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The edge-cracked circular disc under symmetric pin-loading

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Abstract

A circular disc of radius a, made of homogeneous, isotropic, linearly elastic material, contains a radial edge crack of length $b(0 < b < 2a)$. The disc is in equilibrium in a state of generalized plane stress under various loadings, which are motivated by the fact that this geometry is to become a standard test specimen configuration in the fracture testing of materials.

The first cases considered are those in which the disc is loaded by either (i) opposing point forces P normal to the crack, or (ii) opposing point couples M, in each case acting at the crack mouth. The problem of determining the resulting stress field throughout the disc is solved analytically in closed form in each case, and the respective stress intensity factors $\mathscr{K}_P$, $\mathscr{K}_M$ are given exactly by

$$\mathscr{K}_P = P\left(\frac{2a}{\pi b}\right)^{\frac{1}{2}}\left[\frac{b}{K_+(i)\,(2a-b)^{\frac{3}{2}}}+\frac{1}{K_+(0)\,(2a-b)^{\frac{1}{2}}}\right],$$

$$\mathscr{K}_M = \frac{M}{(2\pi ab)^{\frac{1}{2}}}\left[\frac{b}{K_+(i)\,(2a-b)^{\frac{3}{2}}}+\frac{2}{K_+(0)\,(2a-b)^{\frac{1}{2}}}\right],$$

where $K_+(0)$, $K_+(i)$ are constants whose values are

$$K_+(0) = 0{\cdot}966528\ldots,$$
$$K_+(i) = 0{\cdot}355715\ldots,$$

correct to 6 decimal places.

For the case in which the cracked disc is subject to general symmetric 'pin-loading' by opposing point forces P, it is proved that the stress intensity factor $\mathscr{K}_P^C$ is given by the asymptotic formula

$$\mathscr{K}_P^C = \frac{P}{(2\pi ab)^{\frac{1}{2}}}\left[\frac{b(2a-c)}{K_+(i)\,(2a-b)^{\frac{3}{2}}}+\frac{2(a-c)}{K_+(0)\,(2a-b)^{\frac{1}{2}}}\right]+\frac{Pa^{\frac{1}{2}}}{b}O\left\{\left(\frac{R}{R'}\right)^{\beta}\left(\frac{2a-b}{b}\right)^{\beta-\frac{1}{2}}\right\},$$

as $[R(2a-b)]/R'b \to 0$, where c is the distance of the load line from the crack mouth, β is a constant whose value is $\beta = 2{\cdot}739\,593\ldots$ correct to 6 decimal places, and R, R' are shown in Fig. 4.

Corresponding asymptotic formulae are also proved for the opening of the crack at the load-line, and at the crack mouth.

These asymptotic formulae (with the appropriate error terms neglected) are compared with numerical results obtained by boundary collocation and excellent agreement is found.

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It is noted that as $a \to \infty$ with b fixed, $\mathscr{K}_M \to 0$. This result is confirmed in Appendix 1, where we solve the problem of a wedge (of general angle 2α) containing a symmetric edge crack at its vertex, and loaded by opposing point couples at the crack mouth. An exact expression for the resulting stress intensity factor is obtained, and this is shown to change sign at $\alpha = \frac{1}{2}\pi$; this implies that when opposing point couples act at the mouth of a normal edge crack in a *half-plane*, the stress intensity factor is zero.

1. *Introduction*

In order to determine the plane strain fracture toughness of a metallic material it is necessary to carry out a practical fracture test on a specimen of the material which contains a pre-existing crack. In order to interpret the results of such a test it is necessary to have a theoretical formula for the stress intensity factor at the crack tip for the particular shape of the test specimen and the particular choice of loading used. It is also useful to have a corresponding formula for the opening of the crack at some chosen station (e.g. at the load-line or at the crack mouth). This opening may easily be measured experimentally and the theoretical formula used inversely to deduce the crack length; optical measurement of crack lengths is often difficult. Alternatively, when the crack length *can* be measured optically, the displacement formula provides a check on the nature of the applied load. For instance, in the case of Mode 1 pin-loading, it would reveal if friction at the pins were introducing unwanted (and unknown) couples into the loading.

Tada, Paris and Irwin (12), chapter 2, give stress intensity factors and crack openings appropriate to several test specimen configurations. However, since it is often expensive and wasteful of material to fabricate existing test specimens, the American Society for Testing and Materials Committee on Fracture Testing is to introduce the *circular disc with a radial edge crack* as a new standard specimen; this specimen is easily produced from a cylindrical rod.

In a recent paper (Gregory (5)), the author considered the problem of a circular disc of radius a containing a radial edge crack of length b ($0 < b < 2a$). An exact closed form solution was obtained for the case in which the crack is opened by a constant internal pressure; exact expressions were obtained for the stress intensity factor, total strain energy, and the opening at the mouth of the crack.

In the present paper we consider the same disc under different loadings. In Sections 2 and 3 we obtain exact closed form solutions for the cases in which the loading is either (i) opposing point forces P normal to the crack, or (ii) opposing point couples M, in each case acting at the crack mouth. Exact expressions are obtained for the corresponding stress intensity factors $\mathscr{K}_P$, $\mathscr{K}_M$. In Sections 4 and 5 we consider the case of general symmetric (Mode 1) pin-loading, and use the universal function to obtain rigorous asymptotic formulae for the resulting stress intensity factor $\mathscr{K}_P^C$, and the crack openings δ, Δ at the load line and the crack mouth respectively. In Section 6 sharper asymptotic estimates are obtained for Δ via the reciprocal theorem. In Section 7 these asymptotic formulae for $\mathscr{K}_P^C$, δ, Δ (with their respective error terms neglected) are compared with numerical collocation calculations of Gross (6), (7), and Newman (9); excellent agreement is found.

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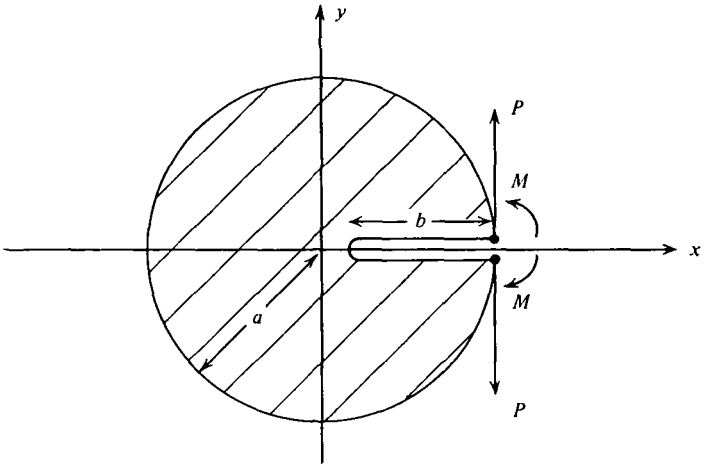


Fig. 1

In Appendix 1 we solve the problem of a wedge (of general angle 2α) containing a symmetric edge crack at its vertex, and loaded by opposing point couples at the crack mouth. An exact expression for the stress intensity factor is obtained, which is found to change sign once at $\alpha = \frac{1}{2}\pi$. This confirms the curious result that $\mathscr{K}_M \to 0$ as $a \to \infty$ with b fixed, and implies that when opposing point couples act at the mouth of a normal edge crack in a *half-plane*, the stress intensity factor is zero.

2. *The problems of point couples and point forces acting at the crack mouth*

The first problems to be solved are depicted in Fig. 1. A circular disc of radius a, made of homogeneous, isotropic, linearly elastic material, contains a radial edge crack of length b ($0 < b < 2a$). The disc is in equilibrium in a state of generalized plane stress caused by loading the mouth of the crack by either (i) opposing point forces P or (ii) opposing point couples M. In a recent paper (Gregory(5)) the author obtained the solution of the problem in which the geometry was the same as here, but the loading was that of constant pressure on the crack faces. The method of solution of the present problems is substantially the same as in (5), and so the details will not be presented here.

The problems are solved in a bipolar co-ordinate system (ξ, η), where $\zeta(= \xi + i\eta)$ is related to $z(= x + iy)$ by

$$\zeta = \log\left(\frac{z-a}{z+a}\right) \tag{2·1}$$

with $0 \leqslant \eta \leqslant 2\pi$. The bi-harmonic equation for the Airy stress function is not, of course, preserved by (2·1), but if we define the Jeffery potential $\phi(\xi, \eta)$ by

$$\phi = \frac{\chi}{J}, \tag{2·2}$$

where

$$J = \left|\frac{dz}{d\zeta}\right| = \frac{a}{\cosh\xi - \cos\eta}, \tag{2·3}$$

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then ϕ satisfies the equation

$$\left(\frac{\partial^4}{\partial\xi^4}+2\frac{\partial^4}{\partial\xi^2\partial\eta^2}+\frac{\partial^4}{\partial\eta^4}-2\frac{\partial^2}{\partial\xi^2}+2\frac{\partial^2}{\partial\eta^2}+1\right)\phi=0, \tag{2·4}$$

an equation with constant coefficients. These boundary value problems for ϕ can be solved by using the Wiener–Hopf technique and, as in (5), it must be verified that the expressions thus obtained do indeed represent the required solutions. All these details will be omitted. In order to express these solutions it is convenient to introduce the following notation:

Let
$$\phi_1(\xi,\eta)=\cos\eta L_1(\xi,\eta)+\sin\eta M_1(\xi,\eta), \tag{2·5}$$

$$\phi_2(\xi,\eta)=\cos\eta L_2(\xi,\eta)+\sin\eta M_2(\xi,\eta), \tag{2·6}$$

where L_1, M_1, L_2, M_2 are harmonic functions of ξ, η defined by

$$L_1(\xi,\eta)=\int_{-\infty}^{\infty}\left\{-\frac{K_+(\alpha)\cosh\alpha(\pi-\eta)}{\alpha-i}+\frac{\sinh\alpha(\pi-\eta)}{\alpha(\alpha-i)(\alpha^2+1)K_-(\alpha)}\right\}e^{-i\alpha\xi}d\alpha, \tag{2·7}$$

$$M_1(\xi,\eta)=\int_{-\infty}^{\infty}\left\{\frac{\cosh\alpha(\pi-\eta)-\coth\frac{1}{2}\pi\alpha\sinh\alpha(\pi-\eta)}{(\alpha-i)(\alpha^2+1)K_-(\alpha)}\right\}e^{-i\alpha\xi}d\alpha, \tag{2·8}$$

$$L_2(\xi,\eta)=\int_{-\infty}^{\infty}\left\{-\frac{K_+(\alpha)\cosh\alpha(\pi-\eta)}{\alpha}+\frac{\sinh\alpha(\pi-\eta)}{\alpha^2(\alpha^2+1)K_-(\alpha)}\right\}e^{-i\alpha\xi}d\alpha, \tag{2·9}$$

$$M_2(\xi,\eta)=\int_{-\infty}^{\infty}\left\{\frac{\cosh\alpha(\pi-\eta)-\coth\frac{1}{2}\pi\alpha\sinh\alpha(\pi-\eta)}{\alpha(\alpha^2+1)K_-(\alpha)}\right\}e^{-i\alpha\xi}d\alpha. \tag{2·10}$$

Notes. (i) In (2·7)–(2·10) the functions $K_\pm(\alpha)$ are the functions obtained by the multiplicative splitting of the meromorphic function

$$K(\alpha)=\frac{\sinh^2\frac{1}{2}\pi\alpha-\alpha^2}{\alpha(\alpha^2+1)\sinh\frac{1}{2}\pi\alpha\cosh\frac{1}{2}\pi\alpha}. \tag{2·11}$$

This splitting is performed in Appendix 1 of (5).

(ii) In (2·9), (2·10), the contour of integration is indented to avoid the pole of the integrand at $\alpha=0$.

The required solutions may now be written as follows:

The point forces problem:

$$\phi_P(\xi,\eta)=\frac{iPe^{\xi_0}}{2\pi K_+(i)}\phi_1(\xi-\xi_0,\eta)+\frac{iP}{2\pi K_+(0)}\phi_2(\xi-\xi_0,\eta). \tag{2·12}$$

The point couples problem:

$$\phi_M(\xi,\eta)=-\frac{M}{4a}e^{-\xi}(1+\cos2\eta)+\frac{iMe^{\xi_0}}{4\pi aK_+(i)}\phi_1(\xi-\xi_0,\eta)+\frac{iM}{2\pi aK_+(0)}\phi_2(\xi-\xi_0,\eta). \tag{2.13}$$

(2·12), (2.13) are the required solutions for a crack of length $b=a(1+\tanh\frac{1}{2}\xi_0)$, where $-\infty<\xi_0<\infty$. They represent the required Jeffery potentials in the upper half-disc $\frac{1}{2}\pi\leqslant\eta\leqslant\pi$, and may be continued by reflexion across the line $\eta=\pi$, $\xi>\xi_0$ into the lower half-disc.

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3. *The stress intensity factors for the two problems in Fig.* 1

In order to deduce the stress intensity factor at the crack tip in (for instance) the point forces problem, it suffices to examine the behaviour of

$$\phi_I(\xi,\pi) = \frac{iPe^{\xi_0}}{2\pi K_+(i)}\int_{-\infty}^{\infty}\frac{K_+(\alpha)\,e^{-i\alpha(\xi-\xi_0)}d\alpha}{\alpha-i} + \frac{iP}{2\pi K_+(0)}\int_{-\infty}^{\infty}\frac{K_+(\alpha)\,e^{-i\alpha(\xi-\xi_0)}d\alpha}{\alpha}, \tag{2·14}$$

as $\xi\to\xi_0+$. The first integral in (2·14) is treated in §6 of (5) and the second may be treated in a similar manner. This leads to the stress intensity factor

$$\mathscr{K}_P = \frac{Pe^{\xi_0}(1+\cosh\xi_0)^{\frac{1}{2}}}{(\pi a)^{\frac{1}{2}}K_+(i)} + \frac{P(1+\cosh\xi_0)^{\frac{1}{2}}}{(\pi a)^{\frac{1}{2}}K_+(0)}. \tag{2·15}$$

In terms of $b(=a(1+\tanh\frac{1}{2}\xi_0))$, (2·15) becomes

$$\mathscr{K}_P = P\left(\frac{2a}{\pi b}\right)^{\frac{1}{2}}\left[\frac{b}{K_+(i)\,(2a-b)^{\frac{1}{2}}} + \frac{1}{K_+(0)\,(2a-b)^{\frac{1}{2}}}\right]. \tag{2·16}$$

The corresponding expression for the stress intensity factor in the point couples problem is

$$\mathscr{K}_M = \frac{M}{(2\pi ab)^{\frac{1}{2}}}\left[\frac{b}{K_+(i)\,(2a-b)^{\frac{1}{2}}} + \frac{2}{K_+(0)\,(2a-b)^{\frac{1}{2}}}\right]. \tag{2.17}$$

Notes. (i) In the formulae (2·16), (2·17)

$$K_+(0) = \left(\frac{\pi^2-4}{2\pi}\right)^{\frac{1}{2}} = 0{\cdot}966\,528\ldots, \tag{2·19}$$

$$K_+(i) = 0{\cdot}355715\ldots. \tag{2·20}$$

The value of $K_+(0)$ follows from the construction of $K_+(\alpha)$ given in Appendix 1 of (5). $K_+(i)$ was evaluated numerically in Appendix 2 of (5), the value (2·20) being correct to 6 decimal places.

(ii) The definition of the stress intensity factor $\mathscr{K}$ used in (2·16), (2·17) is the same as is used in (5), namely that as $x\to(a-b)-$,

$$\sigma_{yy}(x,0) = \frac{\mathscr{K}}{(a-b-x)^{\frac{1}{2}}} + O(1).$$

Care must be excercised in making comparisons with other work since alternative definitions are also used. In particular in 'The Stress Analysis of Cracks Handbook' (Tada, Paris and Irwin(12)), the stress intensity factor is defined to be $(2\pi)^{\frac{1}{2}}\mathscr{K}$.

4. *The cracked disc under general symmetric pin-loading*

Consider now the same disc under general symmetric pin-loading, as depicted in Fig. 2. In this case equal and opposite forces P are applied at the points C, C' (C' is the reflexion of C in the x-axis) thus generating a stress field symmetrical about the crack. The straight line joining the points C, C' will be called the load-line and the distance of this load-line from the crack mouth will be denoted by c.

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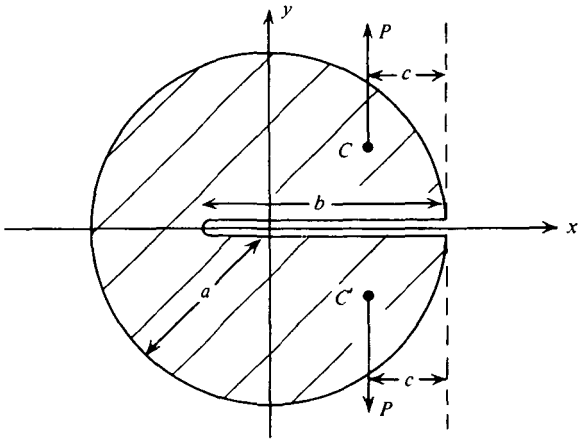


Fig. 2

Now it has been shown by Bueckner(1), and also by Rice(11), that if the cracked disc is placed under a symmetrical loading which produces a stress intensity factor $\mathscr{K}(b)$ and a displacement field $\mathbf{u}(\mathbf{r}, b)$, then the quantity $\mathbf{h}(\mathbf{r}, b)$ defined by

$$\mathbf{h}(\mathbf{r}, b) = \frac{E}{4\pi\mathscr{K}(b)} \frac{\partial}{\partial b}\mathbf{u}(r, b) \tag{4·1}$$

is independent of the particular loading. [**h** is called the *weight function* by Bueckner and the *universal function* by Rice.] Furthermore if the disc is acted upon by a symmetric body force **f** and boundary traction **t**, then the resulting stress intensity factor is given by

$$\mathscr{K} = \int_A \mathbf{f}.\mathbf{h}\, dx\, dy + \int_\Gamma \mathbf{t}.\mathbf{h}\, ds, \tag{4·2}$$

where A is the region of the disc and Γ its boundary.

Notes. (i) The results proved by Bueckner and Rice apply to *any* two-dimensional body with a straight crack where the body and the applied loadings are symmetrical about the crack line. Note that the derivative in (4·1) must be taken at *fixed* values of the applied loads.

(ii) The universal function **h** is only unique to within a rigid body displacement. Such a displacement makes no contribution however in the formula (4·2).

(iii) The universal function given in (4·1) is appropriate to the case of generalised plane stress. In the plane strain case, Young's modulus E must be replaced by $E/1-\nu^2$, where ν is Poisson's ratio.

It is evident that if we are to use (4·2) to calculate $\mathscr{K}$ for general pin-loading, we must first calculate $\mathbf{h}(\mathbf{r}, b)$. This we will do by using the solution obtained in (5), where the loading was that of constant pressure p_0 on the crack faces. The Jeffery potential corresponding to this loading was found to be (in the notation of the present paper)

$$\phi = \frac{ip_0 a}{2\pi K_+(i)} e^{\xi_0}\phi_1(\xi - \xi_0, \eta), \tag{4·3}$$

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where $\phi_1(\xi, \eta)$ is given by (2·5), this being the solution for a crack of length

$$b = a(1 + \tanh \tfrac{1}{2}\xi_0).$$

Formulae for the corresponding displacement field are given in (2), p. 170. In fact

$$Eu_\xi(\xi, \eta) = \frac{1-\nu}{J}\frac{\partial}{\partial \xi}(J\phi) - \frac{1}{J}\frac{\partial Q}{\partial \eta}, \tag{4·4}$$

$$Eu_\eta(\xi, \eta) = \frac{1-\nu}{J}\frac{\partial}{\partial \eta}(J\phi) + \frac{1}{J}\frac{\partial Q}{\partial \xi}, \tag{4·5}$$

where $J(\xi, \eta)$ is given by (2·3) and $Q(\xi, \eta)$ is a displacement potential given by

$$\frac{Q}{J} = -\frac{p_0 a e^{\xi_0}}{\pi K_+(i)}[\cos \eta R_1(\xi - \xi_0, \eta) + \sin \eta S_1(\xi - \xi_0, \eta)] + \frac{p_0 a\, e^{2\xi_0 - \xi}}{2\{K_+(i)\}^2}. \tag{4·6}$$

In (4·6) the functions $R_1(\xi, \eta)$, $S_1(\xi, \eta)$ are harmonic functions of ξ, η given by

$$R_1(\xi, \eta) = \int_{-\infty}^{\infty} \left\{ -\frac{K_+(\alpha)\sinh \alpha(\pi - \eta)}{\alpha - i} + \frac{\cosh \alpha(\pi - \eta)}{\alpha(\alpha - i)(\alpha^2 + 1) K_-(\alpha)} \right\} e^{-i\alpha\xi} d\alpha, \tag{4·7}$$

$$S_1(\xi, \eta) = \int_{-\infty}^{\infty} \left\{ \frac{\sinh \alpha(\pi - \eta) - \coth \tfrac{1}{2}\pi\alpha \cosh \alpha(\pi - \eta)}{(\alpha - i)(\alpha^2 + 1) K_-(\alpha)} \right\} e^{-i\alpha\xi} d\alpha, \tag{4·8}$$

where the contour of integration is indented to avoid the pole of each integrand at $\alpha = 0$.

The formula (4·2), together with the information in (4·3)–(4·8), is sufficient to determine $\mathscr{K}$ exactly for general pin-loading. However, this exact value would be an extremely intractable combination of integrals of the type (4·7), (4·8). Instead useful asymptotic results will be derived in the limit $\xi - \xi_0 \to -\infty$ (that is as $c \to 0$ or $b \to 2a$).

The behaviour of $R_1(\xi, \eta)$ as $\xi \to -\infty$ can be found by deforming the contour of integration in (4·7) into the upper half-plane. The integrand has simple poles at $\alpha = 0$ and $\alpha = i$ (note that $\{K_-(\alpha)\}^{-1}$ has a zero at $\alpha = i$), and simple poles at the zeros of $K_-(\alpha)$. These zeros are just the $\{\alpha_n\}$, $\{-\bar{\alpha}_n\}$, $(n \geqslant 1)$ described in Appendix 1 of (5) and satisfy the equation

$$\sinh^2 \tfrac{1}{2}\pi\alpha - \alpha^2 = 0. \tag{4·9}$$

On calculating the residues at $\alpha = 0, i$, we find that as $\xi \to -\infty$,

$$R_1(\xi, \eta) = -\frac{2\pi}{K_+(0)} - \tfrac{1}{2}\pi K_+(i)(\pi \cos \eta - 4 \sin \eta) e^{\xi} + O(e^{\beta\xi}), \tag{4·10}$$

uniformly for $\frac{1}{2}\pi \leqslant \eta \leqslant \pi$, where $\beta = \operatorname{Im}(\alpha_1) = 2{\cdot}739\,593\ldots$, correct to 6 decimal places.

Similarly

$$S_1(\xi, \eta) = \frac{4}{K_+(0)} - \tfrac{1}{2}\pi^2 K_+(i) \sin \eta\, e^{\xi} + O(e^{\beta\xi}), \tag{4·11}$$

as $\xi \to -\infty$, uniformly for $\frac{1}{2}\pi \leqslant \eta \leqslant \pi$.

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On substituting (4·10), (4·11) into (4·6), we obtain

$$\left(\frac{\pi K_+(i)}{p_0 a e^{\xi_0}}\right)\frac{Q}{J} = \frac{\pi\, e^{-(\xi-\xi_0)}}{2K_+(i)} + \frac{2}{K_+(0)}(\pi\cos\eta - 2\sin\eta) + \tfrac{1}{2}\pi K_+(i)\,(\pi - 4\sin\eta\cos\eta)\,e^{\xi-\xi_0} + O\{e^{\beta(\xi-\xi_0)}\}, \tag{4·12}$$

as $\xi-\xi_0\to-\infty$, uniformly for $\frac{1}{2}\pi \leqslant \eta \leqslant \pi$. Moreover the formula (4·12) may be differentiated any number of times with respect to ξ_0, ξ, η, the error term remaining unchanged.

A similar asymptotic estimate may be made for $\phi(\xi,\eta)$, and by using this together with (4·12) we may substitute into (4·4) (4·5), to find the corresponding estimates for $u_\xi(\xi,\eta)$, $u_\eta(\xi,\eta)$. Knowing these quantities, we may go on to find the displacement of the point (ξ,η) *in the y-direction*, $V(\xi,\eta;\xi_0)$ given by

$$V = \frac{\cosh\xi\cos\eta - 1}{\cosh\xi - \cos\eta}\,u_\eta - \frac{\sinh\xi\sin\eta}{\cosh\xi-\cos\eta}\,u_\xi. \tag{4·13}$$

After a long and tiresome calculation, $V(\xi,\eta;\xi_0)$ is found to be

$$\frac{E}{p_0 a}V(\xi,\eta;\xi_0) = \frac{e^{2\xi_0}}{2\{K_+(i)\}^2}\cdot\frac{e^{-\xi}-\cos\eta}{\cosh\xi-\cos\eta} - \frac{2e^{\xi_0}}{K_+(0)\,K_+(i)}\cdot\frac{\sinh\xi}{\cosh\xi-\cos\eta} + \{\text{terms independent of } \xi_0\} + O\{e^{\beta\xi}e^{-(\beta-1)\xi_0}\}, \tag{4·14}$$

as $\xi-\xi_0\to-\infty$, uniformly for $\frac{1}{2}\pi \leqslant \eta \leqslant \pi$.

A similar expression may be derived for the displacement of the point (ξ,η) in the x-direction, but this will not be needed.

Returning to (4·1), (4·2) we see that when the applied loading is the symmetric pin-loading depicted in Fig. 2, then the resulting stress intensity factor $\mathscr{K}_P^C(b)$ is given by

$$\mathscr{K}_P^C(b) = \frac{PE}{2\pi\mathscr{K}(b)}\frac{\partial}{\partial b}V(\xi,\eta;\xi_0), \tag{4·15}$$

where $\mathscr{K}(b)$ is the stress intensity factor in the pressurized crack case which is derived in (5) to be

$$\mathscr{K}(b) = \frac{p_0 a^{\frac{1}{2}} e^{\xi_0}(1+\cosh\xi_0)^{\frac{1}{2}}}{\pi^{\frac{1}{2}}K_+(i)} \tag{4·16}$$

and where the co-ordinates ξ, η appearing in (4·15) are now those of the loading point C.

Since
$$b = a(1+\tanh\tfrac{1}{2}\xi_0), \tag{4·17}$$

then
$$a\frac{\partial}{\partial b} = (1+\cosh\xi_0)\frac{\partial}{\partial\xi_0}, \tag{4·18}$$

and so by combining (4·14)–(4.18) we obtain

$$\mathscr{K}_P^C = \frac{P}{2(\pi a)^{\frac{1}{2}}}(1+\cosh\xi_0)^{\frac{1}{2}}\left[\frac{e^{\xi_0}}{K_+(i)}\cdot\frac{e^{-\xi}-\cos\eta}{\cosh\xi-\cos\eta} - \frac{2}{K_+(0)}\cdot\frac{\sinh\xi}{\cosh\xi-\cos\eta} + O\{e^{\beta(\xi-\xi_0)}\}\right]. \tag{4·19}$$

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Now ξ_0 is related to the crack length b by (4·17), and it is easily shown that

$$\frac{e^{-\xi}-\cos\eta}{\cosh\xi-\cos\eta}=\frac{2a-c}{a},$$

$$\frac{\sinh\xi}{\cosh\xi-\cos\eta}=-\frac{a-c}{a},$$

where c is the distance of the load line from the crack mouth. Hence (4·19) may be written

$$\mathscr{K}_P^C(b)=\frac{P}{(2\pi ab)^{\frac{1}{2}}}\left[\frac{b(2a-c)}{K_+(i)\,(2a-b)^{\frac{3}{2}}}+\frac{2(a-c)}{K_+(0)\,(2a-b)^{\frac{1}{2}}}\right]+\frac{Pa^{\frac{1}{2}}}{b}O\left\{\left(\frac{R}{R'}\right)^{\beta}\left(\frac{2a-b}{b}\right)^{\beta-\frac{1}{2}}\right\} \quad (4\cdot20)$$

as

$$\frac{R(2a-b)}{R'b}\to 0,$$

where the distances R, R' are shown in Fig. 4.

Notes. (i) The condition $[R(2a-b)]/R'b\to 0$ is certainly satisfied if $c\to 0$ or $b\to 2a$ with $b-c>\epsilon>0$.

(ii) It will be observed that the y-co-ordinates of the points C, C', and the elastic constants E, ν only enter into the expression (4·20) in the error term.

(iii) (4·20) may be written in terms of (2·16), (2·17) as

$$\mathscr{K}_P^C=\mathscr{K}_P+\mathscr{K}_M\big|_{M=-Pc}+\frac{Pa^{\frac{1}{2}}}{b}O\left\{\left(\frac{R}{R'}\right)^{\beta}\left(\frac{2a-b}{b}\right)^{\beta-\frac{1}{2}}\right\}. \quad (4\cdot21)$$

This proves that the stress intensity factor in general pin-loading differs from the stress intensity factor due to a *statically equivalent loading* at the crack mouth by only the error term

$$\frac{Pa^{\frac{1}{2}}}{b}O\left\{\left(\frac{R}{R'}\right)^{\beta}\left(\frac{2a-b}{b}\right)^{\beta-\frac{1}{2}}\right\},$$

a striking example of St Venant's principle.

5. *The displacement field in general pin-loading via the universal function*

By the constancy of the universal function (4·1), it follows that in (4·15) we may replace the V and $\mathscr{K}$ (appropriate to the 'pressurized crack' loading) with the corresponding quantities V_P^C, $\mathscr{K}_P^C$ of pin-loading, to give

$$\frac{\partial}{\partial b}V_P^C(\mathbf{r};b)=\frac{2\pi}{PE}\mathscr{K}_P^C(b)\,\mathscr{K}_P^{\mathbf{r}}(b). \quad (5\cdot1)$$

Since we have the asymptotic estimate (4·20) for $\mathscr{K}_P^C$, $\mathscr{K}_P^{\mathbf{r}}$, an integration with respect to b will give a corresponding estimate for $V_P^C(\mathbf{r};b)$; in particular we may estimate the opening at any station along the crack. (This integration will however introduce a $O(1)$ error term). Of particular interest are the openings δ, Δ of the crack at the

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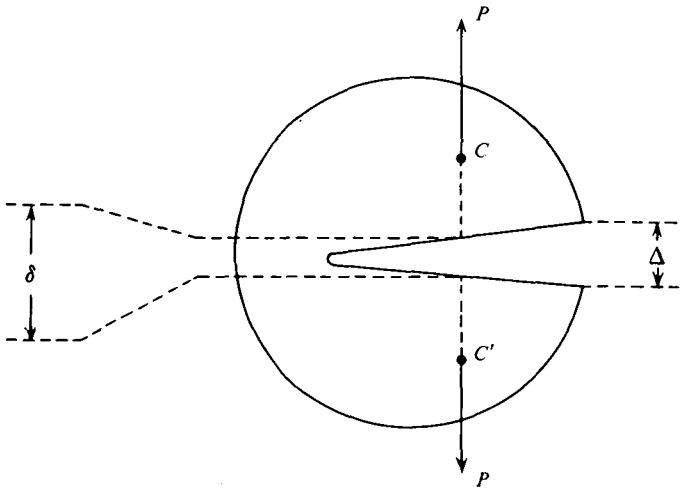


Fig. 3

load-line, and at the crack mouth, as depicted in Fig. 3. The details are straightforward and δ, Δ are found to be given by

$$\frac{E}{P}\delta = \frac{(2a-c)^2}{2a^2\{K_+(i)\}^2}\left(\frac{b}{2a-b}\right)^2 + \frac{4(2a-c)(a-c)}{a^2K_+(i)K_+(0)}\left(\frac{b}{2a-b}\right) + \frac{4(a-c)^2}{a^2\{K_+(0)\}^2}\log\left(\frac{b}{2a-b}\right) + A + O\left\{\left(\frac{2a-b}{b}\right)^{\beta-1}\right\}, \tag{5·2}$$

$$\frac{E}{P}\Delta = \frac{2a-c}{a\{K_+(i)\}^2}\left(\frac{b}{2a-b}\right)^2 + \frac{2(4a-3c)}{aK_+(i)K_+(0)}\left(\frac{b}{2a-b}\right) + \frac{4(a-c)}{a\{K_+(0)\}^2}\log\left(\frac{b}{2a-b}\right) + B + O\left\{\left(\frac{2a-b}{b}\right)^{\beta-1}\right\}, \tag{5·3}$$

as $b/a \to 2$, where A, B depend upon the position of the loading points C, C', but are independent of b.

6. *A more accurate estimate for* Δ *via the 'point forces at the crack mouth' solution*

It follows from the reciprocal theorem that Δ, the opening of the crack mouth when the disc is pin-loaded at C, C', is equal to the increase in separation of C, C' when the disc is similarly loaded at the crack mouth. This last quantity may be calculated from the solution (2·12), i.e. from

$$\phi_1(\xi,\eta), \quad \phi_2(\xi,\eta).$$

Now it follows from Section 4 that $V_1(\xi,\eta)$, the displacement of the point (ξ,η) in the y-direction, corresponding to the potential $\phi_1(\xi,\eta)$ is given by

$$EV_1(\xi,\eta) = \frac{i\pi e^{\xi_0}}{K_+(i)}\cdot\frac{\cos\eta - e^{-\xi}}{\cosh\xi - \cos\eta} + \frac{4i\pi}{K_+(0)}\cdot\frac{\sinh\xi}{\cosh\xi-\cos\eta} + O(e^{\xi-\xi_0}), \tag{6·1}$$

as $\xi - \xi_0 \to -\infty$, uniformly for $\frac{1}{2}\pi \leqslant \eta \leqslant \pi$.

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The ϕ_2 potential is treated in an essentially similar manner. The corresponding displacement potential $Q_2(\xi, \eta)$ is given by

$$\frac{Q_2}{J} = 2i\cos\eta\, R_2(\xi-\xi_0,\eta) + 2i\sin\eta S_2(\xi-\xi_0,\eta) - \frac{2i\pi}{K_+(i)}e^{\xi_0-\xi}, \tag{6·2}$$

where
$$R_2(\xi,\eta) = \int_{-\infty}^{\infty}\left\{-\frac{K_+(\alpha)\sinh\alpha(\pi-\eta)}{\alpha} + \frac{\cosh\alpha(\pi-\eta)}{\alpha^2(\alpha^2+1)K_-(\alpha)}\right\}e^{-i\alpha\xi}d\alpha, \tag{6·3}$$

$$S_2(\xi,\eta) = \int_{-\infty}^{\infty}\left\{\frac{\sinh\alpha(\pi-\eta) - \coth\tfrac{1}{2}\pi\alpha\cosh\alpha(\pi-\eta)}{\alpha(\alpha^2+1)K_-(\alpha)}\right\}e^{-i\alpha\xi}d\alpha. \tag{6·4}$$

The main difference between (6·3), (6·4) and (4·7), (4·8) is that here the integrands have *double* poles at $\alpha = 0$. Proceeding in the same manner as in Section 4 we find that

$$\frac{Q_2}{J} = -\frac{e^{-(\xi-\xi_0)}}{2K_+(i)} + 4\pi i\left(\cos\eta - \frac{2}{\pi}\sin\eta\right)\left(\frac{\xi-\xi_0}{K_+(0)} - D\right) + O\{e^{\beta(\xi-\xi_0)}\}, \tag{6·5}$$

as $\xi-\xi_0 \to -\infty$, uniformly for $\frac{1}{2}\pi \leqslant \eta \leqslant \pi$, where D is a real constant given by

$$D = -i\frac{K_+'(0)}{\{K_+(0)\}^2}. \tag{6·6}$$

D is evaluated numerically in Appendix 2 and found to be

$$D = 1{\cdot}44998\ldots, \tag{6·7}$$

correct to 6 significant figures.

From (6·2) the expression for $V_2(\xi,\eta)$ is eventually found to be

$$EV_2(\xi,\eta) = \frac{2\pi i\, e^{\xi_0}}{K_+(i)}\cdot\frac{\cos\eta - e^{-\xi}}{\cosh\xi - \cos\eta} - 4\pi i\left(\frac{\xi-\xi_0}{K_+(0)} - D\right)\frac{\sinh\xi}{\cosh\xi-\cos\eta}$$
$$+ \frac{4\pi i}{K_+(0)}\cos\eta\left(\cos\eta - \frac{2}{\pi}\sin\eta\right) + O\{e^{\xi-\xi_0}\}, \tag{6·8}$$

as $\xi-\xi_0 \to -\infty$, uniformly for $\frac{1}{2}\pi \leqslant \eta \leqslant \pi$. Now Δ is given by

$$\Delta = \frac{iPe^{\xi_0}}{\pi K_+(i)}V_1(\xi,\eta) + \frac{iP}{\pi K_+(0)}V_2(\xi,\eta), \tag{6·9}$$

where the co-ordinates (ξ,η) appearing in (6·9) are now those of the loading point C. On substituting (6·1), (6·8) into (6·9) we obtain

$$\frac{E}{P}\Delta = \frac{2a-c}{a\{K_+(i)\}^2}\left(\frac{b}{2a-b}\right)^2 + \frac{2(4a-3c)}{aK_+(i)K_+(0)}\left(\frac{b}{2a-b}\right)$$
$$+ \frac{4(a-c)}{a\{K_+(0)\}^2}\log\left(\frac{R'b}{R(2a-b)}\right) + \frac{4}{K_+(0)}\left\{D - \frac{1}{K_+(0)}\cos\eta\left(\cos\eta - \frac{2}{\pi}\sin\eta\right)\right\} + O\left(\frac{R}{R'}\right), \tag{6·10}$$

as
$$\frac{R(2a-b)}{R'b} \to 0.$$

In (6·10) c is, as usual, the distance of the load-line from the crack mouth, and R, R', η are shown in Fig. 4.

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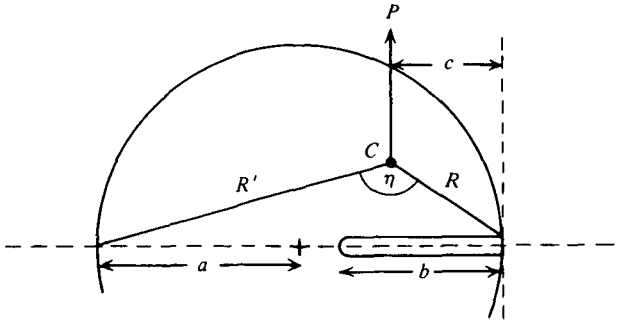


Fig. 4

Notes. (i) (6.10) is consistent with the weaker formula (5·3).

(ii) An even sharper estimate for Δ may be given when the loading points C, C' lie on the crack faces. In this case $R = c$, $R' = 2a - c$, $\eta = \pi$, and it may be shown that the $O(c/2a - c)$ term in (6·10) may be replaced by

$$-\left(\frac{4D}{K_+(0)} + \pi\right)\frac{c}{a} + O\left\{\left(\frac{c}{2a-c}\right)^{\beta}\left(\frac{2a-b}{b}\right)^{\beta-1}\right\}, \tag{6·11}$$

as

$$\frac{c(2a-b)}{b(2a-c)} \to 0.$$

(6·11) is useful in that it gives an indication of the sign and typical magnitude of the error term in (6·10) and of the number B in (5·3).

7. *Comparison with related work*

(i) In the limit as $a \to \infty$ with b fixed,

$$\mathscr{K}_P \to \frac{P}{\pi^{\frac{1}{2}} K_+(0)\, b^{\frac{1}{2}}} = \left(\frac{2}{\pi^2 - 4}\right)^{\frac{1}{2}} \frac{P}{b^{\frac{1}{2}}}, \tag{7·1}$$

$$\mathscr{K}_M \to 0. \tag{7·2}$$

It may be verified that (7·1) is the result (in the point forces problem) for a normal edge crack in a *half-plane*, a problem treated by Eshelby (4) and Ouchterlony (10).

(7·2) is a curious result since it implies that in the point couples problem for a normal edge crack in a half-plane, the stress intensity factor should be zero. This is in fact so, a result which is discussed further in Appendix 1 of this paper.

(ii) The circular disc with a radial edge crack under pin-loading has been investigated numerically by Gross (6), (7) and Newman (9). Gross considered a cracked disc truncated at the required load-line position, and replaced the actual point loadings by a smooth distribution of surface tractions acting over the truncated portion of the boundary. This modified problem was then treated numerically by a boundary collocation technique. Newman (9) has communicated to the author results of numerical boundary collocation calculations for the actual problem with the geometry and loading shown in Fig. 2, in the case when C has co-ordinates $(\frac{1}{2}a, \frac{2}{5}a)$, and Poisson's ratio for the disc is 0·3. It is interesting to compare the values of $\mathscr{K}_P^C$ predicted by (4·20) (with the error term neglected) with these numerical calculations.

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Table 1. *Values of* $a^{\frac{1}{2}}\mathscr{K}_P^C/P$ *to* 4 *significant figures in the case* $c = \frac{1}{2}a$

b/a	Equation (4·20)	Ref. (9)	Ref. (6)
0·8	1·566	1·524	1·546
0·9	1·798	—	1·782
1·0	2·095	2·087	2·073
1·1	2·481	—	2·480
1·2	2·997	2·996	2·998
1·3	3·708	—	3·706
1·4	4·733	4·733	4·734
1·5	6·304	—	6·309
1·6	8·927	8·928	8·929
1·7	13·93	—	13·93
1·8	25.92	—	25·91
1·9	74·28	—	[73·6]

Table 2. *Values of* $E\delta/P$ *to* 4 *significant figures in the case* $c = \frac{1}{2}a$

b/a	Equation (5·2)	Ref. (9)	Ref. (7)
0·8	9·335	7·602	8·43
0·9	12·88	—	12·01
1·0	17·62	15·86	16·76
1·1	24·16	—	23·31
1·2	33·53	31·76	32·68
1·3	47·53	—	46·68
1·4	69·67	67·90	68·80
1·5	107·4	—	106·5
1·6	178·6	176·9	177·8
1·7	336·8	—	336·0
1·8	801·1	—	800·4
1·9	3379	—	[3357]

Table 3. *Values of* $E\Delta/P$ *to* 4 *significant figures in the case* $c = \frac{1}{2}a$

b/a	Equation (5·3)	Ref. (9)
0·8	14·10	13·35
1·0	26·40	25·51
1·2	49·36	48·44
1·4	100·3	99·38
1·6	250·8	250·0

This comparison is set out in Table 1. The maximum discrepancy between equation (4·20) and (9) occurs (predictably) at the least value of b/a considered, $b/a = 0{\cdot}8$, and is less than 3 %; for $b/a \geqslant 1$, the error term in (4·20) appears to be insignificant. It is also apparent that the approximate model used by Gross gives accurate estimates of $\mathscr{K}$ values when the crack length is not too short.

Similar comparisons may be made for δ, the crack opening at the load-line, and Δ, the opening at the crack mouth. These comparisons are set out in Tables 2 and 3.

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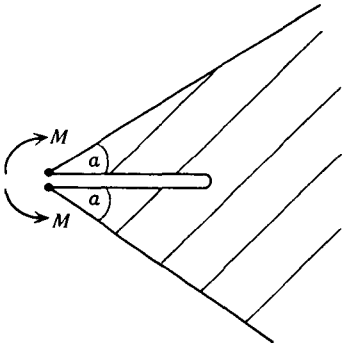


Fig. 5

The comparison between equations (5·2), (5·3) (with the unknown constant and error terms neglected) and (9) as given in Tables 2 and 3 is very satisfactory, showing an almost constant difference in values for the larger b/a. These differences suggest that when C has co-ordinates $(\frac{1}{2}a, \frac{2}{5}a)$ and Poisson's ratio is 0·3, then the constants A, B appearing in (5·2), (5·3) have the approximate values $A \doteqdot -1{\cdot}77$, $B \doteqdot -0{\cdot}9$. Once again the Gross model gives accurate values for δ when b/a is not too small.

8. *Concluding note.* We have seen in Section 7 that the formula (4·20) for $\mathscr{K}_P^C$ and the formulae (5·2), (5·3) for δ, Δ (with the appropriate error terms neglected), will certainly give accurate estimates for these quantities when $c \leqslant \frac{1}{2}a$ and $b \geqslant a$. Moreover Newman (8), in a careful numerical study, has shown (for a similar rectangular specimen) that, within this same range of the parameters, the effect of the finite diameter of the pin-loading holes may be neglected. The author therefore suggests that these formulae should be useful in the callibration of a fracture test specimen of this geometry.

Appendix 1. *A wedge of general angle containing a symmetric edge crack at the vertex, loaded by opposing point couples at the crack mouth*

In Section 7 (i) it was noted that for the couple-loaded disc problem, as $a \to \infty$ with b fixed,

$$\mathscr{K}_M \to 0. \tag{A 1·1}$$

This suggests that in the problem where opposing point couples act at the mouth of a normal edge crack in a *half-plane*, the stress intensity factor is zero.

Consider then the more general geometry shown in Fig. 5. A wedge of angle 2α has a symmetric edge crack of length b emanating from its vertex, and is loaded by opposing point couples M at the crack mouth. The author has been unable to find, in the vast literature on cracks, a published solution of this problem. [Doran (3) considered the pressurized crack; Ouchterlony (10) used Doran's solution to obtain (indirectly) the stress intensity factor when opposing point *forces* act at the crack mouth.] There is little difficulty however in obtaining the required solution by use of the Mellin transform. The details are similar to those in (3), and are therefore omitted. The result for the stress intensity factor $\mathscr{K}_M(\alpha)$ is

$$\mathscr{K}_M(\alpha) = \frac{(8/\pi)^{\frac{1}{2}} \cos\alpha\, k_+(1,\alpha)}{\sin\alpha - \alpha\cos\alpha} \cdot \frac{M}{b^{\frac{3}{2}}}, \tag{A 1·2}$$

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Table 4. *Values of* $M^{-1}b^{\frac{3}{2}}\mathscr{K}_M(\alpha)$ *to 4 decimal places*

α	$M^{-1}b^{\frac{3}{2}}\mathscr{K}_M(\alpha)$
$0{\cdot}2\pi$	1·6237
$0{\cdot}3\pi$	0·5787
$0{\cdot}4\pi$	0·1831
$0{\cdot}5\pi$	0
$0{\cdot}6\pi$	−0·0909
$0{\cdot}7\pi$	−0·1346
$0{\cdot}8\pi$	−0·1528
$0{\cdot}9\pi$	−0·1584
π	−0·1592

where $k_{\pm}(p,\alpha)$ are the functions obtained by Wiener–Hopf multiplicative decomposition of the meromorphic function

$$k(p,\alpha) = \frac{\sin^2\alpha p - p^2\sin^2\alpha}{p(1-p^2)\{\sin 2\alpha p + p\sin 2\alpha\}}. \tag{A 1·3}$$

$k_+(1,\alpha)$ has the properties

$$\text{(i)}\; k_+(1,\alpha) \sim \frac{\alpha^{\frac{3}{2}}}{2\sqrt{3}} \quad \text{as} \quad \alpha \to 0+, \tag{A 1·4}$$

$$\text{(ii)}\; k_+(1,\alpha) > 0 \quad (0 < \alpha \leqslant \pi), \tag{A 1·5}$$

$$\text{(iii)}\; k_+(1,\pi) = \frac{\sqrt{\pi}}{4\sqrt{2}}. \tag{A 1·6}$$

Since $\sin\alpha - \alpha\cos\alpha > 0$ $(0 < \alpha \leqslant \pi)$, it follows from (A 1·2), (A 1·5) that

$$\operatorname{sgn}\{\mathscr{K}_M(\alpha)\} = \operatorname{sgn}\{\cos\alpha\}. \tag{A 1·7}$$

Thus $\mathscr{K}_M(\alpha)$ changes sign once at $\alpha = \frac{1}{2}\pi$ and

$$\mathscr{K}_M(\tfrac{1}{2}\pi) = 0. \tag{A 1·8}$$

This confirms our deduction from (A 1·1) and shows that the vanishing of $\mathscr{K}_M(\alpha)$ occurs only at the single value $\alpha = \frac{1}{2}\pi$.

In fact the function $k_+(1,\alpha)$ is related to the function $H^+(\frac{1}{2})$ tabulated by Doran (3), p. 371, and using Doran's figures together with (A 1·2) we obtain the values of $\mathscr{K}_M(\alpha)$ in Table 4.

Naturally the effect of interference between the crack faces is neglected in the above solution.

Appendix 2. *Numerical evaluation of the constant D*

The constant D was defined by (6·6) to be

$$D = -i\frac{K'_+(0)}{\{K_+(0)\}^2}, \tag{A 2·1}$$

where $K_+(\alpha)$ is defined in (5), Appendix 1. From this definition it follows that

$$-i\frac{K'_+(0)}{K_+(0)} = \log 2 + \sum_{n=1}^{\infty}\left\{\frac{1}{n} - \frac{2\operatorname{Im}(\alpha_n)}{|\alpha_n|^2}\right\}, \tag{A 2·2}$$

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where $\{\alpha_n\}$ $(n \geqslant 1)$ are the roots (lying in the positive quadrant) of

$$\sinh^2 \tfrac{1}{2}\pi\alpha = \alpha^2. \tag{A 2·3}$$

Now the $\{\alpha_n\}$ can be evaluated numerically and substituted in (A 2·2), but the convergence of the series in (A 2·2) is very slow since its terms are $O(1/n^2)$ as $n \to \infty$. However, it may be shown from the known asymptotics of $\{\alpha_n\}$ as $n \to \infty$, that

$$\sum_{n=N+1}^{\infty} \left\{\frac{1}{n} - \frac{2\mathrm{Im}\,(\alpha_n)}{|\alpha_n|^2}\right\} = \frac{1}{2(N+1)} + \frac{1}{8N^2} + 8\left\{\frac{\log(4N+2)}{\pi(4N+2)}\right\}^2 + O\left\{\frac{\log^2 N}{N^3}\right\}, \tag{A 2·4}$$

as $N \to \infty$.

Thus it is much more efficient to combine (A 2·2), (A 2·4) when summing this series numerically. Using this device, D was found to be

$$D = 1{\cdot}44998\ldots, \tag{A 2·5}$$

correct to 6 significant figures, this accuracy being attained at $N = 300$. (Without using (A 2·4) $N = 5000$ yielded only 3 figure accuracy).

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