Problem Corner

Solutions are invited to the following problems. They should be addressed to **Nick Lord** at **Tonbridge School, Tonbridge, Kent TN9 1JP** (e-mail: **njl@tonbridge-school.org**) and should arrive not later than 10 August 2016.

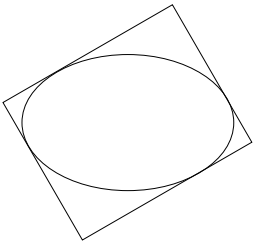
Proposals for problems are equally welcome. They should also be sent to Nick Lord at the above address and should be accompanied by solutions and any relevant background information.

100.A (Jeremy D. King)

Without using a calculator, determine which of $3^{(3^4)}$ or $4^{(4^3)}$ is bigger.

100.B (Stan Dolan)

The diagram illustrates an ellipse and one of its circumscribing rectangles. For a given ellipse, determine the locus of the vertices of the rectangles.



100.C (Muhammed Osman Corbali)

Triangle ABC has side-lengths $a = BC$, $b = AC$, $c = AB$ and ex-radii r_a, r_b, r_c where r_a refers to the escribed circle opposite to vertex A.

Prove that $\dfrac{b^2c^2}{r_a^2} + \dfrac{a^2c^2}{r_b^2} + \dfrac{a^2b^2}{r_c^2} \geqslant \dfrac{4}{3}(a^2 + b^2 + c^2)$.

100.D (Isaac Sofair)

(a) Prove the polynomial identity

$$(1-x)^m \sum_{k=0}^{n-1} \binom{m+k-1}{k} x^k + x^n \sum_{k=0}^{m-1} \binom{n+k-1}{k} (1-x)^k \equiv 1.$$

(b) Show that, for $0 < x < 1$,

$$m(1-x)^m \sum_{k=0}^{n-1} \binom{m+k}{k} x^k + nx^n \sum_{k=0}^{m-1} \binom{n+k}{k} (1-x)^k < m+n-1.$$

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Solutions and comments on **99.E**, **99.F**, **99.G**, **99.H** (July 2015).

99.E (Michael Fox)

The tangents to the parabola $(t^2,2t)$ at points with integer parameters a, b and c intersect to form a triangle whose circumdiameter is d. Prove that for any given a there is an infinity of pairs b, c for which d is an integer, and show how we can obtain as many of these pairs as we please.

All respondents used routine coordinate geometry to derive the key underlying equation, $d^2 = (a^2 + 1)(b^2 + 1)(c^2 + 1)$. Thus the tangent to the parabola at $(t^2,\ 2t)$ has equation $ty - x = t^2$; solving simultanously in pairs shows that the tangents at $t = a, b, c$ intersect to form the triangle with vertices $(ab, a + b)$, $(bc, b + c)$, $(ac, a + c)$. The perpendicular bisectors of the sides meet at the circumcentre

$$\left(\tfrac{1}{2}(1 + ab + bc + ca),\ \tfrac{1}{2}(a + b + c - abc)\right)$$

so that

$$d^2 = 4\left[\tfrac{1}{4}(1 - ab + bc + ca)^2 + \tfrac{1}{4}(a + b - c + abc)^2\right] = (a^2 + 1)(b^2 + 1)(c^2 + 1). \quad (1)$$

Notice that the signs of a, b, c are immaterial in (1), but we may wish to reverse a sign to avoid a degenerate triangle below. In the search for integer solutions, we may restrict to $a \neq 0$ because if $a = 0$ then $c = -b$ gives an infinite family of solutions, as does $c = 4b^3 + 3b$ or, indeed, $c = \sinh(n \sinh^{-1} b)$ for any odd n.

It is now a real challenge to summarise adequately the wealth of ideas submitted to solve (1) !

1. For fixed a, one way to ensure that d is an integer is to insist that $c^2 + 1 = (a^2 + 1)(b^2 + 1)$. This gives the Pell-type equation

$$c^2 - (a^2 + 1)b^2 = a^2. \quad (2)$$

Since $a \neq 0$, $a^2 + 1$ is not a square number so the Pell equation

$$x^2 - (a^2 + 1)y^2 = 1 \quad (3)$$

has an infinite number of solutions (x, y) which provide an infinite number of solutions $c = xa$, $b = ya$ for (2)

Explicitly, these may be generated from $x = 2a^2 + 1$, $y = 2a$ by the iteration $(x, y) \to T(x, y) = ((2a^2 + 1)x + 2a(a^2 + 1)y, 2ax + (2a^2 + 1)y)$.

Mark Hennings showed that, with $m = 2xy + 2ay^2 - 1$, $c = (a^2 + 1)m + a$, $b = (a^2 + 1)m^2 + (a + 1)^2 m + a + 1$ also produces an infinite family of solutions.

2. Alternatively, we could insist that $b = 0$. Then (1) will have integer solutions provided that $c^2 + 1 = k^2(a^2 + 1)$ for integer k, or $c^2 - (a^2 + 1)k^2 = -1$. An infinite number of solutions can be generated from the fundamental solution $c = a$, $k = 1$ by the iteration $(c, k) \to T(c, k)$ with T as above.

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3. Similarly, the solutions $b = 0$, $c = a$ and $b = \pm(a \pm 1)$, $c = \pm(a^2 \pm a + 1)$ of (2) may be used to generate an infinite number of solutions by the same iteration $(c, b) \to T(c, b)$.
4. M. G. Elliott avoided Pell-arguments altogether. As above, $b = a + 1$, $c = a^2 + a + 1$ gives $d = (a^2 + 1)(a^2 + 2a + 2)$ and the iteration $(a, a+1, c, d) \to (a, a+1, 4c^3 + 3c, d(4c^2 + 1))$ provides infinitely more with $b = a + 1$ fixed.
5. The GCHQ Problem Solving Group noted that splicing any solution of $k^2 - (a^2 + 1)b^2 = a^2$ with (for the same k) any of the infinite family of solutions of $c^2 - (k^2 + 1)s^2 = -1$ gives a solution with $d = (k^2 + 1)s$. Jeremy King used a similar approach, splicing together solutions of $b^2 - kp^2 = -1$ and $c^2 - k(a^2 + 1)q^2 = -1$ with $d = kpq(a^2 + 1)$. These both have infinitely many solutions for appropriate choices of the common constant k such as 1 or $a^2 + 2a + 2$.

Correct solutions were received from: M. Bataille, M. V. Channakeshava, S. Dolan, M. G. Elliott, GCHQ Problem Solving Group, M. A. Hennings, G. T. Q. Hoare, G. Howlett, J. D. King, J. A. Mundie, C. Starr, G. Strickland and the proposer M. Fox.

99.F (Michel Bataille)

Let ABC be a triangle with sides $BC = \sqrt{10 - 2\sqrt{2}}$, $CA = \sqrt{5 - \sqrt{2}}$, $AB = 1$ and let P interior to the triangle be such that $\dfrac{BP}{\sin\beta}\cdot\dfrac{CP}{\sin\gamma} = 4\cdot\dfrac{AP^2}{\sin^2\alpha}$ where $\alpha = \angle BPC$, $\beta = \angle CPA$, $\gamma = \angle APB$. Show that the minimal value of AP is $\sqrt{5\sqrt{2} - 7}$.

Most respondents used areal or Cartesian coordinate geometry to resolve this very ingeniously composed problem. Such an approach reveals that the locus of P is a parabola through B and C (tangent to AB and AC at B and C) with axis parallel to the median through A. But Stan Dolan (whose solution follows) achieved succinctness by using vectors and a clever choice of variables.

With notation as in the Figure below,

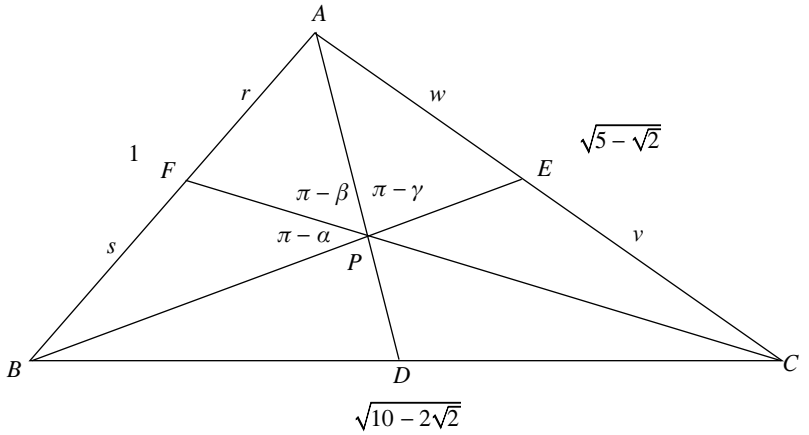
$$\frac{s}{r} = \frac{\text{Area}\,BPF}{\text{Area}\,APF} = \frac{BP\,\sin\alpha}{AP\,\sin\beta}; \text{ similarly } \frac{v}{w} = \frac{CP\,\sin\alpha}{AP\,\sin\gamma}$$

so that $\dfrac{sv}{rw} = \dfrac{BP.CP\,\sin^2\alpha}{AP^2\,\sin\beta\,\sin\gamma} = 4$, by the constraint in **99.F**. A standard vector calculation then shows that

$$\overrightarrow{AP} = \frac{1}{rv + sw + sv}\left(rv\overrightarrow{AB} + sw\overrightarrow{AC}\right) = x^2\overrightarrow{AB} + (1 - x)^2\overrightarrow{AC}$$

where $x = \sqrt{\dfrac{rv}{rv + sw + sv}}$ and $1 - x = \sqrt{\dfrac{sw}{rv + sw + sv}}$ on using $sv = 4rw$.

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Then

$$|AP|^2 = x^4|AB|^2 + 2x^2(1-x)^2\overrightarrow{AB}.\overrightarrow{AC} + (1-x)^4|AC|^2$$
$$= x^4 + (\sqrt{2}-4)x^2(1-x)^2 + (5-\sqrt{2})(1-x)^4$$

on substituting the given side-lengths and using

$$2\overrightarrow{AB}.\overrightarrow{AC} = |AC|^2 + |AB|^2 - |BC|^2 = \sqrt{2} - 4.$$

Since $\angle CAB$ is obtuse, $|AP|$ attains its minimum value at a point strictly inside the triangle where (differentiating)

$$4x^3 - 2(\sqrt{2}-4)x^2(1-x) + 2(\sqrt{2}-4)x(1-x)^2 - 4(5-\sqrt{2})(1-x)^3 = 0$$

or, setting $t = \dfrac{x}{1-x}$,

$$(t-\sqrt{2})(2t^2 + (4+\sqrt{2})t + 5\sqrt{2} - 2) = 0.$$

The only real root is $t = \sqrt{2}$ so that $x = 2 - \sqrt{2}$ and $|AP|^2 = 5\sqrt{2} - 7$, as required.

Correct solutions were received from: S. Dolan, M. G. Elliott, M. A. Hennings, G. Howlett, J. A. Mundie, V. Schindler and the proposer M. Bataille.

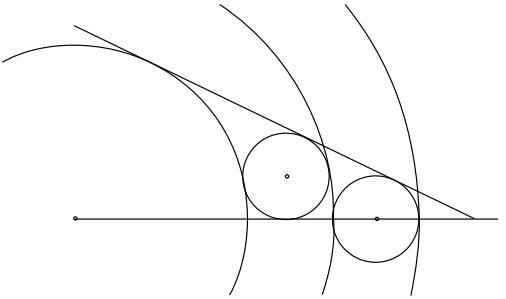
99.G (John D. Mahony)

In the diagram below, the three large arcs are portions of evenly spaced concentric circles: the inner circle (C_1) has radius R. The two small circles have the same radius (r) and satisfy the following conditions:

- They each touch two of the large arcs;
- They share a common tangent with C_1;
- The line joining the centre of C_1 to the centre of the outer small circle is a tangent to the inner small circle.

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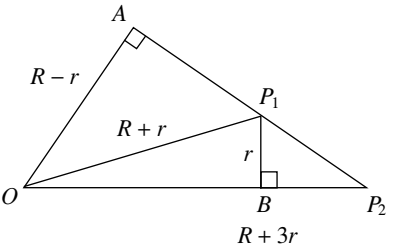
(a) Find a polynomial of lowest degree satisfied by the ratio $\frac{r}{R}$.
(b) How, in practice, would you draw the diagram by hand?



(a) *Answer*: The minimal polynomial (over $\mathbb{Q}$) for $x = \frac{r}{R}$ is

$$f(x) = x^6 - 108x^5 + 518x^4 + 60x^3 - 207x^2 - 24x + 16 = 0. \quad (*)$$

In the Figure below, extracted from the diagram, O is the centre of C_1 and P_1, P_2 the centres of the two smaller circles.



Solvers then used similar triangles and/or trigonometry to derive $(*)$. For example,

$$\frac{P_1P_2}{r} = \frac{R + 3r}{R - r} = \frac{1 + 3x}{1 - x} \qquad \begin{array}{l}\text{(similar triangles } BP_1P_2,\ AOP_2;\\ x \text{ as above)}\end{array}$$

and

$$\begin{aligned} P_1P_2 &= AP_2 - AP_1 \\ &= \sqrt{(R+3r)^2 - (R-r)^2} - \sqrt{(R+r)^2 - (R-r)^2} \qquad \text{(Pythagoras)} \\ &= 2\sqrt{2Rr + 2r^2} - 2\sqrt{Rr} \end{aligned}$$

or

$$\frac{P_1P_2}{r} = 2\sqrt{\frac{2}{x} + 2} - \frac{2}{\sqrt{x}} = \frac{2\sqrt{2 + 2x} - 2}{\sqrt{x}}.$$

Hence $\frac{1 + 3x}{1 - x} = \frac{2\sqrt{2 + 2x} - 2}{\sqrt{x}}$ and squaring twice to eliminate the roots yields $(*)$.

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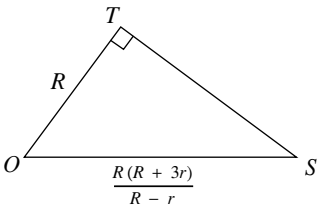
Mark Hennings and Stan Dolan used similar arguments to show that $f(x)$ in $(*)$ is irreducible over $\mathbb{Q}$ and thus of minimal degree. For if f were reducible then, working mod 3, $\bar{f}(x) = x^6 - x^4 + 1$ would be reducible over $\mathbb{Z}_3$. But $\bar{f}$ has no roots in $\mathbb{Z}_3$ and hence no linear factors. Further, $x^2 + 1$ is not a factor of $\bar{f}(x)$; other potential irreducible quadratic factors are of the form $x^2 \pm x + b$. But if $p(x)$ divides $\bar{f}(x)$ then so does $p(-x)$. Quadratic factors would thus pair up, but this cannot happen because three are needed. The only possible factorisation is thus

$$\bar{f}(x) = (x^3 + ax^2 + bx + c)(x^3 - ax^2 + bx - c)$$

where the constant term $-c^2 = 1$ gives a contradiction.

(b) The sextic in $(*)$ has roots 0.249, 0.607, 5.068, 102.964, $-0.443 \pm 0.0791i$. Since from the Figure above, $R - r > r$ we have $x < 0.5$ so that 0.249 is the relevant root. Because $f(x)$ is irreducible of degree 6, an exact ruler and compasses construction is impossible but the fact that the relevant root is very close to $\frac{1}{4}$ means that, for given R, the diagram may be closely approximated by a quick construction. Suggestions offered included:

- In the Figure above, if $r = 0.25R$, then OAP_1 is a 3-4-5 triangle and the rest of the Figure can be built up quickly as a template for the diagram.
- Alternatively, $\tan \angle OAP_2 = \dfrac{2\sqrt{2x + 2x^2}}{1 - x} = 2.10071$ so starting with a ratio $ST : R = 21 : 10$ in the Figure below, the 'outer triangle' in the diagram containing the common tangents OS and ST can be drawn with r obtained from $r = \dfrac{R(OS - R)}{OS + 3R} \approx 0.249R$.



- Using coordinates, draw C_1 with radius 4, centre (0, 0) and the right-most circle with radius 1, centre (0, 7). Then the other small circle has centre at the intersection of $y = 1$ and the line through (0, 7) distance 3 from (0, 0).

Several respondents then displayed their approximations using geometrical software packages and I was challenged to spot the inaccuracy!

Correct solutions were received from: M. V. Channakeshava, S. Dolan, M. G. Elliott, M. A. Hennings, G. T. Q. Hoare, G. Howlett, J. A. Mundie, V. Schindler and the proposer J. D. Mahony.

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99.H (Isaac Sofair)

Prove that the cosines c_1, c_2, c_3, c_4 of the four angles of a plane quadrilateral satisfy the relation

$$S_1^4 - 4S_1^2S_2 + 8S_1S_3 - 4S_1^2S_4 + 4S_3^2 = 0$$

where S_k is the sum of the products of the four cosines taken k at a time; for example, $S_3 = c_1c_2c_3 + c_1c_2c_4 + c_1c_3c_4 + c_2c_3c_4$.

Solvers showed impressive algebraic tenacity in taming this unusual trigonometrical identity. The most common solution went along the following lines. If the angles of the quadrilateral are $\theta_1, \ldots, \theta_4$ with $c_1 = \cos\theta_1$, $s_2 = \sin\theta_2$, etc then, from $\theta_1 + \ldots + \theta_4 = 2\pi$, we have $\cos(\theta_1 + \theta_2) = \cos(\theta_3 + \theta_4)$ or $c_1c_2 - c_3c_4 = s_1s_2 - s_3s_4$ which squares to give

$$\begin{aligned} 2s_1s_2s_3s_4 &= (1-c_1^2)(1-c_2^2) + (1-c_3^2)(1-c_4^2) - c_1^2c_2^2 - c_3^2c_4^2 + 2c_1c_2c_3c_4 \\ &= 2 - (c_1^2 + c_2^2 + c_3^2 + c_4^2) + 2c_1c_2c_3c_4 \\ &= 2 + 2S_2 + 2S_4 - S_1^2. \end{aligned}$$

Many now squared again and used the symmetric sums

$$\sum c_i^2 = S_1^2 - 2S_2, \qquad \sum c_i^2c_j^2 = S_2^2 - 2S_1S_3 + 2S_4, \qquad \sum c_i^2c_j^2c_k^2 = S_3^2 - 2S_2S_4$$

to simplify $4s_1^2s_2^2s_3^2s_4^2 = 4(1-c_1^2)(1-c_2^2)(1-c_3^2)(1-c_4^2)$; routine algebra then yields the required identity.

Stan Dolan neatly shortened the working by noting that

$$\begin{aligned} s_1^2s_2^2s_3^2s_4^2 = \prod(1+c_i)\prod(1-c_i) &= (1+S_1+S_2+S_3+S_4)(1-S_1+S_2-S_3+S_4) \\ &= (1+S_2+S_4)^2 - (S_1+S_3)^2 \end{aligned}$$

so that

$$4(1+S_2+S_4)^2 - 4(S_1+S_3)^2 = \left[2(1+S_2+S_4) - S_1^2\right]^2$$

or

$$S_1^4 + 4(S_1+S_3)^2 - 4S_1^2(1+S_2+S_4) = 0$$

and so

$$S_1^4 - 4S_1^2S_2 + 8S_1S_3 - 4S_1^2S_4 + 4S_3^2 = 0.$$

Other solvers utilised the factor formulae to construct different symmetric functions of the angles. For example, M. G. Elliott employed

$$x = \cos\left(\frac{\theta_2+\theta_3}{2}\right),\ y = \cos\left(\frac{\theta_1+\theta_3}{2}\right),\ z = \cos\left(\frac{\theta_1+\theta_2}{2}\right)$$

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to derive $S_1 = 4xyz$, $S_2 = 4\sum x^2y^2 - 2\sum x^2$, $S_3 = 4xyz\left(\sum x^2 - 2\right)$, $S_4 = \sum x^4 + 4x^2y^2z^2 - 2\sum x^2y^2$ from which the required identity readily follows.

James Mundie used the cosine rule to derive an expression for the LHS of the identity in **99.H** as a rational function of the sides and diagonals of the quadrilateral. The numerator (of degree 16) has a polynomial factor (of degree 6) identical to the Cayley-Menger determinant formula for the volume of a tetrahedron in terms of its edge-lengths and this is zero for a planar quadrilateral. As James noted, this implies that planarity is the only restriction on the shape of the quadrilateral; others commented that all types (such as crossed or degenerate) are also catered for by taking $\theta_1 + \ldots + \theta_4 = 2n\pi$ at the outset of the proof above, the other steps being unchanged.

Correct solutions were received from: S. Dolan, M. G. Elliott, GCHQ Problem Solving Group, M. A. Hennings, G. Howlett, J. A. Mundie, C. Starr, G. B. Trustrum and the proposer I. Sofair.

doi:10.1017/mag.2016.31 N.J.L.

In the pipeline for July 2016

John Stubbs and Jacob Adetunji	UK pension changes in 2015: some mathematical considerations
Peter Shiu	The mean-square value of the divisor function
Martin Josefsson	Properties of Pythagorean quadrilaterals
Fleur T Tehrani	A simple approach to solving cubic equations
Jose Luis Cereceda	An alternative recursive formula for the sum of powers of integers
Geoff Smith	Circles in areals
Osvaldo Marrero	What is Bayesian statistics?
Graham Jameson	Using double integrals to solve single integrals
Nick Lord	Euler, the clothoid and integral of $\frac{\sin x}{x}$
Thomas J Osler, Walter Jacob and Ryo Nishimura	An infinite product of nested radicals for $\log x$ from the Archimedean algorithm

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