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Weak mixing properties for non-singular actions

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Abstract. For a general group G we consider various weak mixing properties of non-singular actions. In the case where the action is actually measure preserving all these properties coincide, and our purpose here is to check which implications persist in the non-singular case.

1. *Introduction*

Let G be a second countable locally compact topological group. We are interested here in various weak mixing properties of non-singular actions of G. A *non-singular* (or *quasi-invariant*) action is a measurable action of G on a standard Lebesgue probability space $(X, \mathcal{B}, \mu)$, where the action preserves the measure class of μ (i.e. $\mu(A) = 0 \iff \mu(gA) = 0$ for every $A \in \mathcal{B}$ and every $g \in G$). In the case where the action is actually measure preserving (i.e. $g\mu = \mu$ for every $g \in G$) all of these weak mixing properties coincide, and our purpose is to check which implications persist in the non-singular case.

More specifically the mixing conditions we consider are defined as follows.

Definition 1.1. Let $\mathbf{X} = (X, \mathcal{B}, \mu, \{T_g\}_{g\in G})$ be a non-singular G action.

(1) $\mathbf{X}$ is *ergodic* if every G-invariant $B \in \mathcal{B}$ (i.e. $\mu(gB \triangle B) = 0$ for every $g \in G$) satisfies $\mu(B) = 0$ or 1.
(2) $\mathbf{X}$ is *doubly ergodic* (DE) if the product system $\mathbf{X} \times \mathbf{X}$, where G acts diagonally on $X \times X$, is ergodic.
(3) $\mathbf{X}$ is *ergodic with isometric coefficients* (EIC) if there is no non-constant Borel measurable equivariant map $\phi : X \to Z$, where (Z, d) is a separable metric space on which G acts by isometries.
(4) $\mathbf{X}$ is *ergodic with unitary coefficients* (EUC) if there is no non-constant Borel measurable equivariant map $\phi : X \to H$, where H is a separable Hilbert space on which G acts by unitary operators.

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(5) $\mathbf{X}$ is *weakly mixing* (WM) if it has the multiplier property for probability measure preserving ergodic G-actions; i.e. for every such probability measure preserving action $\mathbf{Y} = (Y, \mathcal{C}, \nu, \{S_g\}_{g\in G})$ the product system $\mathbf{X} \times \mathbf{Y}$ is ergodic.

(6) $\mathbf{X}$ is *L_∞-weakly mixing* (L_∞-WM) if there are no non-trivial invariant finite-dimensional subspaces of $L_\infty(X, \mu)$.

Remarks 1.2.

(1) The notions of EIC and EUC (and other close variations) appeared first in [**9**], and then also in [**5**, **21**, **26**, **27**].

(2) In a series of related works (e.g. [**8**, **17**, **18**]) the authors treat a notion of 'double ergodicity' which, unfortunately, is different from the one we adopt here (see Remark 4.4 below).

(3) Two more related works [**7**, **16**] appeared recently in the arXiv.

(4) L_∞-eigenfunctions of non-singular transformations were first introduced by Keane who showed that for a non-singular $\mathbb{Z}$-action $(X, \mathcal{B}, \mu, T)$ and a probability measure preserving $\mathbb{Z}$-action $(Y, \mathcal{C}, \nu, S)$ the product system $(X \times Y, \mu \times \nu, \mathcal{B} \times \mathcal{C}, T \times S)$ is ergodic if and only if $\sigma_0(e(T)) = 0$. Here σ_0 is the (restricted) maximal spectral type of S and $e(T)$ is the collection of L_∞-eigenvalues of T (see [**1**, Theorem 2.7.1]). From this theorem one can easily deduce the equivalence of WM with L_∞-WM for $\mathbb{Z}$-actions.

(5) The requirement in the definition of the EIC property that the metric space Z be separable is in fact redundant as it can be shown that the subset cls supp $\phi_*(\mu)$ is necessarily separable.

In the second section we sort these conditions according to their strength, and show that they are all equivalent when μ is measure preserving. In the third section we give a short proof of a theorem of Glücksam's [**16**] which says that when the acting group is a locally compact group all of whose irreducible unitary representations are finite-dimensional (a Moore group), WM is equivalent to L_∞-WM.

Strongly almost transitive (SAT) systems are non-singular systems with the property that every positive set can be blown up to almost full measure. In §4 we show that SAT systems are EIC and give an example of a SAT (hence EIC) system which is not DE.

For a well-behaved probability measure m on a locally compact group G, every m-stationary system $\mathbf{X}$ (i.e. one in which $m * \mu = \mu$) is non-singular. We treat this class of stationary systems in §5 and, using a strengthened form of a theorem of Kaimanovich, show that such systems are DE (in fact doubly EIC).

In §§6 and 7 we use our previous results to show that the free group F_2 and even the group of integers $\mathbb{Z}$ admit non-singular actions which are EIC but not DE. For F_2 this is its Poisson boundary and for $\mathbb{Z}$ the example arises from a standard planar random walk. In the final section we add one more characterization to Furstenberg's list of equivalent conditions on a stationary system to be F_m-proximal.

2. *A chain of implications*

THEOREM 2.1. *For non-singular G systems $\mathbf{X} = (X, \mathcal{B}, \mu, \{T_g\}_{g\in G})$ we have:*

$$DE \Rightarrow EIC \Rightarrow EUC = WM \Rightarrow L_\infty\text{-}WM \Rightarrow \textit{ergodicity}.$$

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If the G action on $\mathbf{X}$ is measure preserving then all these properties (except for the last one, ergodicity) are equivalent.

Proof. $DE \Rightarrow EIC$. Suppose $\mathbf{X}$ is doubly ergodic and let $\phi : X \to Z$ be a metric factor. Then the function

$$(x, x') \to d(\phi(x), \phi(x'))$$

is an invariant measurable function, hence a constant $t_0 \geq 0$. Fix a point z_0 in the support of the measure $\phi_*(\mu)$. For any $r > 0$ the ball $B(z_0, r)$ has positive $\phi_*(\mu)$ measure and it then follows from the constancy $\mu \times \mu$-almost everywhere of the distance function that t_0 is less than $2r$. Since this holds for every $r > 0$ it follows that $t_0 = 0$, and this clearly implies that ϕ is constant.

The implications $EIC \Rightarrow EUC$ is clear. We will show the equivalence of EUC and WM in a separate statement, Theorem 2.2 below.

To see that the implication

$$EUC \Rightarrow L_\infty\text{-}WM$$

holds, observe that if $V \subset L_\infty(X, \mu)$ is a non-trivial (i.e. of dimension at least 1) finite-dimensional invariant subspace, then G acts on the compact unit sphere S_V of V by isometries and thus as a precompact group. (The group of isometries of a compact metric space equipped with the topology of uniform convergence is a compact group.) Let $\tilde{G}$ denote the closure of the image of G in the group Iso (V) and let λ denote the normalized Haar measure on the compact topological group $\tilde{G}$. Pick some Hermitian sesquilinear form ρ on V (of course we can assume that V is defined over $\mathbb{C}$) and set

$$\tilde{\rho}(u, v) = \int_{\tilde{G}} \rho(gu, gv)\, d\lambda(g).$$

Then $\tilde{\rho}$ is a G-invariant inner product on V and G acts on the Hilbert space $(V, \tilde{\rho})$ as a group of unitary operators. Next choose a basis for $(V, \tilde{\rho})$, say $\{f_1, f_2, \ldots, f_k\}$. We can assume that each f_i is defined everywhere and then, for each $x \in X$, the assignment $f_i \mapsto f_i(x)$, $i = 1, 2, \ldots, k$, extends uniquely to a linear functional $\phi(x) \in H := (V, \tilde{\rho})^*$. Via the form $\tilde{\rho}$ the latter vector space becomes a Hilbert space and the unitary G-action on $(V, \tilde{\rho})$ induces a unitary G-action on H. One can easily check that the map $\phi : X \to H$ is a non-constant measurable G-equivariant map, contradicting the EUC property of $\mathbf{X}$.

Suppose now that μ is ergodic and that in the measure class of μ there is an invariant probability measure (a unique one, by ergodicity). So we can assume that μ itself is invariant.

If V is a finite-dimensional G-invariant subspace of $L_2(\mu)$, choose an orthonormal basis $\{f_1, f_2, \ldots, f_n\}$ for V and observe that the function $\phi : X \to \mathbb{C}$ defined by $\phi(x) = \sum_{i=1}^n |f(x)|^2$ is non-zero and G-invariant (the action is via unitary matrices). By ergodicity $\phi(x) = C > 0$ is a constant almost everywhere and it follows that the functions f_i are bounded, so that $V \subset L_\infty(\mu)$. Thus it follows that if $\mathbf{X}$ is L_∞-WM then $L_2(\mu)$ admits no non-trivial finite-dimensional invariant subspaces (one can call this latter property L_2-WM). As is well known, L_2-WM implies weak mixing and, in turn, this implies that $\mathbf{X}$ is DE (as $\mathbf{X}$ is measure preserving). We conclude that for measure preserving systems all the properties DE, EIC, EUC, WM, L_2-WM, and L_∞-WM are equivalent (see, for example, [**15**, Theorem 3.11]).

Finally, the implication L_∞-WM $\Rightarrow$ ergodicity is clear. □

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THEOREM 2.2. *A non-singular system* $\mathbf{X} = (X, \mathcal{B}, \mu, \{T_g\}_{g\in G})$ *is weakly mixing if and only if it is EUC.*

Proof.

(1) Assume first that $\mathbf{X}$ is EUC and let $\mathbf{Y}$ be a probability measure preserving G-action. Suppose that $W \subset X \times Y$ is a G-invariant Borel set and set $F(x, y) = \mathbf{1}_W$. Then the map $\phi : X \to L_2(Y, \nu)$ defined by $\phi(x) = F_x := F(x, \cdot)$ is clearly measurable and, denoting by U_g the Koopman operator induced by S_g on $L_2(Y, \nu)$ (explicitly, $U_g f(y) = f(S_{g^{-1}} y)$) we have:

$$\begin{aligned}\phi(T_g x)(y) &= F(T_g x, y) = F(T_{g^{-1}} T_g x, S_{g^{-1}} y)\\ &= F_x(S_{g^{-1}} y) = (U_g F_x)(y)\\ &= (U_g \phi(x))(y).\end{aligned}$$

Hence, for μ-almost every x,

$$\phi(T_g x) = U_g \phi(x).$$

Since we are assuming that $\mathbf{X}$ is EUC, we conclude that $\phi(x) \equiv f$ is a constant μ-almost everywhere. Then

$$f(S_g y) = F(x, S_g y) = F(T_{g^{-1}} x, y) = f(y)$$

for ν-almost every y, and by the ergodicity of $\mathbf{Y}$ we conclude that either $f \equiv 0$ or $f \equiv 1$; i.e. $\mu \times \nu(W)$ is either 0 or 1. Thus, the $T \times S$ action is indeed ergodic.

(2) We now assume that $\mathbf{X}$ is weakly mixing and suppose that $\phi : X \to H$ is a measurable equivariant map, where H is a separable Hilbert space on which G acts by unitary operators $\{U_g\}_{g\in G}$. Thus μ-almost everywhere,

$$\phi(T_g x) = U_g \phi(x).$$

We assume with no loss in generality that $H = \text{cls span}(\text{supp}\, \phi_*(\mu))$. Let $H_0 = \{v \in H : U_g v = v,\ \forall g \in G\}$ and $H_1 = H_0^{\perp} = H \ominus H_0$. If $H_1 = 0$ then clearly ϕ is a constant (clearly WM implies ergodicity) and we are done. So we can now assume that $H = H_1$, i.e. the representation $g \mapsto U_g$ on H admits no non-zero fixed vectors.

By passing to a cyclic subspace we can assume that there is a vector $v_0 \in H$ such that $H = \text{cls span}\{U_g v_0 : g \in G\}$. Set

$$c(g) = \langle U_g v_0, v_0 \rangle,$$

and let $(\Omega, \mathcal{F}, P, \{X_g\}_{g\in G})$ be an associated Gauss process; i.e. the collection $\{X_g\}_{g\in G}$ is a set of centered random Gauss variables defined on the probability space $(\Omega, \mathcal{F}, P)$ with correlation function

$$c(gh^{-1}) = \mathbb{E}(X_g X_h).$$

Then, the translations defined on the X_g by $h(X_g) = X_{gh}$ define an action of G on $(\Omega, \mathcal{F}, P)$. Denoting this action by $\{R_g\}_{g\in G}$, the system $(\Omega, \mathcal{F}, P, \{R_g\}_{g\in G})$ becomes a probability measure preserving action of G. The fact that H contains no non-zero

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fixed vector implies that the system $(\Omega, \mathcal{F}, P, \{R_g\}_{g\in G})$ is ergodic (see, for example, [**15**, Theorems 3.4 and 3.59]). By our assumption then the product action $(X \times \Omega, \mathcal{B} \otimes \mathcal{F}, \mu \times P, \{T_g \times R_g\}_{g\in G})$ is ergodic.

Now, by the construction of the Gauss process $\{X_g\}_{g\in G}$, there is a unitary equivalence between H and the *first Wiener chaos* $\mathcal{H} := \text{cls span}\{X_g : g \in G\} \subset L_2(\Omega, P)$, say,

$$V : H \to \mathcal{H}$$

with $U_{R_g} V = V U_g$ for every $g \in G$. Set $w(x, \omega) = V\phi(x)(\omega)$. Then

$$\begin{aligned} w(T_{g^{-1}}x, \omega) &= V\phi(T_{g^{-1}}x)(\omega) = VU_g\phi(x)(\omega) \\ &= U_{R_g} V\phi(x)(\omega) = U_{R_g} w(x, \omega) \\ &= w(x, R_g\omega), \end{aligned}$$

whence

$$w(T_{g^{-1}}x, R_{g^{-1}}\omega) = w(x, \omega)$$

for every $g \in G$. By our assumption then $w(x, \omega) = V\phi(x)(\omega)$ is a constant $\mu \times P$-almost everywhere and this leads to a contradiction if H is non-trivial. □

Remark 2.3. Our proof of Theorem 2.2 was motivated by [**6**], and by the proof of [**3**, Proposition 4.6], which is the main ingredient of the proof of [**3**, Theorem 4.7]. Of course Theorem 2.2 is a far reaching generalization of [**3**, Theorem 4.7]. For the implication EUC $\Rightarrow$ WM see also [**27**, Proposition 2.4].

Problem 2.4. Find an example of a system which is EUC but not EIC.

3. *Moore groups*

A locally compact group G which has the property that all of its irreducible unitary representations are finite-dimensional is called a *Moore group* (see [**28**, **31**]). Of course abelian groups are Moore groups, and in [**28**] it is shown that a locally compact group G is a Moore group if and only if $G = \text{proj lim } G_\alpha$ where each G_α is a Lie group which contains an open subgroup H_α of finite index which is a Z-group. Recall that a topological group H is a *Z-group* if $H/Z(H)$ is compact, where $Z(H)$ is the center of H [**28**, Theorem 3].

In a recent work Glücksam [**16**, Theorem 4.3] proves that for a second countable Moore group WM is equivalent to L_∞-WM. We will next use Theorem 2.2 to obtain a much shorter proof of this result.

THEOREM 3.1. *For a second countable Moore group the property EUC (hence also WM) is equivalent to the property L_∞-WM.*

Proof. By Theorem 2.2, EUC = WM. Thus the implication EUC $\Rightarrow$ L_∞-WM has already been proved in Theorem 2.1. For the other direction assume that $\mathbf{X}$ has the property L_∞-WM and assume to the contrary that $\phi : X \to H$ is a non-trivial measurable equivariant map into a separable Hilbert space on which the group G acts via unitary operators $\{U_g : g \in G\}$. Again, with no loss in generality, assume that $H = \text{cls span}(\text{supp } \phi_*(\mu))$. We observe that by the ergodicity of $\mathbf{X}$ and the assumption that G acts on H by unitary

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operators, it follows that the map $x \mapsto \|\phi(x)\|$ is a constant μ-almost everywhere; since this constant is non-zero we can assume that it is one.

Let us first consider the case where H is irreducible and finite-dimensional. We then choose $h_0 \in H$ with $\|h_0\| = 1$ such that the function $f_0 : X \to D$, where D is the unit disk in $\mathbb{C}$, defined by $f_0(x) = \langle \phi(x), h_0 \rangle$ is non-constant. Now, f_0 is a non-constant function in $L_\infty(X, \mu)$ and the space $V = \text{span}\{g \cdot f_0 : g \in G\}$ is a G-invariant subspace of $L_\infty(X, \mu)$. We have

$$(g \cdot f_0)(x) = f_0(g^{-1}x) = \langle \phi(g^{-1}x), h_0 \rangle = \langle U_g(\phi(x)), h_0 \rangle = \langle \phi(x), U_{g^{-1}}h_0 \rangle.$$

Since for an element $h \in H$ the map $f(x) = \langle \phi(x), h \rangle$ satisfies the inequality $\|f\|_\infty \leq \|h\|$ it follows that the natural map from $H = \text{span}\{U_g h_0 : g \in G\}$ onto V, which sends h_0 to f_0, is an isomorphism. Thus V is indeed a non-trivial finite-dimensional G-invariant subspace of $L_\infty(X, \mu)$.

The next step of the proof is based (as in [**16**]) on the fact that for a second countable Moore group G any unitary representation π on a separable Hilbert space H can be represented as a direct integral $\pi \cong \int^\oplus \pi_t \, dP(t)$, on some parameter space T equipped with a measure P, where each π_t is an irreducible representation on a finite-dimensional Hilbert space H_t, so that $H \cong \int^\oplus H_t \, dP(t)$, see [**10**, **24**]. It now follows that if $\phi : X \to H$ is a non-trivial map as above then for a set $A \subset T$ of positive P-measure, for every $t \in A$, the composition ϕ_t of ϕ with the projection onto the component H_t is again a non-trivial map. □

4. *SAT dynamical systems*

Strongly approximately transitive group actions were first introduced and studied by W. Jaworski [**19**].

Definition 4.1. A non-singular action $\mathbf{X} = (X, \mathcal{B}, \mu, \{T_g\}_{g \in G})$ is *strongly almost transitive (SAT)* if for every measurable $A \subset X$ with $\mu(A) > 0$, there is a sequence $\{g_n\} \subset G$ with $\mu(g_n A) \to 1$.

PROPOSITION 4.2.

- *Every SAT system is ergodic.*
- *A SAT system admits no non-identity endomorphism (automorphism).*
- *In every SAT system the center of G acts trivially.*
- *Nilpotent (in particular, abelian) groups admit no non-trivial SAT systems.*

Proof. The first assertion is clear. Suppose that $\phi : X \to X$ is a non-identity endomorphism of the SAT system $\mathbf{X}$. Thus, ϕ is measurable, G equivariant, with $\phi_*(\mu) \sim \mu$. Let $B \in \mathcal{B}$ satisfy $0 < \mu(B)$ and $\mu(B \cap \phi(B)) = 0$. Choose a sequence $g_n \in G$ such that $\mu(g_n B) \to 1$. Then also $\mu(g_n S B) = \mu(S g_n B) \to 1$, hence eventually, $\mu(g_n B \cap S g_n B) = \mu(g_n(B \cap SB)) > 0$; a contradiction. The last two assertions follow readily. □

Note that if $\mathbf{X}$ is non-trivial SAT then the product action on $X \times X$ is never SAT. In fact for $A \subset X$ with $0 < \mu(A) < 1$ we have $\mu \times \mu(T_g A \times T_g A^c) = \mu(T_g A)(1 - \mu(T_g A)) \leq \frac{1}{4}$ for every $g \in G$.

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PROPOSITION 4.3. *If* $\mathbf{X} = (X, \mathcal{B}, \mu, G)$ *is SAT then it is EIC.*

Proof. Let $\mathbf{X}$ be a SAT system and assume to the contrary that $\phi : X \to Z$ is a non-constant, Borel measurable, equivariant map, where (Z, d) is a separable metric space on which G acts by isometries. We assume (as we well may) that $Z = \operatorname{cls} \phi(X)$, so that supp $(\nu) = Z$, where $\nu = \phi_*(\mu)$. We choose an $\epsilon > 0$ and open balls $U = B(z_0, \epsilon)$ and $V = B(z_1, \epsilon)$ in Z whose centers satisfy $d(z_0, z_1) > 2\epsilon$. Let $A = \phi^{-1}(U)$ and $B = \phi^{-1}(V)$. Then the sets $A, B \in \mathcal{B}$ have μ-measures strictly between zero and one and by the SAT property there is $g \in G$ such that $\mu(gA) > \max(1 - \mu(A), 1 - \mu(B))$. It follows that both $\mu(gA \cap A) > 0$ and $\mu(gA \cap B) > 0$. Therefore also $\nu(gU \cap U) > 0$ and $\nu(gU \cap V) > 0$, but this contradicts the fact that g is an isometry on Z. □

Remark 4.4. The condition on $\mathbf{X}$ that for any two positive sets $A, B \in \mathcal{B}$ there is an element $g \in G$ with both $\mu(gA \cap A) > 0$ and $\mu(gA \cap B) > 0$ is called by Silva *et al* 'double ergodicity' (see [**8**, **17**, **18**]).

On the other hand, we will next see that the SAT property does not necessarily imply double ergodicity (see also Propositions 6.2 and 7.1 below).

Example 4.5. Let $G = \mathbb{R} \rtimes \mathbb{R}^+$ be the '$ax + b$' group. As shown in [**19**], the natural action of this group on $\mathbb{R}$ has the property that any absolutely continuous probability measure μ on $\mathbb{R}$ is SAT. (It is also easy to see that the SAT property follows directly from the existence of density points in any Borel measurable subset $A \subset \mathbb{R}$ of positive Lebesgue measure.) Of course such a measure will be SAT also with respect to any countable dense subgroup of G. For concreteness let us consider the group $\Gamma = \mathbb{Q} \rtimes \mathbb{Q}^+$, and let μ be any probability measure which is in the class of Lebesgue measure on $\mathbb{R}$. Taking A, B, C and D to be any four disjoint consecutive open intervals in $\mathbb{R}$, we see that there is no $\gamma \in \Gamma$ for which $\mu(\gamma B \cap D) > 0$ and also $\mu(\gamma C \cap A) > 0$. It is not hard to check that the cartesian square $(\mathbb{R} \times \mathbb{R}, \mu \times \mu, \Gamma)$ has exactly two ergodic components $\{(x, y) : x < y\}$ and $\{(x, y) : x > y\}$.

We thus have the following result.

PROPOSITION 4.6. *The non-singular system* $(\mathbb{R}, \mu, \Gamma)$ *is SAT (hence EIC) but not doubly ergodic.*

Remark 4.7. In the sequel we will exhibit two more examples of systems which are EIC but not DE. It is, however, interesting to note that the reasons for this phenomenon are very different in these three examples. Compare Example 4.5, where the cartesian square decomposes into two ergodic components with Example 6.1, where the cartesian square decomposes into a continuum of ergodic components according to the value of the cross-ratio; and with the example in §7, where the cartesian square is dissipative.

5. *m-Stationary dynamical systems*

Let G be a locally compact second countable topological group. A probability measure m on G is called *spread out* (or *etalée*) if there exists a convolution power $m^{(n)} = m * m * \cdots * m$ (n times) which is not singular with respect to the Haar measure class on G. The

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measure m is *non-degenerate* if the minimal closed semigroup $S \subset G$ with $m(S) = 1$ is G, and it is *symmetric* if it is invariant under the map $x \mapsto x^{-1}$, $x \in G$. The measure m is *admissible* if it is both spread out and non-degenerate.

A probability measure μ on a Borel dynamical system $(X, \mathcal{B}, G)$ is called *m-stationary* when the equation $m * \mu = \mu$ holds. When m is admissible it follows that the system $\mathbf{X} = (X, \mathcal{B}, \mu, G)$ is non-singular, i.e. the measure μ is quasi-invariant: $\mu(gA) = 0 \iff \mu(A) = 0$ for every $A \in \mathcal{B}$ and $g \in G$ [**29**, Lemma 1.1].

We can always find a *topological model* for an m-system $\mathbf{X}$, meaning that X can be chosen to be a compact metric space on which G acts by homeomorphisms (see [**32**, Theorem 3.2]).

With the measure m one associates a random walk on G as follows. Let $\Omega = G^{\mathbb{N}}$ and let $P = m^{\mathbb{N}} = m \times m \times m \ldots$ be the product measure on Ω, so that (Ω, P) is a probability space. We let $\xi_n : \Omega \to G$, denote the projection onto the nth coordinate, $n = 1, 2, \ldots$. We refer to the stochastic process $(\Omega, P, \{\eta_n\}_{n\in\mathbb{N}})$, where $\eta_n = \xi_1\xi_2 \cdots \xi_n$, as the *$m$-random walk on G*.

A real-valued function $f(g)$ for which $\int f(gg')\, dm(g') = f(g)$ for every $g \in G$ is called *harmonic*. For a harmonic f we have

$$\begin{aligned} E(f(g\xi_1\xi_2 \cdots \xi_n\xi_{n+1}|\xi_1\xi_2 \cdots \xi_n) &= \int f(g\xi_1\xi_2 \cdots \xi_n g')\, dm(g') \\ &= f(g\xi_1\xi_2 \cdots \xi_n), \end{aligned}$$

so that the sequence $f(g\xi_1\xi_2 \cdots \xi_n)$ forms a martingale.

If $\mathbf{X} = (X, \mathcal{B}, \mu, G)$ is an m-stationary system on a compact metric space X, for $F \in C(X)$ let $f(g) = \int F(gx)\, d\mu(x)$. Then the equation $m * \mu = \mu$ shows that f is harmonic. It is shown (for example) in [**11**] how these facts combined with the martingale convergence theorem lead to the following result.

THEOREM 5.1. *The limits*

$$\lim_{n\to\infty} \eta_n\mu = \lim_{n\to\infty} \xi_1\xi_2 \cdots \xi_n\mu = \mu_\omega \tag{1}$$

exist for P-almost all $\omega \in \Omega$, and

$$\int \mu_\omega\, dP(\omega) = \mu.$$

The measures μ_ω are the *conditional measures* of the m-system $\mathbf{X}$. We describe the m-system $\mathbf{X}$ as *m-proximal* (or a *boundary* in the terminology of [**11**]) if P-almost surely the conditional measures $\mu_\omega \in M(X)$ are point masses. It can be shown that this property does not depend on the topological model chosen for $\mathbf{X}$. Clearly a factor of a proximal system is proximal as well. There exists a unique m-stationary system $\Pi(G, m)$, called the *Poisson boundary* of (G, m), which is a maximal boundary. Thus an m-stationary system is m-proximal if and only if it is a factor of $\Pi(G, m)$. For more details and basic results concerning general m-stationary dynamical systems and, in particular, m-proximal systems, we refer to [**13**, **14**]. We remind the reader that every m-proximal stationary system is SAT [**19**, Corollary 2.4] (see also [**14**, Proposition 3.7]). For an alternative approach to the Poisson boundary see the seminal work of Kaimanovitch and Vershik [**22**].

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Definition 5.2. A measure preserving G-system $\mathbf{X} = (X, \mathcal{B}, \mu, G)$ is said to have discrete spectrum if $L_2(\mu)$ decomposes as a direct sum of finite-dimensional invariant subspaces.

Recall that an ergodic measure preserving system has discrete spectrum if and only if the group $\{U_g : g \in G\}$, where U_g is the unitary Koopman operator on $L_2(\mu)$ defined by $U_g f(x) = f(g^{-1}x)$, is precompact in the strong operator topology (see, for example, [**15**, §3.1]).

Definition 5.3. A topological dynamical system (X, G) (i.e. X is a compact space and G acts on X via a continuous homomorphism, say ϕ, of G into the group Homeo(X) of self homeomorphisms of X equipped with the uniform convergence topology) is called *weakly almost periodic (WAP)* if the closure of $\{\phi(g) : g \in G\} \subset X^X$, in the pointwise convergence topology on X^X (this closure is called the *enveloping semigroup* of (X, G)), consists of continuous maps.

The following theorem is proved by Björklund in [**7**, Theorem 3.1]. As indicated by him, a slightly different proof, based on the fact that WAP systems are stiff (see [**13**]), is available and is given below.

THEOREM 5.4. *Let m be a non-degenerate spread-out and symmetric probability measure on G. An ergodic m-stationary system $\mathbf{X} = (X, \mathcal{B}, \mu, G)$ is WM (as a non-singular G-system) if and only if it does not admit a non-trivial measure preserving factor which has discrete spectrum.*

Proof. Suppose first that $\mathbf{X}$ admits a non-trivial measure preserving factor $\mathbf{Y} = (Y, \mathcal{C}, \nu, G)$ which has discrete spectrum. Then $\mathbf{Y} \times \mathbf{Y}$ is not ergodic and therefore also $\mathbf{X} \times \mathbf{Y}$, which naturally maps onto $\mathbf{Y} \times \mathbf{Y}$, is not ergodic. Whence $\mathbf{X}$ is not WM.

Conversely, if $\mathbf{X}$ is not WM then there exists a non-trivial probability measure preserving system $\mathbf{Y}$ such that $\mathbf{X} \times \mathbf{Y}$ is not ergodic. Let $W \subset X \times Y$ be a measurable invariant subset of $\mu \times \nu$ measure strictly between zero and one. It is easy to check that the map $\phi : X \to L_2(Y, \nu)$; $x \mapsto 1_{W_x}$, with $W_x = \{y \in Y : (x, y) \in W\}$, is an equivariant, measurable, non-trivial map from X to $L_2(Y, \nu)$, where the action of G on $L_2(Y, \nu)$ is via the (unitary) Koopman representation $g \mapsto U_g$, $g \in G$. Let $\lambda = \phi_*(\mu)$ be the push-forward probability measure on $L_2(Y, \nu)$.

Clearly λ is m-stationary and ergodic. By the ergodicity of λ it follows that the invariant function $x \mapsto \|x\|$ is a constant λ-almost everywhere, and we can therefore assume that $Z := \text{supp}\,(\lambda)$, where the closure is taken with respect to the weak topology, is contained in the weakly compact unit ball of $L_2(Y, \nu)$. Next we note that the action $(Z, \lambda, \{U_g\}_{g \in G})$ is topologically a WAP system. In fact, every element Q of the enveloping semigroup is a pointwise limit (in the weak operator topology) of a net $U_{g_i} \to Q$. It is easy to check that Q is a linear operator with $\|Q\| \leq 1$. In particular, then every element of the enveloping semigroup is continuous (see, for example, [**15**, §§1.9 and 3.1]). Applying [**13**, Theorem 7.4], we conclude that λ is G-invariant. Now in a topologically transitive WAP system the invariant measure is unique and has discrete spectrum and our proof is complete. □

We do not know whether the assumption that $\mathbf{X}$ is an m-stationary system, and not merely a non-singular one, is really necessary.

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Problem 5.5. Clearly a WM non-singular system does not admit a non-trivial measure preserving factor which has discrete spectrum. Is the converse true? For related questions see [**4**, **30**].

The next result is essentially due to Kaimanovich [**21**]. He proves part (2) for the Poisson boundary $\Pi(G, m)$, but then the result holds for all its factors as well. Of course (2) implies (1) (see also [**2**]).

THEOREM 5.6. *Let m be a non-degenerate spread-out and symmetric probability measure on G. Let μ be an m-stationary probability measure on X such that the m-stationary system $\mathbf{X} = (X, \mathcal{B}, \mu, G)$ is m-proximal.*

(1) *The non-singular system $\mathbf{X}$ is doubly ergodic.*

(2) *The product system $\mathbf{X}^2$ is EIC.*

Theorem 5.6 is, in fact, a generalization of Kaimanovich's theorem. Actually the notion of EIC is not even defined in Kaimanovich's paper [**21**]. However, his proof can be easily modified to prove a stronger statement when the target is a separable metric space (Z, d).

In order to understand the way this is done we first recall [**21**, Theorem 6].

THEOREM 5.7. *Let $T : (\Omega, \mu) \to (\Omega, \mu)$ be the bilateral Bernoulli shift over a probability space (X, ν), that is, $\Omega = X^{\mathbb{Z}}$, $\mu = \nu^{\mathbb{Z}}$ and for $\omega = (\ldots, x_{-1}, \dot{x}_0, x_1, \ldots)$, $(T(\omega))_i = x_{i+1}$. If E is a separable Banach space, and $f : \Omega \to E$ and $\pi : X \to \mathrm{Iso}\,(E)$ are measurable maps such that almost everywhere $f(T\omega) = \pi(x_1) f(\omega)$, then f is almost everywhere a constant.*

Note the crucial assumption that π depends only on the first coordinate x_1 of ω. We claim that the same statement holds when we replace E by a separable metric space (Z, d). In fact, all one needs to do is to assume (with no loss in generality) that the metric d is bounded, and then to replace the (linear) space $L_1(\Omega, \mu, E)$ with norm

$$|||f||| = \int \|f(\omega)\|\, d\mu(\omega),$$

which is used by Kaimanovich in his proof of Theorem 5.7, by the (nonlinear) space $M_1(\Omega, \mu, Z)$ of all measurable functions $f : \Omega \to Z$ with the metric

$$D(f_1, f_2) = \int d(f_1(\omega), f_2(\omega))\, d\mu(\omega).$$

We thus have the following result.

THEOREM 5.8. *Let $T : (\Omega, \mu) \to (\Omega, \mu)$ be the bilateral Bernoulli shift over a probability space (X, ν), that is, $\Omega = X^{\mathbb{Z}}$, $\mu = \nu^{\mathbb{Z}}$ and for $\omega = (\ldots, x_{-1}, \dot{x}_0, x_1, \ldots)$, $(T(\omega))_i = x_{i+1}$. If (Z, d) is a separable metric space, and $f : \Omega \to Z$ and $\pi : X \to \mathrm{Iso}\,(Z, d)$ are measurable maps such that almost everywhere $f(T\omega) = \pi(x_1) f(\omega)$, then f is almost everywhere a constant.*

Now from this stronger version of [**21**, Theorem 6], Theorem 5.6 follows exactly as [**21**, Theorem 17] is derived from [**21**, Theorem 6].

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Example 5.9. Consider again the non-singular SAT system $(\mathbb{R}, \mu, \Gamma)$ described in Example 4.5 above. We claim that *there is no non-degenerate and symmetric probability measure m on Γ which admits an m-stationary probability measure, say ν, in the class of μ (i.e. equivalent to Lebesgue measure on $\mathbb{R}$).* In fact if m on G and ν on $\mathbb{R}$ are such measures, then, by [**19**, Proposition 2.2], the system $(\mathbb{R}, \nu, \Gamma)$ would be m-proximal, hence, by Theorem 5.6, this will ensure that the systems $(\mathbb{R}, \nu, \Gamma)$ and therefore also $(\mathbb{R}, \mu, \Gamma)$ are DE, in contradiction to the claim in Example 4.5.

6. *The Poisson boundary for the free group $\mathbb{F}_2$*

Example 6.1. Let G be the free group on two generators, $G = \mathbb{F}_2 = \langle a, b \rangle$, and m the probability measure $m = \frac{1}{4}(\delta_a + \delta_b + \delta_{a^{-1}} + \delta_{b^{-1}})$. Evidently m is spread out, non-degenerate, and symmetric. Let Z be the space of right infinite reduced words on the letters $\{a, a^{-1}, b, b^{-1}\}$. G acts on Z by concatenation on the left and reduction. Let η be the probability measure on Z given by

$$\eta(C(\epsilon_1, \ldots, \epsilon_n)) = \frac{1}{4 \cdot 3^{n-1}},$$

where for $\epsilon_j \in \{a, a^{-1}, b, b^{-1}\}$, $C(\epsilon_1, \ldots, \epsilon_n) = \{z \in Z : z_j = \epsilon_j, \ \ j = 1, \ldots, n\}$. The measure η is m-stationary and the m-system $\mathbf{Z} = (Z, \eta, G)$ is m-proximal. In fact $\mathbf{Z}$ is the Poisson boundary $\Pi(\mathbb{F}_2, m)$ (see e.g. [**11**]). In particular, then the system $\mathbf{Z}$ is SAT (see, for example, [**14**], Proposition 3.7), and by [**20**] it is doubly ergodic. It now follows, by Theorem 5.6, that $\mathbf{X}^2$ is EIC.

It is not hard to see that if we let $\mathbb{F}_2$ act on the unit circle $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ via two appropriately chosen Möbius transformations T_a and T_b, then the quotient system $\mathbf{X} = (X, \mu, G)$, where $X = \mathbb{P} \cong \mathbb{T}/\{\pm 1\}$ is the projective line and μ denotes the image of Lebesgue measure on $\mathbb{T}$, is isomorphic to $\mathbf{Z} = (Z, \eta, G)$.

By choosing four disjoint intervals in $\mathbb{P}$ we can easily see that the system $\mathbf{X}^2$ is not doubly ergodic (i.e. the diagonal G-action on $X \times X \times X \times X$ is not ergodic). Namely, if A, B, C, D are disjoint arcs ordered counterclockwise on the circle then, for any Möbius transformation T, the arcs TA, TB, TC, TD are ordered either clockwise or counterclockwise. Thus, for example,

$$(T \times T \times T \times T)(A \times B \times C \times D) \cap (A \times C \times B \times D) \neq \emptyset,$$

cannot be achieved. Another way of proving this statement is to observe that the cross-ratio relation is preserved by the diagonal G-action on X^4.

As a conclusion we have the following proposition.

PROPOSITION 6.2. *The G-system $\mathbf{X}^2$ is EIC but not DE.*

7. *A $\mathbb{Z}$-system which is EIC but not DE*

PROPOSITION 7.1. *There exists a $\mathbb{Z}$-system which is EIC but not DE.*

Proof. Let $\{Y_n\}$ be a countable state Markov chain on the state space A which is conservative and ergodic with an infinite invariant measure that has no periodic factor. We claim that the corresponding shift (as a non-singular transformation) is EIC. For the

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proof suppose that f is a measurable map from $Y = A^{\mathbb{Z}}$ to a separable metric space (Z, d) such that

$$f(Ty) = Uf(y)$$

where T is the shift and U is an isometry of Z. Now fix a state a_0 in A and look at the induced transformation on the cylinder set $[a_0]$. This becomes a Bernoulli shift where the states are now all the possible blocks $a_0, a_1, a_2, \ldots, a_k$ that represent the states that one visits before the next return to a_0. Let B denote this countable collection of blocks, and of course there is a probability distribution on these blocks given by the transition probabilities of the Markov chain. If S represents the induced transformation then S is the Bernoulli shift on $X = B^{\mathbb{Z}}$. Now if we denote by g the restriction of f to the cylinder set $[a_0]$, then g will satisfy

$$g(Sx) = U^{k(x_0)} g(x)$$

where k is the length of the block $x_0 \in B$. This is exactly the situation of [**21**, Theorem 6] adapted to isometries as in Theorem 5.8 above (except for the inessential change that x_1 is replaced by x_0, the zeroth coordinate of x). It follows that g is a constant almost everywhere. The essential range of f is now a countable set on which U will act as a permutation and, since the chain is assumed to be ergodic and conservative, this set would have to be finite, contradicting the fact that we assumed that the chain has no periodic factor.

Now any such chain whose cartesian square is not conservative will give an example of an EIC system which is not DE. For a simple example consider the simple symmetric planar random walk on $\mathbb{Z}^2$ with a positive probability of remaining at $(0, 0)$. This is to eliminate the two-point factor coming from the parity of the site. As is well known, its cartesian square, which is the random walk on $\mathbb{Z}^4$, is not recurrent, i.e. is not conservative. □

8. *F_m-proximality and mean proximality*

In this section we reconsider some notions of proximality introduced by Furstenberg in [**12**]. Let m be a probability measure on G. For $n \geq 1$ set

$$m_n = \frac{1}{n}(m + m^{(2)} + \cdots + m^{(n)}),$$

where $m^{(j)} = m * m * \cdots * m$ (j times). Given a compact metric G-space X, we define the convolution operators $\mathbb{A}_n : C(X) \to C(X)$ by the formula

$$\mathbb{A}_n f(x) = (m_n * f)(x) = \int f(gx)\, dm_n(g).$$

We write $\mathbb{A}$ for $\mathbb{A}_1$.

The operator $\mathbb{A}$ is a *Markov operator* on $C(X)$, i.e. it is linear, positive and satisfies $\mathbb{A}1_X = 1_X$. Any Markov operator admits an invariant probability measure, and it is called *uniquely ergodic* if there is only one invariant probability measure (see [**23**, §5.1]). The proof of the next theorem is almost identical to that of [**15**, Theorem 4.9].

THEOREM 8.1. *The following conditions are equivalent for a Markov operator $\mathbb{A}$ on a compact metric G-space X.*

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(1) *There is a unique m-stationary probability measure on X, i.e. $\mathbb{A}$ is uniquely ergodic.*
(2) *$C(X) = \mathbb{R} + \bar{B}$, where $B = \{f - \mathbb{A}f : f \in C(X)\}$ and $\bar{B}$ is its closure in the topology of uniform convergence on X.*
(3) *For every continuous function $f \in C(X)$ the sequence of functions $\mathbb{A}_n f$ converges uniformly to a constant function f^*.*
(4) *For every continuous function $f \in C(X)$ the sequence of functions $\mathbb{A}_n f$ converges pointwise to a constant function f^*.*
(5) *For every function $f \in \mathcal{A}$, for a collection $\mathcal{A} \subset C(X)$ which linearly spans a uniformly dense subspace of $C(X)$, the sequence of functions $\mathbb{A}_n f$ converges pointwise to a constant function.*

Definition 8.2. A compact metric G-space X is called *F_m-proximal* if for each $x, y \in X$, $m_n\{g : d(gx, gy) > \epsilon\} \to 0$ as $n \to \infty$ for any $\epsilon > 0$.

Remark 8.3. In his paper [**12**] Furstenberg calls this property m-proximality. Unfortunately, in some later works [**13**, **14**] the name 'm-proximal' was given the meaning we also adopt here, that of a boundary (see the paragraph following Theorem 4.1 above). Thus the reader is warned that F_m-proximal stands in this work for the notion of m-proximal in [**12**].

The next theorem is from [**12**, Theorem 14.1].

THEOREM 8.4. *The following are equivalent for a compact metric G-space X.*
(1) *X is F_m-proximal.*
(2) *Any solution $\nu \in M(X \times X)$ to $m * \nu = \nu$ is concentrated on the diagonal $\Delta(X) \subset X \times X$.*
(3) *For any m-stationary measure $\nu \in M(X)$ the m-stationary system (X, ν) is m-proximal.*
(4) *For any $\theta \in M(X)$ and $\epsilon > 0$,*

$$\lim_{n\to\infty} m_n\{g \in G : d(g\theta, \delta_X) > \epsilon\} = 0.$$

We now extend this list as follows.

THEOREM 8.5. *The following conditions are equivalent for a compact metric G-space X.*
(1) *X is F_m-proximal.*
(2) *There is a unique m-stationary probability measure on X, say ν (i.e. the G-space X is uniquely m-ergodic), and the m-stationary system (X, ν) is m-proximal.*

Proof. (1) $\Rightarrow$ (2) Let ν_1 and ν_2 be two m-stationary measures on X. We consider the 'natural' joining

$$\nu = \nu_1 \curlyvee \nu_2 = \int (\nu_1)_\omega \times (\nu_2)_\omega \, dP(\omega),$$

an element of $M(X \times X)$ (see [**13**, §3]). Clearly $m * \nu = \nu$ (with respect to the diagonal G-action) and by item (2) of the previous theorem we conclude that ν is concentrated on the diagonal $\Delta(X)$. This clearly implies that $\nu_1 = \nu_2$.

(2) $\Rightarrow$ (1) As we assume that ν is the unique m-stationary measure on X and that (X, ν) is m-proximal, Theorem 8.4 (3) implies that X is F_m-proximal. □

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Definition 8.6. A compact metric G-space X is called *mean proximal* if it is F_m-proximal for every $m \in M(G)$ whose closed support is G.

Example 8.7. Since $\mathbb{F}_2$ appears as a finite index subgroup of the group $PSL_2(\mathbb{Z})$, it follows from [**12**, Theorem 16.8] that the G-space $(\mathbb{P}, \mathbb{F}_2)$ is mean proximal (see also [**25**, Ch. VI.2]).

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