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Epimorphism classes and relatively maximal metrics in large-scale geometry

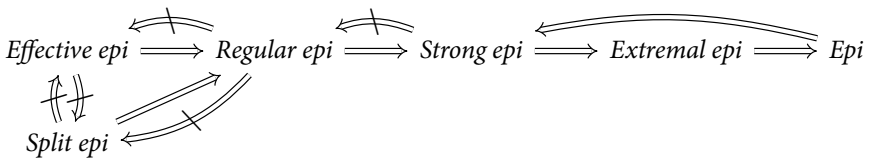
Robert Tang

Abstract. We consider epimorphisms and several variant notions – split, effective, regular, strong, and extremal – and determine which of these coincide in the metric coarse and coarsely Lipschitz categories. In particular, we characterize extremal epis in the coarsely Lipschitz category via a relative maximality condition on the codomain metric; this can be viewed as a morphism-relative analog of Rosendal's maximal metrics for topological groups.

1 Introduction

In this article, we study several classes of epimorphisms from the viewpoint of large-scale geometry. We first consider the metric coarse category **Coarse**, whose objects are extended metric spaces and whose morphisms are closeness classes of controlled maps. In **Coarse**, the epimorphisms are the coarsely surjective maps, and the split epimorphisms are the coarse retractions. Our first main result records which intermediate epi-classes coincide. Here, the straight implication arrows hold in any category; we establish the curved arrows.

Theorem 1.1 *The following implications hold in* **Coarse***:*



Informally, a regular epi behaves like a quotient, while an effective epi is a quotient by a kernel. In **Coarse**, we show regular epis are those maps whose codomain, up to coarse equivalence, is obtained by "coarsely gluing" the domain to itself along some binary relation. Effective epis are a special case of these where the gluing relations coarsely stabilize to a canonical "coarse kernel" at some threshold; for

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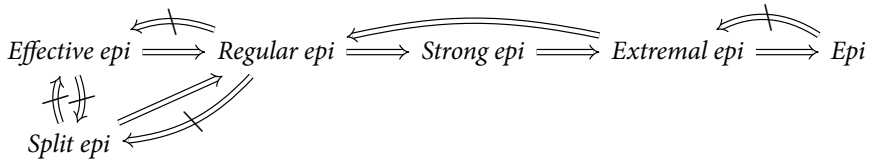
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example, surjective group homomorphisms with finitely generated image and kernel (Lemma 5.6). We characterize these via an explicit stability criterion for the associated coarse quotient and coarse kernel filtrations.

Next, we turn to the coarsely Lipschitz category $\overline{\mathbf{CLip}}$, where morphisms are required to have affine upper control. This is the natural setting for studying finitely generated groups equipped with their word metrics and is thus central to geometric group theory. Our second main result demonstrates a disparity in the behavior of epimorphisms when passing to this category.

Theorem 1.2 *The following implications hold in* $\overline{\mathbf{CLip}}$*:*



Here, extremal epis admit a geometric interpretation analogous to Rosendal's notion of maximal metrics on topological groups [Ros22, Ros23]. Define an ordering of metrics on a set Y by declaring $d \prec d'$ whenever the identity map $(Y, d') \to (Y, d)$ is coarsely Lipschitz. Say a left-invariant metric d on a topological group G is *compatible* if it generates the topology. A set $\Sigma \subseteq G$ is *coarsely bounded* if it has finite diameter under every compatible left-invariant metric on G. A compatible left-invariant metric d on G is *coarsely proper* if the d-bounded sets are coarsely bounded. For a Polish group G, the compatible left-invariant coarsely proper metrics coincide up to coarsely equivalence; a maximal metric, if it exists, picks out a canonical quasi-isometry class among these. This framework has stimulated much recent progress in geometric group theory, especially beyond the locally compactly generated setting [AV20, MRa23, MRo18].

Theorem 1.3 [Ros22, Proposition 2.52] *Let G be a Polish group equipped with a compatible coarsely proper left-invariant metric d. Then, the following are equivalent:*

(1) *(G, d) is quasigeodesic,*
(2) *d is $\prec$-maximal among all compatible left-invariant metrics on G, and*
(3) *G admits a coarsely bounded generating set Σ such that d is quasi-isometric to the word metric induced by Σ.*

Theorem 1.4 [Ros22, Theorem 1.2] *A Polish group G admits a $\prec$-maximal compatible left-invariant metric if and only if it admits a coarsely bounded generating set.*

In a similar vein, we characterize the extremal epis in $\overline{\mathbf{CLip}}$ via a *morphism-relative* maximality condition. Further, we determine exactly when an epi in $\overline{\mathbf{CLip}}$ can be promoted (by replacing the codomain metric with a coarsely equivalent one) to an extremal epi in $\overline{\mathbf{CLip}}$; our proof yields an explicit construction via an augmented Rips graph construction.

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Proposition 1.5 *A coarsely Lipschitz map $f\colon (X, d_X) \to (Y, d_Y)$ represents an extremal epi in $\overline{\mathbf{CLip}}$ if and only if it is coarsely surjective, and d_Y is $\prec$-maximal among all coarsely equivalent metrics d' on Y such that the induced map $(X, d_X) \to (Y, d')$ is coarsely Lipschitz.*

Theorem 1.6 *Let $f\colon (X, d_X) \to (Y, d_Y)$ be a coarsely surjective coarsely Lipschitz map. Then, the following are equivalent:*

(1) *there exists a metric d' coarsely equivalent to d_Y such that the map $f'\colon (X, d_X) \to (Y, d')$ induced by f represents an extremal epi in $\overline{\mathbf{CLip}}$,*
(2) *there exists a $\prec$-maximal element among all metrics d' coarsely equivalent to d_Y such that the induced map $f'\colon (X, d_X) \to (Y, d')$ is coarsely Lipschitz,*
(3) *there exists a $\prec$-greatest element among all metrics d' coarsely equivalent to d_Y such that the induced map $f'\colon (X, d_X) \to (Y, d')$ is coarsely Lipschitz,*
(4) *f factors (up to closeness) as me with e an extremal epi and m a mono in $\overline{\mathbf{CLip}}$, and*
(5) *f represents a regular epi in $\overline{\mathbf{Coarse}}$.*

Moreover, any relatively $\prec$-maximal metric d' is unique up to quasi-isometry.

The principal implication from Theorem 1.2 is that extremal epis are regular in $\overline{\mathbf{CLip}}$. In other words, relative maximality of the codomain metric is equivalent to quotient-like behavior of the map. Since regular epis are stable under pushouts (when they exist) [AHS06, Proposition 11.18], it follows that relative maximality persists under coarse gluings. See Section 2.2 for an explicit construction of pushouts in this setting.

Corollary 1.7 *Let $f\colon A \to X$ and $g\colon A \to Y$ be coarsely Lipschitz maps. If f represents an extremal epi in $\overline{\mathbf{CLip}}$, then so does its pushout $Y \to X \sqcup_A Y$ along g.*

It is worth contrasting Theorems 1.1 and 1.2 with their dual counterparts. In prior work [Tana], it was shown that $\overline{\mathbf{Coarse}}$ and $\overline{\mathbf{CLip}}$ are coregular categories; this implies that the effective, regular, strong, and extremal monos coincide. Moreover, the regular monos in $\overline{\mathbf{Coarse}}$ (resp. $\overline{\mathbf{CLip}}$) are identified with the coarse (resp. quasi-isometric) embeddings.

Theorem 1.8 [Tana, Corollary 1.4, Proposition 1.5, and Theorem 1.8] *In $\overline{\mathbf{Coarse}}$ (or $\overline{\mathbf{CLip}}$), the effective, regular, strong, and extremal monos coincide, while the split monos form a strictly stronger class. Every mono in $\overline{\mathbf{Coarse}}$ is regular, while the same does not hold in $\overline{\mathbf{CLip}}$.*

In summary, our results demonstrate that coarse analogs of quotients and kernels depend more on categorical context than do coarse analogs of embeddings.

Relation to previous work. The notion of a *coarse quotient* between metric spaces was introduced by Zhang [Zha15] and has been subsequently developed in [Bra22, Zha18]. We show that these coarse quotients are effective epis in $\overline{\mathbf{Coarse}}$ (Example 3.10). Higginbotham and Weighill [HW19] introduced a related notion,

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known as *weak coarse quotients*, in the setting of large-scale structures and showed these characterize regular epimorphisms in **Coarse**; they also observed that every weak coarse quotient of a large-scale structure admits a metric realization. Our treatment provides a purely metric approach to characterizing the regular epis. Other important classes of (coarsely) surjective maps in coarse geometry appear in [DZ17, MV13].

It is natural to ask whether maximal metrics on topological groups admit a purely categorical formulation. Our results suggest an approach via extremal epis but this is not immediate: topological group quotients require continuity, while here quotients are coarse. Nevertheless, it could be interesting to explore relative maximality for topological group homomorphisms within an appropriate categorical framework.

2 Preliminaries

2.1 Epimorphism classes

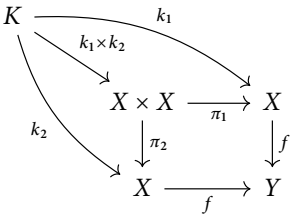
In any category, an *equalizer* of a parallel pair $f, g\colon A \rightrightarrows B$ is a morphism $e\colon E \to A$ such that $fe = ge$ with the universal property: for every morphism $h\colon X \to A$ with $fh = gh$, there exists a unique morphism $h'\colon X \to E$ such that $eh' = h$. Coequalizers are defined dually. Any (co)equalizer, if it exists, is unique up to unique isomorphism. The *kernel pair* of a morphism $f\colon X \to Y$ is the pullback of $f, f\colon X \rightrightarrows Y$ (if it exists). Let us recall some useful constructions of coequalizers and kernel pairs.

Proposition 2.1 [AHS06, Proposition 11.14] *In a category with binary coproducts, if*

$$\begin{array}{ccc} A \sqcup B & \xrightarrow{f\sqcup 1_Y} & B \\ {\scriptstyle g\sqcup 1_Y}\downarrow & & \downarrow{\scriptstyle q} \\ B & \xrightarrow[q]{} & Y \end{array}$$

is a pushout, then $q\colon B \to Y$ is the coequalizer of $f, g\colon A \rightrightarrows B$.

Proposition 2.2 *In a category with binary products, the kernel pair of $f\colon X \to Y$ is given by $k_1, k_2\colon K \rightrightarrows X$ if and only if $k_1 \times k_2\colon K \to X_1 \times X_2$ is the equalizer of $f\pi_1, f\pi_2\colon X \times X \rightrightarrows Y$.*



Proof The forward implication is [AHS06, Proposition 11.11]. The converse is straightforward. ∎

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We now introduce the various epimorphism classes of interest.

Definition 2.3 (Classes of epimorphisms) An epimorphism $f\colon X \to Y$ is:

(1) an *extremal epi* if whenever $f = me$ with m a mono, then m is an isomorphism,
(2) a *strong epi* if for every commutative square

$$\begin{array}{ccc} X & \longrightarrow & A \\ {\scriptstyle f}\downarrow & \overset{d}{\nearrow} & \downarrow{\scriptstyle m} \\ Y & \longrightarrow & B \end{array}$$

with m a mono, there exists a unique $d\colon Y \to A$ making both triangles commute,
(3) a *regular epi* if it is the coequalizer of some pair of parallel arrows,
(4) an *effective epi* if its kernel pair exists and f is the coequalizer of that kernel pair, and
(5) a *split* epi if it admits a right inverse.

Extremal, strong, regular, effective, and split monos are defined dually.

Proposition 2.4 [Bor94, Proposition 4.3.6] *In any category, the following implications hold: extremal epi* $\Longrightarrow$ *regular epi* $\Longrightarrow$ *strong epi* $\Longrightarrow$ *extremal epi* $\Longrightarrow$ *epi.*

Lemma 2.5 [AHS06, Proposition 7.59] *Any split epi is a regular epi.*

Lemma 2.6 [AHS06, Proposition 11.22] *Any regular epimorphism admitting a kernel pair is an effective epimorphism.*

Lemma 2.7 *In any category, every mono is strong if and only if every epi is strong.*

Proof Let $e\colon A \to B$ be an epimorphism and $m\colon X \to Y$ be a monomorphism. Suppose that $f\colon A \to X$ and $g\colon B \to Y$ are morphisms such that $mf = ge$. If m is a strong mono, then there exists a unique morphism $d\colon B \to X$ such that $f = de$ and $md = g$; hence, e is a strong epi. Similarly, if e is a strong epi, then such d exists, and hence m is a strong mono. ∎

2.2 Coarse geometry

Background on coarse geometry can be found in [CH16, LV23, NY12, Roe03].

Given an extended metric space (X, d_X) and $\kappa \geq 0$, we shall write $x \approx_\kappa x'$ to mean $d_X(x, x') \leq \kappa$ for $x, x' \in X$. Say two maps $f, g\colon X \rightrightarrows Y$ are *κ-close* if $fx \approx_\kappa gx$ for all $x \in X$; and *close*, denoted $f \approx g$, if they are κ-close for some $\kappa \geq 0$. Closeness is an equivalence relation on maps between a given pair of metric spaces. Write $\bar{f}$ for the closeness class of f.

An *upper control* for $f\colon X \to Y$ is a proper increasing function $\rho\colon [0, \infty) \to [0, \infty)$ such that $d_Y(fx, fx') \leq \rho d_X(x, x')$ for all $x, x' \in X$ such that $d_X(x, x') < \infty$. We say f is *controlled* if it admits an upper control, and *coarsely Lipschitz* if it has an affine

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upper control. A proper increasing function $\theta\colon [0,\infty) \to [0,\infty)$ is a *lower control* if $\theta d_X(x,x') \le d_Y(fx,fx')$ for all $x,x' \in X$, with the convention that $d_X(x,x') = \infty \implies d_Y(fx,fx') = \infty$. The existence of a lower control is equivalent to f being *uniformly metrically proper*: for every $R \ge 0$, there exists $S \ge 0$ such that $d_Y(fx,fx') \le R \implies d_X(x,x') \le S$ for all $x,x' \in X$. A *quasi-isometric embedding* is a coarsely Lipschitz map with an affine lower control. We say f is *coarsely surjective* if there exists $r \ge 0$ such that $N_r(f(X)) = Y$, where N_r denotes the (closed) metric r-neighborhood. Call f a *coarse equivalence* if it is coarsely surjective and admits a lower control; a coarsely surjective quasi-isometry is called a *quasi-isometry*.

Let **Coarse** be the category whose objects are extended metric spaces, with controlled maps as morphisms. Let $\mathbf{CLip} \subset \mathbf{Coarse}$ be the wide subcategory whose morphisms are coarsely Lipschitz. Closeness of controlled maps is preserved under composition, that is, $\overline{ff'} = \bar{f}\bar{f}'$ whenever defined. The *metric coarse category* $\overline{\mathbf{Coarse}}$ and the *coarsely Lipschitz category* $\overline{\mathbf{CLip}}$ are the respective quotient categories with the same objects, and with closeness classes as morphisms. Since many of our arguments will work for either category, we shall write $\overline{\mathbf{C}}$ to denote either $\overline{\mathbf{Coarse}}$ or $\overline{\mathbf{CLip}}$.

We recall some characterizations of morphisms from [CH16] (see [Tana] for proofs in the setting of extended metric spaces).

Proposition 2.8 [CH16, Propositions 3.A.16 and 3.A.22] *A morphism $\bar{f}$ in $\overline{\mathbf{C}}$ is a mono (resp. epi) if and only if any (hence every) representative $f \in \bar{f}$ admits a lower control (resp. is coarsely surjective). A morphism $\bar{f}$ is an isomorphism in $\overline{\mathbf{Coarse}}$ (resp. $\overline{\mathbf{CLip}}$) if and only if any (hence every) representative $f \in \bar{f}$ is a coarse equivalence (resp. quasi-isometry).*

Theorem 2.9 [Tana, Theorem 1.1] *A morphism $\bar{f}$ in $\overline{\mathbf{CLip}}$ is a regular monomorphism if and only if any (hence every) representative $f \in \bar{f}$ is a quasi-isometric embedding.*

It is worth emphasizing that in $\overline{\mathbf{CLip}}$, the monos (admitting a lower control) form a strictly weaker class than the regular monos (quasi-isometric embeddings).

Lemma 2.10 [Tanb, Lemma 2.10] *Let $f\colon (X,d_X) \to (Y,d_Y)$ be a coarse equivalence. Then, there exists a metric d' on Y such that f factors as $(X,d_X) \xrightarrow{f'} (Y,d') \xrightarrow{m} (Y,d_Y)$, where f' is a quasi-isometry and m is a coarse equivalence given by the underlying identity.*

The empty space is initial in $\overline{\mathbf{C}}$. Given metric spaces X, Y, their binary product $X \times Y$ is realized by the Cartesian product equipped with the ℓ^∞-metric, together with the standard projections to each factor. The binary coproduct $X \sqcup Y$ is realized by the disjoint union equipped with the metric d given by $d(x,x') = d_X(x,x')$, $d(y,y') = d_Y(y,y')$, and $d(x,y) = \infty$ for all $x,x' \in X$ and $y,y' \in Y$; together with the isometric embeddings, $X \hookrightarrow X \sqcup Y$ and $Y \hookrightarrow X \sqcup Y$.

Remark 2.11 In our setting, we permit extended metrics in the definition of $\overline{\mathbf{Coarse}}$ to allow for constructions like binary coproducts or (potentially disconnected) weighted graphs. In other contexts, the metric coarse category treats only metric

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spaces (no infinite distances), for example, in [CH16]. With this restriction, the characterization of epimorphisms via coarse surjectivity requires an assumption that the domain be non-empty [CH16, Propositions 3.A.16 and 3.A.22]; in the extended setting, this assumption can be removed [Tanb, Theorem 2.5].

Coarse gluing. Given maps $f\colon A \to X$ and $g\colon A \to Y$, the space $X \sqcup_A Y$ obtained by *coarsely gluing* X to Y via f and g is defined as follows. Let Γ be the weighted graph on the vertex set $X \sqcup Y$ with two types of edges:

- *Internal edges* – Each pair of distinct $x, x' \in X$ (resp. $y, y' \in Y'$) such that $d_X(x, x') < \infty$ (resp. $d_Y(y, y') < \infty$) are joined by a edge of weight $d_X(x, x')$ (resp. $d_Y(y, y')$),
- *Glued edges* – For each $a \in A$, add an edge of weight 1 between fa and ga.

Equip Γ with the path metric, then define $X \sqcup_A Y$ to be its vertex set endowed with the induced metric. The inclusions yield 1-Lipschitz maps $X \to X \sqcup_A Y$ and $Y \to X \sqcup_A Y$.

Proposition 2.12 [Tana, Proposition 3.3 and Remark 3.5] *Let $\bar{f}\colon A \to X$ and $\bar{g}\colon A \to Y$ be morphisms in $\overline{\mathbf{C}}$. Choose representatives $f \in \bar{f}$ and $g \in \bar{g}$. Then,*

$$\begin{array}{ccc} A & \xrightarrow{\bar{f}} & X \\ {\scriptstyle \bar{g}}\downarrow & & \downarrow \\ Y & \longrightarrow & X \sqcup_A Y \end{array}$$

is a pushout in $\overline{\mathbf{C}}$.

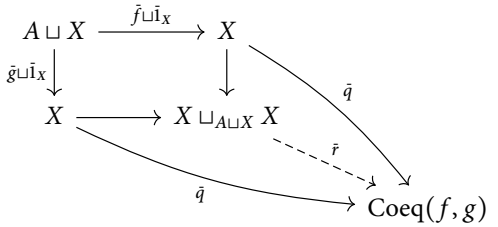
We show that coequalizers can also be computed via a coarse gluing construction. Given a pair of maps $f, g\colon A \to X$, define $\mathrm{Coeq}(f, g)$ to be the space X coarsely glued to itself via f and g; this follows the same construction as given above (taking $Y = \varnothing$), but with the glued edges connecting pairs of points in X. The underlying identity yields a 1-Lipschitz map $q\colon X \to \mathrm{Coeq}(f, g)$. By construction, $qf \approx_1 qg$.

Remark 2.13 Equivalently, $\mathrm{Coeq}(f, g)$ is given by (X, d'), where d' is the $\leq$-greatest metric on X satisfying $d'(x, x') \leq d_X(x, x')$ and $d'(fa, ga) \leq 1$ for all $x, x' \in X$ and $a \in A$ (under the ordering $d'' \leq d'$ defined by $d''(x, x') \leq d'(x, x')$ for all $x, x' \in X$.)

Proposition 2.14 (Coequalizers) *Let $\bar{f}, \bar{g}\colon A \rightrightarrows X$ be a parallel pair in $\overline{\mathbf{C}}$ and choose representatives $f \in \bar{f}$, $g \in \bar{g}$. Then, $\bar{q}\colon X \to \mathrm{Coeq}(f, g)$ is the coequalizer of $\bar{f}, \bar{g}$ in $\overline{\mathbf{C}}$.*

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Proof By Proposition 2.12, the upper left square in the commutative diagram



is a pushout in $\overline{\mathbf{C}}$. Identify the underlying set of $X \sqcup_{A\sqcup X} X$ with $X \times \{0,1\}$; thus, there is a glued edge between $(x,0)$ and $(x,1)$ for each $x \in X$, and a glued edge between $(fa,0)$ and $(ga,1)$ for each $a \in A$. Define a map $r\colon X \sqcup_{A\sqcup X} X \to \mathrm{Coeq}(f,g)$ coinciding with the underlying projection $X \times \{0,1\} \to X$. Any pair of points in $X \sqcup_{A\sqcup X} X$ connected by an edge of weight w are mapped by r either to a common point, or to a pair of points connected by an edge of weight w. Thus, r is 1-Lipschitz. Now, q factors as $X \to X \sqcup_{A\sqcup X} X \xrightarrow{r} \mathrm{Coeq}(f,g)$, where the first arrow is the inclusion to either copy of X. Thus, $\bar{r}\colon X \sqcup_{A\sqcup X} X \to \mathrm{Coeq}(f,g)$ is the canonical morphism in $\overline{\mathbf{C}}$ granted by the universal property of the pushout. We claim that $\bar{r}$ is an isomorphism in $\overline{\mathbf{C}}$. It will then follow that $\bar{q}, \bar{q}\colon X \rightrightarrows \mathrm{Coeq}(f,g)$ yield a pushout; thence, $\bar{q}$ is the desired coequalizer by Proposition 2.1.

Define $s\colon \mathrm{Coeq}(f,g) \to X \sqcup_{A\sqcup X} X$ by $x \mapsto (x,0)$ on underlying sets. If $x, x' \in X$ are connected by an internal edge, then sx, sx' are connected by an internal edge of the same weight; if there is a glued edge between $x, x' \in X$, then there is a path comprising two glued edges passing through the vertices $sx = (x,0), (x',1), (x',0) = sx'$. Therefore, s is 2-Lipschitz. Finally, rs and sr are close to the identity maps on their respective domains; hence, $\bar{r}$ is an isomorphism. ∎

Group actions on metric spaces yield natural examples of coequalizers.

Lemma 2.15 (Quotient by group action) *Let G be a group acting on metric space X by isometries such that every G-orbit is closed. Then, the quotient map $X \to X/G$ is a regular epimorphism in $\overline{\mathbf{C}}$, where X/G is equipped with the metric $d_{X/G}(Gx, Gy) := \inf_{\alpha\in G} d(x, \alpha y)$.*

Proof Let $R \subseteq X \times X$ be the equivalence relation, where $(x,x') \in R$ if and only if $Gx = Gx'$. Let $f_1, f_2\colon R \rightrightarrows X$ be the (restrictions of the) factorwise projections. Then, $Z := \mathrm{Coeq}(f_1, f_2)$ is obtained by coarsely gluing X to itself, where $x, x' \in X$ are connected by a glued edge if and only if they lie in the same G-orbit. We claim that the quotient map $X \to X/G$ induces a quasi-isometry $h\colon Z \to X/G$, and thence the desired result will follow. Note that h is 1-Lipschitz and surjective. Observe that for any $x, x' \in X$ and $\alpha \in G$, the paths $x, x', \alpha x'$ and $x, \alpha x, \alpha x'$ each comprise one glued edge and one internal edge of weight $d(x,x')$. Thus, we can replace any given path in Z by a path of the same length where all internal edges appear before all glued edges. It follows that the distance between $x, y \in Z$ is realized by the infimal length of all paths from x to y consisting of at most one internal and at most one glued edge.

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Since G-orbits are closed, this infimum is realized by a path of the form $x, \alpha y, y$ or x, y (or a trivial path). Therefore, $d_Z(x,y) - 1 \le d_{X/G}(Gx, Gy) \le d_Z(x,y)$. ∎

Example 2.16 (Warped spaces) Let G be a group acting on a proper metric space X and assume G admits a finite generating set S. Roe [Roe05] defines the *warped metric* d_G on X as the $\le$-greatest metric satisfying $d_G(x,x') \le d_X(x,x')$ and $d_G(x,sx) \le 1$ for all $s \in S$ and $x, x' \in X$; the space (X, d_G) is called the *(metric) warped space*. Define a relation

$$R := \{(x, sx) : x \in X, s \in S\} \subseteq X \times X$$

and let $f_1, f_2 \colon R \rightrightarrows X$ be the restricted factorwise projections. Appealing to Remark 2.13, $\mathrm{Coeq}(f_1, f_2)$ coincides with (X, d_G). Therefore, the underlying identity induces a regular epimorphism $(X, d_X) \to (X, d_G)$ in $\overline{\mathbf{C}}$.

Coarse equalizer filtration. Given maps $f, g \colon X \to Y$ and $\kappa \ge 0$, define

$$\mathrm{Eq}_\kappa(f,g) := \{x \in X : d_Y(fx, gx) \le \kappa\} \subseteq X$$

equipped with the induced metric. These form a filtration $\mathrm{Eq}_*(f,g)$ with bonding maps given by the inclusions $\mathrm{Eq}_\sigma(f,g) \hookrightarrow \mathrm{Eq}_\tau(f,g)$ for $\sigma \le \tau$. We say $\mathrm{Eq}_*(f,g)$ *stabilizes* in $\overline{\mathbf{C}}$ if the bonding maps represent isomorphisms in $\overline{\mathbf{C}}$ for all $\sigma \le \tau$ large. Since the inclusions are isometric embeddings, stability is equivalent to the inclusions being coarsely surjective for $\sigma \le \tau$ large; in particular, this does not depend on the choice of the ambient category $\overline{\mathbf{Coarse}}$ or $\overline{\mathbf{CLip}}$.

Proposition 2.17 [Tana, Proposition 4.3] *Let $\bar{f}, \bar{g} \colon X \rightrightarrows Y$ be a parallel pair in $\overline{\mathbf{C}}$. Choose representatives $f \in \bar{f}, g \in \bar{g}$. Then, $\bar{f}, \bar{g}$ admits an equalizer if and only if $\mathrm{Eq}_*(f,g)$ stabilizes. In that case, $\mathrm{Eq}_\kappa(f,g) \hookrightarrow X$ realizes the equalizer for all $\kappa \ge 0$ large.*

3 Epimorphisms in the metric coarse category

In this section, we characterize epimorphism classes in $\overline{\mathbf{Coarse}}$.

3.1 Strong and extremal epis

Proposition 3.1 *Every epimorphism in $\overline{\mathbf{Coarse}}$ is strong.*

Proof This follows from Theorem 1.8 and Lemma 2.7. ∎

3.2 Regular epis

Let $f \colon X \to Y$ be a controlled map and $\pi_1, \pi_2 \colon X \times X \rightrightarrows X$ denote the projections to each factor. We define two filtrations associated with f.

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Coarse kernel filtration. For $\sigma \geq 0$, let

$$K_\sigma(f) := \mathrm{Eq}_\sigma(f\pi_1, f\pi_2) = \{(x, x') \in X \times X : \ d_Y(fx, fx') \leq \sigma\} \subseteq X \times X$$

equipped with the ℓ^∞-metric. These form a filtration $K_*(f)$ whose bonding maps are isometric embeddings $K_\sigma(f) \hookrightarrow K_\tau(f)$ for $\sigma \leq \tau$. Write $\iota_\sigma \colon K_\sigma(f) \hookrightarrow X \times X$ for the inclusion.

Coarse quotient filtration. For $\sigma \geq 0$, let $Q_\sigma(f) := \mathrm{Coeq}(\pi_1\iota_\sigma, \pi_2\iota_\sigma)$. These form a filtration $Q_*(f)$ with surjective 1-Lipschitz maps $Q_\sigma(f) \twoheadrightarrow Q_\tau(f)$ as bonding maps for $\sigma \leq \tau$ induced by the underlying identity on X. Write $q_\sigma \colon X \to Q_\sigma(f)$ for the quotient map.

We have the following diagram in $\overline{\mathbf{C}}$ for each $\sigma \geq 0$, where the upper row is a coequalizer diagram. Up to closeness, the function $f_\sigma \colon Q_\sigma(f) \to Y$ coinciding with f on underlying sets is the unique map making the triangle commute. Therefore, $\bar{f}_\sigma$ is the canonical morphism in $\overline{\mathbf{C}}$ granted by the universal property of the coequalizer

$$K_\sigma(f) \underset{\bar{\pi}_2\bar{\iota}_\sigma}{\overset{\bar{\pi}_1\bar{\iota}_\sigma}{\rightrightarrows}} X \xrightarrow{\bar{q}_\sigma} Q_\sigma(f), \qquad \bar{f} \colon X \to Y, \quad \bar{f}_\sigma \colon Q_\sigma(f) \to Y$$

Let us consider the stability of the filtrations $K_*(f)$ and $Q_*(f)$, that is, when all bonding maps are isomorphisms in $\overline{\mathbf{C}}$ for $\sigma \leq \tau$ large. The stability of $K_*(f)$ is equivalent to $K_\sigma(f) \hookrightarrow K_\tau(f)$ being coarsely surjective for $\sigma \leq \tau$ large; in particular, this does not depend on the choice of the ambient category $\overline{\mathbf{Coarse}}$ or $\overline{\mathbf{CLip}}$. Explicitly, $K_*(f)$ stabilizes precisely when:

- there exists $\sigma_0 \geq 0$ such that for all $\sigma \geq \sigma_0$, there exists $R \geq 0$ such that whenever $x, x' \in X$ satisfy $d_Y(fx, fx') \leq \sigma$, there exist $w, w' \in X$ such that $w \approx_R x$, $w' \approx_R x'$ and $d_Y(fw, fw') \leq \sigma_0$.

For $Q_*(f)$, the bonding maps are surjective, and hence stability in $\overline{\mathbf{Coarse}}$ (resp. $\overline{\mathbf{CLip}}$) is equivalent to the bonding maps being coarse (resp. quasi-isometric) embeddings for $\sigma \leq \tau$ large. An explicit criterion for stability of $Q_*(f)$ in $\overline{\mathbf{Coarse}}$ is given in Proposition 3.4.

Lemma 3.2 *Assume that $Q_*(f)$ stabilizes in $\overline{\mathbf{Coarse}}$. Then, for σ large, $f_\sigma \colon Q_\sigma(f) \to Y$ admits a lower control.*

Proof Assume that $Q_*(f)$ stabilizes at threshold σ_0 in $\overline{\mathbf{Coarse}}$. Then, for each $\tau \geq \sigma \geq \sigma_0$, the map $Q_\sigma(f) \twoheadrightarrow Q_\tau(f)$ admits a lower control. Therefore, for each $\tau \geq \sigma$, there exists $r(\tau) \geq 0$ such that for all $x, x' \in X$, whenever $d_{Q_\tau(f)}(x, x') \leq 1$ holds, then $d_{Q_\sigma(f)}(x, x') \leq r(\tau)$. Consequently,

$$d_Y(fx, fx') \leq \tau \iff (x, x') \in K_\tau(f) \implies d_{Q_\tau(f)}(x, x') \leq 1 \implies d_{Q_\sigma(f)}(x, x') \leq r(\tau)$$

for all $\tau \geq \sigma$ and $x, x' \in X$, and hence f_σ admits a lower control. ∎

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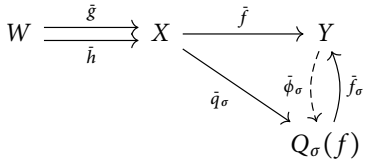
Lemma 3.3 *Let $\bar{f}\colon X \to Y$ be a morphism in $\overline{\mathbf{C}}$. Let f be any representative of $\bar{f}$. Then, the following are equivalent:*

(1) *$\bar{f}$ is a regular epimorphism in $\overline{\mathbf{C}}$,*
(2) *$\bar{f}_\sigma\colon Q_\sigma(f) \to Y$ is an isomorphism in $\overline{\mathbf{C}}$ for σ large, and*
(3) *$K_\sigma(f) \rightrightarrows X \xrightarrow{\bar{f}} Y$ is a coequalizer diagram in $\overline{\mathbf{C}}$ for σ large.*

Moreover, if any of the above hold, then $Q_*(f)$ stabilizes in $\overline{\mathbf{C}}$.

Proof The implications (2) $\implies$ (3) and (3) $\implies$ (1) are immediate.

(1) $\implies$ (2). Assume that $\bar{f}$ is the coequalizer of some parallel pair $\bar{g}, \bar{h}\colon W \rightrightarrows X$ in $\overline{\mathbf{C}}$. Choose representatives $g \in \bar{g}, h \in \bar{h}$. Since $\bar{f}\bar{g} = \bar{f}\bar{h}$, there exists some $\kappa \geq 0$ such that $fg \approx_\kappa fh$. Let $\sigma \geq \kappa$. Then, $(gw, hw) \in K_\kappa(f) \subseteq K_\sigma(f)$ for all $w \in W$ and so $q_\sigma g \approx_1 q_\sigma h$. Consequently, there exists a unique morphism $\bar{\phi}_\sigma\colon Y \to Q_\sigma(f)$ such that $\bar{q}_\sigma = \bar{\phi}_\sigma \bar{f}$. We claim that $\bar{f}_\sigma = \bar{\phi}_\sigma^{-1}$ in $\overline{\mathbf{C}}$.



Observe that $\bar{f}_\sigma\bar{\phi}_\sigma\bar{f} = \bar{f}_\sigma\bar{q}_\sigma = \bar{f}$ satisfies $\bar{f}_\sigma\bar{\phi}_\sigma\bar{f}\bar{g} = \bar{f}_\sigma\bar{\phi}_\sigma\bar{f}\bar{h}$. Appealing to the universal property of $\bar{f}$, we deduce that $\bar{f}_\sigma\bar{\phi}_\sigma = \bar{1}_Y$ and so $\bar{\phi}_\sigma$ admits a left-inverse $\bar{f}_\sigma$. As $\bar{q}_\sigma$ is epic, so is $\bar{\phi}_\sigma$. Since left-invertible epimorphisms are isomorphisms, it follows that $\bar{f}_\sigma = \bar{\phi}_\sigma^{-1}$.

The, moreover, statement then follows since $\bar{f}_\tau\bar{f}_\sigma^{-1}\colon Q_\sigma(f) \to Q_\tau(f)$ is an isomorphism for $\sigma \leq \tau$ large. ∎

With the additional assumption that f is coarsely surjective (i.e., $\bar{f}$ is an epimorphism), the stability of $Q_*(f)$ characterizes the regular epimorphism in $\overline{\mathbf{Coarse}}$. We also give an explicit metric description in terms of sequences.

Proposition 3.4 *Let $\bar{f}\colon X \to Y$ be an epimorphism in $\overline{\mathbf{Coarse}}$. The following are equivalent:*

(1) *$\bar{f}\colon X \to Y$ is a regular epimorphism in $\overline{\mathbf{Coarse}}$;*
(2) *$Q_*(f)$ stabilizes in $\overline{\mathbf{Coarse}}$ for any (hence every) representative f;*
(3) *for some (hence every) representative f, the morphism $\bar{f}_\sigma\colon Q_\sigma(f) \to Y$ is an isomorphism in $\overline{\mathbf{Coarse}}$ for σ large;*
(4) *for some (hence every) representative f, there exists $\sigma \geq 0$ and a proper increasing function $\mu\colon [0,\infty) \to [0,\infty)$ such that whenever $x, x' \in X$ satisfy $d_Y(fx, fx') \leq t < \infty$, there exists $0 \leq n \leq \mu(t)$ and a sequence of points*

$$x = b_0, a_1, b_1, a_2, b_2, \ldots, a_n, b_n, a_{n+1} = x'$$

in X satisfying $fa_i \approx_\sigma fb_i$ and $b_i \approx_{\mu(t)} a_{i+1}$ for all i.

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Proof The implications (1) $\iff$ (3) $\implies$ (2) follow from Lemma 3.3, while (2) $\implies$ (3) follows from Lemma 3.2 and the assumption that $\bar{f}$ is an epimorphism.

(3) $\implies$ (4). Choose $\sigma \geq 0$ so that f_σ is a coarse equivalence and let θ be a lower control for f_σ. Then, $\theta^\perp(t) := \sup\{s \geq 0 : \theta(s) \leq t\}$ is a proper increasing function which satisfies $\theta(s) \leq t \iff s \leq \theta^\perp(t)$. If $x, x' \in X$ satisfy $t = d_Y(fx, fx') < \infty$, then $d_{Q_\sigma(f)}(x, x') \leq \theta^\perp(t)$. This means there exists a path in $Q_\sigma(f)$ from x to x' of length at most $\mu(t) := \theta^\perp(t) + 1$; we may assume that this path does not run over consecutive internal edges. Therefore, this path has $n \in [0, \mu(t)]$ glued edges and at most $n+1$ internal edges. Moreover, each internal edge has weight at most $\mu(t)$. Choosing the vertices along this path yields a desired sequence of points (allowing for the possibility that $b_i = a_{i+1}$).

(4) $\implies$ (3). Choose a representative f and let σ be as given by Condition (4) (note that if Condition (4) holds for some $\sigma \geq 0$, then it also holds for all σ large). Suppose $x, x' \in X$ satisfy $t = d_Y(fx, fx') < \infty$. By assumption, there is a path from x to x' in $Q_\sigma(f)$ of length at most $(n+1)\mu(t) + n \leq (\mu(t)+1)^2$. Consequently, f_σ admits a lower control and is hence a coarse equivalence. ∎

Corollary 3.5 *Any regular epi in* $\overline{\mathbf{CLip}}$ *is also a regular epi in* $\overline{\mathbf{Coarse}}$.

Remark 3.6 Higginbotham and Weighill define weak coarse quotients between large-scale structures in terms of the existence a "quotient scale" on the codomain (this comprises a uniformly bounded family of subsets satisfying certain properties) [HW19, Definition 6.4]. For a controlled map $f\colon X \to Y$ between metric spaces, a quotient scale (if it exists) can be given by the family of all metric balls of radius T in Y for some suitable $T > 0$ [HW19, Proposition 7.6]; moreover, they characterize its existence using a sequence condition equivalent to Condition (4) of Proposition 3.4. By examining their proof, one can show that if $f\colon X \to Y$ is σ-coarsely surjective and $Q_*(f)$ stabilizes at threshold σ, then choosing $T = \sigma$ yields a suitable radius for the quotient scale.

3.3 Effective epis

We characterize the existence of kernel pairs in $\overline{\mathbf{C}}$ using the coarse kernel filtration.

Lemma 3.7 *A morphism $\bar{f}\colon X \to Y$ in $\overline{\mathbf{C}}$ admits a kernel pair if and only if $K_*(f)$ stabilizes for some (hence every) representative f. In that case, $\bar{\pi}_1\bar{\imath}_\sigma, \bar{\pi}_2\bar{\imath}_\sigma\colon K_\sigma(f) \rightrightarrows X$ is the kernel pair of $\bar{f}$ in $\overline{\mathbf{C}}$ for σ large.*

Proof By Proposition 2.17, $\mathrm{Eq}_*(f\pi_1, f\pi_2) = K_*(f)$ stabilizes if and only if the equalizer of $\bar{f}\bar{\pi}_1, \bar{f}\bar{\pi}_2$ exists in $\overline{\mathbf{C}}$; in that case, $K_\sigma(f) = \mathrm{Eq}_\sigma(f\pi_1, f\pi_2) \hookrightarrow X \times X$ realizes the equalizer for σ large. The result then follows from Proposition 2.2. ∎

Lemma 3.8 *If $K_*(f)$ stabilizes at threshold σ in $\overline{\mathbf{C}}$, then so does $Q_*(f)$.*

Proof Assume that $K_*(f)$ stabilizes at threshold $\sigma \geq 0$ and let $\tau \geq \sigma$. Then, there exists a natural isomorphism between the pairs $\bar{\pi}_1\bar{\imath}_\sigma, \bar{\pi}_2\bar{\imath}_\sigma\colon K_\sigma(f) \rightrightarrows X$ and

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$\bar{\pi}_1\bar{\imath}_\tau, \bar{\pi}_2\bar{\imath}_\tau \colon K_\tau(f) \rightrightarrows X$ in $\overline{\mathbf{C}}$ whose components are represented by the inclusion $K_\sigma(f) \hookrightarrow K_\tau(f)$ and identity map $1_X \colon X \to X$.

$$\begin{array}{ccccc} K_\sigma(f) & \overset{\bar{\pi}_1\bar{\imath}_\sigma}{\underset{\bar{\pi}_2\bar{\imath}_\sigma}{\rightrightarrows}} & X & \xrightarrow{\bar{q}_\sigma} & Q_\sigma(f) \\ \downarrow & & \downarrow{\scriptstyle \bar{1}_X} & & \vdots{\scriptstyle \bar{\phi}} \\ K_\tau(f) & \overset{\bar{\pi}_1\bar{\imath}_\tau}{\underset{\bar{\pi}_2\bar{\imath}_\tau}{\rightrightarrows}} & X & \xrightarrow[\bar{q}_\tau]{} & Q_\tau(f) \end{array}$$

Now, $\bar{q}_\sigma \colon X \to Q_\sigma(f)$ and $\bar{q}_\tau \colon X \to Q_\tau(f)$ are coequalizers of these respective parallel pairs. Therefore, this natural isomorphism induces a canonical morphism $\bar{\phi} \colon Q_\sigma(f) \to Q_\tau(f)$ in $\overline{\mathbf{C}}$ which, moreover, is an isomorphism. By construction, the bonding map $Q_\sigma(f) \to Q_\tau(f)$ (coinciding with the underlying identity of X) from the filtration $Q_*(f)$ makes the right-hand square in the diagram commute and is hence a representative of $\bar{\phi}$. Thus, $Q_*(f)$ stabilizes. ∎

Proposition 3.9 *An epimorphism $\bar{f} \colon X \to Y$ is an effective epimorphism in $\overline{\mathbf{Coarse}}$ if and only if $K_*(f)$ stabilizes for some (hence every) representative f.*

Proof The forward implication follows from Lemma 3.7. Conversely, assume that $\bar{f}$ is an epimorphism and that $K_*(f)$ stabilizes. Then, $Q_*(f)$ stabilizes in $\overline{\mathbf{Coarse}}$ by Lemma 3.8; hence, $\bar{f}$ is a regular epimorphism in $\overline{\mathbf{Coarse}}$ by Proposition 3.4. By Lemma 3.7, $\bar{f}$ admits a kernel pair and so it must be an effective epimorphism in $\overline{\mathbf{Coarse}}$ by Lemma 2.6. ∎

Example 3.10 (Coarse quotients) A coarsely surjective controlled map $f \colon X \to Y$ is a *coarse quotient* (in the sense of Zhang [Zha15]) with constant $R \geq 0$ if for all $\varepsilon \geq 0$, there exists $\delta \geq 0$ such that $N_\varepsilon(fx) \subseteq N_R f(N_\delta(x))$ for all $x \in X$. For such a map f, suppose $(x, x') \in K_\varepsilon(f)$ for some $\varepsilon \geq 0$. Then, $fx' \in N_\varepsilon(fx)$, and so there exists $w \in N_\delta(x)$ such that $d_Y(fw, fx') \leq R$. Since $(w, x') \in K_R(f)$, it follows that $(x, x') \in N_\delta K_R(f)$. Thus, $K_*(f)$ stabilizes at threshold R, and hence $\bar{f}$ is an effective epi in $\overline{\mathbf{Coarse}}$.

4 Epimorphisms in the coarsely Lipschitz category

4.1 Regular, strong, and extremal epis

We now work in the category $\overline{\mathbf{CLip}}$. Our goal is to prove that the extremal epis are regular epis and to characterize these in terms of a relative maximality property on the target metric. Define an ordering on the set of metrics on a set Y as follows: declare $d \prec d'$ if and only if the underlying identity $(Y, d') \to (Y, d)$ is coarsely Lipschitz. This is analogous to the ordering of compatible left-invariant metrics on a topological group in the sense of Rosendal [Ros22].

Our strategy is to adapt the weighted Rips graph construction from [Tanb] by adding augmented edges. Let $f \colon X \to Y$ be a coarsely Lipschitz map and assume that $N_r f(X) = Y$ for some $r \geq 0$. Let $\Theta \colon [0, \infty) \to [1, \infty)$ be an increasing function, called

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a *weight function*. Define a weighted graph $\mathrm{Rips}_\infty^\Theta(Y;f)$ with vertex set Y, and with edges coming in two types:

- *Internal edges* – For each distinct pair $y, y' \in Y$ such that $d_Y(y,y') < \infty$, add a edge of weight $\Theta(d_Y(y,y'))$.
- *Augmented edges* – For each distinct pair $y, y' \in f(Y)$ such that $d_Y(y,y') < \infty$, add an edge of weight $d_X(f^{-1}(y), f^{-1}(y')) + 1$.

Augmented weighted Rips filtration. For $\sigma \geq 0$, define $\mathrm{Rips}_\sigma^\Theta(Y;f)$ to be the subgraph of $\mathrm{Rips}_\infty^\Theta(Y;f)$ with vertex set Y, and whose edges comprise all augmented edges together with those internal edges between distinct pairs $y, y' \in Y$ satisfying $d_Y(y,y') \leq \sigma$. These assemble via the inclusions $\mathrm{Rips}_\sigma^\Theta(Y;f) \hookrightarrow \mathrm{Rips}_\tau^\Theta(Y;f)$ for $\sigma \leq \tau$ to form a filtration $\mathrm{Rips}_*^\Theta(Y;f)$.

We shall equip $\mathrm{Rips}_\infty^\Theta(Y;f)$ (resp. $\mathrm{Rips}_\sigma^\Theta(Y;f)$) with the path metric. Let $Y_\infty^\Theta = (Y, \partial_\infty^\Theta)$ (resp. $Y_\sigma^\Theta = (Y, \partial_\sigma^\Theta)$) denote its vertex set endowed with the induced metric. Note that the inclusions $\mathrm{Rips}_\sigma^\Theta(Y;f) \hookrightarrow \mathrm{Rips}_\tau^\Theta(Y;f) \hookrightarrow \mathrm{Rips}_\infty^\Theta(Y;f)$ are 1-Lipschitz, and thus $\partial_\infty \prec \partial_\tau \prec \partial_\sigma$. Let Γ_σ^Θ be the complete subgraph of $\mathrm{Rips}_\sigma^\Theta(Y;f)$ spanned by $f(X) \subseteq Y$, equipped with its path metric, and U_σ^Θ be its vertex set with the induced metric.

Before proceeding with the argument, let us provide some intuition regarding the augmented weighted Rips graph. Informally, the weight function is intended to penalize long internal edges within Y, so that short paths would instead prefer to take multiple shorter edges, while the augmented edges ensure that the maps $X \to Y_\sigma^\Theta \to Y_\infty^\Theta \to Y$ are coarsely Lipschitz under mild assumptions on Θ. If Θ satisfies a suitable lower bound, then Y_∞^Θ turns out to be coarsely equivalent, but not necessarily quasi-isometric, to Y. However, if we impose a maximality assumption on d, then Y_∞^Θ and Y are indeed quasi-isometric; this holds regardless of how expensive we make the weight function. The crucial deduction is that short paths in Y_∞^Θ never use internal edges over a certain weight (as they are too expensive), meaning that Y_∞^Θ is *isometric* to Y_σ^Θ for σ large. By choosing Θ carefully, we can then show that for σ large, $Q_\sigma(f)$ is quasi-isometric to Y_σ^Θ and hence Y.

Consider the factorization $X \xrightarrow{q_\sigma} Q_\sigma(f) \xrightarrow{f_\sigma} Y$ of f for $\sigma \geq 0$. The map f_σ itself factorizes as

$$Q_\sigma(f) \twoheadrightarrow U_\sigma^\Theta \hookrightarrow Y_\sigma^\Theta \to Y_\infty^\Theta \to Y,$$

where the first arrow coincides with f onto its image, and the others with the underlying inclusion or identity maps. We will give sufficient conditions for each arrow to represent a mono in $\overline{\mathbf{CLip}}$ for σ large in terms of the weight function Θ. (For concreteness, consider $\Theta(t) = 2^t$ or $\Theta(t) = 1$.) Extremality of $\bar{f}$ will then imply that $\bar{f}_\sigma$ is an isomorphism, from which regularity would follow using Lemma 3.3.

Lemma 4.1 *For all $\sigma \geq 0$, the map $Q_\sigma(f) \twoheadrightarrow U_\sigma^\Theta$ induced by f is a quasi-isometry.*

Proof Let us regard $Q_\sigma(f)$ as the vertex set of a weighted graph Γ with *internal* and *glued* edges (as in the coarse gluing construction). Let $x, x' \in X$ and $y = fx$, $y' = fx'$. To obtain an upper control, suppose $d_{Q_\sigma(f)}(x,x') < \infty$. Let P be an edge-path in Γ from x to x' of length $L \geq 0$. We may assume that all vertices $x = x_0, \ldots, x_n = x$

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appearing along P are distinct and that P has no consecutive pair of internal edges. Thus, $n \leq L$ since glued edges have unit length. Let $y_i = fx_i$. If the edge of P between x_i, x_{i+1} is

- a glued edge, then, by definition of $Q_\sigma(f)$, we have that $d_Y(y_i, y_{i+1}) \leq \sigma$, and hence y_i, y_{i+1} are connected by an internal edge of weight at most $\Theta(\sigma)$ in Γ_σ^Θ;
- an internal edge, then y_i, y_{i+1} are connected by an augmented edge of weight at most $d_X(f^{-1}(y_i), f^{-1}(y_{i+1})) + 1 \leq d_X(x_i, x_{i+1}) + 1$ in Γ_σ^Θ.

Therefore, there exists an edge-path from y to y' in Γ_σ^Θ of length at most

$$\sum_{i=0}^{n-1} (d_X(x_i, x_{i+1}) + \Theta(\sigma)) = L + \Theta(\sigma)n \leq (\Theta(\sigma) + 1)L.$$

By considering all paths from x to x', we deduce that $Q_\sigma(f) \to U_\sigma^\Theta$ is $(\Theta(\sigma) + 1)$ -Lipschitz.

For the lower control, suppose that P' is an edge-path from y to y' in Γ_σ^Θ of length $L \geq 0$. We may assume all vertices $y = y_0, \ldots, y_n = y$ along P' are distinct. Since all edges of Γ_σ^Θ have weight at least 1, we deduce $n \leq L$. Note that $f^{-1}(y_i)$ is non-empty for all i. If the edge of P' between y_i, y_{i+1} forms

- an augmented edge, then there exist $x_i' \in f^{-1}y_i$ and $x_{i+1} \in f^{-1}y_{i+1}$ such that
$$d_X(x_i', x_{i+1}) \leq d_X(f^{-1}(y_i), f^{-1}(y_{i+1})) + 1;$$
- an internal edge, then $d_Y(y_i, y_{i+1}) \leq \sigma$ by construction; in particular, every pair of distinct points in $f^{-1}y_i \cup f^{-1}y_{i+1}$ are connected by a glued edge in $Q_\sigma(f)$.

In either case, if this edge between y_i, y_{i+1} has weight w, then there exists vertices $x_i' \in f^{-1}y_i$, $x_{i+1} \in f^{-1}y_{i+1}$ connected by an edge of weight at most w in Γ. Therefore, there is a path in Γ passing through vertices $x, x_0', x_1, \ldots, x_{n-1}', x_n, x'$ of length at most $L + n + 1 \leq 2L + 1$. Consequently, $Q_\sigma(f) \to Y_\sigma^\Theta$ has lower control $t \mapsto \frac{t-1}{2}$. ∎

For the next arrow, we ask that Θ grows at most exponentially.

Lemma 4.2 *Assume that there exists $C \geq 1$ such that $\Theta(t + 2r) \leq C\Theta(t)$ for all $t \geq 0$. Then, for all $\sigma \geq r$, the inclusion $\iota\colon U_\sigma^\Theta \hookrightarrow Y_\sigma^\Theta$ is a quasi-isometry.*

Proof For each $y \in Y$, choose some $\phi(y) \in f(X)$ such that $d_Y(y, \phi y) \leq r$; we shall also assume that $\phi u = u$ for all $u \in f(X)$. We claim that $\phi\colon Y_\sigma^\Theta \to U_\sigma^\Theta$ is C-Lipschitz. Since the involved metrics are defined using path metrics, it suffices to check this on edges. Observe that any augmented edge in $\mathrm{Rips}_\sigma^\Theta(Y; f)$ also belongs to the subgraph Γ_σ^Θ. Now, suppose $y, y' \in Y$ are connected by an internal edge of weight w. Then, $d_Y(\phi y, \phi y') \leq d_Y(y, y') + 2r < \infty$, and hence $\phi y, \phi y'$ are connected by an internal edge of weight

$$\Theta(d_Y(\phi y, \phi y')) \leq \Theta(d_Y(y, y') + 2r) \leq C\Theta(d_Y(y, y')) = Cw$$

in Γ_σ^Θ, yielding the claim. Now, ι is 1-Lipschitz and $\phi\iota$ is the identity on U_σ^Θ, and hence ι is a bi-Lipschitz embedding. Since $N_r f(X) = Y$ and $\sigma \geq r$, each $y \in Y$ is connected to

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some $u \in f(X)$ by an internal edge of weight at most $\Theta(r)$ in Γ_σ^Θ, and so ι is coarsely surjective. ∎

For the remaining arrows, we require lower bounds on the growth of Θ.

Lemma 4.3 *Assume $t \le \Theta(t)$ for all $t \ge 0$. Then, the map $Y_\infty^\Theta \to Y$ induced by the identity is coarsely Lipschitz and admits a lower control.*

Proof Let ρ be an (affine) upper control for f. Suppose that $y, y' \in Y$ are connected by an edge of weight w in $\mathrm{Rips}_\infty^\Theta(Y; f)$. If this is an internal edge, then $d_Y(y, y') \le \Theta(d_Y(y, y')) = w$; otherwise, this is an augmented edge, and so there exist $x \in f^{-1}(y)$ and $x' \in f^{-1}(y')$ such that $d_X(x, x') \le d_X(f^{-1}y, f^{-1}y') + 1 = w$; hence,

$$d_Y(y, y') \le \rho(d_X(x, x')) \le \rho(w).$$

Therefore, $t \mapsto \rho(t) + t$ serves as an upper control for $Y_\infty' \to Y$.

If $d_Y(y, y') < \infty$, then $\partial_\infty(y, y') \le \Theta(d_Y(y, y'))$ since there exists an internal edge between y, y'. Moreover, $\partial_\infty(y, y') = \infty$ only if $d_Y(y, y') = \infty$ by construction. Since Θ is proper, we obtain a lower control. ∎

Lemma 4.4 *Assume that for every strictly increasing affine function $\theta\colon [0, \infty) \to \mathbb{R}$, there exists $\sigma \ge 0$ such that $t < \theta(\Theta(t) - 1)$ for all $t \ge \sigma$. If $\bar{f}\colon X \to Y$ is an extremal epi in $\overline{\mathbf{CLip}}$, then for $\sigma \ge 0$ large, the underlying identity induces an isometry $Y_\sigma^\Theta \to Y_\infty^\Theta$.*

Proof By Lemma 4.3, the factorization $X \to Y_\infty^\Theta \to Y$ of f yields an epi–mono factorization of $\bar{f}$ in $\overline{\mathbf{CLip}}$. The extremality of $\bar{f}$ implies that $Y_\infty^\Theta \to Y$ represents an isomorphism in $\overline{\mathbf{CLip}}$ and is hence a quasi-isometry. Therefore, this map admits a (strictly increasing) affine lower control θ. By assumption, there exists $\sigma \ge 0$ such that $t < \theta(\Theta(t) - 1)$ for all $t \ge \sigma$.

Let $y, y' \in Y$ satisfy $\partial_\infty^\Theta(y, y') < \infty$. Consider a weighted edge-path P in $\mathrm{Rips}_\infty^\Theta(Y; f)$ passing through vertices $y = y_0, \dots, y_n = y'$ of length $L \in [\partial_\infty^\Theta(y, y'), \partial_\infty^\Theta(y, y') + \frac{1}{3})$. Note that each $d_Y(y_i, y_{i+1})$ is finite. If there exists some $0 \le i \le n$ such that $d_Y(y_i, y_{i+1}) \ge \sigma$, then

$$\theta\partial_\infty^\Theta(y_i, y_{i+1}) \le d_Y(y_i, y_{i+1}) < \theta\left(\Theta d_Y(y_i, y_{i+1}) - 1\right),$$

and hence $\partial_\infty^\Theta(y_i, y_{i+1}) < \Theta d_Y(y_i, y_{i+1}) - 1$. Therefore, P does not run over the internal edge between y_i and y_{i+1}, for otherwise, we could replace that edge with a detour in $\mathrm{Rips}_\infty^\Theta(Y; f)$ which reduces the length of P by at least $\frac{1}{2}$, contradicting the choice of P. In other words, each edge appearing along P must be an edge in the subgraph $\mathrm{Rips}_\sigma^\Theta(Y; f)$. Thus, for all $y, y' \in Y$, the distance $\partial_\infty(y, y')$ is attained by the infimal length of all weighted edge-paths from y to y' in $\mathrm{Rips}_\sigma^\Theta(Y; f)$. Consequently, $Y_\sigma^\Theta \to Y_\infty^\Theta$ is an isometry. ∎

We now characterize the extremal (and regular) epis in $\overline{\mathbf{CLip}}$ in terms of relatively maximal metrics on the codomain, yielding a proof of Proposition 1.5. Say that d_Y is *maximal relative to f* if condition (4) below holds.

Proposition 4.5 *Let $\bar{f}\colon (X, d_X) \to (Y, d_Y)$ be an epi in $\overline{\mathbf{CLip}}$ and let f be a representative. Then, the following are equivalent:*

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(1) *$\bar{f}$ is an extremal epi in* $\overline{\mathbf{CLip}}$,
(2) *$\bar{f}$ is a regular epi in* $\overline{\mathbf{CLip}}$,
(3) *the metric d_Y is $\prec$-greatest among all coarsely equivalent metrics d' on Y such that the map $f'\colon X \to (Y, d')$ coinciding with f is coarsely Lipschitz, and*
(4) *the metric d_Y is $\prec$-maximal among all coarsely equivalent metrics d' on Y such that the map $f'\colon X \to (Y, d')$ coinciding with f is coarsely Lipschitz.*

Proof (1) $\Longrightarrow$ (2). Consider the factorization $X \xrightarrow{q_\sigma} Q_\sigma(f) \xrightarrow{f_\sigma} Y$ of f for $\sigma \geq 0$. The weight function $\Theta(t) = 2^t$ satisfies the hypotheses of Lemmas 4.1–4.4, and hence $\bar{f}_\sigma$ is monic in $\overline{\mathbf{CLip}}$ for σ large; note that we require the extremality of $\bar{f}$ in order to apply Lemma 4.4. Since $\bar{f}$ is an extremal epi, $\bar{f}_\sigma$ is an isomorphism in $\overline{\mathbf{CLip}}$, and hence $\bar{f}$ is a regular epi by Lemma 3.3. (2) $\Longrightarrow$ (3). Let d' be a metric on Y such that the map $m\colon (Y, d') \to (Y, d_Y)$ given by the underlying identity is a coarse equivalence and $f'\colon X \to (Y, d')$ is coarsely Lipschitz. By Lemma 3.3, $K_\sigma(f) \rightrightarrows X \xrightarrow{\bar{f}} (Y, d)$ is a coequalizer diagram in $\overline{\mathbf{CLip}}$ for σ large. Note that $\bar{f}' = \bar{m}\bar{f}$ coequalizes the parallel pair $K_\sigma(f) \rightrightarrows X$. Therefore, by the universal property, there exists a unique morphism $\bar{g}\colon (Y, d_Y) \to (Y, d')$ in $\overline{\mathbf{CLip}}$ such that $\bar{f}' = \bar{g}\bar{f}$. Since $\bar{f}$ is an epi in $\overline{\mathbf{Coarse}}$, we deduce that $\bar{g} = \bar{m}$. Therefore, m is coarsely Lipschitz, and hence $d' \prec d_Y$.

(3) $\Longrightarrow$ (4). Trivial.

(4) $\Longrightarrow$ (1). Assume that $\bar{f}$ factorizes as $X \xrightarrow{\bar{e}} Z \xrightarrow{\bar{m}} Y$ of $\bar{f}$ in $\overline{\mathbf{CLip}}$, with $\bar{m}$ a mono. Since $\bar{f}$ is an epi, so is $\bar{m}$. Thus, $m\colon Z \to Y$ is a coarse equivalence. By Lemma 2.10, m factorizes as $Z \xrightarrow{g} (Y, d') \xrightarrow{m'} (Y, d)$ for some metric d' with g a quasi-isometry and m' a coarse equivalence coinciding with the underlying identity. Note that $\bar{m}' = \bar{g}^{-1}\bar{m}$ is a morphism in $\overline{\mathbf{CLip}}$. Now, $f' := ge$ is coarsely Lipschitz and coincides with f on underlying sets. By the maximality assumption, $(m')^{-1}$ is coarsely Lipschitz, and hence $\bar{m}'$ is an isomorphism in $\overline{\mathbf{CLip}}$. Therefore, $\bar{m} = \bar{g}\bar{m}'$ is an isomorphism in $\overline{\mathbf{CLip}}$ and so $\bar{f}$ is an extremal epi. ∎

Example 4.6 By [Tanb, Theorem 1.8], a metric space (Y, d) is quasigeodesic (or, equivalently, quasi-isometric to a graph with the combinatorial metric) if and only if d is $\prec$-maximal among all coarsely equivalent metrics d' on Y. Therefore, any epimorphism in $\overline{\mathbf{CLip}}$ to a quasigeodesic space is extremal. For a partial converse, take X to be the space with the underlying set Y equipped with the metric $d_X(y, y') = \infty$ for all distinct $y, y' \in Y$. Then, by Proposition 4.5, if the underlying identity $f\colon X \to Y$ is extremal in $\overline{\mathbf{CLip}}$, then Y is quasigeodesic.

Next, we establish Theorem 1.6 which gives a criterion for promoting an epi in $\overline{\mathbf{CLip}}$ to an extremal epi via a coarsely equivalent remetrization of the codomain metric.

Theorem 4.7 *Let $f\colon (X, d_X) \to (Y, d_Y)$ be a coarsely Lipschitz map and assume $N_r f(X) = Y$ for some $r \geq 0$. Then, the following are equivalent:*

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(1) $\bar{f}$ *is a regular epi in* $\overline{\mathbf{Coarse}}$,
(2) *there exists a* $\prec$*-greatest element among all metrics* d' *on* Y *coarsely equivalent to* d_Y *such that the map* $(X, d_X) \to (Y, d')$ *induced by* f *is coarsely Lipschitz,*
(3) *there exists a* $\prec$*-maximal element among all metrics* d' *on* Y *coarsely equivalent to* d_Y *such that the map* $(X, d_X) \to (Y, d')$ *induced by* f *is coarsely Lipschitz,*
(4) $\bar{f}$ *factors as* $\bar{f} = \bar{m}\bar{e}$ *with* $\bar{e}$ *an extremal epi and* $\bar{m}$ *a mono in* $\overline{\mathbf{CLip}}$*, and*
(5) *there exists a metric* d' *on* Y *coarsely equivalent to* d_Y *such that the map* $(X, d_X) \to (Y, d')$ *induced by* f *represents an extremal epi in* $\overline{\mathbf{CLip}}$.

If any of the above hold, then a relatively maximal metric is realized by ∂_σ^Θ *for* σ *large, where* Θ *is any weight function satisfying the hypothesis of Lemma 4.2.*

Proof (1) $\implies$ (2). Let Θ be any weight function satisfying the hypothesis of Lemma 4.2. For $\sigma \geq 0$, consider the factorization

$$X \xrightarrow{q_\sigma} Q_\sigma(f) \xrightarrow{g_\sigma} (Y, \partial_\sigma^\Theta) \xrightarrow{h_\sigma} (Y, d_Y)$$

of f, where g_σ and h_σ, respectively, coincide with f and the identity as functions. By construction, $\bar{q}_\sigma \colon X \to Q_\sigma(f)$ is a regular epi in $\overline{\mathbf{CLip}}$ for all $\sigma \geq 0$. Appealing to Lemmas 4.1 and 4.2, and coarse surjectivity of f, we deduce that g_σ is a quasi-isometry, and hence $\bar{g}_\sigma$ is an isomorphism in $\overline{\mathbf{CLip}}$. Therefore, $\bar{g}_\sigma \bar{q}_\sigma$ is a regular epi in $\overline{\mathbf{CLip}}$. By assumption, $\bar{f}$ is a regular epi in $\overline{\mathbf{Coarse}}$ and so, by Proposition 3.4, we may choose σ large so that $f_\sigma = h_\sigma g_\sigma \colon Q_\sigma(f) \to Y$ is a coarse equivalence. In particular, ∂_σ^Θ and d_Y are coarsely equivalent. Suppose that d' is a metric on Y such that the map $k \colon (Y, d_Y) \to (Y, d')$ given by the underlying identity is coarse equivalence, and that $f' := kf \colon X \to (Y, d')$ is coarsely Lipschitz. Since $\bar{g}_\sigma \bar{q}_\sigma$ is an extremal epi in $\overline{\mathbf{CLip}}$, it follows from Proposition 4.5 that $d' \prec \partial_\sigma^\Theta$.

(2) $\iff$ (3). This follows from Proposition 4.5.

(2) $\implies$ (4). Let d' be a metric on Y satisfying the given $\prec$-greatest condition. Then, by Proposition 4.5, $X \to (Y, d') \to (Y, d_Y)$ yields an extremal epi–mono factorization of $\bar{f}$ in $\overline{\mathbf{CLip}}$.

(4) $\implies$ (5). Assume that $\bar{f}$ factors as $X \xrightarrow{\bar{e}} Z \xrightarrow{\bar{m}} Y$ with $\bar{e}$ an extremal epi and $\bar{m}$ a regular mono in $\overline{\mathbf{CLip}}$. By the proof of (4) $\implies$ (1) in Proposition 4.5, $\bar{m}$ factors as $Z \xrightarrow{\bar{g}} (Y, d') \xrightarrow{\bar{m}'} (Y, d)$ with $\bar{g}$ an isomorphism in $\overline{\mathbf{CLip}}$ and m a coarse equivalence induced by the underlying identity. Therefore, $\bar{g}\bar{e}$ is an extremal epi in $\overline{\mathbf{CLip}}$ represented by the induced map $(X, d_X) \to (Y, d')$.

(5) $\implies$ (1). By Proposition 4.5, the map $(X, d_X) \to (Y, d')$ represents a regular epi in $\overline{\mathbf{CLip}}$. Appealing to Corollary 3.5, this map also represents a regular epi in $\overline{\mathbf{Coarse}}$. Since the underlying identity $(Y, d') \to (Y, d)$ is a coarse equivalence, it follows that $\bar{f}$ is a regular epi in $\overline{\mathbf{Coarse}}$. ∎

If such a relatively maximal metric exists, then it is unique up to quasi-isometry; this implies that the filtration Y_*^Θ stabilizes in $\overline{\mathbf{CLip}}$. Appealing to Proposition 3.4 and Lemma 4.1, we deduce the following corollary.

Corollary 4.8 *Let* $\bar{f}$ *be an epi in* $\overline{\mathbf{CLip}}$ *and* f *be a representative. Then,* $Q_*(f)$ *stabilizes in* $\overline{\mathbf{Coarse}}$ *if and only if it stabilizes in* $\overline{\mathbf{CLip}}$.

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Remark 4.9 Theorem 4.7 does not require properness of Θ. Indeed, by choosing $\Theta(t) = 1$, we may construct a $\prec$-maximal metric ∂_σ^Θ relative to f by taking the standard Rips graph $\mathrm{Rips}_\sigma Y$ at sufficiently large-scale σ, and then adding the augmented edges.

Example 4.10 Let (Y, d_Y) be a metric space and $f\colon X \to Y$ be as in Example 4.6. By [Tanb, Theorem 1.8], (Y, d_Y) is coarsely geodesic (or coarsely equivalent to a graph with the combinatorial metric) if and only if there exists a $\prec$-maximal metric in the coarse equivalence class of d_Y. Since $\mathrm{Rips}_\sigma^\Theta(Y; f)$ has no augmented edges, this holds precisely when $\bar{f}$ is an extremal epi in $\overline{\mathbf{CLip}}$ by Proposition 1.5. Appealing to Theorem 4.7, if (Y, d_Y) is coarsely geodesic, then a maximal metric relative to f is attained by $\mathrm{Rips}_\sigma(Y, d_Y)$ for σ large; otherwise, no such relatively maximal metric exists.

4.2 Effective epis

Proposition 4.11 *A morphism $\bar{f}\colon X \to Y$ in $\overline{\mathbf{CLip}}$ is an effective epimorphism in $\overline{\mathbf{CLip}}$ if and only $\bar{f}$ is a regular epimorphism in $\overline{\mathbf{CLip}}$ and $K_*(f)$ stabilizes for some (hence every) representative f.*

Proof By Lemma 3.7, $\bar{f}$ admits a kernel pair in $\overline{\mathbf{CLip}}$ if and only if $K_*(f)$ stabilizes for any representative f. The result follows using Lemma 2.6. ∎

5 Non-implications and examples

5.1 Strong does not imply regular in $\overline{\mathbf{Coarse}}$

Example 5.1 (Comb graph) Construct a graph in the integer lattice $\mathbb{Z}^2$ as follows. Begin with the infinite ray $[0, \infty) \times \{0\}$. Set current position at $(x, 0) = (0, 0)$. Then, do the following:

- For each integer $n = 1, 2, \ldots$, do:
  - For $j = 1, \ldots, n$, do: attach a vertical line segment from $(x, 0)$ to (x, n), and then move the current position to $(x + j, 0)$.

Let X be the vertex set of this graph with the standard path metric. Let Y be the underlying set of X equipped with the induced ℓ^1-metric from $\mathbb{Z}^2$. Let $f\colon X \to Y$ be the underlying identity. By Proposition 3.1, $\bar{f}$ is a strong epi in $\overline{\mathbf{Coarse}}$. By construction, for all integers $\sigma, n \geq 1$, there exists an integer $x \geq 0$ such that $v = (x, n)$ and $v' = (x + \sigma + 1, n)$ are the vertices of X appearing on consecutive vertical segments (see Figure 1). Since $d_Y(v, v') = \sigma + 1$, we deduce that $d_{Q_{\sigma+1}(f)}(v, v') = 1$. Now, any path from v to v' in $Q_\sigma(f)$ must pass through some vertex u on the horizontal segment between $(x, 0)$ and $(x + \sigma + 1, 0)$. It follows that

$$d_{Q_\sigma(f)}(v, v') \geq d_{Q_\sigma(f)}(v, u) + d_{Q_\sigma(f)}(u, v') \geq \frac{2n}{\sigma}.$$

Since $n \geq 1$ is arbitrary, we deduce that $Q_\sigma(f) \to Q_{\sigma+1}(f)$ is not a coarse equivalence. Therefore, $Q_*(f)$ does not stabilize in $\overline{\mathbf{Coarse}}$, and so $\bar{f}$ is not a regular epi by Proposition 3.4.

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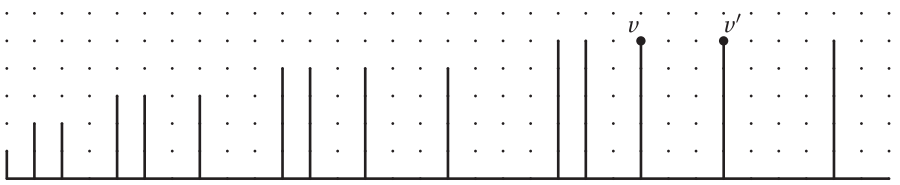


Figure 1: A portion of the comb graph with vertices labeled v, v' as in the case $\sigma = 2$ and $n = 5$.

Remark 5.2 By Theorem 4.7, this also yields an example of an epimorphism in $\overline{\mathbf{CLip}}$ which does not admit a relatively maximal metric on the codomain.

5.2 Epi does not imply extremal in $\overline{\mathbf{CLip}}$

Example 5.3 (Cubes to squares) Let $X = \{n^3 : n \in \mathbb{N}\}$ and $Y = \{n^2 : n \in \mathbb{N}\}$ be equipped with the Euclidean metrics. Define $f\colon X \to Y$ by $n^3 \mapsto n^2$. Then, f is a coarse equivalence but not a quasi-isometry. Thus, $\bar{f}$ is monic and epic in $\overline{\mathbf{CLip}}$, but not an isomorphism. Since $\bar{f}$ factorizes as $\bar{f} = \bar{f}\bar{1}_X$, it follows that $\bar{f}$ is not an extremal epi in $\overline{\mathbf{CLip}}$.

Remark 5.4 Example 5.3 also shows that an isomorphism in $\overline{\mathbf{Coarse}}$ which is coarsely Lipschitz is not necessarily an extremal epi in $\overline{\mathbf{CLip}}$.

5.3 Split does not imply effective

Example 5.5 (Comb retraction) Let X be the comb graph as given in Example 5.1. Equip $[0, \infty)$ with the Euclidean metric and define a 1-Lipschitz map $f\colon X \to [0, \infty)$ by $(x, y) \mapsto x$. Observe that $\bar{f}$ is a split epi in $\overline{\mathbf{C}}$ since the isometric embedding $x \mapsto (x, 0)$ represents a right-inverse. Given integers $\sigma, n \geq 1$, choose vertices v, v' of X as in Example 5.1. Observe that $(v, v') \in K_{\sigma+1}(f)$. Now, any vertex u (resp. u') of X within distance n of v (resp. v') must lie on the vertical segment of X containing v (resp. v'). In particular, $|f(u) - f(u')| = \sigma + 1$ and so $(u, u') \notin K_\sigma(f)$. Therefore, $K_{\sigma+1}(f) \nsubseteq N_n(K_\sigma(f))$. Since n is arbitrary, $K_*(f)$ does not stabilize, and hence $\bar{f}$ is not an effective epi in $\overline{\mathbf{C}}$ by Lemma 3.7.

Another example of a split, but not effective, epi is the closest point projection $f\colon \mathbb{H}^2 \to \gamma$ from the hyperbolic plane to any bi-infinite geodesic.

5.4 Effective does not imply split

Let Z and G be finitely generated groups. Consider a short exact sequence of groups

$$1 \to Z \hookrightarrow E \xrightarrow{f} G \to 1.$$

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Choose finite generating sets S_Z and S_G for Z and G, respectively. Let $\hat{S}_G \subseteq E$ be a set of lifts for S_G under f (i.e., f restricts to a bijection $\hat{S}_G \to S_G$). Then, $S_E := S_Z \cup \hat{S}_G$ is a finite generating set S_E. Equip each group with their corresponding word metrics.

Lemma 5.6 *The morphism $\bar{f}\colon E \to G$ is an effective epimorphism in $\overline{\mathbf{C}}$.*

Proof First, we show that $\bar{f}$ is a regular epimorphism in $\overline{\mathbf{C}}$. Observe that $(a,b) \in K_0(f)$ if and only $aZ = bZ$. Therefore, $Q_0(f)$ is the Cayley graph of E with respect to the generating set $(Z \cup \hat{S}_G)\setminus\{1\}$. The map $Q_0(f) \to G$ induced by f collapses each coset aZ to a point $f(a)$; moreover, distinct cosets aZ and bZ map to adjacent vertices if and only if some vertex of aZ is adjacent to a vertex in bZ. Since each coset has diameter at most 1 in $Q_0(f)$, this map is a quasi-isometry (see [DPRT21, Lemma 4.2]). Therefore, $\bar{f}$ is a regular epi in $\overline{\mathbf{C}}$ by Lemma 3.3.

Next, we show that the coarse kernel filtration $K_*(f)$ stabilizes. Suppose that $(a,b) \in K_\sigma(f)$ for some $\sigma \geq 0$. Then, $a, b \in E$ satisfy $d_G(1, f(a^{-1}b)) = d_G(f(a), f(b)) \leq \sigma$. Thus, $f(a^{-1}b) \in G$ can be expressed as a word in at most σ generators. By replacing each generator in this word with its lift in $\hat{S}_G$, we obtain an element $h \in a^{-1}bZ$ such that $d_E(1,h) \leq \sigma$. Note that $ahZ = bZ$, and hence $(ah, b) \in K_0(f)$. Since $d_E(a, ah) = d_E(1,h) \leq \sigma$, we deduce $(a,b) \in N_\sigma(K_0(f))$. Therefore, $K_*(f)$ stabilizes, and hence $\bar{f}$ is an effective epi in $\overline{\mathbf{C}}$ by Propositions 3.9 and 4.11. ∎

Now, we show that any right inverse of $\bar{f}$ is realized by a Lipschitz section of f.

Lemma 5.7 *Suppose that $\bar{f}\colon E \to G$ admits a right inverse $\bar{s}\colon G \to E$ in $\overline{\mathbf{C}}$. Then, there exists a Lipschitz representative $s \in \bar{s}$ such that fs is the identity map on G.*

Proof Let $s' \in \bar{s}$ be any representative. Then, there exists $\kappa \geq 0$ such that $fs' \approx_\kappa 1_G$. For any $g \in G$, we have that $d_G(fs'g, g) \leq \kappa$. Thus, we may right-multiply $s'g$ by a word of most κ generators from $\hat{S}_G$ to obtain an element $s(g) \in f^{-1}(g)$. Therefore, there exists a section $s\colon G \to E$ of f satisfying $s \approx_\kappa s'$. Since s' is controlled, so is s. The result follows since any controlled map from the vertex set of a graph (with the combinatorial metric) is Lipschitz. ∎

In the case of central extensions, a result of Kleiner–Leeb shows that the existence of a Lipschitz section implies that the extension group is quasi-isometric to a product.

Proposition 5.8 [KL01, Proposition 8.3] *Let $1 \to Z \hookrightarrow E \xrightarrow{f} G \to 1$ be a central extension, where Z and G are finitely generated. Then, f admits a Lipschitz section $s\colon G \to E$ if and only if the extension is quasi-isometrically trivial: there exists a quasi-isometry $\phi\colon E \to Z \times G$ such that $f \approx \phi \circ \pi_G$, where $\pi_G\colon Z \times G \to G$ is the projection to the G-factor.*

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Example 5.9 (Integer Heisenberg group) Let $H = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : x, y, z \in \mathbb{Z} \right\}$ be the integer Heisenberg group and $Z = \left\{ \begin{pmatrix} 1 & 0 & z \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} : z \in \mathbb{Z} \right\} \cong \mathbb{Z}$. These groups fit into a central extension $1 \to Z \hookrightarrow H \xrightarrow{f} \mathbb{Z}^2 \to 1$. By Lemma 5.6, $\bar{f}$ is an effective epi in $\overline{\mathbf{C}}$. Since H has quartic growth rate ([Har00, VII.22] and [Mil68, Lemma 4]), it is not quasi-isometric to $\mathbb{Z}^3 \cong Z \times \mathbb{Z}^2$ (which has a cubic growth rate). Thus, $\bar{f}$ is not a split epi in $\overline{\mathbf{C}}$ by Lemma 5.7 and Proposition 5.8.

5.5 Proofs of the main theorems

We finally establish the (non-)implications that do not follow from Proposition 2.4 and Lemma 2.5.

Proof of Theorem 1.1. In $\overline{\mathbf{Coarse}}$, the implication epi $\implies$ strong epi is Proposition 3.1. Example 5.1 shows that strong epis are not necessarily regular epis. The remaining non-implications follow from Examples 5.5 and 5.9. ∎

Proof of Theorem 1.2. In $\overline{\mathbf{CLip}}$, the implication extremal epi $\implies$ regular epi is Proposition 4.5. Example 5.3 demonstrates that not every epi is an extremal epi. The remaining non-implications follow using the same examples as for the proof of Theorem 1.1. ∎

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