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Extension of Hölder's theorem in $\mathrm{Diff}_+^{1+\epsilon}(I)$

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Abstract. We prove that if Γ is a subgroup of $\mathrm{Diff}_+^{1+\epsilon}(I)$ and N is a natural number such that every non-identity element of Γ has at most N fixed points, then Γ is solvable. If in addition Γ is a subgroup of $\mathrm{Diff}_+^2(I)$, then we can claim that Γ is meta-abelian.

1. *Introduction*

It is a classical result (essentially due to Hölder, cf. [**N**]) that if Γ is a subgroup of $\mathrm{Homeo}_+(\mathbb{R})$ such that every non-trivial element acts freely, then Γ is abelian. A natural question to ask is what if every non-trivial element has at most N fixed points, where N is a fixed natural number. In the case of $N = 1$, we do have a complete answer to this question: it has been proved by Solodov (not published), Barbot [**B**], and Kovacevic [**K**] that in this case the group is meta-abelian; in fact, it is isomorphic to a subgroup of the affine group $\mathrm{Aff}(\mathbb{R})$ (see [**FF**] for the history of this result, where yet another nice proof is presented).

In this paper, we answer this question for an arbitrary N assuming some regularity on the action of the group.

Our main result is the following theorem.

THEOREM 1.1. (Main theorem) *Let* $\epsilon \in (0, 1)$ *and* Γ *be a subgroup of* $\mathrm{Diff}_+^{1+\epsilon}(I)$ *such that every non-trivial element of* Γ *has at most N fixed points. Then* Γ *is solvable.*

Assuming a higher regularity on the action, we obtain a stronger result.

THEOREM 1.2. *Let* Γ *be a subgroup of* $\mathrm{Diff}_+^2(I)$ *such that every non-trivial element of* Γ *has at most N fixed points. Then* Γ *is meta-abelian.*

An important tool in obtaining these results is provided by Theorems B and C from [**A**]. Theorem B (Theorem C) states that a non-solvable (non-meta-abelian) subgroup of $\mathrm{Diff}_+^{1+\epsilon}(I)$ (of $\mathrm{Diff}_+^2(I)$) is non-discrete in the C_0-metric. Existence of C_0-small elements in a group provides effective tools in tackling the problem. Such tools are absent for less regular actions and, for the group $\mathrm{Homeo}_+(I)$ (even for $\mathrm{Diff}_+(I)$), the problem of

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characterizing subgroups where every non-identity element has at most $N \geq 2$ fixed points still remains open.

Basic notation. Throughout this paper, G will denote the group $\mathrm{Diff}_+^{1+\epsilon}(I)$, where $\epsilon \in (0, 1)$. We let $\Gamma \leq G$. For every $g \in \Gamma$, $\mathrm{Fix}(g)$ will denote the set of fixed points of g in $(0, 1)$. A fixed point $x_0 \in \mathrm{Fix}(g)$ is called *tangential* if the function $h(x) = g(x) - x$ does not change its sign near x_0. For $f, g \in G$, we will say that (the graphs of) f and g have *a crossing* at $x_0 \in (0, 1)$ if $f(x_0) = g(x_0)$; this crossing will be called tangential if $f(x) - g(x)$ does not change its sign near x_0. We will write $F(\Gamma) = \sup_{g \in \Gamma \setminus \{1\}} |\mathrm{Fix}(g)|$ (so $F(\Gamma)$ is 0, a positive integer, or infinity). For $f \in G$, we will write

$$d(f) = \sup_{x \in [0,1]} |f(x) - x| \quad \text{and} \quad s(f) = \inf_{x, y \in \mathrm{Fix}(f) \cup \{0,1\}, x \neq y} |x - y|.$$

If $F(\Gamma) < \infty$, then we can introduce the following natural bi-order in Γ: for $f, g \in \Gamma$, we write $g < f$ if $g(x) < f(x)$ near zero. If f is a positive element (with respect to the order), then we will also write $g \ll f$ if $g^n < f$ for every integer n; we will say that g is *infinitesimal* with respect to f. For $f \in \Gamma$, we write $\Gamma_f = \{\gamma \in \Gamma : \gamma \ll f\}$ (so Γ_f consists of diffeomorphisms which are infinitesimal with respect to f). Notice that if Γ is finitely generated with a fixed finite symmetric generating set, and f is the biggest generator, then Γ_f is a normal subgroup of Γ; moreover, Γ / Γ_f is archimedean and hence abelian, and thus we also see that $[\Gamma, \Gamma] \leq \Gamma_f$.

For the rest of the paper, we may and will assume that the action of Γ is irreducible, i.e. Γ has no global fixed point in $(0, 1)$. All the used generating sets of groups will be assumed to be symmetric. Let us also clarify that we define a meta-abelian group as a solvable group of derived length at most two (so abelian groups are special cases of meta-abelian groups).

2. *Preliminary results*

First, we will discuss the case of $N = 1$. Let us first show a quick proof of the fact that $N = 1 \Rightarrow$ meta-abelian.

PROPOSITION 2.1. *If $F(\Gamma) = 1$, then Γ is meta-abelian.*

Proof. (Following Farb and Franks [**FF**]) If all finitely generated subgroups of a group are meta-abelian, then the group is meta-abelian. Hence, we may assume that Γ is finitely generated with a fixed finite generating set.

Let f be the biggest generator of Γ. Let also $h \in \Gamma_f \setminus \{1\}$ be such that h has at least one fixed point (if such h does not exist, then Γ_f is abelian; therefore, Γ is meta-abelian) and $h(x) > x$ near zero. We may also assume that Γ_f has no global fixed point. (If Γ_f has a global fixed point, then it is abelian and hence Γ is meta-abelian.)

If f has no fixed point, then, for sufficiently big n, fh^{-n} will have at least two fixed points, hence giving a contradiction.

Assume now that f has one fixed point. Let a be the fixed point of h. By conjugating f by the element of Γ_f if necessary, we may assume that the fixed point of f is bigger than a. Then, again, for sufficiently big n, fh^{-n} will have at least two fixed points. □

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Notice that the above proof works also in $\mathrm{Homeo}_+(I)$.

We now want to discuss the general case. We will make use of the following claim.

PROPOSITION 2.2. *Let* Γ *be non-solvable,* $F(\Gamma) < \infty$, $\epsilon > 0$, *and* $x_0 \in (0, 1)$. *Then there exist* $\gamma_1, \gamma_2 \in [\Gamma, \Gamma]$ *such that* $0 < \gamma_1(x_0) - x_0 < \epsilon$ *and* $0 < x_0 - \gamma_2(x_0) < \epsilon$.

Using this proposition, we immediately obtain the following lemma.

LEMMA 2.3. *Let* Γ *be non-solvable,* $F(\Gamma) < \infty$, $0 < a < b < 1$, *and* $x_0 \in (0, 1)$. *Then there exists* $\gamma \in [\Gamma, \Gamma]$ *such that* $\gamma(x_0) \in (a, b)$. □

Proposition 2.2 follows from the following more general fact.

PROPOSITION 2.4. *Let* Γ *be non-solvable,* $F(\Gamma) < \infty$, $\epsilon > 0$, *and* $x_1, \ldots, x_n \in (0, 1)$. *Then there exists* $\gamma \in [\Gamma, \Gamma]$ *such that* $|\gamma(x) - x| < \epsilon$ *for all* $x \in [0, 1]$ *and* $\mathrm{Fix}(\gamma) \cap \{x_1, \ldots, x_n\} = \emptyset$.

Proof. Let $F(\Gamma) = N$. Let also $\eta \in \Gamma$ be such that $\eta(x) < x$ near 1, and $\max \mathrm{Fix}(\eta) < \min\{x_1, \ldots, x_n\}$ (such η exists by irreducibility of Γ). We will consider a finite family $\{\eta^k : -nN - 1 \le k \le nN + 1\}$ of diffeomorphisms. By uniform continuity, there exists $\delta > 0$ such that for all $x, y \in [0, 1]$, $-nN - 1 \le k \le nN + 1$, we have $|x - y| < \delta \Rightarrow |\eta^k(x) - \eta^k(y)| < \epsilon$.

By Theorem B in [**A**], there exists $f \in [\Gamma, \Gamma]\backslash\{1\}$ such that $d(f) < \delta$. (Theorem B, as stated in [**A**], claims the existence of such f in Γ, but it is immediately clear from the proof that f can be chosen from the subgroup $[\Gamma, \Gamma]$†.) Then $d(\eta^{-k} f \eta^k) < \epsilon$ for all $k \in \{-nN - 1, -nN, \ldots, n, nN + 1\}$. Moreover, by the pigeonhole principle, there exists $k \in \{1, \ldots, nN, nN + 1\}$ such that $\{x_1, \ldots, x_n\} \cap \mathrm{Fix}(\eta^{-k} f \eta^k) = \emptyset$. □

We would like to emphasize that we do not know how to prove Proposition 2.2 directly, i.e. without using Proposition 2.4, which is significantly stronger.

Now we will prove a somewhat stronger claim, which is more suitable for our purposes.

PROPOSITION 2.5. *Let* Γ *be non-solvable,* $F(\Gamma) < \infty$, $\epsilon > 0$, *and* $0 < a < b < 1$. *Then there exists* $\gamma \in [\Gamma, \Gamma]$ *such that* $|\gamma(x) - x| < \epsilon$ *for all* $x \in [0, 1]$ *and* $[a, b] \cap \mathrm{Fix}(\gamma) = \emptyset$.

Proof. In the proof of Proposition 2.4, it is sufficient to choose $\eta \in \Gamma$ such that $\eta^{-1}(a) > b$. □

Proposition 2.5 in its own turn implies the following stronger claim.

LEMMA 2.6. *Let* Γ *be non-solvable,* $F(\Gamma) < \infty$, $0 < c < a < b < d < 1$, *and* $x_0 \in (c, d)$. *Then there exists* $\gamma \in [\Gamma, \Gamma]$ *such that* γ *has no fixed point in* (c, d) *and* $\gamma(x_0) \in [a, b]$.

Proof. If $x_0 \in [a, b]$, then the claim immediately follows from Proposition 2.5. Then, without loss of generality, we may assume that $x_0 \in (c, a)$.

† The proof of Theorem B relies on the proof of Theorem A. Notice that, in the proof of Theorem A, the maps f and g belong to the commutator subgroup. Hence, the map $h_1^{-1} h_2$ at the end of the proof belongs to the commutator subgroup as well.

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Let $\epsilon_0 \in (0, b-a)$,

$$A = \{f \in G \mid d(f) < \epsilon_0, \mathrm{Fix}(f) \cap (c, d) = \emptyset, \text{ and } f(t) > t \forall t \in (c, d)\},$$

and

$$S = \{x \in [c, d] \mid \exists n \geq 2 \text{ and } z_1, \ldots, z_n \in (c, d), \gamma_1, \ldots, \gamma_{n-1} \in A \text{ s.t.}$$
$$z_1 = x_0, z_n = x, \gamma_i(z_i) = z_{i+1}, \text{ and } |z_{i+1} - z_i| < \epsilon_0, 1 \leq i \leq n-1\}.$$

Notice that by Proposition 2.5 the set A is not empty.

If $S \cap (d, 1) \neq \emptyset$, then we are done. Indeed, let $z \in S \cap (d, 1)$. Then there exist $n \geq 2$ and $z_1, \ldots, z_n \in (c, d), \gamma_1, \ldots, \gamma_{n-1} \in A$ such that $z_1 = x_0, z_n = z$, $|z_{i+1} - z_i| < \epsilon_0$, and $\gamma_i(z_i) = z_{i+1}$, $1 \leq i \leq n-1$. Then, since $\epsilon_0 < b - a$, there exists $m \in \{2, \ldots, n\}$ such that $z_m \in (a, b)$. Now, we can take $\gamma = \gamma_{m-1} \ldots \gamma_1$.

If $S \cap (d, 1) = \emptyset$, then we have $y := \sup S \leq d$. But by Proposition 2.5 there exists $\omega \in G$ such that $\omega(x) > x$ for all $x \in [c, d]$ and $d(\omega) < \epsilon_0$. Then, for sufficiently small $\delta > 0$, $\omega([y - \delta, y]) \subset (y, 1)$. But this contradicts the assumption that $y = \sup S$. □

Remark 2.7. Let us notice that, in Propositions 2.4 and 2.5, Theorem B (from [**A**]) is used only to guarantee the existence of C_0-small elements. But, if $\Gamma \leq \mathrm{Diff}^2_+(I)$, then one can use Theorem C from [**A**] to guarantee the existence of C_0-small elements under a weaker condition that Γ is not meta-abelian. Hence, if $\Gamma \leq \mathrm{Diff}^2_+(I)$, then in Propositions 2.2, 2.4, and 2.5 and Lemmas 2.3 and 2.6 one can replace the condition 'Γ is not solvable' with 'Γ is not meta-abelian'. Moreover, by Theorem B (see the statement in [**A**]), there exists $k(\epsilon)$ such that any C_0 discrete subgroup of G is solvable of derived length at most $k(\epsilon)$. Thus, for subgroup $\Gamma \leq G$, in Propositions 2.2, 2.4, and 2.5 and Lemmas 2.3 and 2.6, the condition 'Γ is not solvable' can be replaced with 'the derived length of Γ is not bigger than $k(\epsilon)$'. We will fix the number $k(\epsilon)$ for the rest of the paper.

3. *Solvability of groups with small N*

Our proof of the main theorem will use the assumption that $N \geq 5$. Therefore, we need to take care of the cases $N \leq 4$ separately. It is also interesting to see how easy it is to prove Theorem 1.1 in the cases of $N \leq 4$, using the results of §2.

The cases of $N = 0, 1$ have been discussed earlier, so it remains to study the cases of $N = 2, 3, 4$. Our proofs for these cases are somewhat different from each other, and the techniques used in these proofs will be useful for the reader as a preparation for the proof of the main theorem.

If $\gamma \in \Gamma, \mathrm{Fix}(\gamma) = \{a_1, \ldots, a_n\}$, where $a_1 < a_2 < \cdots < a_n$, then we will write $p_i(\gamma) = a_i$ for all $1 \leq i \leq n$ (so $p_i(\gamma)$ denotes the ith fixed point of γ).

PROPOSITION 3.1. *Let $\Gamma \leq G$ be such that $F(\Gamma) \leq 4$. Then Γ is solvable.*

Proof. Notice that if, in a group, all finitely generated subgroups are solvable with derived length not bigger than some universal $d \geq 1$, then the group is solvable of derived length at most d. Also, by Theorem B in [**A**], if $\Gamma \leq G$ is C_0 discrete then Γ is solvable, and the derived length of Γ is not bigger than $k(\epsilon)$. We may and will assume that $k(\epsilon) \geq 5$. Keeping this and Remark 2.7 (the fact that in Propositions 2.4 and 2.5, Theorem B from

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[**A**] is used only to guarantee the existence of C_0-small elements) in mind, we will first assume that Γ is finitely generated with a fixed finite generating set.

Let $F(\Gamma) = N$ and $N \in \{2, 3, 4\}$. Let also f be the biggest generator of Γ.

Then $f \notin \Gamma_\gamma$ for all $\gamma \in \Gamma$ (in other words, f is not infinitesimal with respect to any element of Γ) and $[\Gamma, \Gamma] \leq \Gamma_f$.

Without loss of generality, we may assume that f has no tangential fixed point. (Indeed, if f has a tangential fixed point, then for sufficiently small $\delta > 0$ if $\omega \in [\Gamma, \Gamma]$ is such that $d(\omega) < \delta$, $\mathrm{Fix}(\omega) \cap \mathrm{Fix}(f) = \emptyset$ and $\omega(x) > x$ for all $x \in (\delta, 1 - \delta)$ (such ω exists by Proposition 2.5), then there exists $\phi \in \{\omega f, \omega^{-1} f\}$ such that $|\mathrm{Fix}(\phi)| \geq |\mathrm{Fix}(f)|$; moreover, either ϕ has no tangential fixed point or it has more fixed points than f (see Remark 3.2). Notice also that ϕ is positive and $\phi \in \Gamma \backslash \Gamma_f$. Thus, we can replace f by ϕ; and if we still have tangential fixed points we perform the same operation at most four times (since $F(\Gamma) = 4$) until we find a positive element in $\Gamma \backslash \Gamma_f$ (which we still denote by f) without tangential fixed points).

Notice that $f(x) > x$ near zero. We may also assume that f has at least two fixed points. (From the proof of Proposition 2.1, we see that, for a suitable choice of $h \in \Gamma_f$, and for sufficiently big n, the diffeomorphism $h^{-n} f_1$ has at least two fixed points, where $f_1 = \omega f \omega^{-1}$ for some $\omega \in \Gamma_f$. Hence, if necessary, we may replace f with $h^{-n} f_1$.)

(a) Let $N = 2$. Let also $h \in \Gamma_f \backslash \{1\}$ be such that h has two fixed points (if such h does not exist, then Γ_f is meta-abelian by Solodov's theorem; therefore, Γ is solvable with derived length at most three). We may also assume that h has no tangential fixed point, $\mathrm{Fix}(h) \cap \mathrm{Fix}(f) = \emptyset$, and $h(x) > x$ near zero.

Let $\mathrm{Fix}(h) = \{a, b\}$, $a < b$. Let us recall that f has at least two fixed points; hence (since $N = 2$), exactly two fixed points.

Let c be the smallest fixed point of f. If necessary, using Lemma 2.3, by conjugating f by an element of Γ_f, we may assume that $c > b$. Then, for sufficiently big n, the graphs of f and h^n will have at least two crossings in the interval $(0, a)$, and at least one crossing in the interval (b, c). Hence, $h^{-n} f$ will have at least three fixed points, which is a contradiction.

Thus, we proved that if Γ is finitely generated, then it is solvable of derived length at most $\max\{3, k(\epsilon)\} = k(\epsilon)$. Now, to treat the case when Γ is not finitely generated, we notice that all its finitely generated subgroups are solvable of solvability degree at most $k(\epsilon)$; hence, Γ itself is solvable of solvability degree at most $k(\epsilon)$.

(b) $N = 3$. Again, we are assuming that Γ is finitely generated. Let $h \in \Gamma_f \backslash \{1\}$ be such that h has three fixed points. (If such h does not exist, then, by part (a), Γ_f is solvable of solvability degree at most $k(\epsilon)$; therefore, Γ is solvable of derived length at most $k(\epsilon) + 1$.) We may also assume that h has no tangential fixed points, $\mathrm{Fix}(h) \cap \mathrm{Fix}(f) = \emptyset$, and $h(x) > x$ near zero.

Let $\mathrm{Fix}(h) = \{a, b, c\}$, $a < b < c$.

By assumption, f has at least two fixed points. By conjugating f if necessary, and using Lemma 2.3, we may assume that the smallest fixed point of f (let us denote it by p) lies in (b, c). Then, for sufficiently big $n \in \mathbb{N}$, the graphs of f and h^n have at least two crossings in $(0, a)$ and at least one crossing in (b, p). If f has three fixed points, then there will be one more crossing in $(p, 1)$. If f has two fixed points, then there exists one

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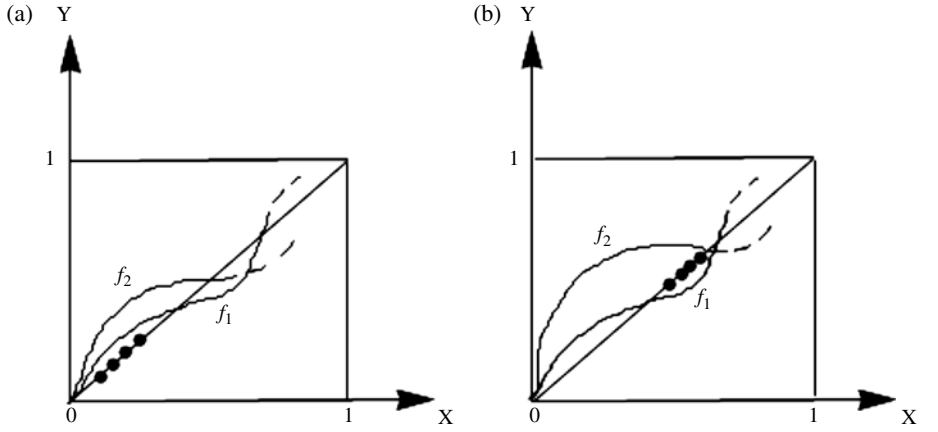


FIGURE 1. The four dots in (a) and (b) represent the fixed points of h_2 and h_3.

more crossing either in (p, c) (if the biggest fixed point of f is less than c) or in $(c, 1)$ (if the biggest fixed point of f is bigger than c). This is a contradiction.

Now, to treat the case when Γ is not finitely generated, we notice that all its finitely generated subgroups are solvable of solvability degree at most $k(\epsilon) + 1$. Hence, Γ will be solvable of derived length at most $k(\epsilon) + 1$.

(c) Let now $N = 4$. First, we are making the assumption that Γ is finitely generated. Again, there exists $h \in \Gamma_f \backslash \{1\}$ such that h has four fixed points. (If such h does not exist, then, by part (b), Γ_f is solvable of solvability degree at most $k(\epsilon) + 1$; therefore, Γ is solvable of derived length at most $k(\epsilon) + 2$.) We may furthermore assume that h has no tangential fixed point, $\mathrm{Fix}(h) \cap \mathrm{Fix}(f) = \emptyset$, and $h(x) > x$ near zero.

By assumption, f has at least two fixed points. We may conjugate h to h_1 by elements of Γ_f such that $\mathrm{Fix}(h_1) \subset (0, p_1(f))$. Then, for sufficiently big n, h_1^n and f have at least three crossings in $(0, p_1(f))$. Hence, $f_1 = h_1^{-n} f$ has at least three fixed points. (Notice that f_1 is positive and $f_1 \notin \Gamma_f$.)

We can conjugate f to f_2 by elements of Γ_f such that $p_1(f_2) \in (p_1(f_1), p_2(f_1))$. Then we can conjugate h to h_2 by elements of Γ_f such that $\mathrm{Fix}(h_2) \subset (0, p_1(f_1))$ (Figure 1(a)). Then, conjugating h_2 by powers of f_2, we obtain h_3, where $\mathrm{Fix}(h_3) \subset (p_1(f_1), p_1(f_2))$ (Figure 1(b)). Using Lemma 2.6, we can conjugate f_1 to f_3 by elements of Γ_f such that the first two fixed points of h_3 lie in $(0, p_1(f_3))$ while the last two fixed points lie in $(p_1(f_3), p_2(f_3))$ (Figure 2).

Let $g_n = f_3^{-n} h_3 f_3^n$, $n \in \mathbb{N}$. Notice that as $n \to \infty$, we have $p_1(g_n) \to 0$, $p_2(g_n) \to 0$ while $p_3(g_n) \to p_2(f_3)$, $p_4(g_n) \to p_2(f_3)$. Then, for sufficiently big n, using Lemma 2.3, g_n can be conjugated to ϕ_n by an element of Γ_f such that ϕ_n has two fixed points in $(0, p_1(f_3))$ and another two fixed points in $(p_2(f_3), p_3(f_3))$. Then, for sufficiently big k, the diffeomorphisms ϕ_n^{-k} and f_3 have at least three crossings in $(0, p_1(f_3))$ and at least two crossings in $(p_2(f_3), p_3(f_3))$. This is a contradiction.

If Γ is not finitely generated, then we notice that all its finitely generated subgroups are solvable of solvability degree at most $k(\epsilon) + 2$. Hence, Γ is solvable of derived length at most $k(\epsilon) + 2$. □

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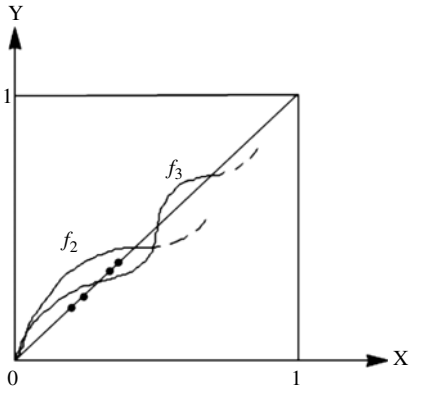


FIGURE 2. The four dots represent the fixed points of h_3.

Let us emphasize that not only have we proved Γ is solvable, but we have also proved that the derived length of Γ is uniformly bounded above by $k(\epsilon) + 2$.

Remark 3.2. It suffices to take

$$\delta = \tfrac{1}{2} \min\{|f(x + sr) - (x + sr)| \mid x \in \mathrm{Fix}(f),\, s \in \{-1, 1\}\},$$

where $r = \frac{1}{4} \min_{u,v \in \mathrm{Fix}(f) \cup \{0,1\}, u \neq v} |u - v|$. To see why the claim holds, call a tangential fixed point $x \in \mathrm{Fix}(f)$ *upper (lower)* if $f(z) \geq z$ ($f(z) \leq z$) near x. Then, for all $x \in \mathrm{Fix}(f)$:

(i) if x is non-tangential, then f and ω will have at least one non-tangential crossing in $(x - r, x + r)$; the same holds also for f and ω^{-1};
(ii) if x is upper-tangential, then f and ω will have at least two non-tangential crossings in $(x - r, x + r)$;
(iii) if x is lower-tangential, then f and ω^{-1} will have at least two non-tangential crossings in $(x - r, x + r)$.

Thus, if the number of upper-tangential fixed points of f is not less than the number of lower-tangential fixed points, then f and ω will have at least $|\mathrm{Fix}(f)|$ non-tangential crossings; hence, f and ω have either more than $|\mathrm{Fix}(f)|$ crossings or exactly $|\mathrm{Fix}(f)|$ non-tangential crossings. Similarly, if the number of lower-tangential fixed points of f is not less than the number of upper-tangential fixed points, then f and ω^{-1} will have at least $|\mathrm{Fix}(f)|$ non-tangential crossings; hence, f and ω^{-1} have either more than $|\mathrm{Fix}(f)|$ crossings or exactly $|\mathrm{Fix}(f)|$ non-tangential crossings.

Remark 3.3. Interestingly, already for $N = 5$, the elementary arguments in the proof of Proposition 3.1 do not seem to work, and one needs significantly newer and more general ideas. (In fact, the value $N = 5$ seems to be too big for 'elementary methods' and too small for 'general methods'. We make extra efforts to cover this particular case.) The essence of the problem is that Lemma 2.3 can be viewed as a dynamical transitivity of the action. However, we do not have a result about dynamical k-transitivity even when $k = 2$. But, as N gets bigger, we need higher transitivity results to control the picture as

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we did in the proof of Proposition 3.1. (*We call the action of* Γ *on* I *k-transitive if for all* $z_1, \dots, z_k \in (0, 1)$*, where* $z_1 < z_2 < \cdots < z_k$*, and for all open non-empty intervals* $I_1, I_2, \dots, I_k$*, where* $x < y$ *if* $x \in I_p$, $y \in I_q$, $1 \le p < q \le k$*, there exists* $\gamma \in \Gamma$ *such that* $\gamma(z_i) \in I_i$, $1 \le i \le k$*. Proposition 2.5 is indeed significantly stronger than 1-transitivity.*)

4. *Proof of the main theorem*

The following lemma is very interesting in itself. It will be used in the proof of Theorem 1.2, and the idea of the proof will be used repeatedly in the proof of Theorem 1.1.

LEMMA 4.1. *Let* Γ *be non-solvable,* $F(\Gamma) < \infty$, $f \in \Gamma$, $|\mathrm{Fix}(f)| = n$*. Then there exists* $\psi \in [\Gamma, \Gamma]$ *such that* $|\mathrm{Fix}(\psi)| \ge n - 1$*. Moreover,* ψ *can be taken arbitrarily close (in the* C_0*-metric) to the identity.*

Proof. Let $\mathrm{Fix}(f) = \{c_1, \dots, c_n\}$, where $c_1 < \cdots < c_n$, $\epsilon < \frac{1}{2}s(f)$, $a = c_1 - \epsilon$, $b = c_n + \epsilon$. By uniform continuity of f and f^{-1}, there exists $\delta > 0$ such that if $d(h) < \delta$, then $d([f, h]) < \epsilon$. By Proposition 2.5, there exists $h \in [\Gamma, \Gamma]$ such that $h(x) > x$ on $[a, b]$, $\mathrm{Fix}(h) \cap [a, b] = \emptyset$, and $d(h) < \min\{\delta, \epsilon\}$.

Let now $g = h^{-1} f h$. Then g has n fixed points $d_1, \dots, d_n$, where $d_1 < c_1 < d_2 < c_2 < \cdots < d_n < c_n$. Then the element $\psi = f^{-1} g$ has at least $n - 1$ fixed points $e_1, \dots, e_{n-1}$, where $e_i \in (c_i, d_{i+1})$, $1 \le i \le n - 1$. Moreover, since $d(h) < \delta$, we also have $d(\psi) < \epsilon$. □

Now we want to observe the following simple lemma.

LEMMA 4.2. *Let* $f, h \in \mathrm{Homeo}_+(I)$, $\mathrm{Fix}(h) \cap [a, b] = \{a, b\}$, $\mathrm{Fix}(f) \cap [a, b] = \emptyset$*. Let also* $f(a) \in (a, b)$ *or* $f(b) \in (a, b)$*. Then the graphs of* $h^{-1} f h$ *and* f *have at least one crossing in* (a, b)*.* □

Remark 4.3. Under the assumptions of the lemma (indeed, under even a slightly more general assumption), f and h are called crossed elements. If f, h are crossed, then the subgroup generated by f, h contains a non-abelian free semigroup [**N**] (we do not use this result here).

In the proof of the main theorem, the case of $N = 5$ requires more effort; we will need the following lemma in treating this case.

LEMMA 4.4. *Let* $\Gamma \le G$ *be a finitely generated subgroup with a fixed finite generating set and with derived length bigger than* $k(\epsilon)$*,* $F(\Gamma) = 5$*, and* f *be the biggest generator. Then at least one of the following conditions holds:*

(i) *there exists* $g \in \Gamma \backslash \Gamma_f$ *with exactly five fixed points;*

(ii) $\Gamma \backslash \Gamma_f$ *contains a diffeomorphism with exactly four fixed points and* Γ_f *does not contain a diffeomorphism with exactly four fixed points.*

Proof. Since Γ is irreducible and Γ_f is a normal subgroup, we may assume that Γ_f has no global fixed point in $(0, 1)$. As we already discussed, we may also assume that f has at least two non-tangential fixed points. Assume that there is no $g \in \Gamma \backslash \Gamma_f$ with five fixed points. Then there exists $h \in \Gamma_f \backslash \{1\}$ with exactly five non-tangential fixed points and $h(x) > x$ near zero. Using Lemma 2.3 (and Remark 2.7), we can conjugate h to h_1 by an element of Γ_f such that the biggest fixed point of h_1 lies in $(p_1(f), p_2(f))$.

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We can also conjugate h to h_2 such that $\mathrm{Fix}(h_2) \subset (p_5(h_1), 1)$ (there is no need to use Lemma 2.3 here; we just use the fact that Γ_f has no global fixed point in $(0, 1)$). Then we can conjugate h_2 to h_3 by powers of h_1 such that $\mathrm{Fix}(h_3) \subset (p_5(h_1), p_2(f))$. Finally, using Lemma 2.6, we can conjugate f to f_1 by an element of Γ_f such that the first two fixed points of h_3 lie in $(0, p_1(f_1))$ and the last three fixed points lie in $(p_1(f_1), p_2(f_1))$. Then, for sufficiently big n, the diffeomorphisms f_1 and h_3^n have at least three crossings in $(0, p_1(f_1))$ and at least one crossing in $(p_1(f_1), p_2(f_1))$. Thus, for sufficiently big n, the diffeomorphism $f_2 = h_3^{-n} f_1$ has at least four fixed points and hence exactly four fixed points. Notice that $f_2 \notin \Gamma_f$ and f_2 is positive.

Now, if there is no diffeomorphism in Γ_f with four fixed points, then we are done. But, if such a diffeomorphism exists, then we fall into the case of the proof of Proposition 3.1, part (c) (i.e. having a diffeomorphism $f_2 \in \Gamma \backslash \Gamma_f$ with at least three fixed points and a diffeomorphism Γ_f with four fixed points), thus obtaining a diffeomorphism with five fixed points in $\Gamma \backslash \Gamma_f$. □

Now we are ready to prove the main result.

Proof of Theorem 1.1. We will assume that $N \geq 5$. Let $g \in \Gamma$ with N fixed points and $g(x) > x$ near zero. We may assume that none of the fixed points of g is tangential. By conjugating g, we obtain g_1 and g_2 such that $\mathrm{Fix}(g_1) \subset (0, \frac{1}{10})$, $\mathrm{Fix}(g_2) \subset (\frac{9}{10}, 1)$. Let $\mathrm{Fix}(g_1) = \{a_1, \ldots, a_N\}$, $\mathrm{Fix}(g_2) = \{b_1, \ldots, b_N\}$, where the elements are listed from the least to the biggest.

We will present a proof by contradiction. Using Proposition 2.5, as in the proof of Lemma 4.1, we obtain $h_1 \in \Gamma$ such that g_2 and $h_1^{-1} g_2 h_1$ have at least one crossing in each of the intervals (b_i, b_{i+1}), $1 \leq i \leq N-1$; moreover, if $d(h_1)$ is sufficiently small, then $d(f) < \min\{\frac{1}{10}, \frac{1}{2} s(g_1)\}$, where $f = g_2^{-1} h_1^{-1} g_2 h_1$. Thus, the diffeomorphism f has at least $N-1$ fixed points in $(b_1, 1) \subset (\frac{9}{10}, 1)$. Then f has at most one fixed point in $(0, \frac{9}{10})$.

Using Proposition 2.5 again, we obtain h_2 such that the diffeomorphisms g_1 and $h_2^{-1} g_1 h_2$ have at least one crossing in each of the intervals (a_i, a_{i+1}), $1 \leq i \leq N-1$; moreover, if $d(h_2)$ is sufficiently small, then $d(h) < \min\{\frac{1}{10}, \frac{1}{2} s(f)\}$, where $h = g_1^{-1} h_2^{-1} g_1 h_2$. Thus, the diffeomorphism h has at least $N-1$ fixed points in $(0, \frac{1}{10})$.

Let us now assume that $N \geq 9$. Then (let us recall that $d(f) < \min\{\frac{1}{10}, \frac{1}{2} s(g_1)\}$), if $d(h_2)$ is sufficiently small, there exist at least four mutually disjoint non-empty intervals $I_i = (p_i, q_i)$, $1 \leq i \leq 4$, in $(0, \frac{1}{5})$ such that for all $i \in \{1, 2, 3, 4\}$:

(a1) $p_i, q_i \in \mathrm{Fix}(h)$;

(a2) $\mathrm{Fix}(h) \cap I_i = \emptyset$;

(a3) $f(p_i) \in I_i$ or $f(q_i) \in I_i$.

Since f has at most one fixed point in $(0, \frac{9}{10})$, f has no fixed points in at least three of these intervals. Thus, applying Lemma 4.2, we obtain that f and $h^{-1} f h$ have at least three crossings in $(0, \frac{9}{10})$. But, since $d(h) < \frac{1}{2} s(f)$, we have at least one crossing in each of the intervals (c_j, c_{j+1}), $1 \leq j \leq M-1$, where $c_1, \ldots, c_M$ are all fixed points of f listed from the least to the biggest, and $M \in \{N-1, N\}$ (we know that f has at least $N-1$ fixed points). Hence, f and $h^{-1} f h$ have at least $3 + (N-2) = N+1$ crossings. This is a contradiction.

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When $N \in \{5, 6, 7, 8\}$, we need to sharpen our observation. Indeed, for $N \in \{6, 7, 8\}$, we have either of the following two cases.

Case 1. There exist four mutually disjoint non-empty intervals $I_i = (p_i, q_i)$, $1 \leq i \leq 4$, in $(0, \frac{1}{5})$ such that for all $i \in \{1, 2, 3, 4\}$ the conditions (a1)–(a3) hold.

Case 2. There exist three mutually disjoint non-empty intervals $I_i = (p_i, q_i)$, $1 \leq i \leq 3$, in $(0, \frac{1}{5})$ such that for all $i \in \{1, 2, 3\}$ the conditions (a1)–(a3) hold; moreover, in two consecutive intervals, either $h(x) > x$ on both of these intervals, or $h(x) < x$.

In both cases, we obtain that f and $h^{-1} f h$ have at least three crossings in $(0, \frac{9}{10})$ and hence at least $N + 1$ crossings in total; this is a contradiction.

The case of $N = 5$ needs special treatment. We may assume that Γ is finitely generated with a fixed generating set and let θ be the biggest generator.

First, let us assume that $\Gamma \backslash \Gamma_\theta$ contains a diffeomorphism with five fixed points. Then we choose g to be this diffeomorphism and we let g_1, g_2 be obtained from g by conjugating by elements from Γ_θ.

Then $f \in \Gamma_\theta$. Hence, if f has a fixed point in $(0, b_1)$, then, for sufficiently big n, f^n and g_2 will have at least two crossings in $(0, b_1)$ and hence at least $(N - 1) + 2$ crossings in total, which is a contradiction. Thus, we may assume that f has no fixed point in $(0, b_1) \supset (0, \frac{9}{10})$; the conclusion follows as in the case of $N \geq 6$.

If $\Gamma \backslash \Gamma_\theta$ does not contain a diffeomorphism with five fixed points, then let H be an arbitrary finitely generated subgroup of Γ_θ with a fixed finite generating set and let $\eta \in H$ be the biggest generator. Then, by Proposition 3.1, either $F(H) = 5$ or H is solvable of derived length at most $k(\epsilon) + 2$. In the former case, if the derived length of H is bigger than $k(\epsilon)$, by Lemma 4.4, the subgroup H contains no diffeomorphism with four fixed points. Then, again by Lemma 4.4, we obtain that $H \backslash H_\eta$ contains a diffeomorphism with five fixed points. Thus, applying the previous argument to this case, we obtain a contradiction. Hence, H is solvable of solvability degree at most $k(\epsilon)$.

Thus, we obtained that an arbitrary finitely generated subgroup of Γ_θ is solvable of derived length at most $k(\epsilon) + 2$. This implies that Γ_θ is solvable. Hence, Γ is solvable. □

For the purposes of clarity, let us emphasize that the proof of Theorem 1.1 does not use an inductive argument on $N \geq 5$.

Now we are ready to prove Theorem 1.2. Taking into account Remark 2.7, the proofs of Proposition 3.1 (for $N \in \{2, 3, 4\}$) and of Theorem 1.1 (for $N \geq 6$) go *mutatis mutandis* to obtain a contradiction under the assumptions that $\Gamma \leq \mathrm{Diff}^2_+(I)$ is non-meta-abelian and $F(\Gamma) = N$, except that we also need to show that the claims in parenthetical remarks in parts (a)–(c) of the proof of Proposition 3.1 hold as well. More precisely, we need to show that if $N \in \{2, 3, 4\}$, then Γ_f contains a diffeomorphism with N fixed points. Indeed, for $N = 2$, if Γ_f does not have such a diffeomorphism, then, by [**FF**, Theorem 1.5], there exists an element of $h \in \mathrm{Homeo}_+(I)$ such that $h \Gamma_f h^{-1}$ is a subgroup of $\mathrm{Aff}_+ I$ (this latter group, by definition, is obtained from $\mathrm{Aff}_+ \mathbb{R}$ by conjugation by a tangent function). Then the subgroup $A = \{g \in h \Gamma_f h^{-1} \mid g = id \text{ or } \mathrm{Fix}(g) = \emptyset\}$ contains elements arbitrarily close to the identity in the C_0-metric. Let $\phi \in h \Gamma h^{-1}$ with two fixed points. Then there exists $\omega \in A$ such that $[\omega, \phi] = \omega \phi \omega^{-1} \phi^{-1}$ has one fixed point. But $\phi \omega^{-1} \phi^{-1}$ has no fixed point; hence, it belongs to A. Then $[\omega, \phi] \in A$; hence, it has no fixed point, which is

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a contradiction. For the case of $N = 3$, assume that Γ_f has no element with three fixed points. If it also has no element with two fixed points, then we run the same argument. But, if $F(\Gamma_f) = 2$, then, by part (a), we obtain a diffeomorphism in Γ_f with three fixed points and thus a contradiction. Similarly, for the case of $N = 4$, assume that Γ_f has no element with four fixed points. If $F(\Gamma_f) = 1$, then we run the previous argument, but, if $F(\Gamma_f) \in \{2, 3\}$, then, by parts (a) and (b), we obtain a contradiction again.

The case of $N = 5$ is again slightly subtle. In this case, we can only conclude that the group H (at the end of the proof of Theorem 1.1) is meta-abelian. Since H is an arbitrary finitely generated subgroup of Γ_θ, then we can conclude that Γ_θ is meta-abelian. This does not automatically imply that Γ is meta-abelian (it only implies that Γ is solvable of derived length at most three). So, let us assume that Γ is not meta-abelian.

If $\Gamma\backslash\Gamma_\theta$ contains a diffeomorphism with five fixed points, then the claim again follows from the proof of Theorem 1.1; we choose g to be a diffeomorphism from $\Gamma\backslash\Gamma_\theta$ with five fixed points and we let g_1, g_2 be obtained from g by conjugating by elements from Γ_θ.

If $\Gamma\backslash\Gamma_\theta$ does not contain a diffeomorphism with five fixed points, then there exists $\xi \in \Gamma_\theta$ with five fixed points.

On the other hand, since Γ is not meta-abelian, $[\Gamma, \Gamma]$ contains a non-trivial element η with $d(\eta) < \frac{1}{2}s(\xi)$ and $\mathrm{Fix}(\eta) \cap \mathrm{Fix}(\xi) = \emptyset$. Then $\mathrm{Fix}(\eta\xi\eta^{-1}) \cap \mathrm{Fix}(\xi) = \emptyset$ and $\gamma = \eta\xi\eta^{-1}\xi^{-1}$ has at least four fixed points, where $\gamma \neq 1$.

Since $[\Gamma, \Gamma] \leq \Gamma_\theta$, we have $\eta \in \Gamma_\theta$. But we also have $\xi \in \Gamma_\theta$; therefore, $\gamma \in [\Gamma_\theta, \Gamma_\theta]$.

Now, since Γ is non-meta-abelian, there exists $\omega \in [\Gamma, \Gamma] \leq \Gamma_\theta$ such that $d(\omega) < \frac{1}{2}s(\gamma)$ and $\mathrm{Fix}(\omega) \cap \mathrm{Fix}(\gamma) = \emptyset$. Then $\mathrm{Fix}(\omega\gamma\omega^{-1}) \cap \mathrm{Fix}(\gamma) = \emptyset$. Thus, we obtain that $\omega\gamma\omega^{-1}$ and γ do not commute (otherwise γ would have to have infinitely many fixed points). But $\gamma \in [\Gamma_\theta, \Gamma_\theta]$; therefore, $\omega\gamma\omega^{-1} \in [\Gamma_\theta, \Gamma_\theta]$. On the other hand, $[\Gamma_\theta, \Gamma_\theta]$ is abelian. Then $[\gamma, \omega\gamma\omega^{-1}] = 1$. This is a contradiction. □

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