100.29 Some odd series for π

Leibniz's alternating series (1673) for π is one of the best known series representations for the number π:

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots + \frac{(-1)^n}{2n+1} + \ldots = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}. \quad (1)$$

Leibniz obtained this series for π by choosing $x = 1$ in the series for the arctangent function:

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots + \frac{(-1)^n x^{2n+1}}{2n+1} + \ldots . \quad (2)$$

He found this series by applying his own method of *arithmetical quadrature* to a circle. He was very pleased with the result, and communicated it almost immediately to Christiaan Huygens. Note that the series (2) was already found in 1670 by James Gregory [1].

The result was published in Leibniz's paper [2], where the series (1) was accompanied by the following figure:

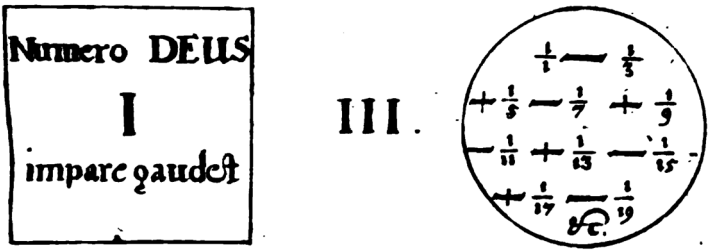


FIGURE 1

probably representing the quadrature of the circle. In the square on the left Leibniz quotes the Roman poet Virgil: God loves odd numbers, and he illustrates this in the circle on the right. Note that the series (1) converges very slowly, something that Leibniz also knew.

It appears that Leibniz's series is but one of an infinite number of similar series for $\pi/4$, using only odd numbers, the next two being

$$\frac{\pi}{4} = \sum_{n=-1}^{\infty} \frac{2!\,(-1)^n}{(2n+1)(2n+3)(2n+5)}, \quad (3)$$

and

$$\frac{\pi}{4} = \sum_{n=-2}^{\infty} \frac{4!\,(-1)^n}{(2n+1)(2n+3)(2n+5)(2n+7)(2n+9)}.$$

The reader should try to prove (3) by expanding the general term of the series in partial fractions and summing the result using (1).

In this note we will prove the following general theorem.

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Theorem 1: We have for $m = 0, 1, 2, \ldots$

$$\frac{\pi}{4} = \sum_{n=-m}^{\infty} \frac{(2m)!\,(-1)^n}{(2n+1)(2n+3)\ldots(2n+4m+1)}. \tag{4}$$

Proof: We use some properties of the digamma function $\psi(x)$, the logarithmic derivative of the gamma function [3]. The digamma function satisfies the recurrence relation

$$\psi(x+1) = \psi(x) + \frac{1}{x}.$$

Furthermore, it has the following series expansion:

$$\psi(x) = -\gamma + \sum_{n=0}^{\infty}\left(\frac{1}{n+1} - \frac{1}{n+x}\right), \qquad x \neq 0, -1, -2, \ldots$$

where γ is Euler's constant. From this we deduce that

$$\psi(x) - \psi(y) = \sum_{n=0}^{\infty}\left(\frac{1}{n+y} - \frac{1}{n+x}\right), \qquad x, y \neq 0, -1, -2, \ldots . \tag{5}$$

By logarithmically differentiating the reflection property of the gamma function we find the reflection formula for the digamma function, which we shall need later on,

$$\psi(1-x) - \psi(x) = \pi \cot \pi x. \tag{6}$$

We start by changing the index of summation $(n \to n - m)$ in the series of the theorem, and we will prove that,

$$\frac{(-1)^m}{(2m)!}\cdot\frac{\pi}{4} = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n-2m+1)(2n-2m+3)\ldots(2n+2m+1)}. \tag{7}$$

Using Heaviside's cover-up method we expand the general term of this series in partial fractions. At this point we don't take the factor $(-1)^n$ into account. The different terms of the partial fraction expansion are

$$\begin{aligned}
&\frac{1}{2\times 4\times 6\times \ldots \times 4m}\times\frac{1}{2n-2m+1}\\
&\frac{1}{(-2)\times 2\times 4\times \ldots \times(4m-2)}\times\frac{1}{2n-2m+3}\\
&\qquad\qquad\ldots\\
&\frac{1}{(-(4m-2))\times(-(4m-4))\times \ldots\times(-2)\times 2}\times\frac{1}{2n+2m-1}\\
&\frac{1}{(-4m)\times(-(4m-2))\times \ldots\times(-4)\times(-2)}\times\frac{1}{2n+2m+1}.
\end{aligned}$$

The general term is given by

$$\frac{1}{2^{2m}(-1)^k(2m-k)!\,k!}\times\frac{1}{2n-2m+2k+1}, \qquad (k = 0, 1, \ldots, 2m).$$

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If n is even, we take $n = 2n'$, and the corresponding term of the series (7) is

$$\sum_{k=0}^{2m} \frac{1}{2^{2m}(-1)^k(2m-k)!\,k!} \times \frac{1}{4n'-2m+2k+1}.$$

If n is odd, $n = 2n'+1$, the corresponding term is

$$-\sum_{k=0}^{2m} \frac{1}{2^{2m}(-1)^k(2m-k)!\,k!} \times \frac{1}{4n'+2m-2k+3}$$

where we have changed the order of the terms $(k \to 2m-k)$. Combining these two consecutive terms, we rewrite the series (7) as

$$\sum_{n'=0}^{\infty}\sum_{k=0}^{2m} \frac{1}{2^{2m}(-1)^k(2m-k)!\,k!}\left(\frac{1}{4n'-2m+2k+1} - \frac{1}{4n'+2m-2k+3}\right).$$

As a consequence of (5), this is equal to

$$\sum_{k=0}^{2m} \frac{(-1)^k}{4\times 2^{2m}(2m-k)!\,k!}\left(\psi\left(\frac{2m-2k+3}{4}\right) - \psi\left(\frac{-2m+2k+1}{4}\right)\right).$$

Note that we had to combine the terms in the partial fraction expansion two by two to get convergent series.

Since $\frac{1}{4}(2m-2k+3) = 1 - \frac{1}{4}(-2m+2k+1)$ we may now use the reflection property (6) of the digamma function for the factor between brackets

$$\psi\left(\frac{2m-2k+3}{4}\right) - \psi\left(\frac{-2m+2k+1}{4}\right) = \pi\cot\frac{(-2m+2k+1)\pi}{4} = \pi(-1)^{m-k}.$$

Hence we find the sum of the series (7) is equal to

$$\sum_{k=0}^{2m} (-1)^m \frac{\pi}{4\times 2^{2m}(2m-k)!\,k!}.$$

Note that this can be rewritten as

$$\frac{(-1)^m}{(2m)!} \times \frac{\pi}{4\times 2^{2m}} \sum_{k=0}^{2m}\binom{2m}{k} = \frac{(-1)^m}{(2m)!} \times \frac{\pi}{4}$$

where we have used a property of the binomial coefficients. This proves the theorem.

We have the following related result, for the case that the number of factors in the denominator is even.

Corollary: For $m = 1, 2, \ldots$, we have

$$\frac{\pi}{4} = -\frac{(2m-1)!}{2(2m-1)!!^2} + \sum_{n=-m}^{\infty} \frac{(2m-1)!(-1)^n}{(2n+1)(2n+3)\ldots(2n+4m-1)}.$$

Note that $(2m-1)!! = (2m-1)\times(2m-3)\times\ \ldots\ \times 3\times 1$.

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Proof: This follows immediately from the following observation.

$$\sum_{n=-m}^{\infty}\left(\frac{(-1)^n}{(2n+1)(2n+3)\dots(2n+4m-1)}-\frac{(-1)^n}{(2n+3)(2n+5)\dots(2n+4m+1)}\right)$$

$$=\sum_{n=-m}^{\infty}\frac{(-1)^n 4m}{(2n+1)(2n+3)\dots(2n+4m+1)}. \tag{8}$$

The sum of the first expression between brackets of the left-hand side is the one we want to calculate. For the second term, we change the index of summation to get

$$\sum_{n=-m}^{\infty}\frac{(-1)^n}{(2n+3)(2n+5)\dots(2n+4m+1)}$$

$$=\sum_{n=-m+1}^{\infty}\frac{(-1)^{n-1}}{(2n+1)(2n+3)\dots(2n+4m-1)}$$

$$=\frac{(-1)^m}{(-2m+1)(-2m+3)\dots(2m-1)}-\sum_{n=-m}^{\infty}\frac{(-1)^n}{(2n+1)(2n+3)\dots(2n+4m-1)}$$

$$=\frac{1}{(2m-1)!!^2}-\sum_{n=-m}^{\infty}\frac{(-1)^n}{(2n+1)(2n+3)\dots(2n+4m-1)}.$$

Combining the two terms, we find that the left-hand side of (8) is equal to

$$-\frac{1}{(2m-1)!!^2}+2\sum_{n=-m}^{\infty}\frac{(-1)^n}{(2n+1)(2n+3)\dots(2n+4m-1)}.$$

Using (4), we find that the right-hand side of (8) is equal to

$$\sum_{n=-m}^{\infty}\frac{(-1)^n 4m}{(2n+1)(2n+3)\dots(2n+4m+1)}=\frac{\pi}{2}\frac{1}{(2m-1)!}.$$

Combining the two results proves the corollary.

In his paper [2] Leibniz notes that combining the terms of the series (1) two by two, leads to a slightly faster converging series:

$$\frac{\pi}{4}=\sum_{n=0}^{\infty}\frac{2}{(4n+1)(4n+3)}. \tag{9}$$

If we do the same with the series (4), we find the following generalisations of (9).

$$\frac{\pi}{4}=\sum_{n=-k}^{\infty}\frac{2(4k+1)!}{(4n+1)(4n+3)\dots(4n+8k+1)(4n+8k+3)}$$

and

$$\frac{\pi}{4}=-\sum_{n=-k}^{\infty}\frac{2(4k+3)!}{(4n-1)(4n+1)\dots(4n+8k+3)(4n+8k+5)}$$

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for $k = 0, 1, 2, \ldots$.

Note that an infinite number of other generalisations of the series (1) can be found, for instance

$$\frac{\pi}{4} = \sum_{n=-3}^{\infty} \frac{18\,(-1)^n}{(2n+1)(2n+7)(2n+13)}$$

and $$\frac{\pi}{4} = \sum_{n=-10}^{\infty} \frac{15000\,(-1)^n}{(2n+1)(2n+11)(2n+21)(2n+31)(2n+41)}.$$

In general, we have the following theorem:

Theorem 2: We have for $k, m = 0, 1, 2, \ldots$

$$\frac{\pi}{4} = \sum_{n=-(2k+1)m}^{\infty} \frac{(-1)^n (2k+1)^{2m} (2m)!}{\prod_{i=0}^{2m} \left(2n + 2(2k+1)i + 1\right)}.$$

For $k = 0, 1, 2, \ldots$ and for $m = 1, 2, 3, \ldots$ we have

$$\frac{\pi}{4} = (-1)^{k+1} \frac{(2m-1)!}{2(2k+1)(2m-1)!!^2} + \sum_{n=-(2k+1)m+k}^{\infty} \frac{(-1)^n (2k+1)^{2m-1} (2m-1)!}{\prod_{i=0}^{2m-1} \left(2n + 2(2k+1)i + 1\right)}.$$

The proof of Theorem 1 can be adapted to this more general case.

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