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Strong stochastic stability for non-uniformly expanding maps

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Abstract. We consider random perturbations of discrete-time dynamical systems. We give sufficient conditions for the stochastic stability of certain classes of maps, in a strong sense. This improves the main result in Alves and Araújo [Random perturbations of non-uniformly expanding maps. *Astérisque* **286** (2003), 25–62], where the stochastic stability in the weak* topology was proved. Here, under slightly weaker assumptions on the random perturbations, we obtain a stronger version of stochastic stability: convergence of the density of the stationary measure to the density of the Sinai–Ruelle–Bowen (SRB) measure of the unperturbed system in the L^1-norm. As an application of our results, we obtain strong stochastic stability for two classes of non-uniformly expanding maps. The first one is an open class of local diffeomorphisms introduced in Alves, Bonatti and Viana [SRB measures for partially hyperbolic systems whose central direction is mostly expanding. *Invent. Math.* **140** (2000), 351–398] and the second one is the class of *Viana maps*.

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1. *Introduction*

Two of the major goals of dynamical systems theory are to study the asymptotic behavior of typical orbits as time goes to infinity, and to understand how stable that behavior is, that is, how the behavior changes when the system is slightly modified or when it is exposed to perturbations during time evolution. Despite the deterministic formulation of dynamical systems, it is easy to find examples whose evolution law is extremely simple and whose dynamics has a high level of complexity and sensitivity to perturbations. This work concerns the stability of systems, in a sense that we will define precisely later, in a broad class of discrete-time dynamical systems—*non-uniformly expanding maps*—when some random noise is introduced in the *deterministic* dynamics.

A successful approach to the study of dynamical systems with complex behavior is given by ergodic theory, which aims at probabilistic descriptions of orbits in a measurable phase space. The existence of an invariant measure for a given dynamics is an important fact in this context, especially if we recall Birkhoff's ergodic theorem, which describes time averages of observable phenomena for typical points with respect to that measure. However, it may happen that an invariant measure lacks physical meaning. *Sinai–Ruelle–Bowen (SRB) measures* play a particularly important role in this context since they provide information about the statistics of orbits for a *large* set of initial states. These are invariant measures which are somewhat compatible with the reference volume measure, when this is not preserved. For some classes of systems they can be obtained as ergodic invariant measures which are absolutely continuous with respect to the volume measure. SRB measures were introduced in the 1970s by Sinai [**Si72**], Ruelle [**Ru76**] and Bowen [**Bo75, BR75**] for Anosov and Axiom A attractors, in both discrete and continuous time systems; see also [**KS69**] for uniformly expanding maps. The definition of SRB measures has been formulated in several ways, essentially motivated by the development of the theory of dynamical systems and the appearance of new examples and subjects of interest, causing some ambiguity in definitions in different contemporary works; for example, see [**Yo02**] for a compilation of related results and historical background, and references therein. The classes of systems studied by Sinai, Ruelle and Bowen exhibit

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uniform expansion/contraction behavior in invariant sub-bundles of the tangent bundle of a Riemannian manifold, and the statistical properties of dynamical systems with these properties were systematically addressed in subsequent work by many different authors. Systems exhibiting expansion only in asymptotic terms were considered in [**Ja81**], where the existence of physical measures for many quadratic transformations of the interval was established; see also [**BeC85, BeY92, CE80**]. Related to [**BeC85**] is the work [**BeC91**] on Hénon maps exhibiting strange attractors. Results for multidimensional non-uniformly expanding systems appear in [**Al00, Vi97**], and, motivated by these results, general conclusions for systems exhibiting non-uniformly expanding behavior are drawn in [**ABV00**].

The introduction of *random perturbations* in dynamical systems has been addressed in several works by slightly different means. One possible approach is to consider at each iterate a map f_t close to an original map f, chosen independently according to some probabilistic law θ_ϵ, where $\epsilon > 0$ is the noise level (for instance, in an ϵ-neighborhood of the original map). We say that μ_ϵ is a *stationary measure* if

$$\iint \varphi(f_t(x))\, d\mu_\epsilon(x)\, d\theta_\epsilon(t) = \int \varphi\, d\mu_\epsilon,$$

for every continuous function $\varphi : M \to \mathbb{R}$. We say that μ_ϵ is a *physical measure* if, for a set with positive Lebesgue measure of initial states $x \in M$, we have

$$\lim_{n\to+\infty} \frac{1}{n} \sum_{j=0}^{n-1} \varphi((f_{t_{j-1}} \circ \cdots \circ f_{t_0})(x)) = \int \varphi\, d\mu_\epsilon,$$

for every continuous $\varphi : M \to \mathbb{R}$ and almost all sequences $(t_0, t_1, \ldots)$ with respect to the product measure $\theta_\epsilon^{\mathbb{N}}$. Physical measures for random perturbations play an equivalent role to that of SRB measures in the deterministic context. In order to distinguish these measures in the deterministic and random perturbation contexts, we will refer to physical measures only in the random perturbation setting and to SRB measures in the deterministic setting.

Stochastic stability is a rather vague notion depending on the nature of the systems under consideration, but it tries to reflect that the introduction of small random noise only slightly affects the statistical description of the dynamical system. We call a system *stochastically stable* if the stationary physical measures converge in the weak* topology to some SRB measure as ϵ goes to zero, and *strongly stochastically stable* if the convergence is with respect to the densities (if they exist) in the L^1-norm. We can also formulate random perturbations and stochastic stability in terms of Markov chains. We refer to [**Ki86, Ki88**] for background and treatment of the topic. For stochastic stability results, see [**Ba97, BaY93, BKS96, Yo86**] for uniformly hyperbolic systems, [**BaV96, BeY92, KK86, Me00**] for non-uniformly expanding interval maps, [**BeV06**] for Hénon-like maps and [**AAV07**] for partially hyperbolic attractors. For related topics see [**CY05**], for example, for an analysis of SRB measures as zero-noise limits, and [**Ara00**] for an important contribution to the stochastic part of a conjecture by Palis [**Pa00**].

Stochastic stability was established in [**AA03**] for a general class of multidimensional non-uniformly expanding maps in the weak sense. The main goal of the present work

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is to improve the result in [**AA03**] to strong stochastic stability, and for this to hold in a more general framework. In particular, no non-degeneracy conditions, as in [**AA03**, §3], are imposed. Our main result is stated in Theorem A, and we discuss whether the result itself or the ideas of proof are quite general and provide useful tools and strategies for the analysis of stochastic stability for dynamical systems with non-uniform expanding behavior.

Overview. This work is organized in the following way. In the remainder of this introduction we formally present the main definitions and results on strong stochastic stability for non-uniformly expanding maps, allowing the presence of critical sets. Sections 2 and 3 are devoted to proving Theorem A. In §2, we initially follow some ideas from [**BBM02**] (and [**BBM10**]) on a random version of Young towers to construct an absolutely continuous stationary probability measure and prove that this stationary measure is ergodic and therefore unique. This approach is based on Theorem 2.9, where we obtain random induced schemes for the stochastic perturbations under consideration. This theorem is a stochastic version of the main result in [**Al04**]. The proof of Theorem 2.9 is given in §4 and it extends ideas from [**ABV00, ALP05**] on deterministic non-uniformly expanding maps to the present situation. It also uses previous material from [**AA03**] which, on its own, extends results from [**ABV00**] to the random situation. In §3, we prove strong stochastic stability, inspired by the approach of [**AV02**], where strong statistical stability is achieved. In §5, we present applications of our main result to two classes of examples which fit our assumptions and for which we obtain strong stochastic stability. The first example is an open class of local diffeomorphisms, introduced in [**ABV00**], and the second is *Viana maps*, an open class of maps with critical sets, introduced in [**Vi97**]. This improves the weaker form of stochastic stability proved in [**AA03**] for both examples.

1.1. *Non-uniformly expanding maps.* Let M be a compact boundaryless manifold endowed with a normalized volume measure m, which we call a Lebesgue measure. Let $f: M \to M$ be a C^2 local diffeomorphism in the whole manifold except, possibly, in a set $\mathcal{C} \subset M$ of critical/singular points. This set $\mathcal{C}$ may be taken as a set of points where the derivative of f is not an isomorphism or simply does not exist.

Definition 1.1. We say that a critical/singular set $\mathcal{C}$ is *non-degenerate* if it has zero Lebesgue measure and the following conditions hold.

(1) There are constants $B > 1$ and $\beta > 0$ such that for every $x \in M \backslash \mathcal{C}$,

$$(c_1) \qquad \frac{1}{B}\,\mathrm{dist}(x, \mathcal{C})^{\beta} \le \frac{\|Df(x)v\|}{\|v\|} \le B\,\mathrm{dist}(x, \mathcal{C})^{-\beta} \quad \text{for all } v \in T_x M.$$

(2) For every $x, y \in M \backslash \mathcal{C}$ with $\mathrm{dist}(x, y) < \mathrm{dist}(x, \mathcal{C})/2$, we have

$$(c_2) \qquad |\log \|Df(x)^{-1}\| - \log \|Df(y)^{-1}\|| \le \frac{B}{\mathrm{dist}(x, \mathcal{C})^{\beta}}\,\mathrm{dist}(x, y); \text{ and}$$

$$(c_3) \qquad |\log |\det Df(x)| - \log |\det Df(y)|| \le \frac{B}{\mathrm{dist}(x, \mathcal{C})^{\beta}}\,\mathrm{dist}(x, y).$$

The first condition says that f *behaves like a power of the distance* to $\mathcal{C}$ and the last two conditions say that the functions $\log |\det Df|$ and $\log \|Df^{-1}\|$ are *locally Lipschitz*

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in $M\backslash\mathcal{C}$, with the Lipschitz constant depending on the distance to $\mathcal{C}$. Given $\delta > 0$ and $x \in M\backslash\mathcal{C}$, we define the *$\delta$-truncated distance* from x to $\mathcal{C}$ as

$$\operatorname{dist}_\delta(x, \mathcal{C}) = \begin{cases} 1 & \text{if } \operatorname{dist}(x, \mathcal{C}) \geq \delta, \\ \operatorname{dist}(x, \mathcal{C}) & \text{otherwise.} \end{cases}$$

Definition 1.2. Let $f : M \to M$ be a C^2 local diffeomorphism outside a non-degenerate critical set $\mathcal{C}$. We say that f is *non-uniformly expanding on a set $H \subset M$* if the following conditions hold.

(1) There is $a_0 > 0$ such that for each $x \in H$

$$\limsup_{n\to+\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log \|Df(f^j(x))^{-1}\| < -a_0. \tag{1}$$

(2) For every $b_0 > 0$, there exists $\delta > 0$ such that for each $x \in H$,

$$\limsup_{n\to+\infty} \frac{1}{n} \sum_{j=0}^{n-1} -\log \operatorname{dist}_\delta(f^j(x), \mathcal{C}) < b_0.$$

We will refer to the second condition above by saying that the orbits of points in H have *slow recurrence* to $\mathcal{C}$. The case $\mathcal{C} = \emptyset$ may also be considered, and in such cases the definition reduces to the first condition. A map is said to be *non-uniformly expanding* if it is non-uniformly expanding on a set of full Lebesgue measure.

1.2. *Random perturbations.* The idea of random perturbations is to replace the original deterministic orbits by *random orbits* generated by an independent and identically distributed random choice of map at each iteration. To be more precise, given a dynamical system $f : M \to M$, consider a family $\mathcal{F}$ of maps from M to M endowed with some metric, a metric space T and a continuous map

$$\begin{aligned} \Phi : T &\longrightarrow \mathcal{F} \\ t &\longmapsto \Phi(t) = f_t \end{aligned}$$

such that $f = f_{t^*}$ for some $t^* \in T$. Moreover, let $(\theta_\epsilon)_{\epsilon>0}$ be a family of Borel probability measures on T. We consider the product space $T^{\mathbb{N}}$ and product probability measure $\theta_\epsilon^{\mathbb{N}}$ on $T^{\mathbb{N}}$. We will refer to such a pair $\{\Phi, (\theta_\epsilon)_{\epsilon>0}\}$ as a *random perturbation* of f.

For a *realization* $\omega = (\omega_0, \omega_1, \ldots) \in T^{\mathbb{N}}$ and $n \geq 0$, we define

$$f_\omega^n(x) = \begin{cases} x & \text{if } n = 0, \\ (f_{\omega_{n-1}} \circ \cdots \circ f_{\omega_1} \circ f_{\omega_0})(x) & \text{if } n > 0. \end{cases}$$

Given $x \in M$ and $\omega \in T^{\mathbb{N}}$, we call the sequence $(f_\omega^n(x))_{n\in\mathbb{N}}$ a *random orbit* of x.

Definition 1.3. A measure μ_ϵ on the Borel sets of M is called a *stationary measure* for $\{\Phi, (\theta_\epsilon)_{\epsilon>0}\}$ if

$$\iint (\varphi \circ f_t)(x)\, d\mu_\epsilon(x)\, d\theta_\epsilon(t) = \int \varphi(x)\, d\mu_\epsilon(x),$$

for all $\varphi : M \to \mathbb{R}$ continuous.

If there is no confusion we will refer to such a measure μ_ϵ as a stationary measure for f.

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1.3. *Non-uniform expansion on random orbits.* Consider a random perturbation $\{\Phi, (\theta_\epsilon)_{\epsilon>0}\}$ of a non-uniformly expanding map f such that

$$\operatorname{supp}(\theta_\epsilon) \to \{t^*\} \quad \text{as } \epsilon \to 0.$$

Because of the presence of the critical set, we will restrict the class of perturbations we are going to consider for maps with critical sets: we take all the maps f_t with the same critical set $\mathcal{C}$ by imposing that

$$Df_t(x) = Df(x) \quad \text{for every } x \in M\backslash\mathcal{C} \text{ and } t \in T. \tag{2}$$

This may be implemented, for instance, in parallelizable manifolds (with an additive group structure, e.g., tori $\mathbb{T}^d$ (or cylinders $\mathbb{T}^{d-k} \times \mathbb{R}^k$), by considering $T = \{t \in \mathbb{R}^d : \|t\| \le \epsilon_0\}$ for some $\epsilon_0 > 0$, and taking $f_t = f + t$, that is, adding at each step a random noise to the unperturbed dynamics.

Definition 1.4. We say that f is *non-uniformly expanding on random orbits* if the following conditions hold, at least for small $\epsilon > 0$.

(1) There is $a_0 > 0$ such that for $\theta_\epsilon^{\mathbb{N}} \times m$ almost every $(\omega, x) \in T^{\mathbb{N}} \times M$,

$$\limsup_{n\to+\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log \|Df(f_\omega^j(x))^{-1}\| < -a_0. \tag{3}$$

(2) Given any small $b_0 > 0$, there is $\delta > 0$ such that for $\theta_\epsilon^{\mathbb{N}} \times m$ almost every $(\omega, x) \in T^{\mathbb{N}} \times M$,

$$\limsup_{n\to+\infty} \frac{1}{n} \sum_{j=0}^{n-1} -\log \operatorname{dist}_\delta(f_\omega^j(x), \mathcal{C}) < b_0. \tag{4}$$

When $\mathcal{C} = \emptyset$, we naturally disregard the second condition in the definition above. In this case, we can remove assumption (2) and replace (3) by the following condition: there is $a_0 > 0$ such that for $\theta_\epsilon^{\mathbb{N}} \times m$ almost every $(\omega, x) \in T^{\mathbb{N}} \times M$,

$$\limsup_{n\to+\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log \|Df_{\sigma^j(\omega)}(f_\omega^j(x))^{-1}\| < -a_0. \tag{5}$$

Condition (3) implies that for $\theta_\epsilon^{\mathbb{N}}$ almost every $\omega \in T^{\mathbb{N}}$, the *expansion time* function

$$\mathcal{E}_\omega(x) = \min\left\{ N \ge 1 : \frac{1}{n} \sum_{j=0}^{n-1} \log \|Df(f_\omega^j(x))^{-1}\| \le -a_0, \text{ for all } n \ge N \right\}$$

is defined and finite Lebesgue almost everywhere in M (if $\mathcal{C} = \emptyset$, the definition of $\mathcal{E}_\omega$ follows from condition (5)).

According to Remark 4.5, condition (4) is not needed in all its strength. Actually, it is enough that it holds for suitable $b_0 > 0$ and $\delta > 0$, chosen in such a way that the proof of Proposition 4.4 works. In view of this, we consider $b_0 > 0$ such that for $\theta_\epsilon^{\mathbb{N}}$ almost every $\omega \in T^{\mathbb{N}}$, we can define the *recurrence time* function Lebesgue almost everywhere in M:

$$\mathcal{R}_\omega(x) = \min\left\{ N \ge 1 : \frac{1}{n} \sum_{j=0}^{n-1} -\log \operatorname{dist}_\delta(f_\omega^j(x), \mathcal{C}) \le b_0, \text{ for all } n \ge N \right\}.$$

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We introduce the *tail set (at time n)*:

$$\Gamma_\omega^n = \{x : \mathcal{E}_\omega(x) > n \text{ or } \mathcal{R}_\omega(x) > n\}. \tag{6}$$

This is the set of points in M whose random orbit at time n has not yet achieved either the uniform exponential growth of a derivative or the slow recurrence given by conditions (3) and (4). If the critical set is empty, we simply ignore the recurrence time function and consider only the expansion time function in the definition of Γ_ω^n.

1.4. *Strong stochastic stability.* It is known that a non-uniformly expanding map f admits a finite number of absolutely continuous ergodic invariant probability measures (*SRB* measures); see [**ABV00**]. Moreover, if f is also topologically transitive, then it has a unique SRB probability measure μ_f; see [**Al03**]. We now state our main result, which asserts the existence of a unique absolutely continuous ergodic stationary probability measure and strong stochastic stability for non-uniformly expanding maps, meaning convergence in the L^1-norm of the density of the stationary measure to the density of the unique SRB probability measure.

THEOREM A. *Let f be a transitive non-uniformly expanding map and non-uniformly expanding on random orbits, for which $p > 1$ and $C > 0$ exist such that $m(\Gamma_\omega^n) < Cn^{-p}$ for $\theta_\epsilon^{\mathbb{N}}$ almost every $\omega \in T^{\mathbb{N}}$. Then:*

(1) *if ϵ is small enough, f admits a unique absolutely continuous ergodic stationary probability measure μ_ϵ; and*

(2) *f is strongly stochastically stable:*

$$\lim_{\epsilon \to 0} \left\| \frac{d\mu_\epsilon}{dm} - \frac{d\mu_f}{dm} \right\|_1 = 0,$$

where μ_f is the unique SRB probability measure for f.

This theorem improves the main result in [**AA03**], where stochastic stability was established in the weak sense (convergence of μ_ϵ to μ_f in the weak* topology). Furthermore, our arguments for the strong stochastic stability can be carried out with no extra assumptions on the probabilities θ_ϵ, such as the assumptions in [**AA03**, §3].

2. *Measures on random perturbations*

In this section, we prove the first item of Theorem A. Our strategy includes extension to a two-sided random perturbation setting, which we introduce in §2.1, together with known results related to both one-sided and two-sided random perturbations. In §2.2, we use this two-sided setting to construct random induced Gibbs–Markov structures. In §2.3, we construct suitable induced measures and use them to obtain an absolutely continuous stationary probability measure, which we prove to be ergodic and unique in §2.4.

2.1. *Generalities on stationary measures.* We introduce two-sided random perturbations, considering also the *past* of the realizations. Similarly to the one-sided case, we consider the product space $T^{\mathbb{Z}}$ and the probability product measures $\theta_\epsilon^{\mathbb{Z}}$. We define the

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two-sided skew-product map as

$$S : T^{\mathbb{Z}} \times M \longrightarrow T^{\mathbb{Z}} \times M$$
$$(\omega, z) \longmapsto (\sigma(\omega), f_{\omega_0}(z)),$$

where $\sigma : T^{\mathbb{Z}} \to T^{\mathbb{Z}}$ is the left-shift map. It is well known that a Borel probability measure μ^* in $T^{\mathbb{Z}} \times M$, invariant under S (in the usual deterministic sense), is characterized by an essentially unique disintegration $d\mu^*(\omega, x) = d\mu_\omega(x)\, d\theta_\epsilon^{\mathbb{Z}}(\omega)$, given by a family $\{\mu_\omega\}_\omega$ of *sample measures* on M with the following properties.

(1) For each Borel set $B \subset M$, $\omega \mapsto \mu_\omega(B)$ is measurable.
(2) For $\theta_\epsilon^{\mathbb{Z}}$ almost every ω, $B \mapsto \mu_\omega(B)$ is a Borel probability measure in M.
(3) For $\theta_\epsilon^{\mathbb{Z}}$ almost every ω, $f_{\omega *}\mu_\omega = \mu_{\sigma(\omega)}$.

The relation between μ^* and the family of sample measures can be expressed as

$$\mu^*(A) = \int \mu_\omega(A_\omega)\, d\theta_\epsilon^{\mathbb{Z}}(\omega),$$

where A is a Borel subset of $T^{\mathbb{Z}} \times M$ and $A_\omega = \{x \in M : (\omega, x) \in A\}$. Given a realization $\omega = (\ldots, \omega_{-1}, \omega_0, \omega_1, \ldots) \in T^{\mathbb{Z}}$, we define the *future* of ω as $\omega^+ = (\omega_0, \omega_1, \ldots)$ and the *past* of ω as $\omega^- = (\ldots, \omega_{-2}, \omega_{-1})$. We say that a Borel measure μ^* on $T^{\mathbb{Z}} \times M$ is a *Markov measure* if for $\theta_\epsilon^{\mathbb{Z}}$ almost every $\omega \in T^{\mathbb{N}}$, the corresponding sample measure μ_w depends only on the past ω^- of ω. Consider the projection map

$$\pi : T^{\mathbb{Z}} \times M \to T^{\mathbb{N}} \times M$$
$$(\omega, x) \mapsto (\omega^+, x).$$

PROPOSITION 2.1. *The stationary probabilities μ_ϵ for $\{\Phi, (\theta_\epsilon)_{\epsilon>0}\}$ are in a one-to-one correspondence with the S-invariant Markov probabilities μ^* through*

$$\mu^* \mapsto \mu_\epsilon := \int \mu_\omega\, d\theta_\epsilon^{\mathbb{Z}}(\omega) \quad \text{and} \quad \mu_\epsilon \mapsto \mu^* := \lim_{n\to+\infty} S_*^n(\theta_\epsilon^{\mathbb{Z}} \times \mu_\epsilon).$$

Moreover, given a stationary probability measure μ_ϵ, the corresponding μ^* is the unique S-invariant probability measure such that $\pi_*\mu^* = \theta_\epsilon^{\mathbb{N}} \times \mu_\epsilon$.

Proof. See [**Arn98**]. □

From now on, we refer to μ_ϵ and μ^* as the corresponding stationary and Markov probability measures, respectively. We define the *one-sided skew-product* map by

$$S^+ : T^{\mathbb{N}} \times X \longrightarrow T^{\mathbb{N}} \times X$$
$$(\omega, z) \longmapsto (\sigma^+(\omega^+), f_{\omega_0}(z)),$$

where $\sigma^+ : T^{\mathbb{N}} \to T^{\mathbb{N}}$ is the one-sided left-shift map. It is easy to see that $S^+ \circ \pi = \pi \circ S$.

PROPOSITION 2.2. *The following conditions are equivalent.*

(i) *μ_ϵ is a stationary probability measure.*
(ii) *μ^* is S-invariant.*
(iii) *$\theta_\epsilon^{\mathbb{N}} \times \mu_\epsilon$ is S^+-invariant.*

Proof. See [**Oh83**] for the equivalence between (i) and (iii). □

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Definition 2.3. A set $A \subset M$ is *random invariant* if for μ_ϵ almost every $x \in M$, we have:
(i) $x \in A \Rightarrow f_t(x) \in A$ for θ_ϵ almost every t; and
(ii) $x \in M \backslash A \Rightarrow f_t(x) \in M \backslash A$ for θ_ϵ almost every t.

Definition 2.4. A stationary measure μ_ϵ is *ergodic* if for every random invariant set A we have $\mu_\epsilon(A) = 0$ or $\mu_\epsilon(A) = \mu_\epsilon(M)$.

PROPOSITION 2.5. *The following conditions are equivalent.*
(i) μ_ϵ *is ergodic.*
(ii) $\theta_\epsilon^{\mathbb{N}} \times \mu_\epsilon$ *is* S^+*-ergodic.*
(iii) μ^* *is* S*-ergodic.*

Proof. See [**Ki86**] for the equivalence between (i) and (ii), and [**LQ95**] for the equivalence between (ii) and (iii). □

2.2. *Random inducing schemes.* From now on, we will consider the two-sided random perturbations scheme. We define non-uniform expansion in random orbits similarly to the previous one-sided definition, just considering two-sided realizations in conditions (3) (or (5) if $\mathcal{C} = \emptyset$) and (4). Analogously, we define the functions $\mathcal{E}_\omega$, $\mathcal{R}_\omega$ and the tail set Γ_ω^n for $\omega \in T^{\mathbb{Z}}$ and $n \geq 0$. It is easy to see that if we assume the hypotheses of the main theorem with respect to the one-sided random perturbations, they still hold in the two-sided environment.

We set Ω_ϵ as the $\theta_\epsilon^{\mathbb{Z}}$ full measure subset of realizations $\omega \in T^{\mathbb{Z}}$ for which conditions (3) (or (5) if $\mathcal{C} = \emptyset$) and (4) are satisfied for all $\sigma^k(\omega)$, $k \in \mathbb{Z}$, and for Lebesgue almost every $x \in M$. Note that if f is itself a non-uniformly expanding map, then $\omega^* = (\ldots, t^*, t^*, t^*, \ldots)$ belongs to Ω_ϵ.

Definition 2.6. We say that $\omega \in T^{\mathbb{Z}}$ induces a piecewise expanding *Gibbs–Markov* map F_ω in a ball $\Delta \subset M$ if there is a countable partition $\mathcal{P}_\omega$ of a full Lebesgue measure subset $\mathcal{D}$ of Δ, and a *return time function* $R_\omega : \mathcal{D} \to \mathbb{N}$, constant in each $U_\omega \in \mathcal{P}_\omega$, such that the map $F_\omega(x) = f_\omega^{R_\omega(x)}(x) : \Delta \to \Delta$ verifies the following.
(1) *Markov*: F_ω is a C^2 diffeomorphism from each $U_\omega \in \mathcal{P}_\omega$ onto Δ.
(2) *Expansion*: there is $0 < \kappa_\omega < 1$ such that for x in the interior of $U_\omega \in \mathcal{P}_\omega$,

$$\|DF_\omega(x)^{-1}\| < \kappa_w.$$

(3) *Bounded distortion*: there is some constant $K_\omega > 0$ such that for every $U_\omega \in \mathcal{P}_\omega$ and $x, y \in U_\omega$,

$$\log \left| \frac{\det DF_\omega(x)}{\det DF_\omega(y)} \right| \leq K_\omega \operatorname{dist}(F_\omega(x), F_\omega(y)).$$

For simplicity of notation we will write $\{R_\omega > n\}$ for the set $\{x \in \Delta : R_\omega(x) > n\}$.

THEOREM 2.7. *Let $f : M \to M$ be a transitive non-uniformly expanding map. The realization ω^*, associated with the deterministic dynamics f, induces a piecewise expanding Gibbs–Markov map $F : \Delta \to \Delta$ for some ball $\Delta \subset M$.*

Proof. See [**ALP05**]. □

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Remark 2.8. It is well known that a Gibbs–Markov map F admits a unique absolutely continuous ergodic invariant probability measure μ_F; see [**Yo99**], for example. From this fact, one easily deduces that the measure

$$\tilde{\mu}_f = \sum_{j=0}^{+\infty} f^j{}_*(\mu_F|\{R > j\})$$

is absolutely continuous ergodic and invariant under the map f. The integrability of the return time function R with respect to m implies that the measure $\tilde{\mu}_f$ is finite. In such cases, we denote the normalization of $\tilde{\mu}_f$ by μ_f.

In what follows, Δ is a ball given by Theorem 2.7. The next theorem ensures that almost all realizations induce a piecewise expanding Gibbs–Markov map with some uniformity on the constants. Most of the auxiliary results we use to prove Theorem 2.9 can be obtained by mimicking the deterministic ones in [**ALP05**], as some of them have already been extended to random perturbations in [**AA03**]. Nevertheless, in §4, we describe their proofs in detail in order to easily track the extension to random perturbations and monitor a certain uniformity on random orbits; this is essential for our purposes.

THEOREM 2.9. *Let $f : M \to M$ be a transitive non-uniformly expanding map and non-uniformly expanding on random orbits. If $\epsilon > 0$ is small enough, then:*

(1) *every $\omega \in \Omega_\epsilon$ induces a piecewise expanding Gibbs–Markov map F_ω in $\Delta \subset M$;*
(2) *if there exist $p > 0$, $C > 0$ such that $m(\Gamma_\omega^n) < Cn^{-p}$ for every $\omega \in \Omega_\epsilon$, then there exists $C' > 0$ such that for every $\omega \in \Omega_\epsilon$, the return time function satisfies*

$$m(\{R_\omega > n\}) \le C'n^{-p}. \tag{7}$$

As we will see later, the proof of this theorem also gives that the following uniformity conditions hold.

(U1) Given an integer $N > 1$ and $\gamma > 0$, if $\epsilon > 0$ is sufficiently small, then

$$m(\{R_{\sigma^{-j}(\omega)} = j\} \triangle \{R_{\sigma^{-j}(\tau)} = j\}) \le \gamma \quad \text{for all } \omega, \tau \in \Omega_\epsilon,$$

for $j = 1, 2, \ldots, N$, where $\triangle$ stands for the symmetric difference of two sets.

(U2) If $\epsilon > 0$ is sufficiently small, the constants K_ω and κ_ω for the induced piecewise expanding Gibbs–Markov maps can be chosen uniformly over ω. We will refer to them as $K > 0$ and $\kappa > 0$, respectively.

(U3) For $\omega, \tau \in \Omega_\epsilon$ with the same past, the sets $\{R_{\sigma^{-j}(\omega)} = j\}$ and $\{R_{\sigma^{-j}(\tau)} = j\}$, for $j \ge 1$, are exactly the same.

From now on, we assume the hypothesis of Theorem A and we consider $\epsilon > 0$ sufficiently small for Theorem 2.9 and conditions (U1)–(U3) to hold.

2.3. *Sample and stationary measures.* We start by defining a random induced dynamical system. The main motivation for the introduction of the two-sided random perturbations comes from this construction. Let us consider disjoint copies Δ_ω of Δ, associated with $\omega \in \Omega_\epsilon$, and their partitions $\mathcal{P}_\omega$. For $x \in \Delta_\omega$, we define $F_\omega(x) = f_\omega^{R_\omega(x)}(x)$ and the dynamics consists in hopping from $x \in \Delta_\omega$ to $F_\omega(x) \in \Delta_{\sigma^{R_\omega(x)}(\omega)}$. However, we

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also can continue to regard this as a dynamical system in Δ. We refine recursively $\mathcal{P}_\omega$ on Δ_ω with the partitions associated with the images of each element of $\mathcal{P}_\omega$:

$$\mathcal{P}_w^{(k)} = \bigvee_{j=0}^{k} \bigvee_{n\in L_\omega^j} (F_\omega^j)^{-1}\mathcal{P}_{\sigma^n(\omega)},$$

where

$$L_\omega^j = \{n \in \mathbb{N}_0 : F_\omega^j(\Delta_\omega) \cap \Delta_{\sigma^n(\omega)} \neq \emptyset\},$$

and set

$$\mathcal{P}_\omega^{(k),n} = \mathcal{P}_\omega^{(k)} \cap (F_\omega^k)^{-1}(\Delta_{\sigma^n(\omega)}) \quad \text{for } n \in \mathbb{N}.$$

Note that if $U \in \mathcal{P}_\omega^{(k),n}$, then $F_\omega^k(U) = \Delta_{\sigma^n(\omega)}$.

Our aim now is to prove that for each $\omega \in \Omega_\epsilon$, there is an absolutely continuous measure ν_ω defined on Δ with some suitable invariance property. Moreover, the density of ν_ω with respect to the Lebesgue measure will belong to a Lipschitz-type space:

$$\mathcal{H} = \{\varphi : \Delta \to \mathbb{R} \mid \exists K_\varphi > 0,\ |\varphi(x) - \varphi(y)| \le K_\varphi\, d(x, y) \forall x, y \in \Delta\}.$$

Given $A \subset \Delta_\omega$, set

$$(F^{-1})_\omega(A) = \bigsqcup_{n\in\mathbb{N}} \{x \in \Delta_{\sigma^{-n}(\omega)} : R_{\sigma^{-n}(\omega)}(x) = n \text{ and } F_{\sigma^{-n}(\omega)}(x) \in A\},$$

and define $[(F^j)^{-1}]_\omega(A)$ by induction. Moreover, given a family $\{\nu_{\sigma^{-n}(\omega)}\}_{n\in\mathbb{N}}$ of measures on $\bigsqcup_{n\in\mathbb{N}} \Delta_{\sigma^{-n}(\omega)}$, we set

$$(F^j)_\omega{}^*\{\nu_{\sigma^{-n}(\omega)}\}_{n\in\mathbb{N}}(A) = \sum_{n\in\mathbb{N}} \nu_{\sigma^{-n}(\omega)}([(F^j)^{-1}]_\omega(A) \cap \Delta_{\sigma^{-n}(\omega)}).$$

Here we use $*$ superscript to distinguish this *push-forward* from the one for deterministic systems, whose notation is usually $*$ subscript.

THEOREM 2.10. *For every $\omega \in \Omega_\epsilon$, there is an absolutely continuous finite measure ν_ω on Δ such that $(F)_\omega{}^*\{\nu_{\sigma^{-n}(\omega)}\}_{n\in\mathbb{N}} = \nu_\omega$ and $\rho_\omega = d\nu_\omega/dm \in \mathcal{H}$. Moreover, there is a constant $K_1 > 0$ such that $K_1{}^{-1} \le \rho_\omega \le K_1$ for all $\omega \in \Omega_\epsilon$.*

Proof. Let m_0 be the probability measure $(m|\Delta)/m(\Delta)$ on Δ. For every $E \subset \bigsqcup_{n\in\mathbb{N}} \Delta_{\sigma^{-n}(\omega)}$, with $E \in \mathcal{P}_{\sigma^{-n}(\omega)}^{(j),n}$, for some $n \in \mathbb{N}$ and $j \le n$, we define on Δ_ω the function

$$\rho_\omega^{j,E} = \frac{d}{dm_0}(F_{\sigma^{-n}(\omega)}^j)_*(m_0|E).$$

Let $x, y \in \Delta_\omega$ be arbitrary points, and let $x', y' \in E \subset \Delta_{\sigma^{-n}(\omega)}$ be such that $x' \in [(F^j)^{-1}]_\omega(x)$ and $y' \in [(F^j)^{-1}]_\omega(y)$. For the decreasing sequence $n = n_0 > \cdots > n_j = 0$ given by

$$n_l = n_{l-1} - R_{\sigma^{n_{l-1}}(\omega)}(F_{\sigma^{-n}(\omega)}^{l-1}(x')), \quad l = 1, \ldots, j,$$

we find that $F_{\sigma^{-n}(\omega)}^l(x')$, $F_{\sigma^{-n}(\omega)}^l(y')$ lies in the same element $U_{\sigma^{-n_l}(\omega)}$ of $\mathcal{P}_{\sigma^{-n_l}(\omega)}$. By Theorem 2.9 (recall the first and second items in Definition 2.6 and also condition (U2)),

$$\log \frac{\rho_\omega^{j,E}(y)}{\rho_\omega^{j,E}(x)} = \log \frac{|\det DF_{\sigma^{-n}(\omega)}^j(x')|}{|\det DF_{\sigma^{-n}(\omega)}^j(y')|} = \sum_{l=0}^{j-1} \log \left|\frac{\det DF_{\sigma^{-n_l}(\omega)}(F_{\sigma^{-n}(\omega)}^l(x'))}{\det DF_{\sigma^{-n_l}(\omega)}(F_{\sigma^{-n}(\omega)}^l(y'))}\right|$$
$$\le C_0 \operatorname{dist}(x, y),$$

with $C_0 = K(1/(1-\kappa))$, which is uniform in ω, j and E.

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We claim that given an inverse branch $G : \Delta_\omega \to \Delta_{\sigma^{-n}(\omega)}$ of $F^k_{\sigma^{-n}(\omega)}$, there is $K_1 > 0$ such that for any any A_1, $A_2 \subset \Delta_\omega$,

$$\frac{1}{K_1}\frac{m_0(A_1)}{m_0(A_2)} \le \frac{m_0(G(A_1))}{m_0(G(A_2))} \le K_1 \frac{m_0(A_1)}{m_0(A_2)}. \tag{8}$$

Indeed, observe that

$$\frac{m_0(A_1)}{m_0(A_2)} = \frac{\int_{G(A_1)} |\det DF^k_{\sigma^{-n}(\omega)}|\, dm_0}{\int_{G(A_2)} |\det DF^k_{\sigma^{-n}(\omega)}|\, dm_0}.$$

Fixing $z \in G(\Delta_\omega)$, we have

$$\begin{aligned}
&\frac{\int_{G(A_1)} |\det DF^k_{\sigma^{-n}(\omega)}|\, dm_0}{\int_{G(A_2)} |\det DF^k_{\sigma^{-n}(\omega)}|\, dm_0} \\
&\quad = \frac{|\det DF^k_{\sigma^{-n}(\omega)}(z)| \int_{G(A_1)} |\det DF^k_{\sigma^{-n}(\omega)}(x)/\det DF^k_{\sigma^{-n}(\omega)}(z)|\, dm_0(x)}{|\det DF^k_{\sigma^{-n}(\omega)}(z)| \int_{G(A_2)} |\det DF^k_{\sigma^{-n}(\omega)}(x)/\det DF^k_{\sigma^{-n}(\omega)}(z)| dm_0(x)} \\
&\quad \le \exp(2C_0 \operatorname{diam}(\Delta)) \frac{m_0(G(A_1))}{m_0(G(A_2))}.
\end{aligned}$$

Using the same arguments for the other inequality gives (8). Now, for $j \ge 1$, we define on Δ_ω the measures

$$\hat{\nu}^j_\omega = \sum_{k=1}^{j} \sum_{U \in \mathcal{P}^{(k),j}_{\sigma^{-j}(\omega)}} F^k_{\sigma^{-j}(\omega)*}(m_0|U),$$

and set $\hat{\nu}^0_\omega = m_0$. Considering $A_2 = \Delta_\omega$ in (8), we have, for any measurable set $A \subset \Delta_\omega$,

$$\begin{aligned}
\hat{\nu}^j_\omega(A) &\le \sum_{k=1}^{j} \sum_{U \in \mathcal{P}^{(k),j}_{\sigma^{-j}(\omega)}} K_1 m_0(A) m_0(U) \\
&\le K_1 m_0(A),
\end{aligned}$$

so ${K_1}^{-1} \le d\hat{\nu}^j_\omega/dm \le K_1$. Consider now

$$\nu^n_\omega = \frac{1}{n}\sum_{j=0}^{n-1} \hat{\nu}^j_\omega,$$

and $\rho^n_\omega = d\nu^n_\omega/dm$. We have ${K_1}^{-1} \le \rho^n_\omega \le K_1$ and, since ρ^n_ω is a linear combination of terms as $\rho^{j,E}_\omega$, one has $\rho^n_\omega(x) \le \exp(C_0 \operatorname{dist}(x, y))\rho^n_\omega(y)$ for all x, y in Δ, and

$$|\rho^n_\omega(x) - \rho^n_\omega(y)| \le ||\rho^n_\omega||_\infty \left|\frac{\rho^n_\omega(x)}{\rho^n_\omega(y)} - 1\right| \le K_1 \text{const.} \left|\log \frac{\rho^n_\omega(x)}{\rho^n_\omega(y)}\right| \le C d(x, y).$$

By the Ascoli–Arzela theorem, the sequence $(\rho^n_\omega)_n$ is relatively compact in $L^\infty(\Delta, m_0)$ and has some subsequence $(\rho^{n^\omega_i}_\omega)_i$ converging to some ρ_ω. By the construction process, we have ${K_1}^{-1} \le \rho_\omega \le K_1$ and $\rho_\omega \in \mathcal{H}$, so the measure $\nu_\omega = \rho_\omega\, dm$ is finite. By a diagonalization argument, we can now choose a suitable sequence $(n_i)_i$ and a

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family $\{\nu_{\sigma^l(\omega)}\}_{l\in\mathbb{Z}}$ of such finite measures such that $d\nu_{\sigma^l(\omega)}/dm$ is the limit of $\rho^{n_i}_{\sigma^l(\omega)}$, as $i \to +\infty$. Moreover, this family of finite measures satisfies the *quasi-invariance* property $(F)_\omega{}^*\{\nu_{\sigma^{-n}(\omega)}\}_{n\in\mathbb{N}}(A) = \nu_\omega(A)$, for $A \subset \Delta_\omega$:

$$
\begin{aligned}
&(F)_\omega{}^*\{\nu_{\sigma^{-n}(\omega)}\}_{n\in\mathbb{N}}(A)\\
&\quad=\sum_{n=1}^{+\infty}\nu_{\sigma^{-n}(\omega)}((F_{\sigma^{-n}(\omega)})^{-1}(A))\\
&\quad=\sum_{n=1}^{+\infty}\lim_{i\to+\infty}\frac{1}{n_i}\sum_{j=0}^{n_i-1}\hat{\nu}^j_{\sigma^{-n}(\omega)}((F_{\sigma^{-n}(\omega)})^{-1}(A))\\
&\quad=\lim_{i\to+\infty}\frac{1}{n_i}\Bigg(\sum_{j=1}^{n_i-1}\hat{\nu}^j_\omega(A)\\
&\qquad\qquad+\sum_{n=1}^{+\infty}\sum_{j=(n_i-n)^+}^{n_i-1}\sum_{k=1}^{j}\sum_{U\in\mathcal{P}^{(k),j}_{\sigma^{-j-n}(\omega)}}F^k_{\sigma^{-j-n}(\omega)_*}(m_0|U)((F_{\sigma^{-n}(\omega)})^{-1}(A))\Bigg)\\
&\quad=\nu_\omega(A),
\end{aligned}
$$

where $(n_i - n)^+ = \max\{0, n_i - n\}$. □

For each $\omega \in \Omega_\epsilon$, the Lipschitz constant K_{ρ_ω} for $\rho_\omega \in \mathcal{H}$ will depend only on K and κ, given by Theorem 2.9. By (U2), considering ϵ small enough, the constants K_{ρ_ω} can be taken to be the same for all $\omega \in \Omega_\epsilon$, which we will refer to as $K_2 > 0$.

We define the family $\{\tilde{\mu}_\omega\}_{\omega\in\Omega_\epsilon}$ of finite Borel measures on M by

$$\tilde{\mu}_\omega = \sum_{j=0}^{+\infty}(f^j_{\sigma^{-j}(\omega)})_*(\nu_{\sigma^{-j}(\omega)}|\{R_{\sigma^{-j}(\omega)} > j\}),$$

where the measures $\nu_{\sigma^{-j}(\omega)}$ are given by Theorem 2.10. Since

$$\tilde{\mu}_\omega(M) = \sum_{j=0}^{+\infty}\nu_{\sigma^{-j}(\omega)}(\{R_{\sigma^{-j}(\omega)} > j\}) \le K_1\sum_{j=0}^{+\infty}m(\{R_{\sigma^{-j}(\omega)} > j\}),$$

the hypothesis on the decay of $m(\Gamma^n_\omega)$ and Theorem 2.9 imply that they are finite measures. The absolute continuity of the measures $\{\nu_w\}_{\omega\in\Omega_\epsilon}$ implies that the measures of the family $\{\mu_\omega\}_{\omega\in\Omega_\epsilon}$ are absolutely continuous, and the quasi-invariance property for $\{\nu_w\}_{\omega\in\Omega_\epsilon}$ implies that $f_{\omega*}\tilde{\mu}_\omega = \tilde{\mu}_{\sigma(\omega)}$.

Remark 2.11. By construction, all the measures in the family $\{\nu_{\sigma^{-n}(\omega)}\}_{n\in\mathbb{N}}$ depend only on the past $\omega^- = (\ldots, \omega_{-2}, \omega_{-1})$ of ω. Moreover, by (U3), for $\omega, \tau \in T^{\mathbb{Z}}$ with the same past, the sets $\{R_{\sigma^{-j}(\omega)} = j\}$ and $\{R_{\sigma^{-j}(\tau)} = j\}$, for $j \ge 1$, are exactly the same (as subsets of $\Delta \subset M$). The measures $\tilde{\mu}_\omega$ involve sums of the type $(f^j_{\sigma^{-j}(\omega)})_*(\nu_{\sigma^{-j}(\omega)}|\{R_{\sigma^{-j}(\omega)} > j\})$, and since

$$m(\{R_{\sigma^{-j}(\omega)} > j\}) = m\left(\Delta\setminus\left\{\bigcup_{k=1}^{j}\{R_{\sigma^{-j}(\omega)} = k\}\right\}\right),$$

and $\nu_{\sigma^{-j}(\omega)} \ll m$, the measures $\tilde{\mu}_\omega$ depend only on the past ω^- of ω.

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LEMMA 2.12. *The measure*

$$\tilde{\mu}_\epsilon = \int \tilde{\mu}_\omega \, d\theta_\epsilon^{\mathbb{Z}}(\omega)$$

is an absolutely continuous stationary finite measure.

Proof. Since $\{\tilde{\mu}_\omega\}_{\omega\in\Omega_\epsilon}$ almost surely depend only on the past, then $\tilde{\mu}_\epsilon = \int \tilde{\mu}_\omega \, d\theta_\epsilon^{\mathbb{Z}}(\omega)$ is a stationary measure. Actually, for every continuous map $\varphi : M \to \mathbb{R}$, we have

$$\begin{aligned}\int \varphi(x)\, d\tilde{\mu}_\epsilon(x) &= \iint \varphi(x)\, d\tilde{\mu}_\omega(x)\, d\theta_\epsilon^{\mathbb{Z}}(\omega)\\ &= \iint \varphi(x)\, d\tilde{\mu}_{\sigma(\omega)}(x)\, d\theta_\epsilon^{\mathbb{Z}}(\omega)\\ &= \iiiint (\varphi\circ f_{\omega_0})(x)\, d\tilde{\mu}_\omega(x)\, d\theta_\epsilon^{\mathbb{Z}^-}(\omega^-)\, d\theta_\epsilon(\omega_0)\, d\theta_\epsilon^{\mathbb{Z}^+}(\sigma^+(\omega^+))\\ &= \iint (\varphi\circ f_{\omega_0})(x)\, d\tilde{\mu}_\epsilon(x)\, d\theta_\epsilon(\omega_0).\end{aligned}$$

Moreover, $\tilde{\mu}_\epsilon$ is absolutely continuous because of the absolute continuity of the measures $\{\tilde{\mu}_\omega\}_{\omega\in\Omega_\epsilon}$. For the finiteness of $\tilde{\mu}_\epsilon$, we have, by Theorem 2.9,

$$\begin{aligned}\tilde{\mu}_\epsilon(M) &= \int \tilde{\mu}_\omega(M)\, d\theta_\epsilon^{\mathbb{Z}}(\omega) = \int \sum_{n=0}^{+\infty} \nu_{\sigma^{-n}(\omega)}(\{R_{\sigma^{-n}(\omega)} > n\})\, d\theta_\epsilon^{\mathbb{Z}}(\omega)\\ &\le K_1 \sum_{n=0}^{+\infty} C' n^{-p} < \infty.\end{aligned}$$

□

We now normalize $\tilde{\mu}_\epsilon$ and define an absolutely continuous stationary probability measure $\mu_\epsilon = \tilde{\mu}_\epsilon/\tilde{\mu}_\epsilon(M)$, and set $h_\epsilon = d\mu_\epsilon/dm$. Next we prove that μ_ϵ is the unique absolutely continuous ergodic stationary probability measure.

2.4. *Ergodicity and uniqueness.* We say that $A \subset M$ is a *random forward invariant* set if, for μ_ϵ almost every $x \in A$, we have $f_t(x) \in A$ for θ_ϵ almost every t.

PROPOSITION 2.13. *There is $\delta_1 > 0$ such that given any random forward invariant set $A \subset M$ with $\mu_\epsilon(A) > 0$, there is a ball B of radius $\delta_1/4$ such that $m(B\backslash A) = 0$.*

The proof is given in §4.

PROPOSITION 2.14. *The stationary measure μ_ϵ is the unique absolutely continuous ergodic stationary probability measure.*

Proof. We prove first that there is a finite partition $H_1, \ldots, H_n$ of a full Lebesgue measure set in M such that the normalized restrictions of μ_ϵ to each H_i, $i = 1, \ldots, n$, are ergodic. Then we use the topological transitivity of f to ensure the uniqueness.

Suppose μ_ϵ is not ergodic. Then we may decompose M into two disjoint random invariant sets H_1 and H_2 $(=M\backslash H_1)$, both with positive μ_ϵ-measure. In particular, both H_1 and H_2 have positive Lebesgue measure. Let μ_ϵ^1 and μ_ϵ^2 be the normalized restrictions of μ_ϵ to H_1 and H_2, respectively. They are also absolutely continuous stationary measures. If they are not ergodic, we continue decomposing them, in the same way as we did for μ_ϵ.

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On the other hand, by Proposition 2.13, each one of the random invariant sets we find in this decomposition has full Lebesgue measure in some disk with fixed radius. Since these disks must be disjoint, and M is compact, there can only be finitely many of them. So, the decomposition must stop after a finite number of steps, giving that μ_ϵ can be written as $\mu_\epsilon = \sum_{i=1}^{p} \mu_\epsilon(H_i)\mu_\epsilon^i$, where $H_1, \ldots, H_p$ is a partition of M into random invariant sets with positive measure and each $\mu_\epsilon^i = (\mu_\epsilon|H_i)/\mu_\epsilon(H_i)$ is an ergodic stationary probability measure.

For the uniqueness, assume that there are two distinct ergodic absolutely continuous invariant measures μ_ϵ^1 and μ_ϵ^2. Since $B(\mu_\epsilon^1)$ and $B(\mu_\epsilon^2)$ are random forward invariant sets, then, by Proposition 2.13, there are disks $\Delta_1 = B(p_1, \delta_1/4)$ and $\Delta_2 = B(p_2, \delta_1/4)$ such that $m(\Delta_i \setminus B(\mu_\epsilon^i)) = 0$ for $i = 1, 2$. The topological transitivity of f, the continuity of Φ and the random invariance of $B(\mu_1)$ and $B(\mu_2)$ imply that, if ϵ is small enough, then $m(B(\mu_\epsilon^1) \cap B(\mu_\epsilon^2)) > 0$. Consider any point x in this intersection. For every continuous function $\varphi\colon M \to \mathbb{R}$ and a $\theta_\epsilon^{\mathbb{Z}}$ full subset of $T^{\mathbb{Z}}$, $(1/n)\sum_{j=0}^{n-1} \varphi(f_\omega^j(x))$ converges to $\int \varphi\, d\mu_\epsilon^1$ as n goes to infinity and, similarly, to $\int \varphi\, d\mu_\epsilon^2$. The uniqueness of the limit implies $\int \varphi\, d\mu_\epsilon^1 = \int \varphi\, d\mu_\epsilon^2$, so $\mu_\epsilon^1 = \mu_\epsilon^2$.

If we consider any stationary probability measure $\hat{\mu}_\epsilon$ on M, we can perform the same procedure as before, and get a finite decomposition of $\hat{\mu}_\epsilon$ in ergodic components, containing (Lebesgue mod 0) disks of a fixed radius. As we saw, by the topological transitivity of f, one should have $\hat{\mu}_\epsilon = \mu_\epsilon$. □

This finishes the proof of the first item of Theorem A.

3. *Strong stochastic stability*

We now prove the convergence of the density of the unique absolutely continuous ergodic stationary probability measure μ_ϵ, for $\{\Phi, (\theta_\epsilon)_{\epsilon>0}\}$, to the density of the unique f-invariant absolutely continuous probability measure μ_f, in the L^1-norm. The strategy is to get an absolutely continuous Borel measure ν_ϵ on Δ, with density ρ_ϵ, by averaging over the previously constructed induced random measures ν_ω on Δ. The family of densities $\{\rho_\epsilon\}_{\epsilon>0}$ has an accumulation point ρ_∞ in $L^1(\Delta)$, as ϵ goes to 0, which give us a measure ν_∞ on Δ. We can project this measure to a probability measure μ_∞ on M, using the dynamics of f, in such a way that we can compare the densities of the measures μ_ϵ and μ_∞ on M, showing their convergence in the L^1-norm. To conclude, we prove in Proposition 3.3 that μ_∞ is f-invariant and is actually equal to μ_f as a result of the uniqueness of the SRB measure.

We start by defining an absolutely continuous measure ν_ϵ on Δ, with density ρ_ϵ, as

$$\nu_\epsilon = \int \nu_\omega\, d\theta_\epsilon^{\mathbb{Z}}(\omega) \quad \text{and} \quad \rho_\epsilon = \int \rho_\omega\, d\theta_\epsilon^{\mathbb{Z}}(\omega).$$

Consider any sequence $(\epsilon_n)_n$, with $\epsilon_n > 0$ and $\epsilon_n \to 0$, as $n \to +\infty$. The family $\{\rho_{\epsilon_n}\}_n$ is uniformly bounded and equicontinuous. Indeed, ${K_1}^{-1} \le \rho_{\epsilon_n} \le K_1$ and, for $x, y \in \Delta$, we have $|\rho_{\epsilon_n}(x) - \rho_{\epsilon_n}(y)| \le K_2 d(x, y)$. By the Ascoli–Arzela theorem, this implies that $(\rho_{\epsilon n})_n$ has a converging subsequence $(\rho_{\epsilon_n'})_n$ to some ρ_∞ in the L^1-norm, with ${K_1}^{-1} \le \rho_\infty \le K_1$. Without loss of generality, we will assume that the whole sequence

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converges; see Remark 3.4. This means that given $\gamma > 0$, there exists $\delta > 0$ such that, if $\epsilon_n < \delta$, then

$$\|\rho_{\epsilon_n} - \rho_\infty\|_1 < \gamma. \tag{9}$$

Let ν_∞ be the Borel measure on Δ with density ρ_∞, and define an absolutely continuous Borel measure $\tilde{\mu}_\infty$ in M by

$$\tilde{\mu}_\infty = \sum_{j=0}^{\infty} (f^j)_*(\nu_\infty|\{R > j\}),$$

and set $\tilde{h}_\infty = d\tilde{\mu}_\infty/dm$. Since f is non-uniformly expanding, the decay of the return time together with the upper bound for ρ_∞ imply that $\tilde{\mu}_\infty$ is finite. We normalize it to obtain an absolutely continuous probability measure μ_∞ with density h_∞.

We introduce the transfer operator acting on $L^1(\Delta)$ by

$$\mathcal{L}_\omega^j \varphi(x) = \sum_{y=(f_\omega^j)^{-1}(x)} \frac{\varphi(y)}{|\det Df_\omega^j(y)|},$$

and, to simplify the notation, we write $\mathcal{L}^j\varphi(x)$ for $\mathcal{L}_{\omega^*}^j\varphi(x)$. Standard results on transfer operators give that

$$\int (\mathcal{L}_\omega^j \varphi)\psi \, dm = \int \varphi(\psi \circ f_\omega^j) \, dm,$$

whenever these integrals make sense, and the operator does not expand:

$$\int |\mathcal{L}_\omega^j \varphi(x)| \, dm \le \int \mathcal{L}_\omega^j |\varphi(x)| \, dm = \int |\varphi| \, dm,$$

for every $\varphi \in L^1(\Delta)$.

LEMMA 3.1. *Given $\gamma > 0$ and $j \in \mathbb{N}$, there is $\alpha_j > 0$ such that if $\epsilon < \alpha_j$, then for every $\omega, \tau \in \Omega_\epsilon$, we have*

$$\int |\mathcal{L}_\omega^j \varphi(x) - \mathcal{L}_\tau^j \varphi(x)| \, dm \le \gamma \|\varphi\|_\infty,$$

for every φ in $\mathcal{H}$.

Proof. Let $\varphi \in \mathcal{H}$. Our assumptions on the critical set imply that the critical set $\mathcal{C}$ intersects Δ in a zero Lebesgue measure set. Given any $\gamma_1 > 0$, define $\mathcal{C}(\gamma_1)$ as the γ_1-neighborhood of this intersection. For every $\omega \in \Omega_\epsilon$, we have $m(f_\omega^j(\mathcal{C}(\gamma_1))) \le \text{const}\, m(\mathcal{C}(\gamma_1))$ for some constant which may be taken uniform over ω if $\epsilon < \alpha_j$ for some small α_j.

By the continuity of f_ω^j with respect to ω, we may fix $\gamma_1 = \gamma_1(j)$ small enough so that if $\epsilon < \alpha_j < \gamma_1$, then

$$m((f_\omega^j)^{-1}(f_\tau^j(\mathcal{C}(\gamma_1)))) \le \frac{1}{2}\left(\frac{\gamma}{8}\right), \tag{10}$$

for every ω, τ in Ω_ϵ.

We decompose $\Delta \backslash \mathcal{C}(\gamma_1)$ into a finite collection $\mathcal{D}(\omega)$ of domains of injectivity for f_ω^j. We may define a corresponding collection $\mathcal{D}(\tau)$ of domains of injectivity for f_τ^j in $\Delta \backslash \mathcal{C}(\gamma_1)$, and there is a natural bijection associating each $D_\omega \in \mathcal{D}(\omega)$ with a

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unique $D_\tau \in \mathcal{D}(\tau)$ such that the Lebesgue measure of $D_\tau \triangle D_\omega$ is small, where $D_\tau \triangle D_\omega$ denotes the symmetric difference of the two sets D_τ and D_ω. Observe that $\mathcal{L}_\tau^j$ is supported in

$$f_\tau^j(\Delta) = f_\tau^j(\mathcal{C}(\gamma_1)) \cup \bigcup_{D_\tau \in \mathcal{D}(\tau)} f_\tau^j(D_\tau),$$

and analogously for $\mathcal{L}_\omega^j$. So,

$$\int |\mathcal{L}_\tau^j \varphi - \mathcal{L}_\omega^j \varphi| \, dm \le \int_{f_\omega^j(\mathcal{C}(\gamma_1)) \cup f_\tau^j(\mathcal{C}(\gamma_1))} (|\mathcal{L}_\tau^j \varphi| + |\mathcal{L}_\omega^j \varphi|) \, dm \tag{11}$$

$$+ \sum_{D_\omega \in \mathcal{D}(\omega)} \int_{f_\omega^j(D_\omega) \cap f_\tau^j(D_\tau)} |\mathcal{L}_\tau^j \varphi - \mathcal{L}_\omega^j \varphi| \, dm \tag{12}$$

$$+ \sum_{D_\omega \in \mathcal{D}(\omega)} \int_{f_\omega^j(D_\omega) \triangle f_\tau^j(D_\tau)} (|\mathcal{L}_\tau^j \varphi| + |\mathcal{L}_\omega^j \varphi|) \, dm, \tag{13}$$

where D_τ always denotes the element of $\mathcal{D}(\tau)$ associated with each $D_\omega \in \mathcal{D}(\omega)$. Let us now estimate the expressions on the right-hand side of this inequality. We start with (11). For notational simplicity, we write $E = f_\omega^j(\mathcal{C}(\gamma_1)) \cup f_\tau^j(\mathcal{C}(\gamma_1))$. Then

$$\int_E |\mathcal{L}_\tau^j \varphi| \, dm \le \int \chi_E (\mathcal{L}_\tau^j |\varphi|) \, dm = \int (\chi_E \circ f_\tau^j) |\varphi| \, dm.$$

It follows from Hölder's inequality and (10) that

$$\int (\chi_E \circ f_\tau^j) |\varphi| \, dm \le m((f_\tau^j)^{-1} E) \|\varphi\|_\infty \le \frac{\gamma}{8} \|\varphi\|_\infty.$$

The case associated with f_ω^j gives a similar bound for the second term in (11). So,

$$\int_{f_\omega^j(\mathcal{C}(\gamma_1)) \cup f_\tau^j(\mathcal{C}(\gamma_1))} (|\mathcal{L}_\tau^j \varphi| + |\mathcal{L}_\omega^j \varphi|) \, dm \le \frac{\gamma}{4} \|\varphi\|_\infty. \tag{14}$$

Making the change of variables $y = f_\omega^j(x)$ in (12), we may rewrite this as

$$\int_{\widehat{D_\omega}} \left| \frac{\varphi}{|\det Df_\tau^j|} \circ ((f_\tau^j)^{-1} \circ f_\omega^j) - \frac{\varphi}{|\det Df_\omega^j|} \right| \cdot |\det Df_\omega^j| \, dm,$$

where

$$\widehat{D_\omega} = (f_\omega^j)^{-1}(f_\omega^j(D_\omega) \cap f_\tau^j(D_\tau)) = D_\omega \cap ((f_\omega^j)^{-1} \circ f_\tau^j)(D_\tau).$$

For notational simplicity, we introduce $g = (f_\tau^j)^{-1} \circ f_\omega^j$. The previous expression is bounded by

$$\int_{\widehat{D}_\omega} \left(|\varphi \circ g - \varphi| \cdot \frac{|\det Df_\omega^j|}{|\det Df_\tau^j| \circ g} + |\varphi| \cdot \left| \frac{|\det Df_\omega^j|}{|\det Df_\tau^j| \circ g} - 1 \right| \right) dm.$$

Choosing $\alpha_j > 0$ sufficiently small, the assumption $\epsilon < \alpha_j$ implies

$$\left| \frac{|\det Df_\omega^j|}{|\det Df_\tau^j| \circ g} - 1 \right| \le \frac{\gamma}{8} \quad \text{and so} \quad \frac{|\det Df_\omega^j|}{|\det Df_\tau^j| \circ g} \le 2$$

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on $\Delta\backslash\mathcal{C}(\gamma_1)$ (which contains $\widehat{D}_\omega$). Hence, since φ belongs to $\mathcal{H}$,

$$\int_{f_\omega^j(D_\omega)\cap f_\tau^j(D_\tau)} |\mathcal{L}_\tau^j\varphi - \mathcal{L}_\omega^j\varphi|\,dm \le 2\int_{\widehat{D}_\omega} |\varphi\circ g - \varphi|\,dm + \frac{\gamma}{8}\int |\varphi|\,dm$$
$$\le 2K_\varphi \int_{\widehat{D}_\omega} \|g - \mathrm{id}_\Delta\|_0\,dm + \frac{\gamma}{8}\|\varphi\|_\infty.$$

Reducing $\alpha_j > 0$, we can make $\|g - \mathrm{id}_\Delta\|_0$ smaller than $\gamma\|\varphi\|_\infty/16K_\varphi m(\Delta)$, so

$$\int_{f_\omega^j(D_\omega)\cap f_\tau^j(D_\tau)} |\mathcal{L}_\tau^j\varphi - \mathcal{L}_\omega^j\varphi|\,dm \le \frac{\gamma}{4}\|\varphi\|_\infty. \tag{15}$$

We estimate the terms in (13) in much the same way as we did for (11). For each D_ω, let E now be $f_\omega^j(D_\omega)\triangle f_\tau^j(D_\tau)$. The properties of the operator, followed by Hölder's inequality, yield

$$\int_E |\mathcal{L}_\tau^j\varphi|\,dm \le \int (\chi_E\circ f_\tau^j)|\varphi|\,dm \le m((f_\tau^j)^{-1}E)\|\varphi\|_\infty.$$

Fix $\gamma_2 > 0$ such that $\#\mathcal{D}(\omega)4\gamma_2 < \gamma$. Taking α_j sufficiently small, we may ensure that the Lebesgue measure of all the sets

$$(f_\tau^j)^{-1}E = (f_\tau^j)^{-1}(f_\omega^j(D_\omega)\triangle f_\tau^j(D_\tau))$$

is small enough that

$$m((f_\tau^j)^{-1}E) < \gamma_2.$$

In this way we get

$$\int_{f_\omega^j(D_\omega)\triangle f_\tau^j(D_\tau)} (|\mathcal{L}_\omega^j\varphi| + |\mathcal{L}_\tau^j\varphi|)\,dm \le 2\gamma_2\|\varphi\|_\infty \tag{16}$$

(the second term on the left is estimated in the same way as the first one). Putting (14), (15) and (16) together, we obtain

$$\int |\mathcal{L}_\tau^j\varphi - \mathcal{L}_\omega^j\varphi|\,dm \le \left(\frac{\gamma}{2} + \#\mathcal{D}(\omega)2\gamma_2\right)\|\varphi\|_\infty < \gamma\|\varphi\|_\infty,$$

which concludes the proof. □

PROPOSITION 3.2. *Let $(\epsilon_n)_n$ be a sequence such that $\epsilon_n > 0$ and $\epsilon_n \to 0$, as $n \to \infty$. The density h_{ϵ_n} converges to h_∞ in the L^1-norm.*

Proof. For simplicity, we prove that $\|\tilde{h}_{\epsilon_n} - \tilde{h}_\infty\|_1$ converges to zero as ϵ_n goes to zero, where $\tilde{h}_{\epsilon_n} = d\tilde{\mu}_{\epsilon_n}/dm$, which implies the desired result. For a given $\gamma > 0$, we are looking for $\alpha > 0$ such that if $\epsilon_n < \alpha$, then $\|\tilde{h}_{\epsilon_n} - \tilde{h}_\infty\|_1 < \gamma$. By (7), there is an integer $N \ge 1$ for which

$$\sum_{j=N+1}^{+\infty} m(\{R_{\sigma^{-j}(\omega)} > j\}) < \frac{\gamma}{4K_1} \quad \text{for all } \omega \in \Omega_{\epsilon_n}. \tag{17}$$

We split the sums

$$\tilde{\mu}_\infty = \sum_{j=0}^{N} \xi_\infty^j + \eta_\infty^N \quad \text{and} \quad \tilde{\mu}_{\epsilon_n} = \sum_{j=0}^{N} \xi_{\epsilon_n}^j + \eta_\infty^N,$$

where, for every $j = 0, 1, 2, \ldots, N$, we have

$$\xi_\infty^j = (f^j)_*(\nu_\infty|\{R > j\}), \quad \xi_{\epsilon_n}^j = \int (f^j_{\sigma^{-j}(\omega)})_*(\nu_{\sigma^{-j}(\omega)}|\{R_{\sigma^{-j}(\omega)} > j\})\,d\theta_{\epsilon_n}^{\mathbb{Z}}(\omega),$$

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and the remaining sums are

$$\eta_\infty^N = \sum_{j=N+1}^{+\infty} (f^j)_*(\nu_\infty|\{R > j\}),$$

$$\eta_{\epsilon_n}^N = \int \sum_{j=N+1}^{+\infty} (f^j_{\sigma^{-j}(\omega)})_*(\nu_{\sigma^{-j}(\omega)}|\{R_{\sigma^{-j}(\omega)} > j\})\, d\theta_{\epsilon_n}^{\mathbb{Z}}(\omega).$$

Recall that the realization ω^*, which reproduces the original deterministic dynamical system, belongs to Ω_{ϵ_n}. By (17), we have

$$\begin{aligned}\eta_\infty^N(M) &= \sum_{j=N+1}^{+\infty} (f^j)_*(\nu_\infty|\{R > j\})(M) \\ &\le K_1 \sum_{j=N+1}^{+\infty} m(\{R > j\}) \\ &\le \frac{\gamma}{4}.\end{aligned}$$

Similarly,

$$\begin{aligned}\eta_{\epsilon_n}^N(M) &= \int \sum_{j=N+1}^{+\infty} (f^j_{\sigma^{-j}(\omega)})_*(\nu_{\sigma^{-j}(\omega)}|\{R_{\sigma^{-j}(\omega)} > j\})(M)\, d\theta_{\epsilon_n}^{\mathbb{Z}} \\ &\le K_1 \int \sum_{j=N+1}^{+\infty} m(\{R_{\sigma^{-j}(\omega)} > j\})\, d\theta_{\epsilon_n}^{\mathbb{Z}} \\ &\le \frac{\gamma}{4}.\end{aligned}$$

Hence, we have

$$\left\| \frac{d\eta_\infty^N}{dm} - \frac{d\eta_{\epsilon_n}^N}{dm} \right\|_1 \le \eta_\infty^N(M) + \eta_{\epsilon_n}^N(M) \le \frac{\gamma}{2}. \tag{18}$$

By (9), there is some $\alpha_0 > 0$ such that $\epsilon_n < \alpha_0$ implies

$$\left\| \frac{d\xi_{\epsilon_n}^0}{dm} - \frac{d\xi_\infty^0}{dm} \right\|_1 = \left\| \rho_{\epsilon_n} - \rho_\infty \right\|_1 < \frac{\gamma}{4}. \tag{19}$$

On the other hand, for every $j = 1, 2, \ldots, N$,

$$\begin{aligned}\left\| \frac{d\xi_{\epsilon_n}^j}{dm} - \frac{d\xi_\infty^j}{dm} \right\|_1 &= \left\| \int \mathcal{L}^j_{\sigma^{-j}(\omega)}(\rho_{\sigma^{-j}(\omega)}\chi_{\{R_{\sigma^{-j}(\omega)} > j\}})\, d\theta_{\epsilon_n}^{\mathbb{Z}} - \mathcal{L}^j(\rho_\infty \chi_{\{R>j\}}) \right\|_1 \\ &\le A_j + B_j + C_j,\end{aligned}$$

where

$$\begin{aligned}A_j &= \left\| \int \mathcal{L}^j_{\sigma^{-j}(\omega)}(\rho_{\sigma^{-j}(\omega)}\chi_{\{R_{\sigma^{-j}(\omega)}>j\}}) - \mathcal{L}^j(\rho_{\sigma^{-j}(\omega)}\chi_{\{R_{\sigma^{-j}(\omega)}>j\}})\, d\theta_{\epsilon_n}^{\mathbb{Z}} \right\|_1, \\ B_j &= \left\| \int \mathcal{L}^j(\rho_{\sigma^{-j}(\omega)}\chi_{\{R_{\sigma^{-j}(\omega)}>j\}} - \rho_{\sigma^{-j}(\omega)}\chi_{\{R>j\}})\, d\theta_{\epsilon_n}^{\mathbb{Z}} \right\|_1, \\ C_j &= \left\| \int \mathcal{L}^j(\rho_{\sigma^{-j}(\omega)}\chi_{\{R>j\}})\, d\theta_{\epsilon_n}^{\mathbb{Z}} - \mathcal{L}^j(\rho_\infty \chi_{\{R>j\}}) \right\|_1.\end{aligned}$$

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Lemma 3.1 implies that there exists some $\alpha_j > 0$ such that if $\epsilon_n < \alpha_j$, then

$$\begin{aligned} A_j &\le \iint |\mathcal{L}^j_{\sigma^{-j}(\omega)}(\rho_{\sigma^{-j}(\omega)}\chi_{\{R_{\sigma^{-j}(\omega)}>j\}}) - \mathcal{L}^j(\rho_{\sigma^{-j}(\omega)}\chi_{\{R_{\sigma^{-j}(\omega)}>j\}})|\, dm\, d\theta^{\mathbb{Z}}_{\epsilon_n} \\ &\le \iint |\mathcal{L}^j_{\sigma^{-j}(\omega)}(\rho_{\sigma^{-j}(\omega)}) - \mathcal{L}^j(\rho_{\sigma^{-j}(\omega)})|\, dm\, d\theta^{\mathbb{Z}}_{\epsilon_n} \\ &\le \frac{\gamma}{12N}. \end{aligned}$$

We also consider α_j small enough that, by condition (U1), if $\epsilon_n < \alpha_j$, then

$$m(\{R_{\sigma^{-j}(\omega)} = l\} \triangle \{R = l\}) \le \frac{\gamma}{12jNK_1},$$

for $l = 1, 2, \ldots, j$ and $\omega \in \Omega_{\epsilon_n}$. Since

$$\begin{aligned} m(\{R_{\sigma^{-j}(\omega)} > j\} \triangle \{R > j\}) &= m\left(\left(\Delta \setminus \bigcup_{l=1}^{j} \{R_{\sigma^{-j}(\omega)} = l\}\right) \triangle \left(\Delta \setminus \bigcup_{l=1}^{j} \{R = l\}\right)\right) \\ &\le \sum_{l=1}^{j} m(\{R_{\sigma^{-j}(\omega)} = l\} \triangle \{R = l\}) \\ &\le \sum_{l=1}^{j} \frac{\gamma}{12jNK_1} = \frac{\gamma}{12NK_1}, \end{aligned}$$

we have

$$\begin{aligned} B_j &\le \iint |\mathcal{L}^j(\rho_{\sigma^{-j}(\omega)}(\chi_{\{R_{\sigma^{-j}(\omega)}>j\}} - \chi_{\{R>j\}}))|\, dm\, d\theta^{\mathbb{Z}}_{\epsilon_n} \\ &\le \iint |\rho_{\sigma^{-j}(\omega)}||\chi_{\{R_{\sigma^{-j}(\omega)}>j\}} - \chi_{\{R>j\}}|\, dm\, d\theta^{\mathbb{Z}}_{\epsilon_n} \\ &\le K_1 \int m(\{R_{\sigma^{-j}(\omega)} > j\} \triangle \{R > j\})\, d\theta^{\mathbb{Z}}_{\epsilon_n} \\ &\le \frac{\gamma}{12N}. \end{aligned}$$

If α_0 is small enough, for $\epsilon_n < \alpha_0$, we also have

$$\begin{aligned} C_j &\le \left\| \int \mathcal{L}^j(\rho_{\sigma^{-j}(\omega)})\, d\theta^{\mathbb{Z}}_{\epsilon_n} - \mathcal{L}^j(\rho_\infty) \right\|_1 \\ &= \|\mathcal{L}^j(\rho_{\epsilon_n}) - \mathcal{L}^j(\rho_\infty)\|_1 \\ &\le \|\rho_{\epsilon_n} - \rho_\infty\|_1 \\ &\le \frac{\gamma}{12N}. \end{aligned}$$

Altogether, considering $\epsilon_n < \min_{j\in\{0,1,\ldots,N\}} \{\alpha_j\}$, we get $A_j + B_j + C_j < \gamma/4N$, for $1 \le j \le N$. The sum over all these js added to the superior limit at (19) is less than $\gamma/2$. The estimate (18) completes the proof. □

PROPOSITION 3.3. *The measure μ_∞ is f-invariant.*

Proof. Take any continuous map $\varphi : M \to \mathbb{R}$. Since

$$\left| \int \varphi\, d\mu_{\epsilon_n} - \int \varphi\, d\mu_\infty \right| \le \|\varphi\|_\infty \|h_{\epsilon_n} - h_\infty\|_1$$

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and $\|h_{\epsilon_n} - h_\infty\|_1 \to 0$, as $\epsilon_n \to 0$, then μ_{ϵ_n} converges to μ_∞ in the weak* sense and

$$\int \varphi \, d\mu_{\epsilon_n} \to \int \varphi \, d\mu_\infty.$$

However, since μ_{ϵ_n} is a stationary measure, we have

$$\int \varphi(x) \, d\mu_{\epsilon_n}(x) = \iint (\varphi \circ f_s)(x) \, d\mu_{\epsilon_n}(x) \, d\theta_{\epsilon_n}(s).$$

It then suffices to prove that

$$\iint (\varphi \circ f_s) \, d\mu_{\epsilon_n} \, d\theta_{\epsilon_n} \to \int (\varphi \circ f) \, d\mu_\infty.$$

So,

$$\begin{aligned}\left| \iint (\varphi \circ f_s) \, d\mu_{\epsilon_n} d\theta_{\epsilon_n} - \int (\varphi \circ f) \, d\mu_\infty \right| \\ \leq \left| \iint (\varphi \circ f_s) \, d\mu_{\epsilon_n} \, d\theta_{\epsilon_n} - \int (\varphi \circ f) \, d\mu_{\epsilon_n} \right| \\ + \left| \int (\varphi \circ f) \, d\mu_{\epsilon_n} - \int (\varphi \circ f) \, d\mu_\infty \right|.\end{aligned}$$

For ϵ_n sufficiently small, $(\varphi \circ f_s - \varphi \circ f)$ is uniformly close to 0 for every $s \in \operatorname{supp} \theta_{\epsilon_n}$. The second term is smaller than $\|\varphi\|_\infty \|h_{\epsilon_n} - h_\infty\|_1$, which is close to zero if ϵ_n is small enough. □

Remark 3.4. The uniqueness of an SRB measure μ_f for f ensures that in the previous arguments we can consider the whole sequence ϵ_n instead of just a subsequence of it. To see this, for every subsequence of ϵ_n, we can repeat the previous process and obtain a new subsequence ϵ'_n for which the corresponding sequence of densities $(h_{\epsilon'_n})_n$ has the limit h'_∞, and $\mu'_f = h'_\infty \, dm$ is also an f-invariant SRB measure (thus equal to μ_f). On the other hand, one knows that if all subsequences of a given sequence admit a subsequence converging to the same limit, then the whole sequence converges to that limit.

This finishes the proof of Theorem A.

4. *Induced Gibbs–Markov maps*

4.1. *Hyperbolic times and bounded distortion.* Hyperbolic times were introduced in [**Al00**] for deterministic systems and extended in [**AA03**] to a random context. Here we recall the definition and the main properties. For the next definition we fix $B > 1$ and $\beta > 0$ as in Definition 1.1, and take a constant $b > 0$ such that $2b < \min\{1, \beta^{-1}\}$.

Definition 4.1. Given $0 < \lambda < 1$ and $\delta > 0$, we say that $n \in \mathbb{N}$ is a (λ, δ)-hyperbolic time for $(\omega, x) \in T^{\mathbb{Z}} \times M$ if, for every $1 \leq k \leq n$,

$$\prod_{j=n-k}^{n-1} \|Df_{\sigma^j(\omega)}(f_\omega^j(x))^{-1}\| \leq \lambda^k \quad \text{and} \quad \operatorname{dist}_\delta(f_\omega^{n-k}(x), \mathcal{C}) \geq \lambda^{bk}. \tag{20}$$

In the case of $\mathcal{C} = \emptyset$, the definition of (λ, δ)-hyperbolic time reduces to the first condition in (20), and we simply call it a λ-*hyperbolic time*.

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We define, for $\omega \in T^{\mathbb{Z}}$ and $n \geq 1$, the set

$$H_\omega^n = \{x \in M : \ n \text{ is a } (\lambda, \delta)\text{-hyperbolic time for } (\omega, x)\}.$$

It follows from the definition that if n is a (λ, δ)-hyperbolic time for $(\omega, x) \in T^{\mathbb{Z}} \times M$, then $(n - j)$ is a (λ, δ)-hyperbolic time for $(\sigma^j(\omega), f_\omega^j(x))$, with $1 \leq j < n$.

LEMMA 4.2. (Pliss) *Given $0 < c \leq A$, let $\zeta = c/A$. Assume that $a_1, \ldots, a_N$ are real numbers satisfying $a_j \leq A$, for every $1 \leq j \leq N$, and $\sum_{j=1}^N a_j \geq cN$. Then there are $\ell \geq \zeta N$ and $1 \leq n_1 < \cdots < n_\ell \leq N$ such that $\sum_{j=n}^{n_i} a_j \geq 0$ for every $1 \leq n \leq n_i$ and $1 \leq i \leq \ell$.*

Proof. See [**ABV00**, Lemma 3.1]. □

Definition 4.3. We say that the *frequency of* (λ, δ)*-hyperbolic times* for $(\omega, x) \in T^{\mathbb{Z}} \times M$ is larger than $\zeta > 0$ if, for large $n \in \mathbb{N}$, there are $\ell \geq \zeta n$ and integers $1 \leq n_1 < n_2 < \cdots < n_\ell \leq n$ which are (λ, δ)-hyperbolic times for (ω, x).

PROPOSITION 4.4. *Assume that f is non-uniformly expanding on random orbits. Then there are $0 < \lambda < 1$, $\delta > 0$ and $\zeta > 0$ (depending only on a_0 in (1) and on the map f) such that for all $\omega \in \Omega_\epsilon$ and Lebesgue almost every point $x \in M$, the frequency of (λ, δ)-hyperbolic times for (ω, x) is larger than ζ.*

Proof. The proof follows from using Lemma 4.2 twice: first for the sequence given by $a_j = -\log \|Df_{\omega_{j-1}}(f_\omega^{j-1}(x))^{-1}\|$ (up to a cut off which makes it bounded from above in the presence of a critical set), and then with $a_j = \log \operatorname{dist}_\delta(f_\omega^{j-1}(x), \mathcal{C})$ for a convenient choice of $\delta > 0$. We prove that there exist many times n_i for which the conclusion of Lemma 4.2 holds simultaneously for both sequences. Then we check that any such n_i is a (λ, δ)-hyperbolic time for (ω, x).

Assuming that (5) and (4) hold for (ω, x), then, for large N, we have

$$\sum_{j=1}^{N} -\log \|Df_{\omega_{j-1}}(f_\omega^{j-1}(x))^{-1}\| \geq a_0 N.$$

If $\mathcal{C} = \emptyset$, we just use Lemma 4.2 for the sequence

$$a_j = -\log \|Df_{\omega_{j-1}}(f_\omega^{j-1}(x))^{-1}\| - \frac{a_0}{2},$$

with $c = a_0/2$ and

$$A = \max_{t \in \operatorname{supp}(\theta_\epsilon)} \max_{x \in M} \{-\log \|Df_t(x)^{-1}\| - a_0/2\},$$

and we obtain the result for $\zeta = a_0/(2A)$ and $\lambda = e^{-a_0/2}$ (δ is not required in this case).

If $\mathcal{C}$ is not empty, we recall assumption (2). Take $B, \beta > 0$ given by Definition 1.1. Condition (c_1) implies that for large $\rho > 0$,

$$|\log \|Df(x)^{-1}\| \,| \leq \rho |\log \operatorname{dist}(x, \mathcal{C})|, \tag{21}$$

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for every $x \in M \backslash \mathcal{C}$. Fix $\alpha_1 > 0$, so that $\rho\alpha_1 \le a_0/2$. The condition (4) of slow recurrence to $\mathcal{C}$ ensures that we may choose $r_1 > 0$, so that for large N,

$$\sum_{j=1}^{N} \log \operatorname{dist}_{r_1}(f_\omega^{j-1}(x), \mathcal{C}) \ge -\alpha_1 N. \tag{22}$$

Fix any open neighborhood V of $\mathcal{C}$ and take $Q \ge \rho\,|\log r_1|$ large enough that it is also an upper bound for $-\log \|Df^{-1}\|$ on $M \backslash V$. Then let

$$J = \{1 \le j \le N : -\|Df(f_\omega^{j-1}(x))^{-1}\| > Q\}.$$

Note that if $j \in J$, then $f_\omega^{j-1}(x) \in V$. Moreover, for each $j \in J$,

$$\rho|\log r_1| \le Q < -\log \|Df(f_\omega^{j-1}(x))^{-1}\| < \rho|\log \operatorname{dist}(f_\omega^{j-1}(x), \mathcal{C})|,$$

which shows that $\operatorname{dist}(f_\omega^{j-1}(x), \mathcal{C}) < r_1$ for every $j \in J$. In particular,

$$\operatorname{dist}_{r_1}(f_\omega^{j-1}(x), \mathcal{C}) = \operatorname{dist}(f_\omega^{j-1}(x), \mathcal{C}) < r_1 \quad \text{for all } j \in J.$$

Therefore, by (21) and (22),

$$\sum_{j\in J} -\log \|Df(f_\omega^{j-1}(x))^{-1}\| \le \rho \sum_{j\in J} |\log \operatorname{dist}(f_\omega^{j-1}(x), \mathcal{C})| \le \rho\,\alpha_1 N \le \frac{a_0}{2} N.$$

Define

$$b_j = \begin{cases} -\log \|Df(f_\omega^{j-1}(x))^{-1}\| & \text{if } j \notin J, \\ 0 & \text{if } j \in J. \end{cases}$$

By definition, $b_j \le Q$ for each $1 \le j \le N$. As a consequence,

$$\sum_{j=1}^{N} b_j = \sum_{j=1}^{N} -\log \|Df(f_\omega^{j-1}(x))^{-1}\| - \sum_{j\in J} -\log \|Df(f_\omega^{j-1}(x))^{-1}\| \ge \frac{a_0}{2} N.$$

Defining $a_j = b_j - a_0/4$, we have

$$\sum_{j=1}^{N} a_j \ge \frac{a_0}{4} N.$$

Thus, we may apply Lemma 4.2 to $a_1, \ldots, a_N$, with $c = a_0/4$ and $A = Q$. The lemma provides $\zeta_1 > 0$ and $\ell_1 \ge \zeta_1 N$ times $1 \le p_1 < \cdots < p_{\ell_1} \le N$ such that for every $0 \le n \le p_i - 1$ and $1 \le i \le \ell_1$,

$$\sum_{j=n+1}^{p_i} -\log \|Df(f_\omega^{j-1}(x))^{-1}\| \ge \sum_{j=n+1}^{p_i} b_j = \sum_{j=n+1}^{p_i} \left(a_j + \frac{a_0}{4}\right) \ge \frac{a_0}{4}(p_i - n). \tag{23}$$

Now fix $\alpha_2 > 0$ small enough that $\alpha_2 < \zeta_1 b a_0/4$, and let $r_2 > 0$ be such that

$$\sum_{j=1}^{N} \log \operatorname{dist}_{r_2}(f_\omega^{j-1}(x), \mathcal{C}) \ge -\alpha_2 N.$$

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Defining

$$a_j = \log \operatorname{dist}_{r_2}(f_\omega^{j-1}(x), \mathcal{C}) + ba_0/4,$$

we have

$$\sum_{j=1}^{N} a_j \geq \left(\frac{ba_0}{4} - \alpha_2\right) N.$$

Applying Lemma 4.2 to $a_1, \ldots, a_N$ with $c = ba_0/4 - \alpha_2$ and $A = ba_0/4$, we conclude that there are $l_2 \geq \zeta_2 N$ times $1 \leq q_1 < \cdots < q_{\ell_2} \leq N$ such that for every $0 \leq n \leq q_i - 1$ and $1 \leq i \leq \ell_2$,

$$\sum_{j=n+1}^{q_i} \log \operatorname{dist}_{r_2}(f_\omega^{j-1}(x), \mathcal{C}) \geq -\frac{ba_0}{4}(q_i - n). \tag{24}$$

Moreover,

$$\zeta_2 = \frac{c}{A} = 1 - \frac{4\alpha_2}{ba_0}.$$

Finally, our condition on α_2 means that $\zeta_1 + \zeta_2 > 1$. Let $\zeta = \zeta_1 + \zeta_2 - 1$. Then there exist $\ell = (\ell_1 + \ell_2 - N) \geq \zeta N$ times $1 \leq n_1 < \cdots < n_\ell \leq N$ at which (23) and (24) occur simultaneously:

$$\sum_{j=n}^{n_i-1} -\log \|Df(f_\omega^j(x))^{-1}\| \geq \frac{a_0}{4}(n_i - n)$$

and

$$\sum_{j=n}^{n_i-1} \log \operatorname{dist}_{r_2}(f_\omega^j(x), \mathcal{C}) \geq -\frac{ba_0}{4}(n_i - n),$$

for every $0 \leq n \leq n_i - 1$ and $1 \leq i \leq \ell$. Letting $\lambda = e^{-a_0/4}$, from the inequalities above we easily obtain

$$\prod_{j=n_i-k}^{n_i-1} \|Df(f_\omega^j(x))^{-1}\| \leq \lambda^k \quad \text{and} \quad \operatorname{dist}_{r_2}(f_\omega^{n_i-k}(x), \mathcal{C}) \geq \lambda^{bk},$$

for every $1 \leq i \leq \ell$ and $1 \leq k \leq n_i$. In other words, all these n_i are (λ, δ)-hyperbolic times for (ω, x), with $\delta = r_2$. □

Remark 4.5. In the presence of a critical set, one can see that condition (4) is not needed in all its strength. Actually, it is enough that (4) holds for some sufficiently small $b_0 > 0$ and some convenient $\delta > 0$ (e.g., $b_0 = \min\{\alpha_1, \alpha_2\}$ and $\delta = \max\{r_1, r_2\}$ in the proof of Proposition 4.4).

Remark 4.6. Observe that the proof of Proposition 4.4 gives more precisely that if for some $(\omega, x) \in T^{\mathbb{Z}} \times M$ and $N \in \mathbb{N}$,

$$\sum_{j=0}^{N-1} -\log \|Df_{\sigma^j(\omega)}(f_\omega^j(x))^{-1}\| \geq a_0 N \quad \text{and} \quad \sum_{j=0}^{N-1} \log \operatorname{dist}_\delta(f_\omega^j(x), \mathcal{C}) \geq -b_0 N$$

(where b_0 and δ are chosen according to Remark 4.5), then there exist integers $0 < n_1 < \cdots < n_l \leq N$ with $l \geq \zeta N$ such that n_i is a (λ, δ)-hyperbolic time for (ω, x), for each $1 \leq i \leq l$.

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The next result gives us property (m_1) in §4.5.2, and is needed to ensure some metric estimates on the algorithm for the random induced partition.

LEMMA 4.7. *Let $A \subset M$ be a set with positive Lebesgue measure, for whose points x we have (ω, x) with a frequency of (λ, δ)-hyperbolic times greater than $\zeta > 0$, for all $\omega \in \Omega_\epsilon$. Then there is $n_0 \in \mathbb{N}$ such that for $n \geq n_0$,*

$$\frac{1}{n}\sum_{j=1}^{n} \frac{m(A \cap H_\omega^j)}{m(A)} \geq \zeta.$$

Proof. Since we are assuming that points (ω, x) for which x is in A have a frequency of (λ, δ)-hyperbolic times greater than $\zeta > 0$, there is $n_0 \in \mathbb{N}$ such that for every $x \in A$ and $n \geq n_0$, there are (λ, δ)-hyperbolic times $0 < n_1 < n_2 < \cdots < n_\ell \leq n$ for x with $\ell \geq \zeta n$. Take $n \geq n_0$ and let ξ_n be the measure on $I_n = \{1, \ldots, n\}$ defined by $\xi_n(J) = \#J/n$ for each $J \subset I_n$. Then, putting $\chi(x, i) = 1$ if $x \in H_\omega^i$, and $\chi(x, i) = 0$ otherwise, by Fubini's theorem,

$$\begin{aligned}\frac{1}{n}\sum_{j=1}^{n} m(A \cap H_\omega^j) &= \int \left(\int_A \chi(x, i)\, dm(x) \right) d\xi_n(i) \\ &= \int_A \left(\int \chi(x, i)\, d\xi_n(i) \right) dm(x).\end{aligned}$$

Since for every $x \in A$ and $n \geq n_0$ there are $0 < n_1 < n_2 < \cdots < n_\ell \leq n$ with $\ell \geq \zeta n$ such that $x \in H_\omega^{n_i}$ for $1 \leq i \leq \ell$, then the integral with respect to $d\xi_n$ is larger than $\zeta > 0$ and the last expression in the formula above is bounded from below by $\zeta m(A)$. □

LEMMA 4.8. *Given $0 < \lambda < 1$ and $\delta > 0$, there is $\delta_1 > 0$ such that if n is a (λ, δ)-hyperbolic time for $(\omega, x) \in T^{\mathbb{Z}} \times M$, then*

$$\|Df_\omega(y)^{-1}\| \leq \lambda^{-1/2} \|Df_\omega(x)^{-1}\|,$$

for any point y in the ball of radius $\delta_1 \lambda^{n/2}$ around x.

Proof. If $\mathcal{C} = \emptyset$, and since f_ω is a local diffeomorphism, for each $x \in M$, there is a radius $\delta_x > 0$ such that f_ω sends a neighborhood of x diffeomorphically onto $B(f_\omega(x), \delta_x)$, the ball of radius δ_x around $f_\omega(x)$. By compactness of M, we may choose a uniform radius $\delta_1 > 0$. We choose $\delta_1 > 0$ small enough that

$$\|Df_\omega(y)^{-1}\| \leq \lambda^{-1/2} \|Df_\omega(x)^{-1}\|,$$

whenever $y \in B(x, \delta_1 \lambda^{1/2})$ and $\omega \in \mathrm{supp}(\theta_\epsilon^{\mathbb{Z}})$.

In the case $\mathcal{C} \neq \emptyset$, if n is a (λ, δ)-hyperbolic time for (ω, x), then

$$\mathrm{dist}_\delta(x, \mathcal{C}) \geq \lambda^{bn}.$$

According to the definition of the truncated distance, this means that

$$\mathrm{dist}(x, \mathcal{C}) = \mathrm{dist}_\delta(x, \mathcal{C}) \geq \lambda^{bn} \quad \text{or else} \quad \mathrm{dist}(x, \mathcal{C}) \geq \delta.$$

In either case, considering $2\delta_1 < \delta$, we have for any point y in the ball of radius $\delta_1 \lambda^{n/2}$ around x,

$$\mathrm{dist}(y, x) < \tfrac{1}{2}\, \mathrm{dist}(x, \mathcal{C}),$$

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because we haven chosen $b < 1/2$ and $\delta_1 < \delta/2 < 1/2$. Therefore, we may use (c_2) to conclude that

$$\log \frac{\|Df(y)^{-1}\|}{\|Df(x)^{-1}\|} \le B\frac{\operatorname{dist}(y, x)}{\operatorname{dist}(x, \mathcal{C})^{\beta}} \le B\frac{\delta_1\lambda^{n/2}}{\min\{\lambda^{b\beta n}, \delta^{\beta}\}}.$$

Since $\delta > 0$, $0 < \lambda < 1$ and we have taken $b\beta < 1/2$, the term on the right-hand side is bounded by $B\delta_1\delta^{-\beta}$. Choosing $\delta_1 > 0$ small, so that $B\delta_1\delta^{-\beta} < \log \lambda^{-1/2}$, we get the conclusion. □

We assume that given (λ, δ) as before, we fix δ_1 small, so that Lemma 4.8 holds. We further require δ_1 small, so that the exponential map is an isometry onto its image in the ball of radius δ_1. This in particular implies that any point in the boundary of a ball of radius δ_1 can be joined to the center of the ball through a smooth curve of minimal length (a geodesic arc).

PROPOSITION 4.9. *If n is a (λ, δ)-hyperbolic time for $(\omega, x) \in T^{\mathbb{Z}} \times M$, then there is a neighborhood $V^n_\omega(x)$ of x in M such that:*

(1) *f^n_ω maps $V^n_\omega(x)$ diffeomorphically onto $B(f^n_\omega(x), \delta_1)$;*
(2) *for every $y \in V^n_\omega(x)$ and $1 \le k \le n$, we have $\|Df^k_{\sigma^{n-k}(\omega)}(f^{n-k}_\omega(y))^{-1}\| \le \lambda^{k/2}$; and*
(3) *for every $y, z \in V^n_\omega(x)$ and $1 \le k \le n$,*

$$\operatorname{dist}(f^{n-k}_\omega(y), f^{n-k}_\omega(z)) \le \lambda^{k/2} \operatorname{dist}(f^n_\omega(y), f^n_\omega(z)).$$

Proof. We will prove the first two items by induction on n. Let $n = 1$ be a (λ, δ)-hyperbolic time for $(\omega, x) \in T^{\mathbb{Z}} \times M$. It follows from Lemma 4.8 and from the definition of hyperbolic times that for any y in the ball $B(x, \delta_1\lambda^{1/2}) \subset M$ of radius $\delta_1\lambda^{1/2}$ around x,

$$\|Df_\omega(y)^{-1}\| \le \lambda^{-1/2}\|Df_\omega(x)^{-1}\| \le \lambda^{1/2}. \tag{25}$$

This means that f_ω is a $\lambda^{-1/2}$-dilation in the ball $B(x, \delta_1\lambda^{1/2})$. Then there exists some neighborhood $V^1_\omega(x)$ of x contained in $B(x, \delta_1\lambda^{1/2})$ which is mapped diffeomorphically onto the ball $B(f_\omega(x), \delta_1)$, and for $y \in V^1_\omega(x)$, condition (25) ensures the second property $\|Df_\omega(y)^{-1}\| \le \lambda^{1/2}$.

Assume now that the conclusion holds for $n \ge 1$. Let $n + 1$ be a (λ, δ)-hyperbolic time for $(\omega, x) \in T^{\mathbb{Z}} \times M$. Take any $z \in \partial B(f^{n+1}_\omega(x), \delta_1)$, and let $\gamma\colon [0, 1] \to M$ be a smooth curve of minimal length joining z to $f^{n+1}_\omega(x)$. The curve γ necessarily lies inside $B(f^{n+1}_\omega(x), \delta_1)$ by the choice of δ_1. Consider γ_n and γ_{n+1} smooth curves which are lifts of γ starting at $f_\omega(x)$ and x, respectively. This means that

$$\gamma = f^n_{\sigma(\omega)} \circ \gamma_n \quad \text{and} \quad \gamma = f^{n+1}_\omega \circ \gamma_{n+1},$$

at least in the domains where the lifts make sense. Since n is a (λ, δ)-hyperbolic time for $(\sigma(\omega), f_\omega(x))$, by the induction hypothesis, there is a neighborhood $V^n_{\sigma(\omega)}(f_\omega(x))$ which is sent diffeomorphically by $f^n_{\sigma(\omega)}$ onto the ball of radius δ_1 around $f^{n+1}_\omega(x)$ with the additional second condition property. One has that γ_n lies inside $V^n_{\sigma(\omega)}(f_\omega(x))$.

Moreover, $n - j$ is a (λ, δ)-hyperbolic time for $(\sigma^{j+1}(\omega), f^{j+1}_\omega(x))$ and $f^j_{\sigma(\omega)}(V^n_{\sigma(\omega)}(f_\omega(x)))$ is a neighborhood $V^{n-j}_{\sigma^{j+1}(\omega)}$ of $f^{j+1}_\omega(x)$ which is mapped by $f^{n-j}_{\sigma^{j+1}(\omega)}$ diffeomorphically onto $B(f^{n+1}_\omega(x), \delta_1)$ and also satisfies the second condition property, for $1 \le j \le n$.

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CLAIM. *The curve γ_{n+1} lies inside the ball of radius $\delta_1\lambda^{(n+1)/2}$ around x.*

Assume, by contradiction, that γ_{n+1} hits the boundary of $B(x, \delta_1\lambda^{(n+1)/2})$ before the end time. Let $0 < t_0 < 1$ be the first moment in such conditions. One necessarily has that $\gamma_{n+1}|[0, t_0]$ is a curve inside the ball $B(x, \delta_1\lambda^{(n+1)/2})$ joining x to a point in the boundary of that ball. Since $n+1-j$ is a (λ, δ)-hyperbolic time for $(\sigma^{j+1}(\omega), f_\omega^{j+1}(x))$, by Lemma 4.8, for each $0 \le t \le t_0$ and $0 \le j \le n$,

$$\|Df_{\sigma^j(\omega)}(f_\omega^j(\gamma_{n+1}(t)))^{-1}\| \le \lambda^{-1/2}\|Df_{\sigma^j(\omega)}(f_\omega^j(x))^{-1}\|$$

and $f_\omega^j(\gamma_{n+1}([0, t_0])) \subset V_{\sigma^j(\omega)}^{n+1-j}(f_\omega^j(x))$, which yields

$$\begin{aligned}
\|Df_\omega^{n+1}(\gamma_{n+1}(t))^{-1}\| &\le \prod_{j=0}^{n} \|Df_{\sigma^{n-j}(\omega)}(\gamma_{j+1}(t))^{-1}\| \\
&= \|Df_\omega(\gamma_{n+1}(t))^{-1}\| \cdot \prod_{j=0}^{n-1} \|Df_{\sigma^{n-j}(\omega)}(\gamma_{j+1}(t))^{-1}\| \\
&\le \lambda^{-1/2}\|Df_\omega(x)^{-1}\| \cdot \prod_{j=0}^{n-1} \lambda^{-1/2}\|Df_{\sigma^{n-j}(\omega)}(f_\omega^{n-j}(x))^{-1}\| \\
&\le \lambda^{-1/2}\lambda\lambda^{-n/2}\lambda^n \\
&= \lambda^{(n+1)/2}. \qquad (26)
\end{aligned}$$

Hence,

$$\begin{aligned}
\int_0^{t_0} \|\gamma'(t)\|\, dt &= \int_0^{t_0} \|Df_\omega^{n+1}(\gamma_{n+1}(t)) \cdot \gamma'_{n+1}(t)\|\, dt \\
&\ge \int_0^{t_0} \lambda^{-(n+1)/2}\|\gamma'_{n+1}(t)\|\, dt \\
&= \delta_1.
\end{aligned}$$

This gives a contradiction since $t_0 < 1$, thus proving the claim.

Let us now finish the proof of the first two items. We simply consider the lifts by f_ω^{n+1} of the geodesics joining $f_\omega^{n+1}(x)$ to the points in the boundary of $B(f_\omega^{n+1}(x), \delta_1)$. This defines a neighborhood $V_\omega^{n+1}(x)$ of x which, by (26), has the required properties.

For the third item, let γ be a curve of minimal length connecting $f_\omega^n(z)$ to $f_\omega^n(y)$. This curve γ must obviously be contained in $B(f_\omega^n(x), \delta_1)$. For $1 \le k \le n$, let γ_k be the (unique) curve in $f_\omega^{n-k}(V_\omega^n(x))$ joining $f_\omega^{n-k}(z)$ to $f_\omega^{n-k}(y)$ such that $f_{\sigma^{n-k}(\omega)}^k(\gamma_k) = \gamma$. We have for every $n \ge 1$,

$$\begin{aligned}
\text{length}(\gamma) &= \int \|\gamma'(t)\|\, dt \\
&= \int \|Df_{\sigma^{n-k}(\omega)}^k(\gamma_k(t)) \cdot \gamma_k'(t)\|\, dt \\
&\ge \lambda^{-(k/2)} \int \|\gamma_k'(t)\|\, dt \\
&= \lambda^{-(k/2)}\, \text{length}(\gamma_k).
\end{aligned}$$

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As a consequence,

$$\operatorname{dist}(f_\omega^{n-k}(y),\, f_\omega^{n-k}(z)) \le \operatorname{length}(\gamma_k) \le \lambda^{k/2} \operatorname{length}(\gamma) = \lambda^{k/2} \operatorname{dist}(f_\omega^n(y),\, f_\omega^n(z)). \quad \square$$

COROLLARY 4.10. (Bounded distortion) *There exists $C_0 > 0$ such that if n is a (λ, δ)-hyperbolic time for $(\omega, x) \in T^{\mathbb{Z}} \times M$, then for every $y, z \in V_\omega^n(x)$,*

$$\log \frac{|\det Df_\omega^n(y)|}{|\det Df_\omega^n(z)|} \le C_0 \operatorname{dist}(f_\omega^n(y),\, f_\omega^n(z)).$$

Proof. If $\mathcal{C} = \emptyset$, since $f_\omega \in C^2(M, M)$, then there is $C_0' > 0$ such that for all $z, y \in M$ and $\omega \in \operatorname{supp}(\theta_\epsilon)$, we have $|\log |\det Df_\omega(z)| - \log |\det Df_\omega(y)|| \le C_0' \operatorname{dist}(z, y)$, and for all $z, y \in V_n(x)$, we have

$$\begin{aligned}
\log \frac{|\det Df_w^n(z)|}{|\det Df_\omega^n(y)|} &= \sum_{j=0}^{n-1} \log \frac{|\det Df_{\sigma^j(\omega)}(f_\omega^j(z))|}{|\det Df_{\sigma^j(\omega)}(f_\omega^j(y))|} \\
&\le \sum_{j=0}^{n-1} C_0' \operatorname{dist}(f_\omega^j(z),\, f_\omega^j(y)) \\
&\le \sum_{j=0}^{n-1} C_0' \lambda^{n-j} \operatorname{dist}(f_\omega^n(z),\, f_\omega^n(y)).
\end{aligned}$$

It is then enough to take

$$C_0 = \exp\left(\sum_{k=0}^{\infty} C_0' \lambda^k\right).$$

If $\mathcal{C}$ is not empty, then let n be a (λ, δ)-hyperbolic time for $(\omega, x) \in T^{\mathbb{Z}} \times M$ with the associated V_ω^n. By Proposition 4.9, we have for each $y, z \in V_\omega^n$ and each $0 \le k \le n-1$,

$$\operatorname{dist}(f_\omega^k(y),\, f_\omega^k(z)) \le \delta_1 \lambda^{(n-k)/2}.$$

On the other hand, since n is a hyperbolic time for (ω, x),

$$\begin{aligned}
\operatorname{dist}(f_\omega^k(y), \mathcal{C}) &\ge \operatorname{dist}(f_\omega^k(x), \mathcal{C}) - \operatorname{dist}(f_\omega^k(x),\, f^k(y)) \\
&\ge \lambda^{b(n-k)} - \delta_1 \lambda^{(n-k)/2} \\
&\ge \tfrac{1}{2} \lambda^{b(n-k)} \\
&\ge 2\delta_1 \lambda^{(n-k)/2}, \qquad (27)
\end{aligned}$$

as long as $\delta_1 < 1/4$; recall that $b < 1/2$. Thus we have

$$\operatorname{dist}(f_\omega^k(y),\, f_\omega^k(z)) \le \tfrac{1}{2} \operatorname{dist}(f_\omega^k(y), \mathcal{C}),$$

and so we may use (c_3) to obtain

$$\log \frac{|\det Df(f_\omega^k(y))|}{|\det Df(f_\omega^k(z))|} \le \frac{B}{\operatorname{dist}(f_\omega^k(y), \mathcal{C})^\beta} \operatorname{dist}(f_\omega^k(y),\, f_\omega^k(z)).$$

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Hence, by (27) and Proposition 4.9,

$$\log \frac{|\det Df_\omega^n(y)|}{|\det Df_\omega^n(z)|} = \sum_{k=0}^{n-1} \log \frac{|\det Df(f_\omega^k(y))|}{|\det Df(f_\omega^k(z))|} \le \sum_{k=0}^{n-1} 2^\beta B \frac{\lambda^{(n-k)/2}}{\lambda^{b\beta(n-k)}} \operatorname{dist}(f_\omega^n(y),\, f_\omega^n(z)).$$

It suffices to take

$$C_0 \ge \sum_{k=1}^{+\infty} 2^\beta B \lambda^{(1/2-b\beta)k};$$

recall that $b\beta < 1/2$. □

The sets V_ω^n are often referred to as as *hyperbolic preballs* and their images $f_\omega^n(V_\omega^n)$ are referred to as *hyperbolic balls*. Notice that the latter are indeed balls of radius $\delta_1 > 0$.

At many points in this paper it will be useful to have the following weaker forms of the previous corollary.

COROLLARY 4.11. *There is a constant $C_1 > 0$ such that if n is a (λ, δ)-hyperbolic time for $(\omega, x) \in T^{\mathbb{Z}} \times M$ and $y, z \in V_\omega^n(x)$, then*

$$\frac{1}{C_1} \le \frac{|\det Df_\omega^n(y)|}{|\det Df_\omega^n(z)|} \le C_1.$$

Proof. By Corollary 4.10, we just have to consider $C_1 = \exp(C_0\delta_1)$. □

COROLLARY 4.12. *There is a constant $C_2 > 0$ such that for any hyperbolic preball $V_\omega^n(x)$ and any $A_1,\, A_2 \subset V_\omega^n(x)$,*

$$\frac{1}{C_2}\frac{m(A_1)}{m(A_2)} \le \frac{m(f_\omega^n(A_1))}{m(f_\omega^n(A_2))} \le C_2 \frac{m(A_1)}{m(A_2)}.$$

Proof. By the change of variable formula for f_ω^n, we may write

$$\frac{m(f_\omega^n(A_1))}{m(f_\omega^n(A_2))} = \frac{\int_{A_1} |\det Df_\omega^n(z)|\, dm(z)}{\int_{A_1} |\det Df_\omega^n(z)|\, dm(z)} = \frac{|\det Df_\omega^n(z_1)| \int_{A_1} |\det Df_\omega^n(z)/\det Df_\omega^n(z_1)|\, dm(z)}{|\det Df_\omega^n(z_2)| \int_{A_2} |\det Df_\omega^n(z)/\det Df_\omega^n(z_2)|\, dm(z)},$$

with z_1 and z_2 chosen arbitrarily in A_1 and A_2, respectively. From Corollary 4.11, we get the desired bounds. □

We are now in a position to prove Proposition 2.13.

Proof. Let $\delta_1 > 0$ be given by Lemma 4.8. It is enough to prove that there exist disks of radius $\delta_1/4$ where the relative measure of A is arbitrarily close to one. For $n \ge 1$, let A_n be the set of points $x \in A$ for which $f_\omega^n(x) \in A$ for $\theta_\epsilon^{\mathbb{Z}}$ almost every ω, and $\tilde{A} = \bigcap_{n=1}^{+\infty} A_n$. Since A is random forward invariant, then $\mu_\epsilon(A \backslash A_n) = 0$ for all n, and thus $\mu_\epsilon(A \backslash \tilde{A}) = 0$. We have $\mu_\epsilon(\tilde{A}) = \mu_\epsilon(A) > 0$, which implies $m(\tilde{A}) > 0$ since $\mu_\epsilon \ll m$. Moreover, for $\theta_\epsilon^{\mathbb{Z}}$ almost every ω, we have $f_\omega^k(\tilde{A}) \subset A$, for all $k \in \mathbb{N}$. Since the set of points $x \in M$

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for which for $\theta_\epsilon^{\mathbb{Z}}$ almost every ω there are infinitely many hyperbolic preballs $V_\omega^n(x)$ is random forward invariant, we may assume, without loss of generality, that every point in A has infinitely many hyperbolic preballs $V_\omega^n(x)$ for $\theta_\epsilon^{\mathbb{Z}}$ almost every ω. Recall that the hyperbolic preballs $V_\omega^n(x)$ are sent diffeomorphically by f_ω^n onto hyperbolic balls with radius δ_1, that is, $f_\omega^n(V_\omega^n(x)) = B(f_\omega^n(x), \delta_1)$.

Let $\gamma > 0$ be some small number. By the regularity of m, there is a compact set $\tilde{A}_c \subset \tilde{A}$ and an open set $\tilde{A}_o \supset \tilde{A}$ such that

$$m(\tilde{A}_o \backslash \tilde{A}_c) < \gamma m(\tilde{A}). \tag{28}$$

Assume that n_0 is large enough that, for every $x \in \tilde{A}_c$ and $\theta_\epsilon^{\mathbb{Z}}$ almost every ω, any hyperbolic preball $V_\omega^n(x)$, associated with the (λ, δ)-hyperbolic time n for (ω, x), with $n \geq n_0$, is contained in $\tilde{A}_o$. Let $W_\omega^n(x)$ be the part of $V_\omega^n(x)$ which is sent diffeomorphically by f_ω^n onto the ball $B(f_\omega^n(x), \delta_1/4)$. By compactness, there are $x_1, \ldots, x_r \in \tilde{A}_c$ and $n(x_1), \ldots, n(x_r) \geq n_0$ such that

$$\tilde{A}_c \subset W_\omega^{n(x_1)}(x_1) \cup \cdots \cup W_\omega^{n(x_r)}(x_r). \tag{29}$$

For the sake of notational simplicity, for each $1 \leq i \leq r$, we will write

$$V_\omega^i = V_\omega^{n(x_i)}(x_i), \quad W_\omega^i = W_\omega^{n(x_i)}(x_i) \quad \text{and} \quad n_i = n(x_i).$$

Assume that

$$\{n_1, \ldots, n_r\} = \{n_1^*, \ldots, n_s^*\}, \quad \text{with } n_1^* < n_2^* < \cdots < n_s^*.$$

Let $I_1 \subset \mathbb{N}$ be a maximal subset of $\{1, \ldots, r\}$ such that for each $i \in I_1$ both $n_i = n_1^*$ and $W_\omega^i \cap W_\omega^j = \emptyset$ for every $j \in I_1$ with $j \neq i$. Inductively, we define I_k for $2 \leq k \leq s$ as follows: supposing that $I_1, \ldots, I_{k-1}$ have already been defined, let I_k be a maximal set of $\{1, \ldots, r\}$ such that for each $i \in I_k$ both $n_i = n_k^*$ and $W_\omega^i \cap W_\omega^j = \emptyset$ for every $j \in I_1 \cup \cdots \cup I_k$ with $i \neq j$.

Define $I = I_1 \cup \cdots \cup I_s$. By construction, we have that $\{W_\omega^i\}_{i \in I}$ is a family of pairwise disjoint sets. We claim that $\{V_\omega^i\}_{i \in I}$ is a covering of $\tilde{A}_c$. To see this, recall that, by construction, given any W_ω^j with $1 \leq j \leq r$, there is some $i \in I$ with $n(x_i) \leq n(x_j)$ such that $W_\omega^{x_j} \cap W_\omega^{x_i} \neq \emptyset$. Taking images by $f_\omega^{n(x_i)}$, we have

$$f_\omega^{n(x_i)}(W_j) \cap B(f_\omega^{n(x_i)}(x_i), \delta_1/4) \neq \emptyset.$$

It follows from Proposition 4.9 that

$$\operatorname{diam}(f_\omega^{n(x_i)}(W_\omega^j)) \leq \frac{\delta_1}{2} \lambda^{(n(x_j) - n(x_i))/2} \leq \frac{\delta_1}{2},$$

and so

$$f_\omega^{n(x_i)}(W_\omega^j) \subset B(f_\omega^{n(x_i)}(x_i), \delta_1).$$

This gives that $W_\omega^j \subset V_\omega^i$. We have proved that given any W_ω^j with $1 \leq j \leq r$, there is $i \in I$ such that $W_\omega^j \subset V_\omega^i$. Taking into account equation (29), this means that $\{V_\omega^i\}_{i \in I}$ is a covering of $\tilde{A}_c$.

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By Corollary 4.12, one may find $\tau > 0$ such that

$$m(W_\omega^i) \geq \tau m(V_\omega^i) \quad \text{for all } i \in I.$$

Hence,

$$\begin{aligned} m\left(\bigcup_{i\in I} W_\omega^i\right) &= \sum_{i\in I} m(W_\omega^i) \\ &\geq \tau \sum_{i\in I} m(V_\omega^i) \\ &\geq \tau m\left(\bigcup_{i\in I} V_\omega^i\right) \\ &\geq \tau m(\tilde{A}_c). \end{aligned}$$

From (28), one easily deduces that $m(\tilde{A}_c) > (1-\gamma)m(\tilde{A})$. Noting that the constant τ does not depend on γ, choosing $\gamma > 0$ small enough, we may have

$$m\left(\bigcup_{i\in I} W_\omega^i\right) > \frac{\tau}{2} m(\tilde{A}). \tag{30}$$

We are going to prove that

$$\frac{m(W_\omega^i \backslash \tilde{A})}{m(W_\omega^i)} < \frac{2\gamma}{\tau} \quad \text{for some } i \in I. \tag{31}$$

This is enough for our purpose. First, since $f_\omega^{n(x_i)}(\tilde{A}) \subset A$ and $f_\omega^{n(x_i)}$ is injective on W_ω^i, we have

$$\begin{aligned} m(f_\omega^{n(x_i)}(W_\omega^i)\backslash A) &\leq m(f_\omega^{n(x_i)}(W_\omega^i)\backslash f_\omega^{n(x_i)}(\tilde{A})) \\ &= m(f_\omega^{n(x_i)}(W_\omega^i \backslash \tilde{A})). \end{aligned}$$

Therefore, by Corollary 4.12, taking $B = f_\omega^{n(x_i)}(W_\omega^i)$ as a ball of radius $\delta_1/4$, we have

$$\frac{m(B\backslash A)}{m(B)} \leq \frac{m(f^{n(x_i)}(W_\omega^i \backslash \tilde{A}))}{m(f_\omega^{n(x_i)}(W_\omega^i))} \leq C_2 \frac{m(W_\omega^i \backslash \tilde{A})}{m(W_\omega^i)} = \frac{2C_2\gamma}{\tau},$$

which can obviously be made arbitrarily small, letting $\gamma \to 0$. From this, one easily deduces that there are disks of radius $\delta_1/4$ where the relative measure of A is arbitrarily close to one.

Finally, let us prove (31). Assume, by contradiction, that it does not hold. Then, using (28) and (30),

$$\begin{aligned} \gamma m(\tilde{A}) &> m(\tilde{A}_o \backslash \tilde{A}_c) \\ &\geq m\left(\left(\bigcup_{i\in I} W_\omega^i\right) \Big\backslash \tilde{A}\right) \\ &\geq \frac{2\gamma}{\tau} m\left(\bigcup_{i\in I} W_\omega^i\right) \\ &> \gamma m(\tilde{A}). \end{aligned}$$

This gives a contradiction. □

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4.2. *Transitivity and growing to large scale.* We do not need transitivity of f in all its strength. Before we say what the weaker form of transitivity sufficient for our purposes is, let us state that given $\delta > 0$, a subset A of M is said to be δ-*dense* if any point in M is at a distance smaller than δ from A. For our purposes, it is enough that there is some point $p \in M$ whose preorbit does not hit the critical set of f and is δ-dense for some sufficiently small $\delta > 0$ (depending on the radius δ_1 of hyperbolic balls for f). As the lemma below shows, in our setting of non-uniformly expanding maps, this is a consequence of the usual transitivity of f.

LEMMA 4.13. *Let $f: M \to M$ be a transitive non-uniformly expanding map. Given $\delta > 0$, there are $p \in M$ and $N_0 \in \mathbb{N}$ such that $\bigcup_{j=0}^{N_0} f^{-j}\{p\}$ is δ-dense in M and disjoint from the critical set $\mathcal{C}$.*

Proof. See [**ALP05**, Lemma 2.5]. □

Assuming that f is non-uniformly expanding and non-uniformly expanding on random orbits, then, by Proposition 4.4, there are λ, δ and ζ such that Lebesgue almost every $x \in M$ has a frequency of (λ, δ)-hyperbolic times greater than ζ. We fix once and for all $p \in M$ and $N_0 \in \mathbb{N}$ for which

$$\bigcup_{j=0}^{N_0} f^{-j}\{p\} \quad \text{is } \delta_1/3\text{-dense in } M \text{ and disjoint from } \mathcal{C},$$

where $\delta_1 > 0$ is the radius of hyperbolic balls, as for Proposition 4.9. Take constants $\alpha > 0$ and $\delta_0 > 0$ such that

$$2\sqrt{\delta_0} \ll \delta_1 \quad \text{and} \quad 0 < \alpha \ll \delta_0.$$

LEMMA 4.14. *There are constants K_0, $D_0 > 0$ depending only on f, λ, δ_1 and the point p such that, if $\epsilon > 0$ is sufficiently small, then for any ball $B \subset M$ of radius δ_1 and every $\omega \in \operatorname{supp}(\theta_\epsilon^{\mathbb{Z}})$, there are an open set $A \subset B$ and an integer $0 \le m \le N_0$ for which:*

(1) *f_ω^m maps A diffeomorphically onto $B(p, 2\sqrt{\delta_0})$; and*

(2) *for each $x, y \in A$,*

$$\log \left| \frac{\det Df_\omega^m(x)}{\det Df_\omega^m(y)} \right| \le D_0 \operatorname{dist}(f_\omega^m(x), f_\omega^m(y)).$$

Moreover, for each $0 \le j \le N_0$, the j-preimages $(f_\omega^j)^{-1}B(p, 2\sqrt{\delta_0})$ are all disjoint from $\mathcal{C}$, and for x belonging to any such j-preimage, we have

$$\frac{1}{K_0} \le \|Df_\omega^j(x)\| \le K_0.$$

Proof. Since $\bigcup_{j=0}^{N_0} f^{-j}\{p\}$ is $\delta_1/3$-dense in M and disjoint from $\mathcal{C}$, choosing $\delta_0 > 0$ sufficiently small, we have that each connected component of the j-preimages $f^{-j}B(p, 2\sqrt{\delta_0})$, with $j \le N_0$, is bounded away from the critical set $\mathcal{C}$ and contained in a ball of radius $\delta_1/3$. Moreover, since we are dealing with a finite number of iterates, less than N_0, and f_t varies continuously with parameter t, if ϵ is sufficiently small, then for every $\omega \in \operatorname{supp}(\theta_\epsilon^{\mathbb{Z}})$, each connected component of the j-preimages $(f_\omega^j)^{-1}B(p, 2\sqrt{\delta_0})$, with $j \le N_0$, is uniformly (on j and ω) bounded away from the critical set $\mathcal{C}$ and contained

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in a ball of radius $\delta_1/3$. This immediately implies that for $\omega \in \mathrm{supp}(\theta_\epsilon^{\mathbb{Z}})$, any ball $B \subset M$ of radius δ_1 contains an m-preimage A of $B(p, 2\sqrt{\delta_0})$ which is mapped diffeomorphically by f_ω^m onto $B(p, 2\sqrt{\delta_0})$ for some $m \le N_0$. Since the number of iterations and the distance to the critical region are uniformly bounded, the volume distortion is uniformly bounded and, moreover, there is some constant $K_0 > 1$ such that for every $\omega \in \mathrm{supp}(\theta_\epsilon^{\mathbb{Z}})$,

$$\frac{1}{K_0} \le \|Df_\omega^m(x)\| \le K_0,$$

for all $1 \le m \le N_0$ and x belonging to an m-preimage of $B(p, 2\sqrt{\delta_0})$ under f_ω^m. □

Next we prove a useful consequence of the existence of hyperbolic times, namely that if we start with some fixed given $\alpha > 0$, then there exists some N_α, depending only on α, such that for $\omega \in \Omega_\epsilon$, any ball in M of radius α has some subset which grows to a fixed size with bounded distortion within N_α iterates.

LEMMA 4.15. *Given $\alpha > 0$, there exists $N_\alpha > 0$ such that if ϵ is sufficiently small, then for every $\omega \in \Omega_\epsilon$, we have that any ball $B \subset M$ of radius α contains a hyperbolic preball $V_\omega^n \subset B$ with $n \le N_\alpha$.*

Proof. Take any $\alpha > 0$ and a ball $B(z, \alpha)$. By Proposition 4.9, we may choose $n_\alpha \in \mathbb{N}$ large enough that any hyperbolic preball V_ω^n associated with a hyperbolic time $n \ge n_\alpha$ will have a diameter not exceeding $\alpha/2$. Notice that, by Proposition 4.4, for Lebesgue almost every $x \in M$, the point (ω^*, x) has an infinite number of hyperbolic times and therefore

$$m\left(M \setminus \bigcup_{j=n_\alpha}^{n} H_{\omega^*}^j\right) \to 0 \quad \text{as } n \to +\infty.$$

Hence, it is possible to choose $N_\alpha \in \mathbb{N}$ such that

$$m\left(M \setminus \bigcup_{j=n_\alpha}^{N_\alpha} H_{\omega^*}^j\right) < m(B(z, \alpha/2)). \tag{32}$$

Observe that if n is a hyperbolic time for (ω^*, x) and ϵ is small enough, then for every $\omega \in \Omega_\epsilon$, the natural n is also a hyperbolic time for (ω, x). Hence, if ϵ is small enough, for a given $\alpha > 0$, we can take an integer N_α, only depending on α, λ and δ_1, such that (32) holds for every $\omega \in \Omega_\epsilon$ in place of ω^*. This ensures that for every $\omega \in \Omega_\epsilon$, there is a point $\hat{x} \in B(z, \alpha/2)$ with a hyperbolic time $n \le N_\alpha$ and associated hyperbolic preball $V_\omega^n(\hat{x})$ contained in $B(z, \alpha)$. □

4.3. *The partitioning algorithm.* We now describe the construction of the partition $\mathcal{P}_\omega$ (mod 0) of $\Delta_0 = B(p, \delta_0)$ for every $\omega \in \Omega_\epsilon$. The basic intuition is that we wait for some iterate $f_\omega^k(\Delta_0)$ to cover Δ_0 completely, and then define the subset $U \subset \Delta_0$, for which $f_\omega^k : U \to \Delta_0$ is a diffeomorphism, as an element with return time k for the partitions corresponding to all elements $\omega' \in \Omega_\epsilon$ with the same first k coordinates as ω: $\omega'_0 = \omega_0, \ldots, \omega'_{k-1} = \omega_{k-1}$. After that, we continue to iterate the complement $\Delta_0 \setminus U$ until this complement again covers Δ_0 and repeat the same procedure to define more elements of the final partition with higher return times. Using the fact that small regions eventually

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become large as a result of the expansivity condition, it follows that this process can be continued and that Lebesgue almost every point eventually belongs to some element of the partition. Moreover, the return time function depends on the time which it takes small regions to become large on average and this turns out to depend precisely on the measure of the tail set. On the other hand, this process avoids undesirable randomness on the *choice* of elements for distinct (but related) partitions of the induced regions. In particular, for different realizations with similar initial non-negative coordinates, the elements in corresponding partitions with return times lower than the number of similar entries are the same, as subsets of Δ_0, providing condition (U3).

Now we introduce neighborhoods of p:

$$\Delta_0 = B(p, \delta_0), \quad \Delta_0^1 = B(p, 2\delta_0), \quad \Delta_0^2 = B(p, \sqrt{\delta_0}) \quad \text{and} \quad \Delta_0^3 = B(p, 2\sqrt{\delta_0}).$$

For $0 < \lambda < 1$ given by Proposition 4.4, let

$$I_s = \{x \in \Delta_0^1 : \delta_0(1 + \lambda^{s/2}) < \operatorname{dist}(x, p) < \delta_0(1 + \lambda^{(s-1)/2})\}, \quad s \geq 1,$$

be a partition (mod 0) into countably many rings of $\Delta_0^1 \backslash \Delta_0$.

The construction of the partition $\mathcal{P}_\omega$ of Δ_0 is inductive and we give the initial and the general step of the induction. For the sake of a better visualization of the process, and to motivate the definitions, we start with the first step. Define

$$[\omega]_k = \{\tau \in \Omega_\epsilon : \omega_0 = \tau_0, \ldots, \omega_{k-1} = \tau_{k-1}\}.$$

The set H_τ^k is the same for any $\tau \in [\omega]_k$ and we will refer to this set as $H_{[\omega]}^k$.

First step of the induction. Take R_0, some large integer to be determined in §4.4 (can be taken independent of ω); we ignore any dynamics occurring up to time R_0. For $\omega \in \Omega_\epsilon$, let $k \geq R_0 + 1$ be the first time that $\Delta_0 \cap H_{[\omega]}^k \neq \emptyset$. For $j < k$, we define formally the objects Λ_ω^j, A_ω^j, $A_\omega^{j,\alpha}$, whose meaning will become clear in what follows, by $A_\omega^j = A_\omega^{j,\alpha} = \Lambda_\omega^j = \Delta_0$.

Let $(U_{k,j}^3)_j$ be the connected components of $A_\omega^{k-1,\alpha} \cap (f_w^k)^{-1}(\Delta_0^3)$ contained in hyperbolic preballs V_ω^{k-m}, with $k - N_0 \leq m \leq k$. Growth of these hyperbolic preballs V_ω^{k-m} in k–m iterates to a hyperbolic ball of radius δ_1 which $f_{\sigma^{k-m}(\omega)}^m$ maps diffeomorphically onto Δ_0^3. Now let

$$U_{k,j}^i = U_{k,j}^3 \cap (f_\omega^k)^{-1}(\Delta_0^i), \quad i = 0, 1, 2, \text{ where } \Delta_0^0 = \Delta_0,$$

and set $R_\omega(x) = k$ for $x \in U_{k,j}^0$. Take

$$\Lambda_w^k = \Lambda_w^{k-1} \backslash \{R_\omega = k\}.$$

We also define a function $t_\omega^k : \Lambda_w^k \to \mathbb{N}$ by

$$t_\omega^k(x) = \begin{cases} s & \text{if } x \in U_{k,j}^1 \text{ and } f_\omega^k(x) \in I_s \text{ for some } j, \\ 0 & \text{otherwise.} \end{cases}$$

Let

$$A_\omega^k = \{x \in \Lambda_w^k : t_\omega^k(x) = 0\}, \quad B_\omega^k = \{x \in \Lambda_w^k : t_\omega^k(x) > 0\}.$$

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We also define

$$A_\omega^{k,\alpha} = \Lambda_w^k \cap \bigcup_{x \in A_\omega^k \cap H_\omega^{k+1}} (f_\omega^{k+1}|_{V_\omega^{k+1}(x)})^{-1} B(f_\omega^{k+1}(x), \alpha).$$

For all $\tau \in [\omega]_k$ and $j \le k$, we define the objects A_τ^j, B_τ^j, Λ_τ^k, $\{R_\tau = j\}$, and t_τ^k to be the same as the corresponding ones, as before. Moreover, for $\tau \in [\omega]_{k+1}$, we also define $A_\tau^{k,\alpha}$ to be the same as $A_\omega^{k,\alpha}$.

General step of the induction. The general inductive step of the construction now follows by repeating the arguments above with minor modifications. More precisely, we assume that the sets Λ_τ^i, A_τ^i, B_τ^i and $\{R_\tau = i\}$ and functions $t_\tau^i : \Lambda_\tau^i \to \mathbb{N}$ are defined for all $i \le n-1$ and are exactly the same for every $\tau \in [\omega]_{n-1}$. We also define $A_\tau^{i,\alpha}$, for $i \le n-1$ to be the same set for all $\tau \in [\omega]_n$. For $i \le R_0$, we just let $A_\omega^i = A_\omega^{i,\alpha} = \Lambda_w^i = \Delta_0$, $B_\omega^i = \{R_\omega = i\} = \emptyset$, and $t_\omega^i \equiv 0$. Now let $(U_{n,j}^3)_j$ be the connected components of $A_\omega^{n-1,\alpha} \cap (f_\omega^n)^{-1}(\Delta_0)$ contained in hyperbolic preballs V_ω^r, with $n - N_0 \le r \le n$, which are mapped onto Δ_0^3 by f_ω^n. Take

$$U_{n,j}^i = U_{n,j}^3 \cap (f_\omega^n)^{-1}(\Delta_0^i), \quad i = 0, 1, 2,$$

and set $R_\omega(x) = n$ for $x \in U_{n,j}^0$. Also take

$$\Lambda_w^n = \Lambda_w^{n-1} \setminus \{R_\omega = n\}.$$

The definition of the function $t_\omega^n : \Lambda_w^n \to \mathbb{N}$ is slightly different in the general case:

$$t_\omega^n(x) = \begin{cases} s & \text{if } x \in U_{n,j}^1 \setminus U_{n,j}^0 \text{ and } f_\omega^n(x) \in I_s \text{ for some } j, \\ 0 & \text{if } x \in A_\omega^{n-1} \setminus \bigcup_j U_{n,j}^1, \\ t_\omega^{n-1}(x) - 1 & \text{if } x \in B_\omega^{n-1} \setminus \bigcup_j U_{n,j}^1. \end{cases}$$

Finally, let

$$A_\omega^n = \{x \in \Lambda_w^n : t_\omega^n(x) = 0\}, \quad B_\omega^n = \{x \in \Lambda_w^n : t_\omega^n(x) > 0\},$$

and

$$A_\omega^{n,\alpha} = \Lambda_w^n \cap \bigcup_{x \in A_\omega^n \cap H_\omega^{n+1}} (f_\omega^{n+1}|_{V_\omega^{n+1}(x)})^{-1} B(f_\omega^{n+1}(x), \alpha).$$

Once more, for all $\tau \in [\omega]_n$, we define the objects A_τ^n, B_τ^n, Λ_τ^n, $\{R_\tau = n\}$ and t_τ^n to be, respectively, A_ω^n, B_ω^n, Λ_ω^n, $\{R_\omega = n\}$ and t_ω^n, and for $\tau \in [\omega]_{n+1}$ we also define $A_\tau^{n,\alpha}$ as $A_\omega^{n,\alpha}$.

Remark 4.16. Associated with each component U_{n-k}^0 of $\{R_\omega = n-k\}$, for some $k > 0$, we have a collar $U_{n-k}^1 \setminus U_{n-k}^0$ around it; knowing that the new components of $\{R_\omega = n\}$ do not intersect $U_{n-k}^1 \setminus U_{n-k}^0$ 'too much' is important for preventing overlaps on sets of the partition. By [**Al04**, Lemma 4.5], it is enough to consider

$$\alpha < K_0^{-1} \lambda^{N_0/2} \delta_0 (\lambda^{-1/2} - 1),$$

so that $U_n^1 \cap \{t_\omega^{n-1} > 1\} = \emptyset$ for each component U_n^1.

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4.4. *Expansion, bounded distortion and uniformity.* The inductive construction we detailed before provides a family of topological balls contained in Δ_0 which, as we will see, define a Lebesgue modulo zero partition $\mathcal{P}_\omega$ of Δ_0. We start, however, by checking conditions (1)–(3) in the definition of the induced piecewise expanding Gibbs–Markov map in order to prove Theorem 2.9.

Recall that, by construction, the return time R_ω for an element U of the partition $\mathcal{P}_\omega$ of Δ_0 is formed by a certain number n of iterations given by the hyperbolic time of a hyperbolic preball $V_\omega^n \supset U$, and a certain number $m \le N_0$ of additional iterates, which is the time it takes to go from $f_\omega^n(V_\omega^n)$, which could be anywhere in M, to $f_\omega^{n+m}(V_\omega^n)$ which covers Δ_0 completely. The map $F_\omega = f_\omega^{R_w} : \Delta \to \Delta$ is indeed a C^2 diffeomorphism from each component U onto Δ.

It follows from Proposition 4.9 and Lemma 4.14 that

$$\|Df_\omega^{n+m}(x)^{-1}\| \le \|Df_{\sigma^n(\omega)}^m(f_\omega^n(x))^{-1}\| \cdot \|Df_\omega^n(x)^{-1}\| < K_0\lambda^{n/2} \le K_0\lambda^{(R_0-N_0)/2}.$$

By taking R_0 sufficiently large, we can make this last expression smaller than some κ_ω, with $0 < \kappa_\omega < 1$. Since K_0 and N_0 are independent of ω, then R_0 (and hence k_w) can be the same for all $\omega \in \Omega_\epsilon$, proving part of condition (U2).

For the bounded distortion estimate, we need to show that there exists a constant $K_\omega > 0$ such that for any x, y belonging to an element $U_\omega \in \mathcal{P}_\omega$ with return time R_ω, we have

$$\log \left| \frac{\det Df_\omega^{R_\omega}(x)}{\det Df_\omega^{R_\omega}(y)} \right| \le K_\omega \operatorname{dist}(f_\omega^{R_\omega}(x),\, f_\omega^{R_\omega}(y)).$$

By the chain rule,

$$\log \left| \frac{\det Df_\omega^{R_\omega}(x)}{\det Df_\omega^{R_\omega}(y)} \right| = \log \left| \frac{\det Df_{\sigma^n(\omega)}^{R_\omega - n}(f_\omega^n(x))}{\det Df_{\sigma^n(\omega)}^{R_\omega - n}(f_\omega^n(y))} \right| + \log \left| \frac{\det Df_\omega^n(x)}{\det Df_\omega^n(y)} \right|.$$

For the first term in this last sum, we observe that, by Lemma 4.14, we have

$$\log \left| \frac{\det Df_{\sigma^n(\omega)}^{R_\omega - n}(f_\omega^n(x))}{\det Df_{\sigma^n(\omega)}^{R_\omega - n}(f_\omega^n(y))} \right| \le D_0 \operatorname{dist}(f_\omega^{R_\omega}(x),\, f_\omega^{R_\omega}(y)).$$

For the second term in the sum above, we may apply Corollary 4.10 and obtain

$$\log \left| \frac{\det Df_\omega^n(x)}{\det Df_\omega^n(y)} \right| \le C_0 \operatorname{dist}(f_\omega^n(x),\, f_\omega^n(y)).$$

Also, by Lemma 4.14, we may write

$$\operatorname{dist}(f_\omega^n(x),\, f_\omega^n(y)) \le K_0 \operatorname{dist}(f_\omega^{R_\omega}(x),\, f_\omega^{R_\omega}(y)).$$

We just have to take $K_\omega = D_0 + C_0K_0$, which, clearly, can be uniformly chosen on w, which completes condition (U2).

For the uniformity condition (U1), given $N > 1$ and $\varrho > 0$, we define, for every $\omega \in \Omega_\epsilon$, the sets $\{R_\omega = j\}$, A_j^ω and B_j^ω, with $j \le N$, as described in §4.3. The process which leads to the construction of these sets is based on the fact that small domains in Δ_0 become large (in balls of radius δ_1) under f_ω^k, for some $0 \le k \le N_\alpha$, and then under $f_{\sigma^k(\omega)}^i$,

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with $0 \le i \le N_0$, they cover the ball $B(p, 2\sqrt{\delta_0}) \supset \Delta_0$ completely. Hence, just by the continuity of Φ, associated with the random perturbation $\{\Phi, \{\theta_\epsilon\}_{\epsilon>0}\}$, we can guarantee that, for any two realizations ω, ω' in Ω_ϵ, the Lebesgue measure of the symmetric difference of respective sets $\{R_\omega = j\}$, A^j_ω and B_j, for $j \le N$, is smaller than ϱ, as long as we take ϵ sufficiently small. In particular, this holds for $\{R_{\sigma^{-j}(\omega)} = j\}$ and $\{R_{\sigma^{-j}(\omega')} = j\}$, for every $j = 1, 2, \ldots, N$.

4.5. *Metric estimates.* We now compute some estimates to show that the previous algorithm indeed produces a partition (Lebesgue mod 0) of Δ_0.

4.5.1. *Estimates obtained from the construction.* In this first part, we obtain some estimates relating the Lebesgue measure of the sets A^n_ω, B^n_ω and $\{R_\omega > n\}$, with the help of specific information extracted from the inductive construction we performed in §4.3.

LEMMA 4.17. *There exists a constant $c_0 > 0$ (not depending on δ_0) such that for every $\omega \in \Omega_\epsilon$ and $n \ge 1$,*

$$m(B^{n-1}_\omega \cap A^n_\omega) \ge c_0 m(B^{n-1}_\omega).$$

Proof. It is enough to see that this holds for each connected component of B^{n-1}_ω at a time. Let C be a component of B^{n-1}_ω and Q be its outer ring corresponding to $t^{n-1}_\omega = 1$. Observe that, by Remark 4.16, we have $Q \subset C \cap A^n_\omega$. Moreover, there must be some $k < n$ and a component U^0_k of $\{R_\omega = k\}$ such that f^k_ω maps C diffeomorphically onto $\bigcup_{i=k}^{+\infty} I_i$ and Q onto I_k, both with distortion bounded by C_1 and $e^{D_0 D}$, where D is the diameter of M; see Corollary 4.11 and Lemma 4.14. Thus, it is sufficient to compare the Lebesgue measures of $\bigcup_{i=k}^{+\infty} I_i$ and I_k. We have

$$\frac{m(I_k)}{m(\bigcup_{i=k}^{+\infty} I_i)} \approx \frac{[\delta_0(1+\lambda^{(k-1)/2})]^d - [\delta_0(1+\lambda^{k/2})]^d}{[\delta_0(1+\lambda^{(k-1)/2})]^d - \delta_0^d} \approx 1 - \lambda^{1/2}.$$

Clearly, this proportion depends on neither δ_0 nor ω. □

LEMMA 4.18. *There exist $d_0, r_0 > 0$ with $d_0 + r_0 < 1$ such that for every $\omega \in \Omega_\epsilon$ and $n \ge 1$ we have:*

(1) $m(A^{n-1}_\omega \cap B^n_\omega) \le d_0 m(A^{n-1}_\omega)$*; and*

(2) $m(A^{n-1}_\omega \cap \{R_\omega = n\}) \le r_0 m(A^{n-1}_\omega)$.

Moreover $d_0 \to 0$ and $r_0 \to 0$ as $\delta_0 \to 0$.

Proof. It is enough to prove these estimates for each neighborhood of a component U^0_n of $\{R_\omega = n\}$. Observe that by construction we have $U^3_n \subset A^{n-1,\alpha}_\omega$, which means that $U^2_n \subset A^{n-1}_\omega$, because $\alpha < \delta_0 < \sqrt{\delta_0}$. Using the distortion bounds of f^n_ω on U^3_n given by Corollary 4.12 and Lemma 4.14, we obtain

$$\frac{m(U^1_n \backslash U^0_n)}{m(U^2_n \backslash U^1_n)} \approx \frac{m(\Delta^1_0 \backslash \Delta_0)}{m(\Delta^2_0 \backslash \Delta^1_0)} \approx \frac{\delta_0^d}{\delta_0^{d/2}} \ll 1,$$

which gives the first estimate. Moreover,

$$\frac{m(U^0_n)}{m(U^2_n \backslash U^1_n)} \approx \frac{m(\Delta_0)}{m(\Delta^2_0 \backslash \Delta^1_0)} \approx \frac{\delta_0^d}{\delta_0^{d/2}} \ll 1,$$

and this gives the second one. □

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We state a lemma useful in proving the following proposition.

LEMMA 4.19. *There exists L, depending only on the manifold M, such that for every finite Borel measure ϑ and every measurable subset $G \subset M$ with compact closure, there is a finite subset $I \subset G$ such that the balls $B(z, \delta_1/4)$ around the points $z \in I$ are pairwise disjoint, and*

$$\sum_{z\in I} \vartheta\left(B\left(z, \frac{\delta_1}{4}\right) \cap G\right) \geq L\vartheta(G).$$

Proof. See [**Al04**, Lemma 4.9]. □

The next proposition asserts that a fixed proportion of $A_\omega^{n-1} \cap H_\omega^n$ gives rise to new elements of the partition within a finite number of steps (not depending on n). We first state an auxiliary result.

PROPOSITION 4.20. *There exist $s_0 > 0$ and a positive integer $N = N(\alpha)$ such that for every $\omega \in \Omega_\epsilon$ and $n \geq 1$,*

$$m\left(\bigcup_{i=0}^{N} \{R_\omega = n + i\}\right) \geq s_0 m(A_\omega^{n-1} \cap H_\omega^n).$$

Proof. We use Lemma 4.19 with $G = f_\omega^n(A_\omega^{n-1} \cap H_\omega^n)$ and $\vartheta = (f_\omega^n)_* m$, thus obtaining a finite subset I of points $z \in f_\omega^n(A_\omega^{n-1} \cap H_\omega^n)$ for which the conclusion of the lemma in particular implies

$$\sum_{z\in I} m\left(\left((f_\omega^n)^{-1} B\left(z, \frac{\delta_1}{4}\right)\right) \cap A_\omega^{n-1} \cap H_\omega^n\right) \geq Lm(A_\omega^{n-1} \cap H_\omega^n). \tag{33}$$

Fix $z \in I$. Consider $\{C_j\}_j$, the set of connected components of $(f_\omega^n)^{-1}B(z, \delta_1/4)$, which intersect $A_\omega^{n-1} \cap H_\omega^n$. Note that each C_j is contained in a hyperbolic preball $V_\omega^n(x_j)$ associated with some point $x_j \in ((f_\omega^n)^{-1}B(z, \delta_1/4)) \cap A_\omega^{n-1} \cap H_\omega^n$ as in Proposition 4.9. In what follows, given $A \subset B(z_j, \delta_1/4)$, we will simply denote $(f_\omega^n|_{V_\omega^n(x_j)})^{-1}(A)$ by $(f_\omega^n)_j^{-1}(A)$. Note that the sets $\{(f_\omega^n)_j^{-1}B(z, \delta_1/4)\}_j$ are pairwise disjoint as long as δ_1 is sufficiently small (only depending on the manifold). In fact, f_ω^n sends each of them onto $B(z, \delta_1/4)$ and f_w^n is a diffeomorphism restricted to each one of them. In particular, their union does not contain points of $\mathcal{C}$.

CLAIM 1. *There is $0 \leq k_j \leq N_\alpha + N_0$ such that $t_\omega^{n+k_j}$ is not identically 0 in $(f_\omega^n)_j^{-1}B(z, \alpha)$.*

Assume by contradiction that $t_\omega^{n+k_j}|_{(f_\omega^n)_j^{-1}B(z,\alpha)} = 0$ for all $0 \leq k_j \leq N_\alpha + N_0$. This implies that $(f_\omega^n)_j^{-1}B(z, \alpha) \subset A_\omega^{n+k_j,\alpha}$ for all $0 \leq k_j \leq N_\alpha + N_0$. Using Lemma 4.15, we may find a hyperbolic preball $V_{\sigma^n(\omega)}^m \subset B(z, \alpha) \subset A_\omega^{n+k_j}$ with $m \leq N_\alpha$. Now, since $f_{\sigma^n(\omega)}^m(V_{\sigma^n(\omega)}^m)$ is a ball of radius δ_1, it follows from Lemma 4.14 that there is some $V \subset f_{\sigma^n(\omega)}^m(V_{\sigma^n(\omega)}^m)$ and $m' \leq N_0$ with $f_{\sigma^{n+m}(\omega)}^{m'}(V) = \Delta_0$. Thus, taking $k_j = m + m'$, we have that $0 \leq k_j \leq N_\alpha + N_0$ and $(f_\omega^n)_j^{-1}(V_{\sigma^n(\omega)}^m)$ contains an element of $\{R_\omega = n + k_j\}$ inside $(f_\omega^n)_j^{-1}B(z, \alpha)$. This contradicts the fact that $t_\omega^{n+k_j}|_{(f_\omega^n)_j^{-1}B(z_j,\alpha)} = 0$ for all $0 \leq k_j \leq N_\alpha + N_0$.

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CLAIM 2. *$(f_\omega^n)_j^{-1}B(z,\delta_1/4)$ contains a component $\{R_w = n+k_j\}$, with $0 \le k_j \le N_\alpha + N_0$.*

Let k_j be the smallest integer $0 \le k_j \le N_\alpha + N_0$ for which $t_\omega^{n+k_j}$ is not identically zero in $(f_\omega^n)_j^{-1}B(z,\alpha)$. Since $(f_\omega^n)_j^{-1}B(z,\alpha) \subset A_\omega^{n-1,\alpha} \subset \{t_\omega^{n-1} \le 1\}$, there must be some element $U_{n+k_j}^0$ of $\{R_\omega = n+k_j\}$ for which $((f_\omega^n)_j^{-1}B(z,\alpha)) \cap U_{n+k_j}^1 \ne \emptyset$. Recall that by definition $f_\omega^{n+k_j}$ sends $U_{n+k_j}^1$ diffeomorphically onto Δ_0^1, the ball of radius $2\delta_0$ around p. From time n to $n+k_j$, we may have some final 'bad' period of length at most N_0 where the derivative of f may contract, but being bounded from below by $1/K_0$ in each step. Thus, the diameter of $f_\omega^n(U_{n+k_j}^1)$ is at most $4\delta_0 K_0^{N_0}$. Since $B(z,\alpha)$ intersects $f_\omega^n(U_{n+k_j}^1)$ and $\alpha < \delta_0 < \delta_0 K_0^{N_0}$, we have $(f_\omega^n)_j^{-1}B(z,\delta_1/4) \supset U_{n+k_j}^0$, as long as we take $\delta_0 > 0$ small enough, so that

$$5\delta_0 K_0^{N_0} < \frac{\delta_1}{4}.$$

Thus, we have shown that $(f_\omega^n)_j^{-1}B(z,\delta_1/4)$ contains some component of $\{R_\omega = n+k_j\}$ with $0 \le k_j \le N_\alpha + N_0$, and proved the claim.

Since n is a hyperbolic time for each x_j, we have, by the distortion control given by Corollary 4.12,

$$\frac{m((f_\omega^n)_j^{-1}B(z,\delta_1/4))}{m(U_{n+k_j}^0)} \le C_2 \frac{m(B(z,\delta_1/4))}{m(f_\omega^n(U_{n+k_j}^0))}. \tag{34}$$

From time n to time $n+k_j$, we have at most $k_j = m_1 + m_2$ iterates with $m_1 \le N_\alpha$, $m_2 \le N_0$ and $f_\omega^n(U_{n+k_j}^0)$ containing some point $y_j \in H_{\sigma^n(\omega)}^{m_1}$. By the definition of (λ,δ)-hyperbolic time, we have $\mathrm{dist}_\delta(f_{\sigma^n(\omega)}^i(x), \mathcal{C}) \ge \lambda^{bN_\alpha}$, for every $0 \le i \le m_1$, which implies that there is some constant $D_1 = D_1(\alpha) > 0$ such that $|\det(Df_{\sigma^n(\omega)}^i(x))| \le D_1$ for $0 \le i \le m_1$ and $x \in f_\omega^n(U_{n+k_j}^0)$. On the other hand, since the first N_0 preimages of Δ_0 are uniformly bounded away from $\mathcal{C}$, we also have some $D_2 > 0$ such that $|\det(Df_{\sigma^{n+m_1}(\omega)}^i(x))| \le D_2$, for every $0 \le i \le m_2$ and x belonging to an i-preimage $(f_{\sigma^{n+k_j-i}(\omega)}^i)^{-1}\Delta_0$ of Δ_0. Hence,

$$m(f_\omega^n(U_{n+k_j}^0)) \ge \frac{1}{D_1 D_2} m(\Delta_0),$$

which, combined with (34), gives

$$m((f_\omega^n)_j^{-1}B(z_j,\delta_1/4)) \le Dm(U_{n+k_j}^0),$$

with D only depending on C_2, D_1, D_2, δ_0 and δ_1. Moreover, if ϵ is small enough, D_1 and D_2 can be taken to be uniform over ω.

We are now able to compare the Lebesgue measures of $\bigcup_{i=0}^N \{R_\omega = n+i\}$ and $A_\omega^{n-1} \cap H_\omega^n$. Using (33), we get

$$\begin{aligned} m(A_\omega^{n-1} \cap H_\omega^n) &\le L^{-1} \sum_{z\in I} m(((f_\omega^n)_j^{-1}B(z_j,\delta_1/4)) \cap A_\omega^{n-1} \cap H_\omega^n) \\ &\le L^{-1} \sum_{z\in I} \sum_j m((f_\omega^n)_j^{-1}B(z_j,\delta_1/4)) \\ &\le DL^{-1} \sum_{z\in I} \sum_j m(U_{n+k_j}^0). \end{aligned}$$

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One should mention that the sets $U^0_{n+k_j}$ also depend on $z \in I$. They are disjoint for different values of $z \in I$. Hence, putting $N = N_0 + N_\alpha$, we have

$$m(A^{n-1}_\omega \cap H^n_\omega) \le DL^{-1} m\Big(\bigcup_{i=0}^{N} \{R_\omega = n+i\}\Big).$$

To finish the proof, we only have to take $s_0 = DL^{-1}$. □

Remark 4.21. It follows from the choice of the constants D_1, D_2 and D that the constant s_0 depends only on the constants λ, b, N_α, N_0, C_2, δ_0 and δ_1. Recall that L is an absolute constant depending only on M.

4.5.2. *Independent metric estimates.* We have taken a disk Δ_0 of radius $\delta_0 > 0$ around a point $p \in M$ with certain properties and, for every $\omega \in \Omega_\epsilon$, we defined inductively the subsets A^n_ω, B^n_ω, $\{R_\omega = n\}$ and Λ^n_ω, which are related in the following way:

$$\Lambda^n_\omega = \Delta_0 \backslash \{R_\omega \le n\} = A^n_\omega \dot\cup B^n_\omega.$$

Since we are dealing with a non-uniformly expanding on random orbits system, for each $\omega \in \Omega_\epsilon$ and each $n \in \mathbb{N}$, we have also defined the set $H^n_\omega \subset M$ of points which have n as a (λ, δ)-hyperbolic time, and the tail set Γ^n_ω as in (6). From the definition of Γ^n_ω, Remark 4.6 and Lemma 4.7 we deduce that:

(m_1) there is $\zeta > 0$ such that for every $\omega \in \Omega_\epsilon$, $n \ge 1$ and every $A \subset M \backslash \Gamma^n_\omega$ with $m(A) > 0$,

$$\frac{1}{n} \sum_{j=1}^{n} \frac{m(A \cap H^j_\omega)}{m(A)} \ge \zeta.$$

Moreover, Lemmas 4.17 and 4.18 and Proposition 4.20 give us a random version of the metric relations (m_2)–(m_4) in [**Al04**, §4.5.2]. In the inductive process of construction of the sets A^n_ω, B^n_ω, $\{R_\omega = n\}$ and Λ^n_ω, we have fixed some large integer R_0, this being the first step at which the construction began. Recall that $A^n_\omega = \Lambda^n_\omega = \Delta_0$ and $B^n_\omega = \{R_\omega = n\} = \emptyset$ for $n \le R_0$. We will assume that

$$R_0 > \max\{2(N+1), 12/\zeta\}.$$

Note that since N and ζ do not depend on R_0, this is always possible, so we can follow [**Al04**, §4.5.2] to conclude that for every $\omega \in \Omega_\epsilon$, this process indeed produces a partition $\mathcal{P}_\omega = \{R_\omega = n\}_n$ of Δ_0. Moreover, it also follows that if there exist C, $p > 0$ such that for every $\omega \in \Omega_\epsilon$ we have $m(\Gamma^n_\omega) \le Cn^{-p}$, then there exists C' such that for every $\omega \in \Omega_\epsilon$ the return time function satisfies

$$m(R_\omega > n) \le C' n^{-p}.$$

It is possible to check that the constant C' depends ultimately on the constants B, β and b_0 associated with the non-uniform expanding condition in Definition 1.4. This implies that C' can be considered the same for every $\omega \in \Omega_\epsilon$. This ends the proof of Theorem 2.9.

5. *Applications*

5.1. *Local diffeomorphisms.* One example of a transformation which fits our hypothesis was introduced in [**ABV00**] and consists of robust (C^1 open) classes of local

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diffeomorphisms (with no critical sets) which are non-uniformly expanding. The existence and uniqueness of SRB probability measures for this map was proved in [**ABV00, Al03**]. Random perturbations for this map were considered in [**AA03**], where it was proved to be a weak form of stochastic stability—the convergence in the weak* topology of the density of the unique stationary probability measure to the density of the unique SRB probability measure. Here we improve it to the strong version of stochastic stability. We follow closely the constructions and results in [**ABV00, AA03**] and introduce some extras to give the required transitivity.

This class of maps and can be obtained through deformation of a uniformly expanding map by isotopy inside some small region. In general, these maps are not uniformly expanding: deformation can be made in such a way that the new map has periodic saddles.

Let M be the d-dimensional torus $\mathbb{T}^d$, for some $d \geq 2$, and m the normalized Riemannian volume form. Let $f_0 \colon M \to M$ be a uniformly expanding map and $V \subset M$ be a small neighborhood of a fixed point p of f_0, so that the restriction of f_0 to V is injective. Consider a C^1-neighborhood $\mathcal{U}$ of f_0 sufficiently small that any map $f \in \mathcal{U}$ satisfies the following.

(i) f is *expanding outside* V: there exists $\lambda_0 < 1$ such that

$$\|Df(x)^{-1}\| < \lambda_0 \quad \text{for every } x \in M \backslash V.$$

(ii) f is *volume expanding everywhere*: there exists $\lambda_1 > 1$ such that

$$|\det Df(x)| > \lambda_1 \quad \text{for every } x \in M.$$

(iii) f is *not too contracting on* V: there is some small $\gamma > 0$ such that

$$\|Df(x)^{-1}\| < 1 + \gamma \quad \text{for every } x \in V.$$

Moreover, the constants λ_0, λ_1 and γ are the same for all $f \in \mathcal{U}$. For $f \in \mathcal{U}$, we introduce random perturbations $\{\Phi, (\theta_\epsilon)_{\epsilon>0}\}$. In particular, we consider a continuous map

$$\begin{aligned} \Phi : T &\longrightarrow \mathcal{U} \\ t &\longmapsto f_t, \end{aligned}$$

where T is a metric space and $f \equiv f_{t^*}$ for some $t^* \in T$. Consider a family $(\theta_\epsilon)_{\epsilon>0}$ of probability measures on T such that their supports are non-empty and satisfy $\operatorname{supp}(\theta_\epsilon) \to \{t^*\}$, when $\epsilon \to 0$. We can choose appropriately the constants λ_0, λ_1 and γ, so that every map $f \in \mathcal{U}$ is non-uniformly expanding on *all* random orbits with uniform exponential decay of the Lebesgue measure of the tail sets Γ_ω^n given by (6), naturally ignoring the recurrence time function.

PROPOSITION 5.1. *Consider f_0, $\mathcal{U}$, $f \in \mathcal{U}$ and $\{\Phi, (\theta_\epsilon)_{\epsilon>0}\}$ as before. There exists $a_0 > 0$ such that for every $\omega \in \operatorname{supp}(\theta_\epsilon^{\mathbb{N}})$ and Lebesgue almost every $x \in M$,*

$$\limsup_{n\to+\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log \|Df_{\sigma^j(\omega)}(f_\omega^j(x))^{-1}\| \leq -a_0.$$

Moreover, there is $0 < \tau < 1$ such that $m(\Gamma_\omega^n) \leq \tau^n$, for $n \geq 1$ and $\omega \in \operatorname{supp}(\theta_\epsilon^{\mathbb{N}})$.

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Proof. See [**AA03**]. □

We now show that by performing the construction a bit more carefully we have maps which are also transitive, which in particular implies that each map has a unique SRB probability measure. We will actually prove that these maps are topologically mixing. We start by considering a map $\bar{f}\colon M \to M$ (in the boundary of the set of uniformly expanding maps) which satisfies (i), (ii) and (iii) as the Cartesian product of one-dimensional maps $\varphi_1 \times \cdots \times \varphi_d$, with $\varphi_1, \ldots, \varphi_{d-1}$ uniformly expanding in S^1, and φ_d the *intermittent* map in S^1: it can be written as

$$\varphi_d(x) = x + x^{1+\alpha} \quad \text{for some } 0 < \alpha < 1,$$

in a neighborhood of 0 and $\varphi_d'(x) > 1$ for every $x \in S^1 \backslash \{0\}$. One already has that any f in a sufficiently small C^1-neighborhood $\bar{\mathcal{U}}$ of $\bar{f}$ satisfies (i), (ii) and (iii) for convenient choice of constants $\lambda_0, \lambda_1, \gamma$, and neighborhood V of the fixed point $p = 0 \in \mathbb{T}^d$. The next lemma ensures that $\bar{f}$ is topologically mixing, and thus topologically transitive. We show moreover that if $\bar{\mathcal{U}}$ is sufficiently small, then all the maps in $\bar{\mathcal{U}}$ are topologically mixing.

LEMMA 5.2. *Given $\alpha > 0$, there is $N_\alpha \geq 1$ such that $\bar{f}^{N_\alpha}(B_\alpha(x)) = \mathbb{T}^d$ for any $x \in M$.*

Proof. This is an immediate consequence of the fact that a similar conclusion holds for the maps $\varphi_1, \ldots, \varphi_d$ in S^1. This is standard for the uniformly expanding maps $\varphi_1, \ldots, \varphi_{d-1}$, and also for the intermittent map φ_d, as it is topologically conjugate to a uniformly expanding map of the same degree. □

Let us now obtain a similar conclusion for any map f in $\bar{\mathcal{U}}$. This cannot be done by a simple continuity argument, since for smaller radii $\alpha > 0$, in principle we need to diminish the size of the C^1-neighborhood. However, a continuity argument works if one just needs to consider balls of some fixed radius. By Proposition 5.1, any map $f \in \bar{\mathcal{U}}$ is non-uniformly expanding and, if we consider a random perturbation of f as before, then f is also non-uniformly expanding on (all) random orbits (naturally, for sufficiently small noise level), with uniform exponential decay of the Lebesgue measure of the tail set. By Propositions 5.1 and 4.4, Lebesgue almost every point $x \in M$ has infinitely many λ-hyperbolic times and, moreover, we may take $\lambda = e^{-a_0/2}$. Lemmas 4.8 and Proposition 4.9 imply that there exists $\delta_1 > 0$ (uniform for the maps in $\bar{\mathcal{U}}$) such that Lebesgue almost every point in $\mathbb{T}^d$ has arbitrarily small neighborhoods which are sent onto balls of radius $\delta_1 > 0$. Taking $\alpha = \delta_1/2$ in Lemma 5.2, there is some positive integer N for which every ball of radius $\delta_1/2$ is sent onto M by $\bar{f}^N$. Then, just by continuity, one has that any ball of radius δ_1 is sent onto M by f^N for any $f \in \bar{\mathcal{U}}$, provided this C^1-neighborhood is sufficiently small. Then, in particular, each $f \in \bar{\mathcal{U}}$ is topologically transitive. The next theorem is now a direct application of Theorem A.

THEOREM 5.3. *Let $f \in \bar{\mathcal{U}}$. Then:*

(1) *if ϵ is small enough, then f admits a unique absolutely continuous ergodic stationary probability measure; and*
(2) *f if strongly stochastically stable.*

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5.2. *Viana maps.* We now consider an important open class of non-uniformly expanding maps with critical sets in higher dimensions introduced in [**Vi97**]. This example features the hypothesis of Theorem A, resulting in the proof of their strong stochastic stability. The existence of a unique absolutely continuous ergodic invariant probability measure and strong statistical stability were proved in [**AV02**]. A weaker form of stochastic stability (weak* convergence of the stationary measure to μ_f) was established in [**AA03**]. In order to check the hypothesis of Theorem A, we use essentially the results in [**AA03**] for non-uniform expansion, slow recurrence to the critical set and uniform decay of the Lebesgue measure of the tail set, both for deterministic and random cases. Without loss of generality, we discuss the two-dimensional case and we refer to [**AA03, AV02, Vi97**] for details.

Let $p_0 \in (1, 2)$ be such that the critical point $x = 0$ is preperiodic for the quadratic map $Q(x) = p_0 - x^2$. Let $S^1 = \mathbb{R}/\mathbb{Z}$ and $b : S^1 \to \mathbb{R}$ be a Morse function, for instance, $b(s) = \sin(2\pi s)$. For fixed small $\alpha > 0$, consider the map

$$\begin{aligned} \hat{f} : S^1 \times \mathbb{R} &\longrightarrow S^1 \times \mathbb{R} \\ (s, x) &\longmapsto (\hat{g}(s), \hat{q}(s, x)), \end{aligned}$$

where $\hat{g}$ is the uniformly expanding map of the circle defined by $\hat{g}(s) = ds \pmod{\mathbb{Z}}$ for some $d \geq 16$, and $\hat{q}(s, x) = a(s) - x^2$ with $a(s) = p_0 + \alpha b(s)$. As shown in [**AA03**], it is not a restriction to assume that $\mathcal{C} = \{(s, x) \in S^1 \times I : x = 0\}$ is the critical set of $\hat{f}$, and we do so. If $\alpha > 0$ is small enough, there is an interval $I \subset (-2, 2)$ for which $\hat{f}(S^1 \times I)$ is contained in the interior of $S^1 \times I$. Any map f sufficiently close to $\hat{f}$ in the C^3 topology has $S^1 \times I$ as a forward invariant region (in fact, here it suffices to be C^1 close). We consider a small C^3-neighborhood $\mathcal{V}$ of $\hat{f}$ as before, and will refer to maps in $\mathcal{V}$ as *Viana maps*. Thus, any *Viana map* $f \in \mathcal{V}$ has $S^1 \times I$ as a forward invariant region, and so an *attractor* inside it, which is precisely

$$\Lambda = \bigcap_{n \geq 0} f^n(S^1 \times I).$$

We introduce the random perturbations $\{\Phi, (\theta_\epsilon)_\epsilon\}$ for these maps. We set $T \subset \mathcal{V}$ to be a C^3 neighborhood of $\hat{f}$ consisting of maps f restricted to the forward invariant region $S^1 \times I$ for which $Df(x) = D\hat{f}(x)$ if $x \notin \mathcal{C}$, the map Φ to be the identity map at T and $(\theta_\epsilon)_\epsilon$ a family of Borel measures on T such that their supports are non-empty and satisfy $\operatorname{supp}(\theta_\epsilon) \to \{f\}$, when $\epsilon \to 0$, for $f \in T$. Let h_f be the density of the unique absolutely continuous invariant probability measure μ_f for f. We will show that such Viana maps $f \in \mathcal{V}$ satisfy the hypothesis of Theorem A so that we may conclude the following theorem.

THEOREM 5.4. *Let $f \in \mathcal{V}$ be a Viana map. Then:*

(1) *if ϵ is small enough, f admits a unique absolutely continuous ergodic stationary probability measure; and*
(2) *f is strongly stochastically stable.*

5.2.1. *Deterministic estimates.* The results in [**Vi97**] show that if $\mathcal{V}$ is sufficiently small (in the C^3 topology), then $f \in \mathcal{V}$ has two positive Lyapunov exponents almost everywhere:

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there is a constant $\eta > 0$ for which

$$\liminf_{n\to+\infty} \frac{1}{n} \log \|Df^n(s, x)v\| \geq \eta$$

for Lebesgue almost every $(s, x) \in S^1 \times I$ and every non-zero $v \in T_{(s,x)}(S^1 \times I)$. This does not necessarily imply that f is non-uniformly expanding. However, as was shown in [**AA03**], a slightly deeper use of Viana's arguments enables us to prove non-uniform expansion and slow recurrence to the critical set of any C^2 map f such that

$$\|f - \hat{f}\|_{C^2} < \alpha.$$

In particular, they proved that there exist $C, \zeta > 0$ such that for f, as before, and $n \geq 1$ there is a set $\Gamma_n \subset S^1 \times I$ for which

$$m(\Gamma_n) \leq Ce^{-\zeta\sqrt{n}},$$

and such that, for each $(s, x) \notin \Gamma_n$,

(1) there is $a_0 > 0$ such that

$$\frac{1}{k} \sum_{j=0}^{k-1} \log \|Df(f^j(s, x))^{-1}\| \leq -a_0 \quad \text{for all } k \geq n; \text{ and}$$

(2) for a given small $b_0 > 0$ there is $\delta > 0$ such that

$$\frac{1}{k} \sum_{j=0}^{k-1} -\log \operatorname{dist}_\delta(f^j(s, x), \mathcal{C}) \leq b_0 \quad \text{for all } k \geq n.$$

Moreover, the constants ζ, a_0 and δ only depend on the quadratic map Q and $\alpha > 0$.

In [**AV02**], a topological mixing property was also proved.

THEOREM 5.5. *For every $f \in \mathcal{V}$ and every open set $A \subset S^1 \times I$, there is some $n_A \in \mathbb{N}$ for which $f^{n_A}(A) = \Lambda$.*

Proof. See [**AV02**, Theorem C]. □

5.2.2. *Estimates for random perturbations.* Let f be close to $\hat{f}$ in the C^3 topology and fix a random perturbation $\{\Phi, (\theta_\epsilon)_{\epsilon>0}\}$ as before. We want to show that if $\epsilon > 0$ is small enough, then f is non-uniformly expanding on random orbits and Γ_ω^n decays sufficiently fast and uniformly in ω. The estimates in [**AA03**] for $\log \operatorname{dist}_\delta(f^j(s, x), \mathcal{C})$ and $\log \|Df(f^j(s, x))^{-1}\|$ over the orbit of a given point $(s, x) \in S^1 \times I$ can easily be obtained by replacing the iterates $f^j(s, x)$ by random iterates $f_\omega^j(s, x)$. Briefly, these estimates rely on a delicate decomposition of the orbit of the point (s, x), from time 0 until time n, into finite pieces according to its returns to the neighborhood $S^1 \times (-\sqrt{\alpha}, \sqrt{\alpha})$ of the critical set. The main tools for these estimates were from [**Vi97**, Lemma 2.4] and [**Vi97**, Lemma 2.5], and the proofs may easily be mimicked for random orbits. Indeed, the important fact in the proof of the lemmas referred to is that orbits of points in the central direction stay close to orbits of the quadratic map Q for long periods, as long as $\alpha > 0$ is taken sufficiently small. Hence, such results can easily be obtained for random orbits as

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long as we take $\epsilon > 0$ satisfying $\epsilon \ll \alpha$. It was also proved in [**AA03**] that $C > 0$ and $\zeta > 0$ exist such that $m(\Gamma_\omega^n) < Ce^{-\zeta\sqrt{n}}$, for almost every $\omega \in \mathrm{supp}(\theta_\epsilon^{\mathbb{N}})$, which clearly is enough for our purposes. Moreover, the constants for the estimates on the the tail set, non-uniform expansion and slow recurrence depend only on the quadratic map Q and α. In particular, they are uniform over ω.

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