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Distribution of intersection lengths of a random geodesic with a geodesic lamination

MARTIN BRIDGEMAN† and DAVID DUMAS‡

† *Department of Mathematics, Boston College, Chestnut Hill, MA 02467, USA*
(e-mail: bridgem@bc.edu)

‡ *Department of Mathematics, Brown University, Providence, RI 02912, USA*
(e-mail: ddumas@math.brown.edu)

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Abstract. We investigate the distribution of lengths obtained by intersecting a random geodesic with a geodesic lamination. We give an explicit formula for the distribution for the case of a maximal lamination and show that the distribution is independent of the surface and lamination. We also show how the moments of the distribution are related to the Riemann zeta function.

1. *Introduction*

Let S be a closed hyperbolic surface and λ a maximal geodesic lamination on S. If we take an infinite geodesic α on S, then $\lambda \cap \alpha$ decomposes α into geodesic arcs. We consider the distribution of the lengths of these arcs and we show that for almost every geodesic the distribution is the same and we explicitly calculate this distribution.

Let $v \in T_1(S)$ be a unit tangent vector on S and $\alpha_v : \mathbb{R} \to S$ the geodesic parameterized by arc length such that $v = \alpha_v'(0)$. In the complement of λ, α_v is a countable union of open intervals (see Figure 1). We let

$$\alpha_v^{-1}(S - \lambda) = \bigcup_{j=1}^{\infty} I_j$$

where I_j are open disjoint intervals. We denote the length $|I_j|$ of I_j by l_j.

Definition. The *length distribution* $D_\lambda(v)$ is a measure on $(0, \infty)$ given by

$$D_{\lambda,t}(v) = \frac{1}{t} \sum_{I_i \cap [0,t] \neq \emptyset} \delta(l_i),$$

$$D_\lambda(v) = \overline{\lim_{t \to \infty}} D_{\lambda,t}(v)$$

where $\delta(x)$ is the Dirac measure at x and the limit is taken with respect to the weak* topology.

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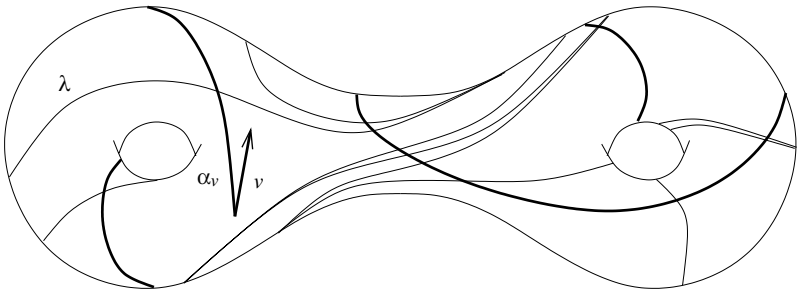


FIGURE 1. Intersection of lamination λ with ray α_v in direction v.

The main result is that for almost all vectors v, the distribution $D_\lambda(v)$ is independent of λ and v and can be explicitly calculated.

THEOREM 1. *Let λ be a maximal geodesic lamination. Then there exists a measure M such that for almost all $v \in T_1(S)$ with respect to the volume measure on $T_1(S)$ we have*

$$D_\lambda(v) = \lim_{t\to\infty} D_{\lambda,t}(v) = M,$$

where

$$dM = \frac{6x\,dx}{\pi^2 \sinh^2 x}.$$

We define measure $P_\lambda(v)$ on $(0, \infty)$ similarly by

$$P_{\lambda,t}(v) = \frac{1}{t} \sum_{I_i \cap [0,t] \neq \emptyset} l_i \delta(l_i),$$

$$P_\lambda(v) = \overline{\lim_{t\to\infty}} P_{\lambda,t}(v).$$

Note that if $\phi : (0, \infty) \to \mathbb{R}$ is a continuous function with compact support then

$$P_{\lambda,t}(v)(\phi(x)) = D_{\lambda,t}(v)(x\,\phi(x)).$$

Then we obtain the following corollary.

COROLLARY 2. *Let λ be a maximal geodesic lamination. Then for almost all $v \in T_1(S)$ with respect to the volume measure on $T_1(S)$, the measure $P_\lambda(v)$ is equal to the probability measure P given by*

$$dP = \frac{6x^2\,dx}{\pi^2 \sinh^2 x}.$$

Furthermore, the probability measure P has expected value

$$E_P(x) = \frac{9}{\pi^2}\zeta(3),$$

and nth moment given by

$$E_P(x^n) = \frac{3(n+2)!}{2^n \pi^2}\zeta(n+2),$$

where ζ is the Riemann zeta function.

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Before we give a proof of the corollary, let us describe a simple interpretation of the probability measure P in terms of the geometry of an ideal triangle. Consider picking a random tangent vector p in an ideal triangle T and let $g(p)$ be the geodesic through p. Then P is the same probability distribution as the probability distribution of lengths of $T \cap g(p)$. In particular, this corollary gives a good description of $\zeta(3)$ in terms of the average length of a segment in an ideal triangle.

Proof of Corollary 2. Let $\phi : (0, \infty) \to \mathbb{R}$ be a continuous function with compact support. Then by definition, $P_{\lambda,t}(\alpha_v)(\phi(x)) = D_{\lambda,t}(v)(x\,\phi(x))$. Therefore, as $x\,\phi(x)$ is continuous with compact support, it follows from the definition of the weak* topology that $P_\lambda(v)(\phi(x)) = D_\lambda(v)(x\,\phi(x))$. Therefore, for almost all $v \in T_1(S)$,

$$P_\lambda(v)(\phi(x)) = \int_0^\infty x\,\phi(x)\,d(D_\lambda(v)) = \int_0^\infty \frac{6x^2\phi(x)\,dx}{\pi^2 \sinh^2 x}.$$

It follows that for almost all $v \in T_1(S)$,

$$d(P_\lambda(v)) = \frac{6x^2\,dx}{\pi^2 \sinh^2 x}.$$

We let P be the measure on $(0, \infty)$ with the distribution given above. To show that P is a probability distribution and to evaluate its moments, we calculate the integral

$$\int_0^\infty \frac{6x^n\,dx}{\pi^2 \sinh^2 x}$$

for $n \geq 2$. The antiderivative we describe below was found using *Mathematica* (see [**7**]).

Let $\mathrm{li}_n(z)$ be the principal branch of the nth polylog function defined on the complex plane minus the real ray $[1, \infty)$. The power series expansion about 0 is given by

$$\mathrm{li}_n(x) = \sum_{k=1}^\infty \frac{x^k}{k^n}$$

and has radius of convergence 1.

For $n = 0$, $\mathrm{li}_0(x) = 1/(1-x)$, and for $n = 1$, $\mathrm{li}_1(x) = \log(1-x)$. Also for $n > 1$ we have convergence at $x = 1$ and $\mathrm{li}_n(1) = \zeta(n)$. Taking the derivative of the power series we obtain the relation

$$\mathrm{li}_n'(x) = \frac{\mathrm{li}_{n-1}(x)}{x}.$$

Let n be a positive integer greater than 1. We define a function $F : (0, \infty) \to \mathbb{R}$ on the positive real axis by

$$F(x) = -\sum_{k=0}^n \frac{n!}{k!2^{n-k-1}} x^k \mathrm{li}_{n-k}(e^{-2x}).$$

We now find the derivative of F. We have for $k \neq n$

$$\frac{d}{dx} x^k \mathrm{li}_{n-k}(e^{-2x}) = kx^{k-1}\mathrm{li}_{n-k}(e^{-2x}) - 2x^k \mathrm{li}_{n-k-1}(e^{-2x}).$$

For $k = n$, as $x^n \mathrm{li}_0(e^{-2x}) = x^n/(1-e^{-2x})$, we have

$$\frac{d}{dx} x^n \mathrm{li}_0(e^{-2x}) = nx^{n-1}\mathrm{li}_0(e^{-2x}) - \frac{2x^n e^{-2x}}{(1-e^{-2x})^2} = nx^{n-1}\mathrm{li}_0(e^{-2x}) - \frac{x^n}{2\sinh^2 x}.$$

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Putting all these derivatives together we have

$$F'(x) = \frac{x^n}{\sinh^2 x}.$$

Then

$$\int_0^\infty \frac{x^n\,dx}{\sinh^2 x} = F(\infty) - F(0) = 0 - \left(-\frac{n!}{2^{n-1}}\mathrm{li}_n(1)\right) = \frac{n!}{2^{n-1}}\zeta(n).$$

Therefore, as $\zeta(2) = \pi^2/6$,

$$\int_0^\infty \frac{6x^2\,dx}{\pi^2 \sinh^2 x} = 1.$$

Therefore the nth moment of this probability distribution is

$$\int_0^\infty \frac{6x^{n+2}\,dx}{\pi^2 \sinh^2 x} = \frac{3(n+2)!}{2^n\pi^2}\zeta(n+2).$$

In particular, P is a probability measure and has expected value $(9/\pi^2)\zeta(3)$. □

Given a geodesic current m (a generalization of a closed geodesic described formally in the next section) we define a measure $D_\lambda(m)$ on $\mathbb{R}_+$ associated with the intersection of the geodesic current m with the geodesic lamination λ. We prove the following properties of the function $m \to D_\lambda(m)$.

THEOREM 3. *Let μ be the Liouville geodesic current for the closed hyperbolic surface S. Then:*

(1) *$D_\lambda(\mu) = M$, where M is as in Theorem 1;*

(2) *if m_i are discrete geodesic currents such that $m_i \to \mu$ then*

$$\lim_{i\to\infty} D_\lambda(m_i) = D_\lambda(\mu) = M.$$

Note that sequences m_i as in (2) above are abundant: Bonahon shows in [**1**] that for almost any $v \in T_1(S)$, such a sequence can be constructed from the geodesic ray α_v determined by v (by closing up long segments). The application of Theorem 3 to this construction is discussed in Corollary 17.

2. *Geodesic laminations and geodesic currents*

Let S be a closed hyperbolic surface. A *geodesic lamination* is a closed subset of S that is the union of a disjoint collection of geodesics on S (see [**6**]). A geodesic lamination λ is *maximal* if $S - \lambda$ is a disjoint collection of ideal triangles.

Let $\mathbf{H}^2$ be the hyperbolic plane and $G(\mathbf{H}^2)$ be the set of oriented geodesics in $\mathbf{H}^2$. We take the upper half-plane model for $\mathbf{H}^2$ in the Riemann sphere $\hat{\mathbf{C}}$. Then the boundary of $\mathbf{H}^2$ in $\hat{\mathbf{C}}$ is $\overline{\mathbb{R}} = \mathbb{R}\cup\{\infty\}$ and is called the circle at infinity of $\mathbf{H}^2$. Identifying an oriented geodesic with its endpoints, we have that $G(\mathbf{H}^2) = \overline{\mathbb{R}} \times \overline{\mathbb{R}} - \Lambda$ where Λ is the diagonal in the product space. We give $G(\mathbf{H}^2)$ the subspace topology in $\overline{\mathbb{R}} \times \overline{\mathbb{R}}$.

In general, if X is a topological space, we let $C_0(X)$ be the space of continuous functions on X with compact support. If m is a measure on X we obtain a linear function $m : C_0(X) \to \mathbb{R}$ by defining

$$m(\phi) = \int_X \phi\,dm.$$

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The set of non-negative measures on X is denoted $M(X)$. The weak* topology on $M(X)$ has basis at $m \in M(X)$ given by the sets $U(m, \phi, \epsilon) = \{n \in M(X) | \, |n(\phi) - m(\phi)| < \epsilon\}$ where $\phi \in C_0(X), \epsilon > 0$. Thus $m_i \to m$ if $m_i(\phi) \to m(\phi)$ for all $\phi \in C_0(X)$. If $A \subseteq X$ and $m \in M(X)$ we define $m|_A$ to be the restriction of the measure m to A.

Let $S = \mathbf{H}^2/\Gamma$ be a closed hyperbolic surface, and α a closed geodesic in S of length $l(\alpha)$. Then the preimage of α in $G(\mathbf{H}^2)$ is a Γ-invariant discrete subset. If α is primitive, we obtain a Γ-invariant measure $m(\alpha)$ on $G(\mathbf{H}^2)$, by taking the Dirac measure on this set. If α is not primitive and $\alpha = \beta^k$, where β is a primitive closed geodesic, we define $m(\alpha) = k\, m(\beta)$. This measure is an example of a *geodesic current*.

Definition. A *geodesic current* on $S = \mathbf{H}^2/\Gamma$ is a Γ-invariant positive measure on $G(\mathbf{H}^2)$. The space of geodesic currents on S is denoted $\mathcal{C}(S)$ and given the weak* topology.

If a geodesic current m is a positive real multiple of $m(\alpha)$ for some closed geodesic α in S, then m is called a *discrete geodesic current*. If m is a discrete geodesic current with $m = \lambda m(\alpha), \lambda \in \mathbb{R}_+$, we define the length $l(m)$ of m by $l(m) = \lambda l(\alpha)$ where $l(\alpha)$ is the length of the closed geodesic α.

Another geodesic current is the *Liouville* measure given by

$$\mu([a, b] \times [c, d]) = \left| \log \left| \frac{(a - c)(b - d)}{(a - d)(b - c)} \right| \right|.$$

By the invariance of the cross-ratio under Möbius transformations, μ is invariant under the group of isometries of $\mathbf{H}^2$. In particular, μ is invariant under the Fuchsian group Γ and is therefore a geodesic current. In fact, μ is the unique measure on $G(\mathbf{H}^2)$ (up to constant multiple) that is invariant under the full Möbius group. Differentially μ is given by

$$d\mu_{(a,b)} = \frac{dx\, dy}{|a - b|^2}$$

where $dx\, dy$ is the standard area measure on $\mathbb{R} \times \mathbb{R}$. For a full description of geodesic currents and their properties see Bonahon [**1**].

The following lemmas show that geodesic currents are a natural extension of closed geodesics.

LEMMA 4. (Bonahon [**1**]) *The set of discrete geodesic currents is dense in* $\mathcal{C}(S)$.

LEMMA 5. (Bonahon [**1**]) *There exists a continuous function* $l : \mathcal{C}(S) \to \mathbb{R}_+$ *such that* $l(m(\alpha)) = l(\alpha)$ *for all closed geodesics* α, *and* $l(km) = kl(m)$ *for all* $k \in \mathbb{R}_+, m \in \mathcal{C}(S)$.

3. *Ergodic theory*

Let S be a closed hyperbolic surface and λ a maximal lamination in S. Let $T_1(S)$ be the unit tangent bundle of S and Ω the standard volume measure on $T_1(S)$. We normalize Ω to obtain the *unit volume measure* V on $T_1(S)$. Then as $\Omega(T_1(S)) = 2\pi \operatorname{Area}(S) = 4\pi^2|\chi(S)|$,

$$V = \frac{\Omega}{4\pi^2|\chi(S)|}.$$

As before, if $v \in T_1(S)$, we let $\alpha_v : \mathbb{R} \to S$ be the geodesic in S parameterized by arc length such that $\alpha_v'(0) = v$. Then $\alpha_v^{-1}(S - \lambda) = \bigcup_i I_i$ with I_i having length l_i.

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We define a length function $L_\lambda : T_1(S) \to [0, \infty]$ where $L_\lambda(v) = l_j$ if $\alpha_v(0) \in I_j$, and otherwise $L_\lambda(v) = 0$. Then $(L_\lambda)_* V \in M(\mathbb{R}_+)$ is defined by $((L_\lambda)_* V)(\phi) = V(\phi \circ L)$.

THEOREM 6. *For almost all $v \in T_1(S)$ (with respect to Ω), $\lim_{t\to\infty} P_{\lambda,t}(v)$ exists and satisfies*

$$\lim_{t\to\infty} P_{\lambda,t}(v) = P_\lambda(v) = (L_\lambda)_* V.$$

Proof. Let $\phi \in C_0(\mathbb{R}_+)$, a continuous function on $\mathbb{R}_+$ with compact support. Then

$$P_{\lambda,t}(v)(\phi) = \frac{1}{t} \sum_{I_i \cap [0,t] \neq \emptyset} l_i \phi(l_i).$$

Let $L_\lambda : T_1(S) \to \mathbb{R}_+$ be the function described above. Then if $\alpha_v(t) \in I_i$, we have $L_\lambda(\alpha_v'(t)) = l_i$. Therefore

$$\begin{aligned} P_{\lambda,t}(v)(\phi) &= \frac{1}{t} \sum_{I_i \cap [0,t] \neq \emptyset} l_i \phi(l_i) \\ &= \frac{1}{t} \sum_{I_i \cap [0,t] \neq \emptyset} \int_{I_i} \phi(L_\lambda(\alpha_v'(t)))\, dt = \frac{1}{t} \int_{J_t} \phi(L_\lambda(\alpha_v'(t)))\, dt \end{aligned}$$

where J_t is the union of the intervals I_i that intersect $[0, t]$. We compare $P_{\lambda,t}(v)(\phi)$ to the time average, i.e.

$$Q_{\lambda,t}(v)(\phi) = \frac{1}{t} \int_0^t \phi(L_\lambda(\alpha_v'(s)))\, ds.$$

If $\alpha_v(0) \notin \lambda$, then there exists an interval $I_{i_0} = (-a_0, b_0)$ with $0 \in I_{i_0}$. If $\alpha_v(0) \in \lambda$, we set $a_0 = 0$. Similarly, if $\alpha_v(t) \notin \lambda$, then there exists an interval $I_{i_t} = (t - a_t, t + b_t)$ with $t \in I_{i_t}$. If $\alpha_v(t) \in \lambda$, we set $b_t = 0$. Then

$$P_{\lambda,t}(v)(\phi) - Q_{\lambda,t}(v)(\phi) = \frac{1}{t}(a_0 \phi(l_{i_0}) + b_t \phi(l_{i_t})).$$

As ϕ has compact support, let $\operatorname{supp}(\phi) \subseteq [x, y]$. Then we have

$$|P_{\lambda,t}(v)(\phi) - Q_{\lambda,t}(v)(\phi)| \leq \frac{(a_0 + y)\|\phi\|_\infty}{t}, \tag{1}$$

where $\|\cdot\|_\infty$ is the sup norm.

The integral $Q_{\lambda,t}(v)(\phi)$ is the time average of the function $\phi \circ L_\lambda$ along the ray $t \mapsto \alpha_v'(t)$ in $T_1(S)$. The function $\phi \circ L_\lambda$ is easily shown to be in $L^1(V)$. As the geodesic flow on $T_1(S)$ is ergodic (see [**3**]), by the Birkhoff ergodic theorem (see [**2**]) we have that for almost all v with respect to the volume measure V,

$$\lim_{t\to\infty} Q_{\lambda,t}(v)(\phi) = \int_{T_1(S)} \phi(L_\lambda(v))\, dV = ((L_\lambda)_* V)(\phi).$$

Therefore, for any $\phi \in C_0(\mathbb{R}_+)$ there exists a set of full measure $A_\phi \subset T_1(S)$ such that

$$\lim_{t\to\infty} Q_{\lambda,t}(v)(\phi) = ((L_\lambda)_* V)(\phi) \quad \text{for all } v \in A_\phi.$$

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By the Weierstrass approximation theorem (see [**5**]), we can choose a countable basis $\{\phi_i\}$ for the sup norm topology on $C_0(\mathbb{R}_+)$. Then $A = \bigcap A_{\phi_i}$ is a set of full measure on which $\lim_{t\to\infty} Q_{\lambda,t}(v)(\phi_i) = ((L_\lambda)_* V)(\phi_i)$ for all i.

Now let $\phi \in C_0(\mathbb{R}_+)$ and $v \in A$. Then there exists a subsequence ϕ_{i_j} converging uniformly to ϕ on $\mathbb{R}_+$. By linearity, we have

$$|Q_{\lambda,t}(v)(\phi) - Q_{\lambda,t}(v)(\phi_{i_j})| \le \|\phi_{i_j} - \phi\|_\infty,$$
$$|((L_\lambda)_* V)(\phi_{i_j}) - ((L_\lambda)_* V)(\phi)| \le \|\phi_{i_j} - \phi\|_\infty.$$

By uniform convergence of ϕ_{i_j}, for any $\epsilon > 0$, there exists an $m > 0$ such that $\|\phi_{i_j} - \phi\|_\infty < \epsilon$ for all $j > m$. We choose a $j > m$, then

$$\begin{aligned}|Q_{\lambda,t}(v)(\phi) - ((L_\lambda)_* V)(\phi)| \le{}& |Q_{\lambda,t}(v)(\phi) - Q_{\lambda,t}(v)(\phi_{i_j})| \\ &+ |Q_{\lambda,t}(v)(\phi_{i_j}) - ((L_\lambda)_* V)(\phi_{i_j})| \\ &+ |((L_\lambda)_* V)(\phi_{i_j}) - ((L_\lambda)_* V)(\phi)|.\end{aligned}$$

Therefore,

$$|Q_{\lambda,t}(v)(\phi) - ((L_\lambda)_* V)(\phi)| \le |Q_{\lambda,t}(v)(\phi_{i_j}) - ((L_\lambda)_* V)(\phi_{i_j})| + 2\epsilon.$$

As $v \in A$, $\lim_{t\to\infty} Q_{\lambda,t}(v)(\phi_{i_j}) = ((L_\lambda)_* V)(\phi_{i_j})$. Thus there exists a $T > 0$ such that

$$|Q_{\lambda,t}(v)(\phi_{i_j}) - ((L_\lambda)_* V)(\phi_{i_j})| \le \epsilon \quad \text{for all } t \ge T.$$

Therefore

$$|Q_{\lambda,t}(v)(\phi) - (L_\lambda)_*(V)(\phi)| \le 3\epsilon \quad \text{for all } t \ge T$$

giving

$$\lim_{t\to\infty} Q_{\lambda,t}(v)(\phi) = ((L_\lambda)_* V)(\phi).$$

By (1) there are constants a_0, y such that

$$|P_{\lambda,t}(v)(\phi) - Q_{\lambda,t}(v)(\phi)| \le \frac{(a_0 + y)\|\phi\|_\infty}{t}.$$

Therefore, we have

$$\lim_{t\to\infty} P_{\lambda,t}(v)(\phi) = \lim_{t\to\infty} Q_{\lambda,t}(v)(\phi) = ((L_\lambda)_* V)(\phi).$$

As ϕ is arbitrary,

$$\lim_{t\to\infty} P_{\lambda,t}(v) = (L_\lambda)_* V \quad \text{for almost every } v \in T_1(S). \qquad \square$$

Let $M(\mathbb{R}_+)$ be the space of measures on the open interval $(0, \infty)$ with the weak* topology. If $f : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous function, then we obtain a map $\hat{f} : M(\mathbb{R}_+) \to M(\mathbb{R}_+)$ given by

$$\hat{f}(m)(\phi) = m(f \cdot \phi) = \int_0^\infty f(x)\phi(x)\, dm$$

for any $\phi \in C_0(\mathbb{R}_+)$ with compact support. On the level of infinitesimals, the map $\hat{f}$ is just multiplication by the function f and thus has inverse $\widehat{(1/f)}$.

Taking $r(x) = 1/x$, we have by definition that

$$\hat{r}(P_{\lambda,t}(v)) = D_{\lambda,t}(v).$$

Therefore we have the following corollary.

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COROLLARY 7. *For almost all $v \in T_1(S)$ (with respect to Ω), $\lim_{t\to\infty} D_{\lambda,t}(v)$ exists and satisfies*

$$\lim_{t\to\infty} D_{\lambda,t}(v) = D_\lambda(v) = \hat{r}((L_\lambda)_* V).$$

From the above corollary, Theorem 1 follows from calculating $(L_\lambda)_* V$. Before we perform the calculation, we consider the simpler case of geodesics and geodesic currents intersecting a single ideal triangle in $\mathbf{H}^2$.

4. *Intersection of a geodesic current with an ideal triangle*

We first consider the length of intersection of a geodesic with a fixed ideal triangle. Let T be an ideal triangle in $\mathbf{H}^2$. We define $L : G(\mathbf{H}^2) \to [0, \infty]$ by $L(g) = \text{Length}(g \cap T)$, with $L(g) = 0$ if $g \cap T = \emptyset$. We observe that if $L(g)$ is non-zero and finite, then L is continuous at g. Thus letting $O_T = L^{-1}((0, \infty))$ we have that L is continuous when restricted to O_T.

If m is a geodesic current, we define a measure $L_* m$ on $(0, \infty)$ as follows. Let $\phi : (0, \infty) \to \mathbb{R}$ be a continuous map with compact support in $(0, \infty)$. We extend ϕ to a continuous map $\overline{\phi} : [0, \infty] \to \mathbb{R}$ by defining $\overline{\phi}(0) = \overline{\phi}(\infty) = 0$ and $\overline{\phi}(x) = \phi(x)$ otherwise. We then define the measure $L_* m$ on $(0, \infty)$ by letting $(L_* m)(\phi) = m(\overline{\phi} \circ L)$. In the following we will suppress the extension notation and identify ϕ with $\overline{\phi}$.

We now describe the length function L in terms of the endpoints of the geodesic. Let T be the ideal triangle in the upper half space $\mathbf{H}^2$ with endpoints $\{0, 1, \infty\}$ in $\overline{\mathbb{R}}$. We denote the intervals $I_1 = (-\infty, 0)$, $I_2 = (0, 1)$ and $I_3 = (1, \infty)$. We consider the nine subsets $I_i \times I_j \subseteq G(\mathbf{H}^2)$. If $g \in I_i \times I_i$ then $g \cap T = \emptyset$ and therefore $L = 0$ on the subsets $I_i \times I_i$. If $g \notin I_i \times I_j$ for any i, j, then g has one endpoint in $\{0, 1, \infty\}$ and therefore has either zero or infinite length. Therefore, the set $O_T = L^{-1}((0, \infty))$ is given by

$$O_T = \bigcup_{\substack{i \neq j \\ i,j}}^{3} I_i \times I_j.$$

We define $L_{ij} : I_i \times I_j \to (0, \infty)$ to be the restriction of L to $I_i \times I_j$.

LEMMA 8. *The length function $L_{12} : I_1 \times I_2 \to (0, \infty)$ is given by the formula*

$$L_{12}(u, v) = \frac{1}{2} \ln\left(\frac{1-u}{1-v}\right).$$

Proof. We will first find L_{13}. Let $g = (u, v) \in I_1 \times I_3$ be a geodesic. We drop a perpendicular P from ∞ to g, with vertex p. We will first assume that $p \in T$. Then P and g decompose T into two triangles and a quadrilateral. We label the triangles T_0 and T_1 so that the vertical line $x = i$ is a side of T_i. Both triangles have a right angle and an ideal vertex. We label the other angle of T_i by θ_i and the length of the only finite length side of T_i by l_i (see Figure 2). Then $L_{13}(u, v) = l_0 + l_1$. By hyperbolic trigonometry we have $\tanh l_i = \cos \theta_i$ (see [**6**]). We denote the Euclidean center and radius of g by c and r, respectively. Then $u = c - r$, $v = c + r$. From Euclidean trigonometry we have

$$\tanh l_0 = \cos \theta_0 = \frac{c}{r}, \quad \tanh l_1 = \cos \theta_1 = \frac{1-c}{r}.$$

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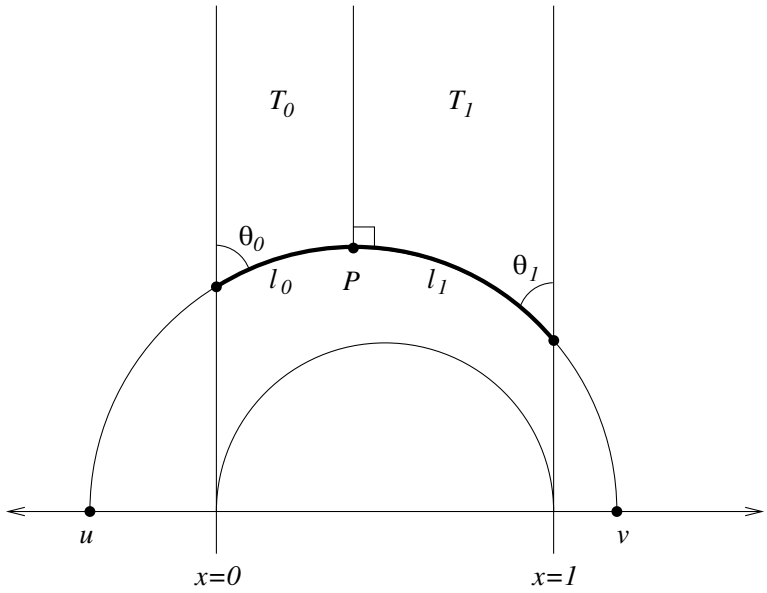


FIGURE 2. The intersection of a geodesic g with the standard ideal triangle T in $\mathbf{H}^2$.

As $\tanh^{-1} x = \frac{1}{2}\ln((1+x)/(1-x))$, we have

$$l_0 = \frac{1}{2}\ln\left(\frac{1+(c/r)}{1-(c/r)}\right) = \frac{1}{2}\ln\left(-\frac{v}{u}\right), \quad l_1 = \frac{1}{2}\ln\left(\frac{1+(1-c)/r}{1-(1-c)/r}\right) = \frac{1}{2}\ln\left(\frac{-u+1}{v-1}\right).$$

Combining the previous equations we obtain

$$L_{13}(u,v) = \frac{1}{2}\ln\left(\frac{v(u-1)}{u(v-1)}\right).$$

The case when $p \notin T$ gives the same formula.

The isometry $f(z) = 1/\overline{z}$ maps $I_1 \times I_3$ to $I_1 \times I_2$. Therefore on $I_1 \times I_2$ we have

$$L_{12}(u,v) = L_{13}(1/u, 1/v) = \frac{1}{2}\ln\left(\frac{1-u}{1-v}\right). \qquad \square$$

Recall that μ is the Liouville measure on $G(\mathbf{H}^2)$. We now use Lemma 8 to calculate its pushforward by L.

THEOREM 9. *The measure $M_T = L_*\mu$ is given differentially by*

$$dM_T = \frac{6x\,dx}{\sinh^2 x}.$$

Proof. As μ is invariant under Möbius transformations, it suffices to consider the standard ideal triangle T with endpoints $\{0, 1, \infty\}$ in $\overline{\mathbb{R}}$. As before we note that L is supported in $O_T = \bigcup_{i\neq j} I_i \times I_j$ where $I_1 = (-\infty, 0)$, $I_2 = (0, 1)$ and $I_3 = (1, \infty)$.

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Let $\phi : (0, \infty) \to \mathbb{R}$ be a continuous function with compact support on $(0, \infty)$. Then by definition of the pushforward, we have

$$(L_*\mu)(\phi) = \mu(\phi \circ L) = \sum_{\substack{i \neq j \\ i,j=1}}^{3} \left(\int_{I_i \times I_j} \phi \circ L \, d\mu \right) = \sum_{\substack{i \neq j \\ i,j=1}}^{3} M_{ij}(\phi).$$

Thus $L_*\mu$ on $(0, \infty)$ is the sum of the six measures M_{ij}, $i \neq j$.

Let $i \neq j$ and $k \neq l$, then there exists a Möbius transformation p sending T to itself and mapping $I_i \times I_j$ homeomorphically to $I_k \times I_l$. By invariance of μ, we have $\mu = p_*\mu$. The map p is a hyperbolic isometry and sends T to itself, and thus $L \circ p = L$. Therefore, by a change of variable in the integral M_{ij} we have $M_{kl} = M_{ij}$. Thus

$$L_*\mu = 6\, M_{12}.$$

By Lemma 8 we have that for $(u, v) \in I_1 \times I_2$

$$L(u, v) = \frac{1}{2} \ln\left(\frac{1-u}{1-v} \right).$$

Letting $x = L(u, v)$, we solve for u and v to obtain $u = 1 - (1 - v)e^{2x}$ and $v = 1 - (1 - u)e^{-2x}$. The measure μ is given differentially by

$$d\mu = \frac{du\, dv}{|u - v|^2}$$

Therefore,

$$M_{12}(\phi) = \int_{I_1} \int_{I_2} \frac{\phi(L(u, v))\, du\, dv}{|u - v|^2}.$$

We make the change of variable $x = L(u, v)$, $v = v$. Then $u = f(v, x) = 1 - (1 - v)e^{2x}$ and $v = g(v, x) = v$. Thus the Jacobian is

$$J(v, x) = \begin{vmatrix} \dfrac{\partial f}{\partial x} & \dfrac{\partial g}{\partial x} \\ \dfrac{\partial f}{\partial v} & \dfrac{\partial g}{\partial v} \end{vmatrix} = \begin{vmatrix} -2(1 - v)e^{2x} & 0 \\ e^{2x} & 1 \end{vmatrix} = 2(1 - v)e^{2x}.$$

Then we have

$$d\mu = \frac{J(v, x)\, dv\, dx}{|u - v|^2} = \frac{2(1 - v)e^{2x}\, dv\, dx}{|1 - (1 - v)e^{2x} - v|^2} = \frac{dv\, dx}{2(1 - v)\sinh^2 x}.$$

Let $F(u, v) = (v, x)$. Then we have that

$$M_{12}(\phi) = \int_{F(I_1 \times I_2)} \frac{\phi(x)\, dv\, dx}{2(1 - v)\sinh^2 x}.$$

We now need to find $F(I_1 \times I_2)$. We fix a $v \in (0, 1)$. Then from the formula for L in terms of u, v it follows that L has range

$$\frac{1}{2} \ln\left(\frac{1}{1 - v} \right) < L(u, v) < \infty.$$

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Therefore,

$$F(I_1 \times I_2) = \left\{ (v, x) \;\middle|\; 0 < v < 1, \frac{1}{2}\ln\left(\frac{1}{1-v}\right) < x < \infty \right\}$$

rewriting we obtain

$$F(I_1 \times I_2) = \left\{ (v, x) \mid 0 < v < 1 - e^{-2x}, 0 < x < \infty \right\}.$$

Therefore,

$$M_{12}(\phi) = \int_0^\infty \left(\int_0^{1-e^{-2x}} \frac{dv}{2(1-v)} \right) \frac{\phi(x)\,dx}{\sinh^2 x}.$$

Substituting $t = 1 - v$ we obtain

$$\int_0^{1-e^{-2x}} \frac{dv}{2(1-v)} = \frac{1}{2}\int_{e^{-2x}}^1 \frac{dt}{t} = \frac{1}{2}(\ln(1) - \ln(e^{-2x})) = x.$$

Therefore we have

$$M_{12}(\phi) = \int_0^\infty \frac{x\phi(x)\,dx}{\sinh^2 x},$$

giving

$$dM_T = \frac{6x\,dx}{\sinh^2 x}.$$ □

Related to the measure M_T is a probability measure P_T on $(0, \infty)$ given by the pushforward of the unit volume measure on the unit tangent bundle $T_1(T)$ to an ideal triangle T. If $p \in T_1(T)$ then associated to p we let $g(p)$ be the oriented geodesic with tangent vector p. Then $L(g(p))$ is the length of the segment through p in T. The measure P_T is defined to be the probability distribution of lengths for a randomly chosen tangent vector in T.

If we let Ω be the standard volume measure on $T_1(T)$, then the probability measure P_T is given by

$$P_T = \frac{(L \circ g)_* \Omega}{\Omega(T_1(T))}.$$

THEOREM 10. *The measure P_T is given differentially by*

$$dP_T = \frac{6x^2 dx}{\pi^2 \sinh^2 x}.$$

Proof. The map $g : T_1(\mathbf{H}^2) \to G(\mathbf{H}^2)$ is a trivial fiber bundle. Thus we have $T_1(\mathbf{H}^2) = G(\mathbf{H}^2) \times \mathbb{R}$. In terms of this representation, the volume form Ω is given by $d\Omega = 2d\mu \times dt$ where μ is the Liouville geodesic current and dt is hyperbolic length along the fiber (see [**4**] for details). Thus

$$(L \circ g)_* \Omega(\phi) = \int_{T_1(T)} \phi(L(g(p)))\,d\Omega.$$

As $T_1(\mathbf{H}^2) = G(\mathbf{H}^2) \times \mathbb{R}$, we have p represented by (x, y, t) where (x, y) are the endpoints of $g(p)$. Furthermore, $d\mu = dx\,dy/|x - y|^2$. Also, for a given geodesic (x, y),

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the parameter t takes values from $c(x, y)$ to $c(x, y) + L(x, y)$, for some constant $c(x, y)$. Thus

$$(L \circ g)_* \Omega(\phi) = \sum_{\substack{i,j=1 \\ i \neq j}}^{3} \int_{I_i \times I_j} \frac{2\,dx\,dy}{|x-y|^2} \int_{c(x,y)}^{c(x,y)+L(x,y)} \phi(L(x, y))\,dt.$$

Therefore, integrating over t

$$(L \circ g)_* \Omega(\phi) = \sum_{\substack{i,j=1 \\ i \neq j}}^{3} \int_{I_i \times I_j} \frac{2\phi(L(x, y))L(x, y)\,dx\,dy}{|x-y|^2}.$$

Using Theorem 9 and performing the change of variables $l = L(x, y)$, $y = y$, we obtain

$$(L \circ g)_* \Omega(\phi) = \sum_{\substack{i,j=1 \\ i \neq j}}^{3} \int_0^\infty \frac{2l^2\phi(l)}{\sinh^2 l}\,dl = \int_0^\infty \frac{12l^2\phi(l)}{\sinh^2 l}\,dl.$$

Therefore, normalizing by $\Omega(T_1(T)) = 2\pi^2$, we obtain

$$dP_T(x) = \frac{6x^2}{\pi^2 \sinh^2 x}\,dx. \qquad \square$$

An immediate corollary relates the expected value of P_T to the Riemann zeta function.

COROLLARY 11. *The expected value $E(P_T)$ of P_T is given by*

$$E(P_T) = \int_0^\infty \frac{6x^3}{\pi^2 \sinh^2 x}\,dx = \frac{9}{\pi^2}\zeta(3) = 1.09614\ldots$$

To put this expected value in perspective, we observe that it is quite close to the diameter of the hyperbolic disk inscribed in T, which is $\log(3) = 1.09861\ldots$.

5. *Calculating* $(L_\lambda)_* V$

Let $S = \mathbf{H}^2/\Gamma$ be a closed hyperbolic surface and let λ be a maximal geodesic lamination on S. Then $S - \lambda = \bigcup_{i=1}^k T_i$ where T_i are disjoint ideal triangles. Then as the area of S is $2\pi|\chi(S)|$, we have $k = 2|\chi(S)|$.

We let T_i' be an ideal triangle in $\mathbf{H}^2$ which is a lift of T_i. Then as before, we define $L_i'(g) = \text{Length}(g \cap T_i')$. If T_i'' is another lift of T_i then there exists a Möbius transformation $p \in \Gamma$ with $p(T_i') = T_i''$. Let $L_i''(g) = \text{Length}(g \cap T_i'')$. Then $L_i'(g) = \text{Length}(g \cap p^{-1}(T_i'')) = \text{Length}(p(g) \cap T_i'') = L_i''(p(g))$. Therefore, $L_i' = L_i'' \circ p$. Thus if $m \in \mathcal{C}(S)$ is a geodesic current, then $(L_i')_* m = (L_i'' \circ p)_* m = (L_i'')_*(p_* m)$. As m is invariant under Γ, $p_* m = m$. Therefore, $(L_i')_* m = (L_i'')_* m$ and therefore can define $(L_i)_* m = (L_i')_* m$.

Definition. If m is a geodesic current of length $l(m)$ then the *distribution of lengths of intersections of m and λ*, denoted $D_\lambda(m)$, is the measure

$$D_\lambda(m) = \frac{1}{l(m)} \sum_{i=1}^{k} (L_i)_* m.$$

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From Theorem 9 we have that on $(0, \infty)$

$$d(D_\lambda(\mu)) = \frac{12|\chi(S)|x\,dx}{l(\mu)\sinh^2 x}.$$

Then as $l(\mu) = 2\pi^2|\chi(S)|$ (see Bonahon [**1**]), we have

$$d(D_\lambda(\mu)) = \frac{6x\,dx}{\pi^2\sinh^2 x} = dM.$$

Thus $D_\lambda(\mu) = M$ and we call M the *distribution of lengths of intersections* for a maximal lamination. This proves (1) of Theorem 3.

THEOREM 12. *For any maximal geodesic lamination λ, the pushforward of the normalized volume measure V on $T_1(S)$ by L_λ is given by*

$$(L_\lambda)_* V = P_T.$$

Proof. As V is the unit volume measure obtained by normalizing Ω, and $\Omega(T_1(S)) = 4\pi^2|\chi(S)|$, we have

$$((L_\lambda)_* V)(\phi) = \frac{1}{4\pi^2|\chi(S)|}\int_{T_1(S)} \phi(L_\lambda(v))\,d\Omega.$$

As λ is a maximal lamination, we have $S - \lambda = \bigcup_{i=1}^k T_i$ where T_i are disjoint ideal triangles. As $\lambda \subset S$ is measure zero, we have

$$((L_\lambda)_* V)(\phi) = \frac{1}{4\pi^2|\chi(S)|}\sum_{i=1}^k \int_{T_1(T_i)} \phi(L_\lambda(v))\,d\Omega.$$

Let $q : T_1(\mathbf{H}^2) \to T_1(S)$ be the standard covering map. Let T_i' be a lift of T_i to $\mathbf{H}^2$. Then we lift the integrals over T_i using q to integrals over T_i', also using Ω to denote the volume measure on $T_1(\mathbf{H}^2)$, and obtain

$$\int_{T_1(T_i)} \phi(L_\lambda(v))\,d\Omega = \int_{T_1(T_i')} \phi(L_\lambda(q(v)))\,d\Omega.$$

Let $g : T_1(\mathbf{H}^2) \to G(\mathbf{H}^2)$ be the fiber bundle such that v is tangent to geodesic $g(v)$. As above, let $L_i : G(\mathbf{H}^2) \to [0, \infty]$ where $L_i(g) = \text{Length}(g \cap T_i')$. Then for $v \in T_1(T_i')$, $L_\lambda(q(v)) = L_i(g(v))$ and

$$\int_{T_1(T_i')} \phi(L_\lambda(q(v)))\,d\Omega = \int_{T_1(T_i')} \phi(L_i(g(v)))\,d\Omega.$$

By Theorem 10,

$$\int_{T_1(T_i')} \phi(L_i(g(v)))\,d\Omega = 2\pi^2 P_T(\phi).$$

Therefore, as there are $2|\chi(S)|$ triangles T_i, we combine to obtain

$$((L_\lambda)_* V)(\phi) = \frac{1}{4\pi^2|\chi(S)|}\sum_{i=1}^k (2\pi^2 P_T(\phi)) = P_T(\phi).$$

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Finally we have

$$d((L_\lambda)_* V) = dP_T = \frac{6x^2\,dx}{\pi^2 \sinh^2 x}. \qquad \square$$

By application of Corollary 7 we obtain the following.

COROLLARY 13. *For almost all $v \in T_1(S)$ (with respect to Ω), we have $D_\lambda(v) = M$ and thus*

$$d(D_\lambda(v)) = \frac{6x\,dx}{\pi^2 \sinh^2 x}.$$

The above corollary completes the proof of Theorem 1.

6. *Continuity of D_λ at μ*

In this section we prove (2) of Theorem 3. We recall the definition of $D_\lambda(m)$:

$$D_\lambda(m) = \frac{1}{l(m)} \sum_{i=1}^{k} (L_i)_* m.$$

We first prove a lemma about weak* convergence.

LEMMA 14. *Let $X \subseteq \mathbb{R}^n$ be open and $K \subseteq X$ a compact domain with piecewise smooth boundary ∂K. If $m \in M(X)$ we define $\overline{m} \in M(K)$ to be the restriction of m to K. Let $m_i \to m$ in $M(X)$ and $\phi : K \to \mathbb{R}$ continuous on K. Then*

$$\overline{\lim_{i\to\infty}} |\overline{m}_i(\phi) - \overline{m}(\phi)| \le \int_{\partial K} |\phi|\,dm.$$

Proof. Let $\phi : K \to \mathbb{R}$ be a continuous function. We let $\phi = \phi^+ - \phi^-$ where ϕ^+ and ϕ^- are both continuous non-negative functions on K. If the lemma is true for non-negative functions then we have

$$\begin{aligned}
\overline{\lim_{i\to\infty}} |\overline{m}_i(\phi) - \overline{m}(\phi)| &= \overline{\lim_{i\to\infty}} |\overline{m}_i(\phi^+ - \phi^-) - \overline{m}(\phi^+ - \phi^-)| \\
&\le \overline{\lim_{i\to\infty}} |\overline{m}_i(\phi^+) - \overline{m}(\phi^+)| + \overline{\lim_{i\to\infty}} |\overline{m}_i(\phi^-) - \overline{m}(\phi^-)| \\
&\le \int_{\partial K} \phi^+\,dm + \int_{\partial K} \phi^-\,dm = \int_{\partial K} |\phi|\,dm.
\end{aligned}$$

We let $\phi : K \to \mathbb{R}$ be a continuous non-negative function. Given an $\epsilon > 0$, we let $N_\epsilon(\partial K)$ be the ϵ neighborhood of the boundary of K. For ϵ small, we obtain non-empty compact domains $K_{-\epsilon}, K_{+\epsilon}$ given by $K_{-\epsilon} = K - N_\epsilon(\partial K)$ and $K_{+\epsilon} = K \cup \overline{N_\epsilon(\partial K)}$. Then $K_{-\epsilon} \subset K \subset K_{+\epsilon}$. We define continuous non-negative functions with compact support $\phi_{-\epsilon} : X \to \mathbb{R}$, $\phi_{+\epsilon} : X \to \mathbb{R}$ which will approximate ϕ from above and below.

The function $\phi_{-\epsilon}$ is chosen to be pointwise monotonically increasing in ϵ and satisfy

$$\operatorname{supp}(\phi_{-\epsilon}) \subseteq K, \quad \phi_{-\epsilon}(x) = \phi(x) \text{ for } x \in K_{-\epsilon}, \quad \phi_{-\epsilon}(x) \le \phi(x) \text{ for } x \in K.$$

The function $\phi_{+\epsilon}$ is similarly chosen to be pointwise monotonically decreasing in ϵ and satisfy

$$\operatorname{supp}(\phi_{+\epsilon}) \subseteq K_{+\epsilon}, \quad \phi_{+\epsilon}(x) = \phi(x) \text{ for } x \in K, \quad \|\phi_{+\epsilon}\|_X \le \|\phi\|_K.$$

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Then as the measures are positive, we have

$$m_i(\phi_{-\epsilon}) - m(\phi_{+\epsilon}) \leq \overline{m}_i(\phi) - \overline{m}(\phi) \leq m_i(\phi_{+\epsilon}) - m(\phi_{-\epsilon}).$$

As $m_i \to m$ we take limits to obtain

$$m(\phi_{-\epsilon}) - m(\phi_{+\epsilon}) \leq \underline{\lim_{i\to\infty}}(\overline{m}_i(\phi) - \overline{m}(\phi)) \leq \overline{\lim_{i\to\infty}}(\overline{m}_i(\phi) - \overline{m}(\phi)) \leq m(\phi_{+\epsilon}) - m(\phi_{-\epsilon}).$$

Thus letting $\psi_\epsilon = \phi_{+\epsilon} - \phi_{-\epsilon}$, we have

$$\overline{\lim_{i\to\infty}}|\overline{m}_i(\phi) - \overline{m}(\phi)| \leq |m(\psi_\epsilon)|.$$

The function ψ_ϵ is non-negative, monotonically decreasing in ϵ, and has $\operatorname{supp}(\psi_\epsilon) \subseteq \overline{N_\epsilon(\partial K)}$. Furthermore, if $x \notin \partial K$, then $\lim_{\epsilon\to 0}\psi_\epsilon(x) = 0$ and if $x \in \partial K$ then $\lim_{\epsilon\to 0}\psi_\epsilon(x) = \phi(x)$. Therefore, by Lebesgue dominated convergence (see [**5**]) we have

$$\lim_{\epsilon\to 0} m(\psi_\epsilon) = \int_{\partial K} \phi\, dm.$$

Thus

$$\overline{\lim_{i\to\infty}}|\overline{m}_i(\phi) - \overline{m}(\phi)| \leq \int_{\partial K} \phi\, dm. \qquad \square$$

LEMMA 15. *Let T be the lift of an ideal triangle embedded in S and μ be the Liouville geodesic current of S. If m_i is a sequence of discrete geodesic currents such that $m_i \to \mu$, then $L_* m_i \to L_*\mu$.*

Proof. We need to show that if $\phi : (0,\infty) \to \mathbb{R}$ is continuous with compact support then $L_* m_i(\phi) \to L_*\mu(\phi)$, or equivalently that $m_i(\phi \circ L) \to \mu(\phi \circ L)$. This would follow directly from the definition of the weak* topology if $\phi \circ L$ were continuous and compactly supported; however, this is not the case. To establish the convergence we will instead describe a continuous function $\hat{L}$ such that $\phi \circ \hat{L}$ has compact support and $m(\phi \circ L) = m(\phi \circ \hat{L})$. Then continuity follows from the definition of the weak* topology.

Let $\operatorname{supp}(\phi) = [a,b] \subseteq (0,\infty)$. As before we need only integrate $\phi \circ L$ over $O_T = L^{-1}((0,\infty))$. We move T by a Möbius transformation to have endpoints $\{0, 1, \infty\}$ in the upper half space model of $\mathbf{H}^2$. Then as in Lemma 8, the set O_T is composed of six open sets $I_i \times I_j, i, j = 1, \ldots, 3, i \neq j$.

Let $A = L^{-1}([a,b])$ be the set of all geodesics in O_T with length between a and b. We let $A_{ij} = A \cap (I_i \times I_j)$ and $L_{ij} : I_i \times I_j \to (0,\infty)$ be the restriction of L to $I_i \times I_j$. Then for $m \in \mathcal{C}(S)$,

$$L_* m(\phi) = m(\phi \circ L) = \sum_{\substack{i\neq j \\ i,i=1}}^{3} \left(\int_{I_i \times I_j} \phi \circ L_{ij}\, dm \right) = \sum_{\substack{i\neq j \\ i,i=1}}^{3} m|_{I_i\times I_j}(\phi \circ L_{ij}).$$

By the formula for L_{12} we have if $(u, v) \in A_{12}$ then

$$a \leq \frac{1}{2}\ln\left(\frac{1-u}{1-v}\right) \leq b.$$

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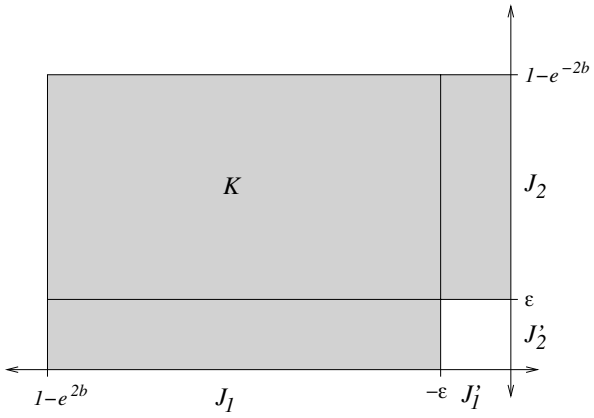


FIGURE 3. The region K as a subset of $\mathbb{R}^2$.

By solving this inequality we obtain

$$(1-v)e^{2a} \leq 1-u \leq (1-v)e^{2b}.$$

As $v > 0$, we obtain $1-u \leq e^{2b}$ giving $1-e^{2b} \leq u$. Similarly, as $u < 0$ we obtain $1 \leq (1-v)e^{2b}$ giving $v \leq 1-e^{-2b}$. Thus we have $1-e^{2b} \leq u$ and $v \leq 1-e^{-2b}$. Thus $A_{12} \subseteq [1-e^{2b}, 0) \times (0, 1-e^{-2b}]$.

We choose $\epsilon > 0$ so that if both u and v are within ϵ of 0, then $L_{12}(u, v) < a$. Using the formula for L_{12}, we can choose $\epsilon = (e^{2a}-1)/(e^{2a}+1)$.

Thus we consider the intervals

$$J_1 = [1-e^{2b}, -\epsilon], \quad J_1' = [-\epsilon, 0] \quad \text{and} \quad J_2 = [\epsilon, 1-e^{-2b}], \quad J_2' = [0, \epsilon].$$

Then the set $K = (J_1 \times J_2) \cup (J_1 \times J_2') \cup (J_1' \times J_2)$ is a compact subset of $G(\mathbf{H}^2)$ (see Figure 3). Also, we have that $\operatorname{supp}(\phi \circ L_{12}) \subseteq K$.

The function $\phi \circ L_{12}$ is not continuous on K, being discontinuous at the points $D = (J_1 \times \{0\}) \cup (\{0\} \times J_2) \subseteq \partial K$. We define the function $\hat{L}_{12} : K \to \mathbb{R}$ by

$$\hat{L}_{12}(u, v) = \frac{1}{2} \ln\left(\frac{1-u}{1-v}\right).$$

Note that this is the same formula as we obtained for L_{12} in Lemma 8; however, here we allow any $(u, v) \in K$ whereas L_{12} is supported on $I_1 \times I_2$. The key difference is that K includes some geodesics (i.e. D) sharing one ideal endpoint with T and L_{12} extends continuously to these points, giving $\hat{L}_{12}$. Thus $\hat{L}_{12}$ is continuous on K, with $\hat{L}_{12} = L$ on $K - D$, and we have

$$|m|_K(\phi \circ \hat{L}_{12}) - m|_{I_1 \times I_2}(\phi \circ L_{12})| = \left| \int_D \phi \circ \hat{L}_{12} \, dm \right| \leq m(D)\|\phi\|.$$

If $m = \mu$ then $m(D) = 0$ by the definition of μ. As T is a lift of an embedded ideal triangle in S, none of the vertices of T can be an endpoint of a closed geodesic on S. Therefore, for m discrete, $m(D) = 0$. Thus for $m = \mu$ or m_i,

$$m|_{I_1 \times I_2}(\phi \circ L_{12}) = m|_K(\phi \circ \hat{L}_{12}).$$

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As $m_i \to \mu$, K is a compact domain with piecewise smooth boundary, and, $\phi \circ \hat{L}_{12}$ is continuous, we have by Lemma 14

$$\overline{\lim_{i\to\infty}} |m_i|_K(\phi \circ \hat{L}_{12}) - \mu|_K(\phi \circ \hat{L}_{12})| \leq \int_{\partial K} |\phi \circ \hat{L}_{12}|\, d\mu.$$

Therefore,

$$\overline{\lim_{i\to\infty}} |m_i|_{I_1\times I_2}(\phi \circ L_{12}) - \mu|_{I_1\times I_2}(\phi \circ L_{12})| \leq \int_{\partial K} |\phi \circ \hat{L}_{12}|\, d\mu.$$

We have $D \subseteq \partial K$ and by our choice of K we have $(\phi \circ \hat{L}_{12})(x) = 0$ for $x \in \partial K - D$. Therefore,

$$\overline{\lim_{i\to\infty}} |m_i|_{I_1\times I_2}(\phi \circ L_{12}) - \mu|_{I_1\times I_2}(\phi \circ L_{12})| \leq \int_{D} |\phi \circ \hat{L}_{12}|\, d\mu \leq \mu(D)\|\phi\|.$$

Therefore, as $\mu(D) = 0$ we have

$$\lim_{i\to\infty} |m_i|_{I_1\times I_2}(\phi \circ L_{12}) - \mu|_{I_1\times I_2}(\phi \circ L_{12})| = 0.$$

Combining the contributions from each of the $I_i \times I_j$ we obtain $m_i(\phi \circ L) \to \mu(\phi \circ L)$. □

We now use Lemma 15 to prove the continuity property of D_λ described in (2) of Theorem 3.

COROLLARY 16. *Let μ be the Liouville geodesic current of S. If m_i is a sequence of discrete geodesic currents such that $m_i \to \mu$, then $D_\lambda(m_i) \to D_\lambda(\mu)$.*

Proof. By definition

$$D_\lambda(m_i) = \frac{1}{l(m_i)} \sum_{j=1}^{k} (L_j)_*(m_i).$$

By Lemma 15, for each j we have $\lim_{i\to\infty}(L_j)_*(m_i) = (L_j)_*(\mu)$. Also, by continuity of the length function l, we have $\lim_{i\to\infty} l(m_i) = l(\mu)$ and

$$\lim_{i\to\infty} D_\lambda(m_i) = \lim_{i\to\infty} \frac{1}{l(m_i)} \sum_{j=1}^{k} (L_j)_*(m_i) = \frac{1}{l(\mu)} \sum_{j=1}^{k} (L_j)_*(\mu) = D_\lambda(\mu). \qquad \square$$

We now apply Theorem 3 to Bonahon's construction of sequences of discrete geodesic currents m_i such that $m_i \to \mu$. Recall that for any $v \in T_1(S)$ we defined the geodesic α_v by the map $\alpha_v : [0, \infty) \to S$ where $\alpha'(0) = v$ and α is parameterized by arc length. For any $L > 0$, we form a closed path by joining the endpoints of $\alpha_v([0, L])$ by a shortest arc (not necessarily unique). If this curve is homotopic to a geodesic we call this closed geodesic α_v^L.

COROLLARY 17. *For almost every $v \in T_1(S)$ with respect to the volume measure Ω, there exists a sequence $\{L_n\}$ monotonically increasing to infinity, such that the geodesics $\beta_n = \alpha_v^{L_n}$ satisfy*

$$\lim_{n\to\infty} D_\lambda(\beta_n) = D_\lambda(\mu) = M.$$

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Proof. By Bonahon [**1**], for almost every $v \in T_1(S)$ with respect to the volume measure Ω, the geodesics α_v^L can be used to approximate the Liouville geodesic current μ. Specifically, for almost every $v \in T_1(S)$ there exists a sequence $\{L_n\}$ monotonically increasing to infinity, such that the geodesics $\beta_n = \alpha_v^{L_n}$ exist and satisfy

$$\mu = (2\pi^2|\chi(S)|) \lim_{n\to\infty} \frac{m(\beta_n)}{l(\beta_n)}.$$

Let $m_i = (2\pi^2|\chi(S)|)m(\beta_i)/l(\beta_i)$. By definition of D_λ for geodesic currents, $D_\lambda(\beta_i) = D_\lambda(m_i)$. Therefore, by applying Corollary 16 to the sequence $m_i \to \mu$ we have

$$\lim_{i\to\infty} D_\lambda(\beta_i) = \lim_{i\to\infty} D_\lambda(m_i) = D_\lambda(\mu) = M. \qquad \square$$

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