}$; (3) $\|(Df^n(x_{-n}))^{-1} \cdot v\| \leq C\lambda^n \|v\|$ for all $v \in E^u_{\hat{x}}$ [**15**]. It is clear that this class of endomorphisms fits in the class of dynamics considered here, hence there exist finitely many SRB measures for f and it is unique whenever f is transitive [**7**, Theorem A]. Observe that the uniform expansion along the cone field C implies that all times are c-cone-hyperbolic times. Consequently, the non-lacunarity assumption or the sequence of c-cone-hyperbolic times is trivially satisfied. Therefore, Theorem A and Corollary A provide large-deviation upper bounds for the SRB measure of the Anosov endomorphism. This provides an alternative proof for the large-deviation estimates for uniformly

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hyperbolic endomorphisms obtained in [**8**, Theorem 1.2] with respect to continuous potentials for the SRB measure, which explores the Hölder continuity of the unstable Jacobian on the natural extension M^f.

7.2. *Endomorphisms derived from Anosov.* Dynamical systems in the isotopy class of uniformly hyperbolic dynamical systems have been intensively studied in recent decades, as candidates for robustly transitive dynamics. The class of C^1-robustly transitive non-Anosov diffeomorphisms considered by Mañé is among the first classes of examples of this kind, and their SRB measures were constructed by Carvalho in [**5**]. In this subsection we illustrate how some partial hyperbolic endomorphisms can be obtained, by local perturbations, in the isotopy class of hyperbolic endomorphisms. In [**20**] Sumi constructed non-hyperbolic topologically mixing partially hyperbolic endomorphisms on $\mathbb{T}^2$. We construct the example in dimension 3 for simplicity, although similar statements hold in higher dimensions. In what follows we give an example of a dynamical system with hyperbolic periodic points with different index, but we could also consider the existence of periodic points with an indifferent direction.

Consider $M = \mathbb{T}^3$, and let $g : M \to M$ be a linear Anosov endomorphism induced by a hyperbolic matrix $A \in \mathcal{M}_{3\times 3}(\mathbb{Z})$ displaying three real eigenvalues and such that $TM = E^s \oplus E^u$ is the Dg-invariant splitting, where $\dim E^u = 2$. For instance, take a matrix of the form

$$A = \begin{pmatrix} n & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

for an integer $n \geq 2$. The map g is a special Anosov endomorphism, meaning that the unstable space is independent of the pre-orbits and it admits a finest dominated splitting $E^s \oplus E^u \oplus E^{uu}$. If $p \in M$ is a fixed point for g and $\delta > 0$ is sufficiently small, one can write g on the ball $B(p, \delta)$ (in terms of local coordinates in $E^s_p \oplus E^u_p$) as $g(x, y, z) = (f(x), h(y, z))$, where $x \in E^s_p$, $(y, z) \in E^u_p$, f is a contraction along E^s_p, and h is expanding along E^u_p. Let $\lambda_2, \lambda_3 \in \mathbb{R}$ be the eigenvalues of $Dg(p)\,|_{E^u_p}$. Suppose that $|\lambda_2| \geq |\lambda_3| > 1$. Consider an isotopy $[0, 1] \ni t \mapsto h_t$ satisfying: (i) $h_0 = h$; (ii) for every $t \in [0, 1]$ the diffeomorphism h_t has a fixed point p_t (continuation of p) which, without loss of generality, we assume to coincide with p; (iii) $h_t : E^c_{t,p} \to E^c_{t,p}$ is a $C^{1+\alpha}$ map, where $E^c_{t,p} := E^u_p$ for every $t \in [0, 1]$; (iv) the eigenvalues of Dh_1 at p are λ_2 and $\rho \in \mathbb{R}$ with $|\rho| < 1$ (determined by (7.1) and (7.2) below); (v) if $g_t(x, y, z) = (f(x), h_t(y, z))$ then $g_t\,|_{M\setminus B(p,\delta)} = g\,|_{M\setminus B(p,\delta)}$ for every $t \in [0, 1]$. Reducing $\delta > 0$ if necessary, we may assume that $g_t\,|_{B(p,\delta)}$ is injective for every $t \in [0, 1]$.

Roughly, the fixed point p goes through a pitchfork bifurcation along a one-dimensional subspace contained in the unstable subspace associated to the original dynamics on the open set $\mathcal{O} = B(p, \delta)$. Given $a \in (0, 1)$ and the splitting $TM = E^s_t \oplus E^c_t$, consider the families of cone fields

$$\mathcal{C}^s_a(x) := \{v = v_s \oplus v_c : \ \|v_c\| \leq a\|v_s\|\},$$
$$\mathcal{C}^u_a(x) := \{v = v_s \oplus v_c : \ \|v_s\| \leq a\|v_c\|\},$$

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and assume without loss of generality that $\|(u, v, w)\| = |u| + |v| + |w|$ for $(u, v, w) \in E^s \oplus E^u \oplus E^{uu}$ (local coordinates for g). If $a > 0$ is small then the families of cone fields are Dg_1-invariant. Indeed, one can write $v = v_s \oplus v_c \in \mathcal{C}^s(g_1(p))$, with $v_* \in E^*_{1,g_1(p)}$, $* \in \{s, c\}$. Assume ρ is such that

$$|\lambda_1 \cdot \rho^{-1}| < 1. \tag{7.1}$$

Then $\|Dg_1(g_1(p))^{-1} \cdot v_c\| \le |\rho|^{-1}\|v_c\| \le |\rho|^{-1}a\|v_s\| \le |\lambda_1||\rho|^{-1} \cdot a\|Dg_1(p)^{-1} \cdot v_s\|$, which proves that $Dg_1(p)^{-1} \cdot \mathcal{C}^s_a(g_1(p)) \subset \mathcal{C}^s_{|\lambda_1\rho^{-1}|a}(p)$. Analogously, if $v \in \mathcal{C}^u_1(p)$ then $\|Dg_1(p) \cdot v_s\| \le |\lambda_2^{-1}\lambda_1|a\|Dg_1(p) \cdot v_c\|$ and, consequently, $Dg_1(p) \cdot \mathcal{C}^u_a(p) \subset \mathcal{C}^u_{|\lambda_1\lambda_2^{-1}|a}(g_1(p))$. By continuity, since δ is assumed small, we get the invariance of the cone fields for all points on the ball $B(p, \delta)$. Now observe that $g_1 \mid_{B(p,\delta)}$ expands volume along $E^c_{1,x}$ as it coincides with $g \mid_{B(p,\delta)}$. Moreover, if ρ is such that

$$|\lambda_2 \cdot \rho| > 1 \tag{7.2}$$

then $|\det(Dg_1(p) \mid_{E^c_{1,p}})| = |\lambda_2|\,|\rho| > 1$ and, using the fact that $a > 0$ is small and $x \mapsto \det(Dg_1(x) \mid_{E^c_{1,x}})$ is continuous, we conclude that $\inf_{x\in M} |\det(Dg_1(x) \mid_{E^c_{1,x}})| > 1$ and that the same property holds for the Jacobian along disks tangent to the cone field. Simple computations show that $\|Dg_1(p) \cdot v\| \ge (1 - a) \cdot \rho\|v\|$ for $v \in \mathcal{C}^u_a(p)$ and, by continuity of the derivative, we get that $\|Dg_1(x) \cdot v\| \ge L\|v\|$ for every $x \in B(p, \delta)$ and $v \in C^u(x)$ with $L = L(\rho)$ close to $(1 - a)\rho$, hence close to 1. If $x \notin B(p, \delta)$, then $g_1 \mid_{M\setminus B(p,\delta)} = g \mid_{M\setminus B(p,\delta)}$ and so $\|Dg_1(x) \cdot v\| \ge (1 - a) \cdot \lambda_3\|v\|$ for every $v \in \mathcal{C}^u_a(x)$. Finally, since δ is assumed small, any partition $\{V_1, V_2, \ldots, V_k, V_{k+1}\}$ of $\mathbb{T}^3$ such that $V_{k+1} = B(p, \delta)$ and each V_i contains a ball of radius δ and is contained in a ball of radius 2δ satisfies that for every disk D tangent to the cone field, the image $g_1(D \cap V_i) \cap V_j$ has at most one connected component for all $i, j \in \{1, 2, \ldots, k + 1\}$. The estimates in [**7**, §7] ensure that g_1 satisfies hypotheses (H1)–(H4) and that there exist $K, \varepsilon > 0$ uniform so that, for every disk D tangent to the cone field,

$$\mathrm{Leb}_D(n_1(\cdot) > n) \le Ke^{-\varepsilon n} \tag{7.3}$$

for every $n \ge 1$. Then, a calculation identical to [**24**, §5] guarantees that the density of μ with respect to the Lebesgue measure along local unstable manifolds is bounded away from infinity. This is a consequence of the fact that $\mu = \lim_{n\to\infty} \mu_n$ and that one can write

$$\begin{aligned}\mu_n = \frac{1}{n}\sum_{j=0}^{n-1} f_*^j\,\mathrm{Leb}_D \le \frac{1}{n}\sum_{j=0}^{n-1} f_*^j\,\mathrm{Leb}_{D\cap\{n_1=j\}} \\ + \frac{1}{n}\sum_{j=0}^{n-1}\sum_{\ell=0}^{+\infty} f_*^\ell(f_*^j\,\mathrm{Leb}_{D\cap\{n_1=j\}} \mid \{n_1 > \ell\}).\end{aligned} \tag{7.4}$$

For each $j \ge 1$ the measures $f_*^j\,\mathrm{Leb}_{D\cap\{n_1=j\}}$ are absolutely continuous with respect to Lebesgue along disks tangent to the cone field with density uniformly bounded from above. The densities of the measures in the second term in (7.4) combine the previous observation with the information on the tail of the Lebesgue measure given by (7.3) (this argument

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closely follows the computation in [**24**, Lemma 5.5]). The latter implies that the first cone-hyperbolic time is integrable with respect to μ, hence the sequence of c-cone-hyperbolic times is non-lacunar μ-almost everywhere [**24**, Corollary 3.8]. As a consequence of our results the volume lemmas hold and we obtain exponential convergence of Birkhoff averages for g_1 and all continuous observables with respect to the SRB measure.

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