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On embedding of repetitive Meyer multiple sets into model multiple sets

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Abstract. Model sets are always Meyer sets but the converse is generally not true. In this work we show that for a repetitive Meyer multiple set of $\mathbb{R}^d$ with associated dynamical system $(\mathbb{X}, \mathbb{R}^d)$, the property of being a model multiple set is equivalent to $(\mathbb{X}, \mathbb{R}^d)$ being almost automorphic. We deduce this by showing that a repetitive Meyer multiple set can always be embedded into a repetitive model multiple set having a smaller group of topological eigenvalues.

1. *Outline*

In this paper we address a study of particular point patterns of a Euclidean space. From a general point of view, a point pattern is a collection of points inside some space $\mathbb{R}^d$ (or some locally compact Abelian group) that obeys some discreteness and relative density properties. In this work we consider a slight generalization of *multiple point patterns*, that is, a finite collection of point patterns which may overlap. Thanks to this minor generalization, our results become true for symbolic sequences and arrays as well. We note that the concept of point patterns can be much further generalized by considering for instance weighted Dirac combs as done in [**6**] or the even more general concept of translation-bounded measures [**3**].

In the topic of point patterns, probably the most understood and studied objects are the so-called *model sets* or model multiple sets according to our point of view. Formally, a model multiple set is a point set of a Euclidean space that arises as follows: we consider a 'total space' to be the product of $\mathbb{R}^d$ (the ambient space) with a locally compact Abelian group H (the internal group), together with a lattice Σ in this product. A model multiple set is then a finite family $(\Lambda_i)_{i\in I}$ of patterns, each Λ_i being obtained as projections on $\mathbb{R}^d$ of the points of the lattice whose projection in H falls in some compact and topologically

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regular subset W_i of H. The data of the group H together with the lattice Σ is called a cut and project scheme and the family $\{W_i\}_{i\in I}$ is called a window. Any *coding of a rotation* is a model multiple set: such arrays may be defined as the model multiple sets associated to cut and project schemes and window $\{W_i\}_{i\in I}$ obeying the following three conditions:

(i) the internal space H is compact;

(ii) the window $\{W_i\}_{i\in I}$ covers H;

(iii) the sets W_i have pairwise-disjoint interiors.

It is also often required that the window admits a trivial redundancy subgroup. Famous examples of such arrays are the *Toeplitz sequences* and arrays [**11**], also called elsewhere *limit-periodic point sets* [**7**], which are uniquely characterized as being the coding of a rotation over an *odometer* H.

A great challenge for point patterns and symbolic sequences is to characterize intrinsic properties of the considered pattern in terms of a natural dynamical system attached to it. Such a dynamical system is provided, given a multiple point pattern Λ, by a compact metric space $\mathbb{X}_\Lambda$ called the *hull* of Λ, together with a natural action of $\mathbb{R}^d$, the space into which Λ fits, by homeomorphisms. It turned out that for symbolic sequences, and later generalized for arrays [**9**, **24**], the property to be the coding of a rotation is characterizable in such a way: an array Λ in $\mathbb{R}^d$ is the coding of a rotation if and only if its associated dynamical system $(\mathbb{X}_\Lambda, \mathbb{R}^d)$ is *almost automorphic* (an equivalent statement can be set on the subshift Ξ_Λ generated by the array, with the group $\mathbb{Z}^d$ acting on it [**9**]). This property deals with a certain factor map

$$\pi : \mathbb{X}_\Lambda \twoheadrightarrow \mathbb{X}_{\mathrm{eq}},$$

where the factor space $\mathbb{X}_{\mathrm{eq}}$ is called the *maximal equicontinuous factor* of $(\mathbb{X}_\Lambda, \mathbb{R}^d)$. For any point pattern, as well as for any topologically transitive dynamical system over a compact space with Abelian acting group, it is a compact Abelian group. A dynamical system is called almost automorphic whenever its factor map π admits a one-point fiber [**26**].

If one wishes to provide such a characterization from arrays to general multiple point patterns, one has to consider a property, which to our knowledge does not admit any counterparts on the dynamical system $(\mathbb{X}_\Lambda, \mathbb{R}^d)$ (see however [**17**]), namely the so-called *Meyer property*. This property admits several different but equivalent forms [**17**, **18**, **21**] and is satisfied by model multiple sets. Then a powerful result of Baake *et al* [**4**], generalized in the multiple set setting in [**19**], asserts that a Meyer set Λ is a regular model set if and only if its associated dynamical system is almost automorphic with the one-point fiber elements of $\mathbb{X}_{\mathrm{eq}}$ having full Haar measure. Here the regularity property means that each W_i used to construct the considered model multiple set has a boundary of Haar measure zero in H. However, this result gets rid of the class of non-necessarily regular model multiple sets, and thus of many interesting examples.

The strategy given in [**4**] remains on the use of the so-called *autocorrelation hull* $\mathbb{A}_\Lambda$ of a point pattern [**6**, **19**, **22**, **23**], in a context where it is compact and a factor of the original hull $\mathbb{X}_\Lambda$. This is not necessarily satisfied for model sets which are not regular. On one hand regular model sets are all uniquely ergodic, with pure point dynamical spectrum (with respect to the unique ergodic probability measure) and of entropy zero. On the other hand

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one can find examples of model sets, appearing as subsets of lattices or equivalently as 0–1 arrays, with an arbitrarily large positive topological entropy [**20**] (such examples can even be chosen uniquely ergodic [**10**]). By a result of [**5**], these examples cannot have a pure point dynamical spectrum, which implies by the characterization of purely diffractive point patterns provided in [**6**] that the autocorrelation hull cannot be compact, and certainly not a factor of $\mathbb{X}_\Lambda$. This observation shows that the techniques developed in [**4**] do not generalize to not necessarily regular model sets. Still, one has the following theorem.

THEOREM 1. *A repetitive Meyer multiple set Λ of $\mathbb{R}^d$ is a model multiple set if and only if its associated dynamical system $(\mathbb{X}_\Lambda, \mathbb{R}^d)$ is almost automorphic.*

The method developed here greatly differs from the one of [**4**]. There is in particular no question of autocorrelation, and to some extent it is not related to any measure aspect. The earlier statement will in fact show up as a consequence of a more general fact concerning the embedding of Meyer multiple sets into model multiple sets. A Meyer set of $\mathbb{R}^d$ is always a subset of some model set [**21**]. We revisit this fact here in the case of repetitive Meyer multiple sets.

THEOREM 2. *Let Λ_0 be a repetitive Meyer multiple set of $\mathbb{R}^d$ with dynamical system $(\mathbb{X}_{\Lambda_0}, \mathbb{R}^d)$ and factor map $\pi : \mathbb{X}_{\Lambda_0} \twoheadrightarrow \mathbb{X}_{\mathrm{eq}}$ onto its maximal equicontinuous factor. Then we have the following.*

(i) *There is a cut and project scheme $(H, \Sigma, \mathbb{R}^d)$ with window $\{W_i\}_{i\in I}$ in the internal space, such that $H \times_\Sigma \mathbb{R}^d \simeq \mathbb{X}_{\mathrm{eq}}$ as compact Abelian groups. We denote $\mathbb{X}_{\mathrm{MS}}$ to be the unique hull of repetitive model multiple sets issuing from this cut and project scheme and window.*

(ii) *Any Λ in $\mathbb{X}_{\Lambda_0}$ symbol-wise embeds in a repetitive model multiple set Δ_Λ in $\mathbb{X}_{\mathrm{MS}}$. Moreover, for residually many $\Delta \in \mathbb{X}_{\mathrm{MS}}$, there exists a subset C_Δ of $\mathbb{X}_{\Lambda_0}$ such that*

$$\Delta_i = \bigcup_{\Lambda\in C_\Delta} \Lambda_i.$$

(iii) *The group of topological eigenvalues $\mathcal{E}(\mathbb{X}_{\mathrm{MS}}, \mathbb{R}^d)$ is a subgroup of $\mathcal{E}(\mathbb{X}_{\Lambda_0}, \mathbb{R}^d)$.*

Theorem 1 is a very direct consequence of Theorem 2, and consists in showing that the dynamical system $(\mathbb{X}_{\Lambda_0}, \mathbb{R}^d)$ is almost automorphic if and only if the hulls $\mathbb{X}_{\Lambda_0}$ and $\mathbb{X}_{\mathrm{MS}}$ are the same collection of point patterns. The question whether the dynamical systems $(\mathbb{X}_{\mathrm{MS}}, \mathbb{R}^d)$ and $(\mathbb{X}_{\Lambda_0}, \mathbb{R}^d)$ have the same group of topological eigenvalues can be addressed in terms of the redundancy subgroup $\mathcal{R}$ of the multiple window: this is a compact subgroup of H whose Pontryagin dual is precisely equal to the quotient

$$\widehat{\mathcal{R}} = \mathcal{E}(\mathbb{X}_{\Lambda_0}, \mathbb{R}^d) \big/ \mathcal{E}(\mathbb{X}_{\mathrm{MS}}, \mathbb{R}^d).$$

It is also interesting to note that, given Λ_0 a repetitive Meyer set, any $\Delta_{\Lambda_0} \in \mathbb{X}_{\mathrm{MS}}$ containing Λ_0 generates a subgroup $\langle \Delta_{\Lambda_0} \rangle$ equal to the subgroup $\langle \Lambda_0 \rangle$ generated by Λ_0. This means that if Λ_0 is not supported on a lattice in $\mathbb{R}^d$, then Δ_{Λ_0} is not supported on a lattice as well, and in particular cannot be fully periodic.

This paper is organized as follows: in §2 we make a presentation of point patterns and their associated dynamical systems, and introduce the so-called model sets. We provide

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in §3 a short survey of the necessary notions of topological dynamics which are involved in this article. In §4 we present a new topology, which we call combinatorial topology, and show that in the case of a repetitive Meyer multiple set this topology naturally gives rise to a locally compact Abelian group. In §5 we briefly introduce what we call there a subsystem of the hull of a point pattern, and from the results of §§4 and 5 we provide in §6 a construction of a cut and project scheme with multiple window. We provide in §7 the statement and the proof of Theorem 2, and we deduce the proof of Theorem 1 in §8.

2. *Point patterns and their dynamical systems*

2.1. *Point patterns and the Meyer property.* Let us consider a finite collection $\Lambda = (\Lambda_i)_{i\in I}$ of point sets in a Euclidean space $\mathbb{R}^d$, whose support is set as the finite union $\underline{\Lambda} := \bigcup_{i\in I} \Lambda_i$. We say that Λ is *uniformly discrete* if each open ball of some radius $r_0 > 0$ in $\mathbb{R}^d$ contains at most one point of $\underline{\Lambda}$. We say that Λ is *relatively dense* if each open ball of some radius R_0 contains at least one point of each subset Λ_i. We then say that Λ is a *Delone* multiple set whenever it is both uniformly discrete and relatively dense in $\mathbb{R}^d$. Such multiple sets will also be referred to as *point patterns*.

An important notion affiliated with a point pattern Λ is its *language*, that is, the collection of all 'circular-shaped' patterns appearing at some site of Λ:

$$\mathcal{L}_\Lambda := \{(\Lambda - \gamma) \cap B(0,\, R) \mid \gamma \in \underline{\Lambda},\, R > 0\}.$$

A point pattern Λ_0 is said to have *finite local complexity* whenever it has only a finite number of circular patterns for any fixed radius. The language of a point pattern Λ_0 can be used to set a collection of other point patterns called its *hull*, which is defined as

$$\mathbb{X}_{\Lambda_0} := \{\Lambda \subset \mathbb{R}^d \mid \mathcal{L}_\Lambda \subseteq \mathcal{L}_{\Lambda_0}\}.$$

Thus, from its very construction, the hull of Λ_0 is nothing but the collection of point sets whose bounded patterns appear somewhere in Λ_0. It admits a natural topology, sometimes called the *local topology*, which arises from the metric

$$d(\Lambda,\, \Lambda') := \inf\left\{ \frac{1}{1+R} \;\middle|\; \exists |t|,\, |t'| < \frac{1}{1+R},\, (\Lambda - t) \cap B(0,\, R) = (\Lambda' - t') \cap B(0,\, R) \right\}.$$

This metric roughly means that two point sets are close if they agree on a large domain about the origin up to small shifts. Now, if we are given a point set Λ, then any vector $t \in \mathbb{R}^d$ defines a point set $\Lambda \cdot t$ by simply shifting any site of Λ by $-t$. Obviously, if some point set lies within the hull $\mathbb{X}_{\Lambda_0}$, then any of its translates remains in $\mathbb{X}_{\Lambda_0}$, providing so a jointly continuous action of $\mathbb{R}^d$ on $\mathbb{X}_{\Lambda_0}$ and thus a dynamical system $(\mathbb{X}_{\Lambda_0}, \mathbb{R}^d)$. It can be observed that the space $\mathbb{X}_{\Lambda_0}$ topologized in the above way is nothing but the completion of the $\mathbb{R}^d$-orbit of Λ_0 with respect to the metric d. In particular, the dynamical system $(\mathbb{X}_{\Lambda_0}, \mathbb{R}^d)$ is topologically transitive.

We consider the *canonical transversal* of the hull of Λ_0 to be the subset

$$\Xi := \{\Lambda \in \mathbb{X}_{\Lambda_0} \mid 0 \in \underline{\Lambda}\}. \tag{1}$$

The terminology comes from the fact that this set intersects each orbit under the $\mathbb{R}^d$-action, and that any radius of uniform discreteness r_0 for $\underline{\Lambda_0}$ is such that if $\Lambda \in \Xi$, then $\Lambda \cdot t \notin \Xi$ for any $t \in B(0,\, r_0)$. Now, using this set, we can turn the finite local complexity condition into the topological setting.

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PROPOSITION 2.1. *Let Λ_0 be a point pattern of $\mathbb{R}^d$ with hull $\mathbb{X}_{\Lambda_0}$. The following assertions are equivalent:*

(i) *Λ_0 is of finite local complexity;*
(ii) *the set of differences $\underline{\Lambda_0} - \underline{\Lambda_0} := \{\gamma - \gamma' \mid \gamma, \gamma' \in \underline{\Lambda_0}\}$ is discrete and closed in $\mathbb{R}^d$;*
(iii) *the metric space $\mathbb{X}_{\Lambda_0}$ is compact;*
(iv) *the transversal Ξ with induced topology is compact.*

All of this follows from the fact that finite local complexity means exactly the precompacity of the transversal Ξ with respect to the metric d. A Delone multiple set Λ of finite local complexity is *repetitive* if, for all $t_0 \in \mathbb{R}^d$ and $R > 0$, the set of $t \in \mathbb{R}^d$ such that $(\Lambda + t) \cap B(t_0, R) \equiv \Lambda \cap B(t_0, R)$ is relatively dense in $\mathbb{R}^d$. This notion is equivalent to the minimality of the dynamical system $(\mathbb{X}_\Lambda, \mathbb{R}^d)$.

Definition 2.2. [**18**] A Delone multiple set Λ admits the Meyer property if the set of differences $\underline{\Lambda} - \underline{\Lambda}$ is uniformly discrete in $\mathbb{R}^d$, or equivalently if there exists a finite set F with $\underline{\Lambda} - \underline{\Lambda} \subset \underline{\Lambda} + F$.

The Meyer property on patterns admits several different, but equivalent, formulations [**18**, **21**]. As the above definition suggests, this property appears as a strengthened version of the finite local complexity property. In fact, we have even more.

PROPOSITION 2.3. [**21**] *If Λ is a Meyer multiple set, then all finite combinations $\underline{\Lambda} \pm \underline{\Lambda} \pm \cdots \pm \underline{\Lambda}$ (with any choice of sign) are uniformly discrete.*

2.2. *Model sets.* Now we restrict our attention to the so-called *model sets* of $\mathbb{R}^d$.

2.2.1. *Cut and project schemes and window.*

Definition 2.4. A cut and project scheme is a triple $(H, \Sigma, \mathbb{R}^d)$, where H is a locally compact Abelian group, with some diagram of the form

$$\begin{array}{ccccc} H & \twoheadleftarrow & H \times \mathbb{R}^d & \twoheadrightarrow & \mathbb{R}^d \\ \cup & & \cup & & \cup \\ \Gamma^* & \twoheadleftarrow & \Sigma & \twoheadrightarrow & \Gamma \end{array}$$

such that:

– Σ is a discrete and co-compact subgroup of $H \times \mathbb{R}^d$;
– the canonical projection onto $\mathbb{R}^d$ is bijective from Σ to its image Γ;
– the image Γ^* of Σ under the canonical projection onto H is a dense subgroup of H.

There is a well-established formalism associated to a cut and project scheme: the space $\mathbb{R}^d$ is usually called the *physical space* and H the *internal space*, and the subgroup Γ of $\mathbb{R}^d$ is called the *structure group*. Also, the group morphism $\Gamma \longrightarrow H$, which to any γ associates $\gamma^* := \pi_H(\pi_{\mathbb{R}^d}^{-1}(\gamma)) \in \Gamma^*$, is usually called the *-map of the cut and project scheme, whose graph is precisely the subgroup Σ.

Definition 2.5. Let $(H, \Sigma, \mathbb{R}^d)$ be a cut and project scheme. A window is a finite collection $\{W_j\}_{j \in I}$ of compact topologically regular subsets of H (a subset of a topological space is

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topologically regular if it is the closure of its interior), subject to the condition that the subset $NS^H := H\backslash[\Gamma^* - \bigcup_{i\in I} \partial W_i]$ is non-empty in H.

Note that if Γ is countable, then NS^H is automatically non-empty from the Baire category theorem, but otherwise one has to assume its non-emptiness.

2.2.2. *Model sets.* Given a cut and project scheme $(H, \Sigma, \mathbb{R}^d)$ with window $\{W_i\}_{i\in I}$ in H, we consider a *model multiple set* to be the point pattern $(\mathfrak{P}(W_i))_{i\in I}$ of $\mathbb{R}^d$, where

$$\mathfrak{P}(W_i) := \{\gamma \in \Gamma \mid \gamma^* \in W_i\}.$$

We may also consider translates of the resulting point pattern by any vector $t \in \mathbb{R}^d$, or translates of the window $\{W_j\}_{j\in I}$ by any element $w \in H$, which in both cases leads to a new point pattern of $\mathbb{R}^d$.

Definition 2.6. An inter-model multiple set Λ associated to a cut and project scheme $(H, \Sigma, \mathbb{R}^d)$ with a window $\{W_i\}_{i\in I}$ in H is a point pattern of $\mathbb{R}^d$ of the form

$$\mathfrak{P}(\overset{\circ}{W}_i + w) - t \subseteq \Lambda_i \subseteq \mathfrak{P}(W_i + w) - t.$$

In the particular case where the window admits boundary sets ∂W_i of null Haar measure in H, any resulting inter-model multiple set is said to be *regular*. We will not assume this property in what follows. An inter-model multiple set is called *non-singular* or sometimes *generic* when there are equalities

$$\mathfrak{P}(\overset{\circ}{W}_i + w) - t = \Lambda_i = \mathfrak{P}(W_i + w) - t. \tag{2}$$

The situation where such equality occurs for a given couple (w, t) clearly depends only on the choice of $w \in H$, and happens precisely when there are no points of Γ^* within any boundary $w + \partial W_i$: in other words, equality occurs exactly when $w \in NS^H$. The next statement is folklore in the topic of model sets.

THEOREM 2.7. *Let $(H, \Sigma, \mathbb{R}^d)$ be a cut and project scheme with window $\{W_i\}_{i\in I}$ in H. Consider the full family $\mathbb{X}_{(H,\Sigma,\mathbb{R}^d),\{W_i\}_{i\in I}}$ of inter-model multiple sets that emerge from this data. Then:*

(i) *$\mathbb{X}_{(H,\Sigma,\mathbb{R}^d),\{W_i\}_{i\in I}}$ is complete for the metric d, $\mathbb{R}^d$-invariant and all point patterns of this family have the Meyer property;*

(ii) *the class of non-singular inter-model multiple sets generates a unique hull, denoted $\mathbb{X}_{\mathrm{MS}}$, included in $\mathbb{X}_{(H,\Sigma,\mathbb{R}^d),\{W_i\}_{i\in I}}$;*

(iii) *an inter-model multiple set of $\mathbb{X}_{(H,\Sigma,\mathbb{R}^d),\{W_i\}_{i\in I}}$ is repetitive if and only if it lies within $\mathbb{X}_{\mathrm{MS}}$.*

In the sequel, we will mainly focus on repetitive point patterns, and thereby in the context of inter-model multiple sets on the unique minimal hull arising from a given cut and project scheme and window. One could equivalently define $\mathbb{X}_{\mathrm{MS}}$ to be the hull of any given non-singular model multiple set arising from the data.

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2.2.3. *Redundancies.* It is possible, given a cut and project scheme $(H, \Sigma, \mathbb{R}^d)$ with window $\{W_i\}_{i\in I}$, to tighten up the data without altering the repetitive model multiple sets which may emerge. We call the *redundancies group* of the multiple window the subgroup of H given by

$$\mathcal{R} := \{w \in H \mid W_i + w = W_i \ \forall i \in I\}$$

and we say that a multiple window is *irredundant* if its associated redundancies group is trivial. In any case, $\mathcal{R}$ is a closed subgroup of H, and is moreover compact, as any redundancy satisfies $w \in W_i - W_i$, the latter being compact. Thus, we may consider the quotient locally compact Abelian group $H_{\mathcal{R}}$. Then, as argued in [**19**], the triple $(H_{\mathcal{R}}, \Sigma_{\mathcal{R}}, \mathbb{R}^d)$ with associated diagram

$$\begin{array}{ccccc} H_{\mathcal{R}} & \twoheadleftarrow & H_{\mathcal{R}} \times \mathbb{R}^d & \twoheadrightarrow & \mathbb{R}^d \\ \cup & & \cup & & \cup \\ [\Gamma^*]_{\mathcal{R}} & \twoheadleftarrow & \Sigma_{\mathcal{R}} & \twoheadrightarrow & \Gamma \end{array}$$

where $\Sigma_{\mathcal{R}} := \{([\gamma^*]_{\mathcal{R}}, \gamma) \in H_{\mathcal{R}} \times \mathbb{R}^d \mid \gamma \in \Gamma\}$, is a cut and project scheme and $\{[W_i]_{\mathcal{R}}\}_{i\in I}$ is an irredundant window in the internal space. Using the equality $W_i + \mathcal{R} = W_i$ for each index $i \in I$, one may remark that any inter-model multiple set of the latter data arises as an inter-model multiple set of the former, but the converse is not true in general. However, it is true for the non-singular model multiple sets: due to the equality $W_i + \mathcal{R} = W_i$, one has $\partial W_i + \mathcal{R} = \partial W_i$, so that a $w \in H$ lies within NS^H if and only if its class $[w]_{\mathcal{R}}$ lies within $NS^{H_{\mathcal{R}}}$. Then, for each such element, we have

$$\mathfrak{P}(W_i + w) - t = \mathfrak{P}([W_i]_{\mathcal{R}} + [w]_{\mathcal{R}}) - t,$$

that is, the cut and project scheme with irredundant window gives rise to the same collection of non-singular model multiple sets. It directly follows that it gives rise to the same hull of repetitive inter-model multiple sets.

2.2.4. *Parameterization map for model sets.* Given a cut and project scheme $(H, \Sigma, \mathbb{R}^d)$ with window $\{W_i\}_{i\in I}$, any element $w \in NS^H$ and vector $t \in \mathbb{R}^d$ define a unique model multiple set according to the equalities (2). If moreover after possibly modding out some subgroup of the internal space the considered window is irredundant, then there is only one possible pair $(w, t) \in H \times \mathbb{R}^d$ yielding a given model set, up to an element of Σ. Thus, one would ask for the existence of some mapping from $H \times_\Sigma \mathbb{R}^d$, the compact Abelian group obtained as the quotient of $H \times \mathbb{R}^d$ by the lattice Σ, into the space of point patterns $\mathbb{X}_{(H,\Sigma,\mathbb{R}^d),\{W_i\}_{i\in I}}$. It is in fact the inverse phenomenon which occurs (we restrict ourselves to the space $\mathbb{X}_{\mathrm{MS}}$ of repetitive inter-model multiple sets; see [**19**] for a more general version).

THEOREM 2.8. [**19**, **25**] *Let $\mathbb{X}_{\mathrm{MS}}$ be the hull of repetitive inter-model multiple sets arising from a cut and project scheme $(H, \Sigma, \mathbb{R}^d)$ and window $\{W_i\}_{i\in I}$, which can be supposed irredundant. Then there exists a factor map*

$$\mathbb{X}_{\mathrm{MS}} \twoheadrightarrow H \times_\Sigma \mathbb{R}^d$$

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which satisfies $\pi(\Lambda) = [w_\Lambda, t_\Lambda]_\Sigma$ *if and only if there are inclusions*

$$\mathfrak{P}(\mathring{W}_i + w_\Lambda) - t_\Lambda \subseteq \Lambda_i \subseteq \mathfrak{P}(W_i + w_\Lambda) - t_\Lambda.$$

Moreover, π *is injective precisely on the subset of non-singular point patterns of* $\mathbb{X}_{\mathrm{MS}}$.

By a factor map, we mean here a continuous, onto and $\mathbb{R}^d$-equivariant map, where on $H \times_\Sigma \mathbb{R}^d$ the space $\mathbb{R}^d$ acts through $[w, t]_\Sigma \cdot s := [w, t + s]_\Sigma$. The mapping of Theorem 2.8 is called the *parameterization map* [**2**] and, having a window with boundaries of null Haar measure, is equivalent to have π injective above a full Haar measure subset of $H \times_\Sigma \mathbb{R}^d$ [**19**]. As a result, we have (see §3 for definitions) the following corollary.

COROLLARY 2.9. *The dynamical system* $(\mathbb{X}_{\mathrm{MS}}, \mathbb{R}^d)$ *associated to a hull of repetitive model multiple sets of* $\mathbb{R}^d$ *is almost automorphic, with maximal equicontinuous factor given by the compact Abelian group* $H \times_\Sigma \mathbb{R}^d$.

3. *General aspects of dynamical systems*

In this section we provide a short presentation of the notions of equicontinuity, proximality and Ellis semigroup for general dynamical systems. A complete exposition of this topic can be found in the book [**1**]. In this section we consider a compact dynamical system $(\mathbb{X}, T)$, that is, a compact (Hausdorff) space $\mathbb{X}$ with an action of a group T by homeomorphism. We assume moreover that T is Abelian and for simplicity that $\mathbb{X}$ is metrizable, although results also hold by using uniformities. Denote the collection of homeomorphisms issued from the group action by

$$T^* := \{t^* \in \mathrm{Homeo}(\mathbb{X}) \mid t \in T\}.$$

Definition 3.1. A compact dynamical system $(\mathbb{X}, T)$ is equicontinuous if the family T^* is equicontinuous, that is,

$$\text{for all } \varepsilon \text{ there exists } \delta \text{ such that } d(x, x') < \delta \Rightarrow d(x \cdot t, x' \cdot t) < \varepsilon \quad \text{for all } t \in T.$$

Such dynamical systems are minimal whether they are topologically transitive, and in this case are *Kronecker systems*: from the Ascoli theorem, the closure $T_{\mathbb{X}}$ of the group T^* in $(\mathrm{Homeo}(\mathbb{X}), d_\infty)$, d_∞ being the distance of uniform convergence on $\mathbb{X}$, is a compact Abelian group acting on $\mathbb{X}$ freely and transitively by homeomorphism. There is an obvious group morphism

$$T \longrightarrow T^* \subseteq T_{\mathbb{X}} \tag{3}$$

such that T acts by composition (right or left) on $T_{\mathbb{X}}$, which we note additively by $\chi \mapsto \chi + t^*$, yielding a compact dynamical system $(T_{\mathbb{X}}, T)$. Given a transitive point $\mathfrak{o} \in \mathbb{X}$, one has a homeomorphism $T_{\mathbb{X}} \simeq \mathbb{X}$, $\chi \mapsto \chi(\mathfrak{o})$, which conjugates the T-action. Therefore, the space $\mathbb{X}$ admits a compact Abelian group structure with $\mathfrak{o}$ as unit and T-action *by rotation*, that is, coming from a group morphism of the form (3).

To any compact dynamical system is naturally associated an equicontinuous compact dynamical system.

THEOREM 3.2. *Let* $(\mathbb{X}, T)$ *be a compact dynamical system. There exists a unique closed* T*-invariant equivalence relation* $\sim_{\mathrm{eq}}$ *on* $\mathbb{X}$*, such that the quotient space* $\mathbb{X}_{\mathrm{eq}} := \mathbb{X}/_{\sim_{\mathrm{eq}}}$ *with* T*-action is an equicontinuous flow, which is maximal in the sense that any equicontinuous factor of* $(\mathbb{X}, T)$ *factors through* $\mathbb{X}_{\mathrm{eq}}$.

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This is an existential result, whose proof can be found in [**1**]. We generically denote the quotient map under the relation $\sim_{\mathrm{eq}}$ by

$$\pi : \mathbb{X} \longrightarrow \mathbb{X}_{\mathrm{eq}} .$$

The relation $\sim_{\mathrm{eq}}$ is called the *equicontinuous structure relation* and the space $\mathbb{X}_{\mathrm{eq}}$ the *maximal equicontinuous factor* of $(\mathbb{X}, T)$. If the dynamical system $(\mathbb{X}, T)$ is topologically transitive, with some $x_0 \in \mathbb{X}$ having dense T-orbit, then so is $(\mathbb{X}_{\mathrm{eq}}, T)$, which is thus a Kronecker system with unit $\mathfrak{o} = \pi(x_0)$. In this case $\mathbb{X}_{\mathrm{eq}}$ is a compact Abelian group related to the spectral analysis of $(\mathbb{X}, T)$ (see [**4**, **8**]).

A (algebraic) character ω on the group T is called a *topological eigenvalue* for the system $(\mathbb{X}, T)$ if there exists a non-trivial continuous complex-valued function χ_ω on $\mathbb{X}$, called an *eigenfunction* associated to ω, satisfying $\chi_\omega(x \cdot t) = \omega(t)\chi_\omega(x)$ for all $x \in \mathbb{X}$ and all $t \in T$. We remark that if T is endowed with a topology so that the action becomes separately continuous, then a topological eigenvalue must be a continuous character on T. The collection of topological eigenvalues forms an Abelian group which we denote by $\mathcal{E}(\mathbb{X}, T)$. If we let x_0 be a transitive point of $(\mathbb{X}, T)$, then each topological eigenvalue admits a unique associated eigenfunction χ_ω having constant modulus and such that $\chi_\omega(x_0) = 1$ in the unit circle $\mathbb{T}$.

PROPOSITION 3.3. [**8**] *Let $(\mathbb{X}, T)$ be a compact topologically transitive dynamical system with T Abelian. Then*

$$\pi(x) = \pi(x') \Longleftrightarrow \chi_\omega(x) = \chi_\omega(x') \quad \textit{for all } \omega \in \mathcal{E}(\mathbb{X}, T).$$

What this proposition shows is that the group of topological eigenvalues of $(\mathbb{X}, T)$ is naturally isomorphic with the Pontryagin dual of the compact Abelian group $\mathbb{X}_{\mathrm{eq}}$,

$$\widehat{\mathbb{X}_{\mathrm{eq}}} \simeq \mathcal{E}(\mathbb{X}, T),$$

where a continuous character ω on $(\mathbb{X}, T)$ has its eigenfunction χ_ω written as $\varpi \circ \pi$ with $\varpi \in \widehat{\mathbb{X}_{\mathrm{eq}}}$. In the context where $(\mathbb{X}, T)$ is minimal, one can give a fairly different formulation of the equicontinuous structure relation, that is, whether or not two points x and x' are identified under π.

THEOREM 3.4. [**1**] *Let $(\mathbb{X}, T)$ be a minimal compact dynamical system. Then $\pi(x) = \pi(x')$ if and only if x and x' are regionally proximal, that is, for any $\varepsilon > 0$, there are x_ε, x'_ε in $\mathbb{X}$ and $t \in T$ such that*

$$d(x, x_\varepsilon) < \varepsilon, \quad d(x', x'_\varepsilon) < \varepsilon, \quad d(x_\varepsilon \cdot t, x'_\varepsilon \cdot t) < \varepsilon.$$

In this article we are especially interested in a particular type of dynamical systems, namely the *almost automorphic* systems. In the case where $(\mathbb{X}, T)$ is minimal, this is equivalent to saying that the continuous eigenfunctions associated to $\mathcal{E}(\mathbb{X}, T)$ separate *at least* one point of $\mathbb{X}$.

Definition 3.5. [**26**] A compact dynamical system $(\mathbb{X}, T)$ is almost automorphic if the factor map π admits a one-point fiber.

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Even if a dynamical system $(\mathbb{X}, T)$ is not equicontinuous, one can still consider the completion $T_{\mathbb{X}}$ of T^*, which is an Abelian group of homeomorphisms on $\mathbb{X}$, but by doing so we lose compacity. Instead, one is led towards a completion of T^* which is compact, but where the group structure is lost: this is called the *Ellis enveloping semigroup* of $(\mathbb{X}, T)$.

Definition 3.6. Let $(\mathbb{X}, T)$ be a compact dynamical system, and consider T^* as a subset of $\mathbb{X}^{\mathbb{X}}$, the product space with product topology, or equivalently the space of all maps from $\mathbb{X}$ into itself with pointwise convergence topology. The Ellis semigroup $E(\mathbb{X}, T)$ of this system is the closure of T^* in $\mathbb{X}^{\mathbb{X}}$, endowed with composition of maps.

The Ellis semigroup has a well-defined semigroup structure and is compact (Hausdorff), consisting of transformations on $\mathbb{X}$ obtained as pointwise limits of homeomorphisms in T^*. It acts on $\mathbb{X}$, here with action written on the right-hand side, meaning that $x \cdot g$ stands for the image of x under the map g, and with this convention it is always a *right-topological semigroup*: if a net g_λ converges (pointwise) to g, then $h \cdot g_\lambda$ converges (pointwise) to $h \cdot g$. Usually, the transformations in $E(\mathbb{X}, T)$ are neither continuous nor invertible on $\mathbb{X}$. The Ellis semigroup for dynamical systems is still not completely understood and admits a fairly developed literature [**1**, **12–16**]. However, we shall use it only as a tool for proving the results of the present article.

We will also need the notion of Ellis semigroup for locally compact dynamical systems $(\mathbb{X}, T)$, which we define below. Denote by $\hat{\mathbb{X}}$ the one-point compactification of $\mathbb{X}$ (with neighborhood basis at the point at infinity ∞ given by the complementary sets of compact sets in $\mathbb{X}$), endowed with the extended T-action by homeomorphisms defined by keeping the point at infinity fixed.

Definition 3.7. The Ellis semigroup of a locally compact dynamical system $(\mathbb{X}, T)$ is defined to be

$$E(\mathbb{X}, T) := E(\hat{\mathbb{X}}, T) \cap \mathcal{F}_{\mathbb{X}},$$

where $\mathcal{F}_{\mathbb{X}}$ is the semigroup of mappings in $\hat{X}^{\hat{\mathbb{X}}}$ which map $\mathbb{X}$ into itself and keep the point at infinity fixed, with topology induced from $\hat{\mathbb{X}}^{\hat{\mathbb{X}}}$.

We can view $E(\mathbb{X}, T)$ as a subsemigroup of $\mathbb{X}^{\mathbb{X}}$ simply by restricting transformations on $\mathbb{X}$, and the latter give rise to the same topology on $E(\mathbb{X}, T)$. Again it is a right-topological semigroup which contains T^* as a dense subgroup. To show the difference between $E(\mathbb{X}, T)$ and $E(\widehat{\mathbb{X}}, T)$, we show the following proposition.

PROPOSITION 3.8. *Suppose that $(\mathbb{X}, T)$ is a non-compact locally compact minimal dynamical system, with T Abelian. Then the transformation $\bowtie_{\mathbb{X}}$ identically equal to ∞ on $\hat{\mathbb{X}}$ lies in $E(\hat{\mathbb{X}}, T)$ but not in $E(\mathbb{X}, T)$.*

Proof. Let x be some point of $\mathbb{X}$: its T-orbit is dense in $\mathbb{X}$ and, since the latter is non-compact, the point ∞ is not isolated, so that the T-orbit of x is in fact dense in $\hat{\mathbb{X}}$. As ∞ is invariant under the extended T-action on $\hat{\mathbb{X}}$, the only T-invariant compact subset of $\hat{\mathbb{X}}$ is the singleton $\{\infty\}$. From the Zorn lemma, the dynamical system $(E(\hat{\mathbb{X}}, T), T)$ with T acting by composition admits some T-minimal subsystem (M, T), into which we may choose a transformation g. Then, for each $x \in \hat{\mathbb{X}}$, the evaluation map ev_x :

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$E(\hat{\mathbb{X}}, T) \longrightarrow \hat{\mathbb{X}}$ is continuous and T-equivariant (as T is Abelian, any $t \in T$ commutes with any transformation in $E(\hat{\mathbb{X}}, T)$), so maps M onto a minimal subsystem of $\hat{\mathbb{X}}$, which must then be $\{\infty\}$. It follows that $x \cdot g = \infty$ for each $x \in \hat{\mathbb{X}}$, as desired. It is clear that $\bowtie_{\mathbb{X}}$ cannot be a transformation in $E(\mathbb{X}, T)$ since it does not preserve $\mathbb{X}$ in $\widehat{\mathbb{X}}$, concluding the proof. □

4. *Combinatorial topology on a hull of point patterns*

Given some point pattern Λ of $\mathbb{R}^d$, we wish to consider here a topology on its associated hull $\mathbb{X}_\Lambda$, which we call *combinatorial topology*, obtained from the metric

$$\mathsf{d}(\Lambda', \Lambda'') := \inf\left\{ \frac{1}{1+R} \;\middle|\; \Lambda' \cap B(0, R) = \Lambda'' \cap B(0, R) \right\}.$$

This metric is in fact an ultrametric on the hull $\mathbb{X}_\Lambda$, setting that two point patterns are close whenever they exactly match on a large domain about the origin in $\mathbb{R}^d$. We refer to the space $\mathbf{X}_\Lambda$ as the collection $\mathbb{X}_\Lambda$ itself, endowed with the above ultrametric, and call it the *combinatorial hull* of Λ. The action of $\mathbb{R}^d$ by translation site by site on any point patterns yields an action by homeomorphisms on $\mathbf{X}_\Lambda$ and thus a dynamical system $(\mathbf{X}_\Lambda, \mathbb{R}^d)$. Observe that on the transversal Ξ given in (1) the ultrametric d coincides with the usual metric d and, with respect to this new topology, Ξ is a *clopen* set, that is, is both closed and open in $\mathbf{X}_\Lambda$.

PROPOSITION 4.1. *Let Λ be a point pattern of $\mathbb{R}^d$. Then, if Λ is of finite local complexity, the space $\mathbf{X}_\Lambda$ is locally compact. Moreover, the dynamical system $(\mathbf{X}_\Lambda, \mathbb{R}^d)$ is minimal if and only if Λ is repetitive.*

Consider now a point pattern Λ_0 of finite local complexity with hull $\mathbb{X}_{\Lambda_0}$ and let $\mathbb{X}_{\mathrm{eq}}$ be the maximal equicontinuous factor of $(\mathbb{X}_{\Lambda_0}, \mathbb{R}^d)$. The combinatorial topology on $\mathbf{X}_{\Lambda_0}$ is naturally finer than the topology of $\mathbb{X}_{\Lambda_0}$ and thus the closed $\mathbb{R}^d$-invariant equivalence relation induced on $\mathbb{X}_{\Lambda_0}$ from the factor map π onto $\mathbb{X}_{\mathrm{eq}}$ is also closed in $\mathbf{X}_{\Lambda_0}$. One may thus define the space $\mathbf{X}_{\mathrm{eq}}$ as the quotient space of $\mathbf{X}_{\Lambda_0}$ under this relation. In other words, we have a commuting diagram

$$\begin{array}{ccc} \mathbf{X}_{\Lambda_0} & \xrightarrow{\;\Pi\;} & \mathbf{X}_{\mathrm{eq}} \\ \updownarrow & & \updownarrow \\ \mathbb{X}_{\Lambda_0} & \xrightarrow{\;\pi\;} & \mathbb{X}_{\mathrm{eq}} \end{array}$$

where vertical maps are the identity maps, continuous from top to bottom. The space $\mathbf{X}_{\mathrm{eq}}$ admits an induced $\mathbb{R}^d$-action, yielding so a dynamical system $(\mathbf{X}_{\mathrm{eq}}, \mathbb{R}^d)$. The quotient map, denoted here by Π, becomes equivariant with respect to the $\mathbb{R}^d$-actions. When Λ_0 is repetitive, the dynamical system is minimal.

Consider the particular element $\mathfrak{o} := \pi(\Lambda_0) \in \mathbb{X}_{\mathrm{eq}}$. The space $\mathbb{X}_{\mathrm{eq}}$ is (see §3) a compact Abelian group with $\mathfrak{o}$ as unit, and we endow $\mathbf{X}_{\mathrm{eq}}$ with this Abelian group structure. We have a (non-continuous) group morphism

$$\begin{array}{ccc} \mathbb{R}^d & \longrightarrow & \mathbf{X}_{\mathrm{eq}} \\ t & \longmapsto & t^* \end{array} \tag{4}$$

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such that the image under the action of a $t \in \mathbb{R}^d$ on an element $\mathfrak{e} \in \mathbf{X}_{\mathrm{eq}}$ is written $\mathfrak{e} + t^*$. The equivariance property of Π is formulated as $\Pi(\Lambda + t) = \Pi(\Lambda) + t^*$ for any $\Lambda \in \mathbb{X}_{\Lambda_0}$ and any $t \in \mathbb{R}^d$. Then our aim in this section is to show that, under the Meyer and repetitivity assumptions, the topology and the Abelian group structure on $\mathbf{X}_{\mathrm{eq}}$ are compatible. More precisely, we have the following theorem.

THEOREM 4.2. *Let $\mathbf{X}_{\Lambda_0}$ be the combinatorial hull of a repetitive Meyer multiple set Λ_0 and let $\mathbf{X}_{\mathrm{eq}}$ be as above. Then $\mathbf{X}_{\mathrm{eq}}$ is a locally compact Abelian group.*

The great difficulty in proving this statement is to point out how the Meyer property is related to the quotient topology of $\mathbf{X}_{\mathrm{eq}}$. To that end, a main tool is Proposition 4.4 stated below. First we introduce the regional proximality relation of the system $(\mathbf{X}_{\Lambda_0}, \mathbb{R}^d)$, called after [**8**] the *strong regional proximality*.

Definition 4.3. Two multiple sets Λ and Λ' of a combinatorial hull $\mathbf{X}$ are strongly regionally proximal if for any given $R > 0$ there exist Λ_R, Λ'_R in $\mathbf{X}$ and $t \in \mathbb{R}^d$ such that one has

$$\begin{aligned} \Lambda \cap B(0, R) &\equiv \Lambda_R \cap B(0, R), \\ \Lambda' \cap B(0, R) &\equiv \Lambda'_R \cap B(0, R), \\ \Lambda_R \cap B(t, R) &\equiv \Lambda'_R \cap B(t, R). \end{aligned}$$

This relation means that, although Λ and Λ' may not agree anywhere, each always agrees with some respective Λ_R and Λ'_R on a patch of radius R around the origin, with these two multiple sets in turn agreeing on some patch of radius R. In order to prove Theorem 4.2, we will make use of the following fact.

PROPOSITION 4.4. [**8**] *Let Λ and Λ' be two multiple sets of the hull of a repetitive Meyer multiple set. Then $\Pi(\Lambda) = \Pi(\Lambda')$ if and only if Λ and Λ' are strongly regionally proximal.*

We add here a simple but useful lemma.

LEMMA 4.5. *Let Λ_0 be a point pattern of $\mathbb{R}^d$ with hull $\mathbf{X}_{\Lambda_0}$. If two point patterns Λ, Λ' in $\mathbf{X}_{\Lambda_0}$ are strongly regionally proximal, then $\underline{\Lambda} - \underline{\Lambda'} \subseteq \underline{\Lambda_0} - \underline{\Lambda_0} + \underline{\Lambda_0} - \underline{\Lambda_0} + \underline{\Lambda_0} - \underline{\Lambda_0}$.*

Proof. Given some point pattern $\Lambda \in \mathbf{X}_{\Lambda_0}$, it is clear since any local configuration of Λ appears somewhere in Λ_0 that $\underline{\Lambda} - \underline{\Lambda} \subseteq \underline{\Lambda_0} - \underline{\Lambda_0}$. Now, if we suppose that Λ and Λ' are strongly regionally proximal, then there exists, for R_0 to be some radius of relative density of any multiple set in $\mathbf{X}_{\Lambda_0}$, some Λ_1 and Λ_2 together with a vector $t \in \mathbb{R}^d$ such that one can find points p_1, p_2 and p_3 satisfying

$$\begin{aligned} p_1 \in \Lambda \cap B(0, R_0) &\equiv \Lambda_1 \cap B(0, R_0), \\ p_2 \in \Lambda' \cap B(0, R_0) &\equiv \Lambda_2 \cap B(0, R_0), \\ p_3 \in \Lambda_1 \cap B(t, R_0) &\equiv \Lambda_2 \cap B(t, R_0). \end{aligned}$$

It ensures that $\Lambda = \Lambda - p_1 + (p_1 - p_3) + (p_3 - p_2) + p_2$ is supported in $(\underline{\Lambda} - \underline{\Lambda}) + (\underline{\Lambda_1} - \underline{\Lambda_1}) + (\underline{\Lambda_2} - \underline{\Lambda_2}) + \underline{\Lambda'}$, yielding the proof. □

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Proof of the theorem. Let us start by supposing that Λ_0 is a Delone multiple set of finite local complexity in $\mathbb{R}^d$. The space $\mathbf{X}_{\Lambda_0}$ is consequently locally compact and one can form the Ellis semigroup $E(\mathbf{X}_{\Lambda_0}, \mathbb{R}^d)$ (see Definition 3.7), which is a right-topological semigroup.

PROPOSITION 4.6. *The mapping given by the composition* $\Pi \circ ev_{\Lambda_0}$, ev_{Λ_0} *denoting the evaluation map at* Λ_0 *on the Ellis semigroup* $E(\mathbf{X}_{\Lambda_0}, \mathbb{R}^d)$,

$$\chi : E(\mathbf{X}_{\Lambda_0}, \mathbb{R}^d) \longrightarrow\!\!\!\!\!\rightarrow \mathbf{X}_{\mathrm{eq}},$$

is a continuous semigroup morphism.

Proof. The map χ is continuous, being the composition of two continuous maps. The image under χ of the identity map in $E(\mathbf{X}_{\Lambda_0}, \mathbb{R}^d)$ is $\mathfrak{o}$ from the choice of the latter and, for each $t \in \mathbb{R}^d$ and $g \in E(\mathbf{X}_{\Lambda_0}, \mathbb{R}^d)$, we have the equality $\chi(g \cdot t) = \Pi(\Lambda_0 \cdot g + t) = \Pi(\Lambda_0 \cdot g) + t^* = \chi(g) + t^*$. Select then two transformations g and h in $E(\mathbf{X}_{\Lambda_0}, \mathbb{R}^d)$, and consider a net $(t_\lambda)_\lambda \subset \mathbb{R}^d$ converging to h in $E(\mathbf{X}_{\Lambda_0}, \mathbb{R}^d)$ (that is, pointwise on $\mathbf{X}_{\Lambda_0}$). From the right-continuity of the composition law on $E(\mathbf{X}_{\Lambda_0}, \mathbb{R}^d)$, we have on one hand that $\chi(g \cdot h)$ is the limit of $\chi(g \cdot t_\lambda) = \chi(g) + t_\lambda^*$ in $\mathbf{X}_{\mathrm{eq}}$, and on the other hand that $\chi(h) = \Pi(\Lambda_0 \cdot h)$ is the limit of $\Pi(\Lambda_0 + t_\lambda) = t_\lambda^*$ in $\mathbf{X}_{\mathrm{eq}}$. But the identity map from $\mathbf{X}_{\mathrm{eq}}$ onto $\mathbb{X}_{\mathrm{eq}}$ is continuous, so these convergences also hold in $\mathbb{X}_{\mathrm{eq}}$. Since for the latter the sum is continuous, we deduce that $\chi(g) + t_\lambda^*$ converges to $\chi(g) + \chi(h)$ in $\mathbb{X}_{\mathrm{eq}}$, giving $\chi(g \cdot h) = \chi(g) + \chi(h)$, as desired. □

Consider the one-point compactification $\hat{\mathbf{X}}_{\Lambda_0}$ of the space $\mathbf{X}_{\Lambda_0}$. It is written as a disjoint union $\mathbf{X}_{\Lambda_0} \cup \{\infty\}$, with basis of open neighborhoods of ∞ given by the complementary sets of the compact subsets of $\mathbf{X}_{\Lambda_0}$. Consider the extended $\mathbb{R}^d$-action by homeomorphisms keeping the point at infinity ∞ fixed, and consider the compact right-topological semigroup $E(\hat{\mathbf{X}}_{\Lambda_0}, \mathbb{R}^d)$. The latter naturally contains an isomorphic copy of $E(\mathbf{X}_{\Lambda_0}, \mathbb{R}^d)$, as well as the map $\bowtie_{\mathbf{X}_{\Lambda_0}}$ identically equal to ∞ on $\hat{\mathbf{X}}_{\Lambda_0}$ by Proposition 3.8.

PROPOSITION 4.7. *Suppose that* Λ_0 *has the Meyer property. If a transformation* g *in* $E(\hat{\mathbf{X}}_{\Lambda_0}, \mathbb{R}^d)$ *gives* $\Lambda \cdot g \neq \infty$ *for some* $\Lambda \in \mathbf{X}_{\Lambda_0}$, *then* $\Lambda \cdot g \neq \infty$ *for any other* $\Lambda \in \mathbf{X}_{\Lambda_0}$. *In other words, we have*

$$E(\mathbf{X}_{\Lambda_0}, \mathbb{R}^d) = E(\hat{\mathbf{X}}_{\Lambda_0}, \mathbb{R}^d) \backslash \{\bowtie_{\mathbf{X}_{\Lambda_0}}\}.$$

Proof. Consider a transformation $g \in E(\hat{\mathbf{X}}_{\Lambda_0}, \mathbb{R}^d)$ with some $\Lambda_1 \in \mathbf{X}_{\Lambda_0}$ with $\Lambda_1 \cdot g \neq \infty$, and let $\Lambda_2 \in \mathbf{X}_{\Lambda_0}$. We can choose a net $(t_\lambda)_\lambda \subset \mathbb{R}^d$ converging to g in $E(\mathbf{X}_{\Lambda_0}, \mathbb{R}^d)$, which has to satisfy $\begin{smallmatrix} \Lambda_1 + t_\lambda \longrightarrow \Lambda_1 \cdot g \\ \Lambda_2 + t_\lambda \longrightarrow \Lambda_2 \cdot g \end{smallmatrix}$ in $\mathbf{X}_{\Lambda_0}$. Let r_0 and R_0 be some radii of uniform discreteness and relative density of any multiple set of $\mathbf{X}_{\Lambda_0}$. As $\Lambda_1 + t_\lambda$ converges to $\Lambda_1 \cdot g$ with respect to the combinatorial metric, there exists an index λ_0 so that for any $\lambda > \lambda_0$ we may find a common point

$$p \in (\Lambda_1 + t_\lambda) \cap B(0, R_0) \equiv (\Lambda_1 + t_{\lambda'}) \cap B(0, R_0) \equiv (\Lambda_1 \cdot g) \cap B(0, R_0).$$

We hence have $t_\lambda - t_{\lambda'} \in \Lambda_1 - \Lambda_1$ for any $\lambda, \lambda' > \lambda_0$. Select then a net v_λ of points, each taken within the patch $(\Lambda_2 + t_\lambda) \cap B(0, R_0)$. Each v_λ can be written as $p_\lambda + t_\lambda$ for some

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point $p_\lambda \in \Lambda_2$. Consider then the collection $\{v_\lambda\}_{\lambda>\lambda_0}$, lying inside the relatively compact set $B(0, 2R_0)$: the difference $v_\lambda - v_{\lambda'}$ is equal to $(p_\lambda + t_\lambda) - (p_{\lambda'} + t_{\lambda'}) = (p_\lambda - p_{\lambda'}) + (t_\lambda - t_{\lambda'})$ and so falls in $\Lambda_2 - \Lambda_2 + \Lambda_1 - \Lambda_1 \subset \Lambda_0 - \Lambda_0 + \Lambda_0 - \Lambda_0$, uniformly discrete due to the Meyer property. The collection $\{v_\lambda\}_{\lambda>\lambda_0}$ is consequently uniformly discrete, and so is finite. Pick some finite set of representatives $v_{\lambda_1}, \ldots, v_{\lambda_l}$: the net $(\Lambda_2 + t_\lambda)_{\lambda>\lambda_0}$ stays within the compact subset $\bigcup_{i=1}^{l} \Xi + v_{\lambda_i}$ of $\mathbf{X}_{\Lambda_0}$, where Ξ is the canonical transversal. Thus, the limit $\Lambda_2 \cdot g$ lies inside $\mathbf{X}_{\Lambda_0}$ and, as Λ_2 has been chosen arbitrarily in $\mathbf{X}_{\Lambda_0}$, the statement is proven. □

COROLLARY 4.8. *Suppose that Λ_0 has the Meyer property. Then the space $E(\mathbf{X}_{\Lambda_0}, \mathbb{R}^d)$ is locally compact and the evaluation map*

$$ev_{\Lambda_0} : E(\mathbf{X}_{\Lambda_0}, \mathbb{R}^d) \longrightarrow\!\!\!\!\!\rightarrow \mathbf{X}_{\Lambda_0}$$

is onto and proper.

Proof. The evaluation map $ev_{\Lambda_0} : E(\hat{\mathbf{X}}_{\Lambda_0}, \mathbb{R}^d) \longrightarrow \hat{\mathbf{X}}_{\Lambda_0}$ is continuous from the very definition of the topology on $E(\hat{\mathbf{X}}_{\Lambda_0}, \mathbb{R}^d)$, and is onto since its image is a compact set containing the $\mathbb{R}^d$-orbit of Λ_0, dense in $\mathbf{X}_{\Lambda_0}$ and consequently dense in $\hat{\mathbf{X}}_{\Lambda_0}$ (the space $\mathbf{X}_{\Lambda_0}$ being never compact, even for Λ_0 a lattice, the point at infinity $\hat{\mathbf{X}}_{\Lambda_0}$ is always an accumulation point of $\mathbf{X}_{\Lambda_0}$). Any antecedent transformation of the point ∞ under the map ev_{Λ_0} in $E(\hat{\mathbf{X}}_{\Lambda_0}, \mathbb{R}^d)$ must map Λ_0 onto ∞, and so has to be $\bowtie_{\mathbf{X}_{\Lambda_0}}$ from the previous proposition. Hence, the restriction of ev_{Λ_0} outside $\bowtie_{\mathbf{X}_{\Lambda_0}}$ takes its values in $\mathbf{X}_{\Lambda_0}$ and is onto. Properness easily follows. □

PROPOSITION 4.9. *Suppose that Λ_0 has the Meyer property and is repetitive. Then the space $\mathbf{X}_{\mathrm{eq}}$ is locally compact and Π is a proper map.*

Proof. Let $\sim$ be the closed $\mathbb{R}^d$-invariant equivalence relation induced by Π on $\mathbf{X}_{\Lambda_0}$ with graph $\mathcal{G}(\sim)$ in $\mathbf{X}_{\Lambda_0} \times \mathbf{X}_{\Lambda_0}$. This relation coincides with the strong regional proximality by Proposition 4.4. Define the relation $\hat{\sim}$ on the one-point compactification $\hat{\mathbf{X}}_{\Lambda_0}$ as the relation of the graph $\mathcal{G}(\hat{\sim}) := \mathcal{G}(\sim) \cup \{(\infty, \infty)\}$ in $\hat{\mathbf{X}}_{\Lambda_0} \times \hat{\mathbf{X}}_{\Lambda_0}$. Clearly, $\hat{\sim}$ is an equivalence relation and is moreover $\mathbb{R}^d$-invariant on $\hat{\mathbf{X}}_{\Lambda_0}$. Let us show that it is closed.

Let $\mathcal{G}(\hat{\sim}) \supset (\Lambda_\lambda^1, \Lambda_\lambda^2)_\lambda \longrightarrow (x, y)$ be some net converging in $\hat{\mathbf{X}}_{\Lambda_0} \times \hat{\mathbf{X}}_{\Lambda_0}$. We may suppose that $(\Lambda_\lambda^1, \Lambda_\lambda^2) \in \mathcal{G}(\sim)$ and, for a contradiction, we may also suppose that the limit pair (x, y) falls outside the graph $\mathcal{G}(\hat{\sim})$. From the very construction of $\hat{\sim}$, we hence have (up to a switch of x and y) $x = \Lambda \in \mathbf{X}_{\Lambda_0}$ and $y = \infty$. Let r_0 and R_0 be some radii of uniform discreteness and relative density of any multiple set of $\mathbf{X}_{\Lambda_0}$. From Proposition 4.4, any Λ_λ^1 and Λ_λ^2 are strongly regionally proximal, and thus by Lemma 4.5 satisfy

$$\underline{\Lambda_\lambda^1} - \underline{\Lambda_\lambda^2} \subseteq 3(\underline{\Lambda_0} - \underline{\Lambda_0}). \tag{5}$$

Now, as Λ_λ^1 converges to $x = \Lambda$ in $\mathbf{X}$, there is some index λ_0 such that for any indices $\lambda, \lambda' > \lambda_0$ there is some common point p within both Λ_λ^1, $\Lambda_{\lambda'}^1$ and Λ, and consequently for any $\lambda, \lambda' > \lambda_0$ one obtains

$$\underline{\Lambda_\lambda^1} - \underline{\Lambda_{\lambda'}^1} = \underline{\Lambda_\lambda^1} - p + p - \underline{\Lambda_{\lambda'}^1} \subseteq (\underline{\Lambda_\lambda^1} - \underline{\Lambda_\lambda^1}) + (\underline{\Lambda_{\lambda'}^1} - \underline{\Lambda_{\lambda'}^1}) \subseteq 2(\underline{\Lambda_0} - \underline{\Lambda_0}). \tag{6}$$

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Let then $\{v_\lambda\}_\lambda$ be a collection of points each contained in $\Lambda_\lambda^2 \cap B(0, R_0)$. Then, by (5) and (6),

$$v_\lambda - v_{\lambda'} \in \underline{\Lambda}_\lambda^2 - \underline{\Lambda}_{\lambda'}^2 \subseteq [3(\underline{\Lambda}_0 - \underline{\Lambda}_0) + \underline{\Lambda}_\lambda^1] - [3(\underline{\Lambda}_0 - \underline{\Lambda}_0) + \underline{\Lambda}_{\lambda'}^1] \subseteq 8(\underline{\Lambda}_0 - \underline{\Lambda}_0),$$

the right-hand term being uniformly discrete by Proposition 2.3. The collection $\{v_\lambda\}_\lambda$ is consequently uniformly discrete and lies inside the precompact set $B(0, R_0)$, so is a finite set. Taking a finite set of representatives $v_{\lambda_1}, \ldots, v_{\lambda_l}$, we obtain that the net $(\Lambda_\lambda^2)_\lambda$ remains in the compact subset $\bigcup_{i=1}^{l} \Xi + v_{\lambda_i}$ of $\mathbf{X}_{\Lambda_0}$. This contradicts the fact that $(\Lambda_\lambda^2)_\lambda$ converges to ∞, giving that $\hat{\sim}$ is a closed relation.

Let $Z := \hat{\mathbf{X}}_{\Lambda_0}/_{\hat{\sim}}$ with $\hat{\pi}_Z : \hat{\mathbf{X}}_{\Lambda_0} \twoheadrightarrow Z$ onto a continuous $\mathbb{R}^d$-equivariant map of compact spaces. It is clear that $Z\backslash\{\pi_Z(\infty)\}$ is locally compact, and the restriction $\pi_Z : \mathbf{X}_{\Lambda_0} \twoheadrightarrow Z\backslash\{\pi_Z(\infty)\}$ is continuous, onto and proper. But, from the construction of $\hat{\sim}$, we have $Z\backslash\{\pi_Z(\infty)\}$ under bijective correspondence with $\mathbf{X}_{\mathrm{eq}}$, so by definition of the quotient topology we get a continuous bijective map $i : \mathbf{X}_{\mathrm{eq}} \longrightarrow Z\backslash\{\pi_Z(\infty)\}$. The map i is proper: for, if K is compact in $\mathbf{X}_{\mathrm{eq}}$, its preimage $(\Pi)^{-1}(K)$ is written as $(\pi_Z)^{-1}(i(K))$, so is compact. It follows that i is an homeomorphism; thus, $\mathbf{X}_{\mathrm{eq}}$ is a locally compact space and the map Π is proper. □

Now we can establish the proof of Theorem 4.2.

Proof. From Corollary 4.8 together with Proposition 4.9, we know that the morphism

$$\chi : E(\mathbf{X}_{\Lambda_0}, \mathbb{R}^d) \longrightarrow\!\!\!\!\!\rightarrow \mathbf{X}_{\mathrm{eq}}$$

is onto and proper between locally compact spaces. Let now $(v_\lambda)_\lambda$ be a net in $\mathbf{X}_{\mathrm{eq}}$ converging to some v, and consider $w \in \mathbf{X}_{\mathrm{eq}}$: we have to show that $v_\lambda + w$ converges to $v + w$ in $\mathbf{X}_{\mathrm{eq}}$. This is equivalent to the fact that each subnet $v_{\lambda'} + w$ accumulates at $v + w$. Thus, let us consider a subnet $v_{\lambda'} + w$. Then $v_{\lambda'}$ still converges to v. If we consider $(g_{\lambda'})_{\lambda'}$ and h to be liftings in $E(\mathbf{X}_{\Lambda_0}, \mathbb{R}^d)$ of the subnet and w, respectively, then, for any compact neighborhood V of v, the net $(g_{\lambda'})_{\lambda'}$ eventually lies in the compact set $\chi^{-1}(V)$. Consequently, we may select an accumulation point g (that is, a transformation lying in $\bigcap_{\lambda_0} \overline{(g_{\lambda'})_{\lambda' > \lambda_0}}$ contained in $\bigcap_{V \ni v} \chi^{-1}(V) = \chi^{-1}(v)$, so that $\chi(g) = v$). Since $(g_{\lambda'})_{\lambda'}$ accumulates at g, the net $(h \cdot g_{\lambda'})_{\lambda'}$, as image of the net $(g_{\lambda'})_{\lambda'}$ through the continuous map $s \mapsto h \cdot s$ of $E(\mathbf{X}_{\Lambda_0}, \mathbb{R}^d)$, accumulates at $h \cdot g$. Taking the image under the continuous morphism χ, we obtain that $\chi(h \cdot g_{\lambda'}) = w + v_{\lambda'}$ accumulates at $\chi(h \cdot g) = w + v$ in $\mathbf{X}_{\mathrm{eq}}$. This shows that addition is right-continuous on $\mathbf{X}_{\mathrm{eq}}$ and, since it is Abelian, addition is then separately continuous. It implies by [**1**], since the topology on $\mathbf{X}_{\mathrm{eq}}$ is locally compact, that addition is jointly continuous and inversion is continuous as well, so that $\mathbf{X}_{\mathrm{eq}}$ is a locally compact Abelian group. □

5. *The internal system of a hull of point patterns*

Given a point pattern Λ_0 of $\mathbb{R}^d$, its generated group is the countable subgroup $\langle \underline{\Lambda}_0 \rangle$ of $\mathbb{R}^d$ generated by the support of Λ_0. It naturally contains the difference set $\underline{\Lambda}_0 - \underline{\Lambda}_0$, and thus the difference set $\underline{\Lambda} - \underline{\Lambda}$ of any other point pattern $\Lambda \in \mathbb{X}_{\Lambda_0}$. From now on, we shall consider a subgroup

$$\langle \underline{\Lambda}_0 \rangle \leqslant \Gamma \leqslant \mathbb{R}^d$$

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endowed with discrete topology, which we keep fixed. We do not assume any countability assumption, so in particular it may be the entire Euclidean space.

Definition 5.1. Given a point pattern Λ_0 with combinatorial hull $\mathbf{X}_{\Lambda_0}$, the subsystem associated to Γ is the space with combinatorial metric and restricted Γ-action

$$\Xi^\Gamma := \{\Lambda \in \mathbb{X}_{\Lambda_0} \mid \underline{\Lambda} \subset \Gamma\}.$$

It is easy to observe that Ξ^Γ remains stable under the Γ-action on $\mathbf{X}_{\Lambda_0}$ and is, by homeomorphisms, providing so a dynamical system (Ξ^Γ, Γ). From the very choice of Γ, the point pattern Λ_0 lies in Ξ^Γ, and it can be verified that its Γ-orbit is dense in the latter space. The dynamical system (Ξ^Γ, Γ) is minimal if and only if Λ_0 is repetitive.

LEMMA 5.2. *The transversal Ξ is a clopen subset of the space Ξ^Γ, which is in turn a clopen subset of the combinatorial hull $\mathbf{X}_{\Lambda_0}$.*

Proof. The fact that Ξ is a clopen subset of the space Ξ^Γ is obvious. Now the space Ξ^Γ is closed in $\mathbf{X}_{\Lambda_0}$ since any net of $\mathbf{X}_{\Lambda_0}$ supported on Γ and converging for the combinatorial metric admits a limit point pattern also supported on Γ. On the other hand, for each $\Lambda \in \Xi^\Gamma$, the set $\Xi + \gamma$, with γ some point taken in the support of Λ, is an open neighborhood of Λ in $\mathbf{X}_{\Lambda_0}$ which is included in Ξ^Γ: for, if $\Lambda' \in \Xi + \gamma$, then $\underline{\Lambda'} = \underline{\Lambda'} - \gamma + \gamma \subset (\underline{\Lambda'} - \underline{\Lambda'}) + \underline{\Lambda} \subset \langle \underline{\Lambda_0} \rangle + \Gamma = \Gamma$. Thus, Ξ^Γ is also open in $\mathbf{X}_{\Lambda_0}$. □

The following result is more or less folklore: it asserts that under the finite local complexity assumption the hull $\mathbb{X}_{\Lambda_0}$ is the $\mathbb{R}^d$-*suspension* of the subsystem (Ξ^Γ, Γ).

PROPOSITION 5.3. *If Λ_0 is of finite local complexity, there is a topological conjugacy*

$$\begin{aligned} \Xi^\Gamma \times_\Gamma \mathbb{R}^d &\longrightarrow\!\!\!\!\!\rightarrow \mathbb{X} \\ [\Lambda, t]_\Gamma &\longmapsto \Lambda - t \end{aligned}$$

where the left-hand side is the quotient of $\Xi^\Gamma \times \mathbb{R}^d$ under the diagonal Γ-action $(\Lambda, t).\gamma := (\Lambda + \gamma, t + \gamma)$, and equipped with an $\mathbb{R}^d$-action via $[\Lambda, t]_\Gamma \cdot t' := [\Lambda, t + t']_\Gamma$.

Proof. Consider the map $\Xi^\Gamma \times \mathbb{R}^d \longrightarrow \mathbb{X}_{\Lambda_0}$, which to any pair (Λ, t) associates $\Lambda - t$. It is clearly continuous, and is $\mathbb{R}^d$-equivariant if on the respective spaces the $\mathbb{R}^d$-actions are written $(\Lambda, t) \cdot t' := (\Lambda, t + t')$ and $\Lambda \cdot t := \Lambda - t$. Moreover, if for a given $\Lambda \in \mathbb{X}_{\Lambda_0}$ we take a vector t such that $\Lambda + t$ contains 0 in its support, then $\Lambda = (\Lambda + t) - t$ with $\Lambda + t \in \Xi \subset \Xi^\Gamma$, which shows that this map is onto. It is also invariant under the Γ-diagonal action, so taking the quotient under this Γ-action provides the continuous, onto and $\mathbb{R}^d$-equivariant map of the statement. This map is one-to-one: for, if $\Lambda - t = \Lambda' - t'$, then $t' - t \in \underline{\Lambda'} - \underline{\Lambda} \subset \Gamma - \Gamma = \Gamma$, and thus $[\Lambda, t]_\Gamma = [\Lambda + (t' - t), t + (t' - t)]_\Gamma = [\Lambda', t']_\Gamma$. To show that it is a homeomorphism, it suffices to show that the quotient space $\Xi^\Gamma \times_\Gamma \mathbb{R}^d$ is compact. Let Ξ be the canonical transversal of the hull $\mathbb{X}_{\Lambda_0}$. It is a compact subset of Ξ^Γ and, if R_0 stands for some radius of relative density for each multiple set in $\mathbb{X}_{\Lambda_0}$, then $\Xi^\Gamma \times_\Gamma \mathbb{R}^d = [\Xi \times \overline{B}(0, R_0)]_\Gamma$: for, any $(\Lambda, t) \in \Xi^\Gamma \times \mathbb{R}^d$ has some $\gamma \in \Lambda \cap B(t, R_0)$, yielding $(\Lambda - \gamma, t - \gamma) \in \Xi \times \overline{B}(0, R_0)$ with $[\Lambda - \gamma, t - \gamma]_\Gamma = [\Lambda, t]_\Gamma$. This shows that the space $\Xi^\Gamma \times_\Gamma \mathbb{R}^d$ is compact, completing the proof. □

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6. *Construction of a cut and project scheme*

Throughout this section we assume that Λ_0 is a repetitive Meyer multiple set of $\mathbb{R}^d$, and we let Γ be as before. Recall the existence of the group morphism (4) from $\mathbb{R}^d$ into the locally compact Abelian group $\mathbf{X}_{\mathrm{eq}}$. We now set the ingredients in order to construct a cut and project scheme.

PROPOSITION 6.1. *The set* $H := \Pi(\Xi^\Gamma)$ *is a clopen locally compact subgroup of* $\mathbf{X}_{\mathrm{eq}}$ *such that* $H = \overline{\Gamma^*}^{\mathbf{X}_{\mathrm{eq}}}$.

Proof. Let us show first that Ξ^Γ is saturated over the set H with respect to the mapping Π: for, if Λ lies in Ξ^Γ, then any Λ' with $\Pi(\Lambda) = \Pi(\Lambda')$ must be strongly regionally proximal to Λ from Proposition 4.4, and thus by Lemma 4.5 one gets that $\underline{\Lambda'} \subset 3(\underline{\Lambda_0} - \underline{\Lambda_0}) + \underline{\Lambda} \subset \langle \underline{\Lambda_0} \rangle + \Gamma = \Gamma$. This shows that $\Lambda' \in \Xi^\Gamma$, as wished. Now, as the space Ξ^Γ is clopen in $\mathbf{X}_{\Lambda_0}$, we obtain, by definition of the quotient topology, that H is also clopen in $\mathbf{X}_{\mathrm{eq}}$. Since Ξ^Γ is Γ-invariant and Π is $\mathbb{R}^d$-equivariant, we thus obtain that H is Γ-invariant in $\mathbf{X}_{\Lambda_0}$ as well. As Λ_0 is supposed repetitive, the Γ-action is minimal on Ξ^Γ, and thus the translation action of Γ on H is also minimal. Since $\Lambda_0 \in \Xi^\Gamma$ from the very choice of Γ, the neutral element $\mathfrak{o} = \Pi(\Lambda_0)$ of the group $\mathbf{X}_{\Lambda_0}$ lies in H. By minimality, $H = \overline{\mathfrak{o} \cdot \Gamma}^{\mathbf{X}_{\mathrm{eq}}} = \overline{\Gamma^*}^{\mathbf{X}_{\mathrm{eq}}}$. Therefore, H is the closure of a subgroup of $\mathbf{X}_{\mathrm{eq}}$, and thus is itself a locally compact Abelian group. □

From this process, we have extracted a locally compact Abelian group which will be the internal space for our construction of a cut and project scheme. Note that if we consider two different groups with $\Gamma \leqslant \Gamma'$ then we get two different internal spaces, say H^Γ and $H^{\Gamma'}$, in $\mathbf{X}_{\mathrm{eq}}$. It is clear from the previous proposition that H^Γ appears as a clopen subgroup of $H^{\Gamma'}$. The two extreme cases about this construction are when $\Gamma = \langle \underline{\Lambda_0} \rangle$ with associated locally compact Abelian group $H^{\mathfrak{o}}$, and when $\Gamma = \mathbb{R}^d$, which in this case gives rise to the full locally compact Abelian group $\mathbf{X}_{\mathrm{eq}}$. Each resulting H^Γ is then interpolated by these two groups.

Definition 6.2. The restriction $\Gamma \longrightarrow H$ of the morphism (4) on Γ is the *-map associated to Γ.

From Proposition 6.1, the *-map as defined here takes its values in H and has dense range in the latter. We will generically write γ for an element of Γ, as well as γ^* for its image under the *-map. Having constructed an internal space and a $*$-map as well, we now need to find a window in this internal space. This may be done by considering for each index $i \in I$ the associated transversal $\Xi_i := \{\Lambda \in \mathbb{X}_{\Lambda_0} \mid 0 \in \Lambda_i\}$ of the hull. Each is compact and open in $\mathbf{X}_{\Lambda_0}$, and is contained in the canonical transversal Ξ and thus in the subsystem Ξ^Γ. We thus consider the compact subsets of H:

$$W_i := -\Pi(\Xi_i) \quad \text{for all } i \in I.$$

THEOREM 6.3. *The data* $(H, \Sigma, \mathbb{R}^d)$ *with associated diagram*

$$\begin{array}{ccccc} H & \twoheadleftarrow & H \times \mathbb{R}^d & \twoheadrightarrow & \mathbb{R}^d \\ \cup & & \cup & & \cup \\ \Gamma^* & \twoheadleftarrow & \Sigma & \twoheadrightarrow & \Gamma \end{array}$$

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where $\Sigma := \{(\gamma^*, \gamma) \in H \times \mathbb{R}^d \mid \gamma \in \Gamma\}$, *is a cut and project scheme, with quotient* $H \times_\Sigma \mathbb{R}^d$ *isomorphic, as compact group, with* $\mathbb{X}_{\text{eq}}$. *Moreover, the family* $\{W_i\}_{i\in I}$ *is a window in* H.

Proof of the theorem. The map Π restricts in a mapping

$$\Pi^\Gamma : \Xi^\Gamma \longrightarrow\!\!\!\!\!\rightarrow H, \tag{7}$$

which is continuous and proper as Π was, Γ-equivariant and onto since Π was onto and Ξ^Γ is saturated over H. Since it is the restriction of Π, one has $\Pi^\Gamma(\Lambda) = \Pi^\Gamma(\Lambda')$ if and only if Λ and Λ' are strongly regionally proximal.

LEMMA 6.4. *The group* $H \times_\Sigma \mathbb{R}^d$ *obtained as the quotient of* $H \times \mathbb{R}^d$ *by the subgroup* Σ *is a compact Abelian group conjugated with* $\mathbb{X}_{\text{eq}}$ *through the isomorphism*

$$\begin{aligned} H \times_\Sigma \mathbb{R}^d &\longleftrightarrow \mathbb{X}_{\text{eq}} \\ [w, t]_\Sigma &\longleftrightarrow w - t^* \end{aligned}$$

Proof. The mapping Π^Γ naturally provides a factor map from $\Xi^\Gamma \times_\Gamma \mathbb{R}^d$ onto $H \times_\Sigma \mathbb{R}^d$ by letting $[\Lambda, t]_\Gamma$ be mapped onto $[\Pi^\Gamma(\Lambda), t]_\Sigma$. In addition, the map from $H \times \mathbb{R}^d$ to $\mathbb{X}_{\text{eq}}$ mapping (w, t) onto $w - t^*$ is well defined, and an onto continuous group morphism which is obviously Σ-invariant, and thus defines a continuous group morphism from $H \times_\Sigma \mathbb{R}^d$ onto $\mathbb{X}_{\text{eq}}$. This shows that $H \times_\Sigma \mathbb{R}^d$ is a compact (Hausdorff) Abelian group. Now observe that from the conjugacy of $\mathbb{X}_{\Lambda_0}$ with $\Xi^\Gamma \times_\Gamma \mathbb{R}^d$ of Proposition 5.3, one has $H \times_\Sigma \mathbb{R}^d$ as an equicontinuous factor of $\mathbb{X}_{\Lambda_0}$ an thus must also be a factor of $\mathbb{X}_{\text{eq}}$. With this argument, one can show that the morphism from $H \times_\Sigma \mathbb{R}^d$ onto $\mathbb{X}_{\text{eq}}$ is one-to-one, setting the proof. □

Our wish now is to show that the compact subsets W_i have non-empty interior in H, and that they are in fact the closure of their interior. Our strategy is to show that there are point patterns $\Lambda \in \Xi^\Gamma$ where the mapping Π^Γ is *open* (recall that a map $\pi : X \longrightarrow Y$ is open at $x \in X$ if any neighborhood of $x \in X$ is mapped onto a neighborhood of $\pi(x) \in Y$). If we find such point pattern Λ within for instance Ξ_i, which is an open subset of Ξ^Γ, then $\Pi(\Lambda)$ will automatically remain in the interior of $\Pi(\Xi_i) = -W_i$, showing the non-emptiness of the latter interior and thus the non-emptiness of the interior of W_i. The next proposition asserts that such point patterns do exist.

PROPOSITION 6.5. [**26**] *Let* $(\mathbb{X}, \mathbb{R}^d)$ *be a minimal dynamical system over* $\mathbb{X}$, *a compact metric space, and* $\pi : \mathbb{X} \twoheadrightarrow \mathbb{X}_{\text{eq}}$ *the factor map onto its maximal equicontinuous factor. Then there exists a dense residual* $\mathbb{R}^d$*-invariant subset* $\mathbb{X}^0_{\text{eq}} \subseteq \mathbb{X}_{\text{eq}}$ *such that* π *is open at any point of the saturated* $\mathbb{R}^d$*-invariant subset* $\mathbb{X}^0 := \pi^{-1}(\mathbb{X}^0_{\text{eq}}) \subset \mathbb{X}$.

PROPOSITION 6.6. *For* Λ_0 *a repetitive Meyer multiple set of* $\mathbb{R}^d$, *the factor map* $\pi : \mathbb{X}_{\Lambda_0} \twoheadrightarrow \mathbb{X}_{\text{eq}}$ *is open exactly where* $\Pi : \mathbf{X}_{\Lambda_0} \twoheadrightarrow \mathbf{X}_{\text{eq}}$ *is.*

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Proof. Due to the conjugacy maps of Proposition 5.3 and of Lemma 6.4, one has, when applied to the case $\Gamma = \mathbb{R}^d$ so that we have $\Xi^\Gamma = \mathbf{X}_{\Lambda_0}$ and $H = \mathbf{X}_{\mathrm{eq}}$ in this case, a commutative diagram

$$\begin{array}{ccccccc}
\mathbf{X}_{\Lambda_0} & \twoheadleftarrow & \mathbf{X}_{\Lambda_0} \times \mathbb{R}^d & \twoheadrightarrow & \mathbf{X}_{\Lambda_0} \times_\Gamma \mathbb{R}^d & \xleftarrow{\simeq} & \mathbb{X}_{\Lambda_0} \\
\downarrow{\scriptstyle \Pi} & & \downarrow{\scriptstyle \Pi \times \mathrm{id}} & & \downarrow{\scriptstyle \Pi \times_\Gamma \mathrm{id}} & & \downarrow{\scriptstyle \pi} \\
\mathbf{X}_{\mathrm{eq}} & \twoheadleftarrow & \mathbf{X}_{\mathrm{eq}} \times \mathbb{R}^d & \twoheadrightarrow & \mathbf{X}_{\mathrm{eq}} \times_\Gamma \mathbb{R}^d & \xleftarrow{\simeq} & \mathbb{X}_{\mathrm{eq}}
\end{array}$$

Now the right-hand-side horizontal maps are conjugacies, and the center horizontal maps are open maps, as quotient maps through group actions. Obviously, the left-hand-side horizontal maps are open maps, and it easily follows that the right-hand-side vertical map π is open exactly where the left-hand-side map Π is. □

As a consequence of the two previous propositions, the factor map $\Pi^\Gamma : \Xi^\Gamma \twoheadrightarrow H$ admits a dense residual Γ-invariant subset $\Xi^{\Gamma,0} \subseteq \Xi^\Gamma$, which is saturated and with dense image $H^0 \subseteq H$, of elements where Π^Γ is open. This is because Ξ^Γ is saturated over H and both of these are open in their respective spaces $\mathbf{X}_{\Lambda_0}$ and $\mathbf{X}_{\mathrm{eq}}$.

PROPOSITION 6.7. *Each W_i is the closure of its interior in H. Moreover, the non-empty subset $H^0 \subseteq H$ is included in $NS^H := H \backslash [\Gamma^* - \bigcup_{i \in I} \partial W_i]$.*

Proof. Given an open subset U of Ξ^Γ, let us show the equalities

$$\Pi^\Gamma(U \cap \Xi^{\Gamma,0}) = \mathrm{int}(\Pi^\Gamma(U)) \cap H^0 = \Pi^\Gamma(U) \cap H^0. \tag{8}$$

This is true because each $\Lambda \in U \cap \Xi^{\Gamma,0}$ admits U as neighborhood, so have an image $\Pi(\Lambda) \in H^0$ admitting $\Pi(U)$ as neighborhood in H, giving $\Pi(U \cap \Xi^{\Gamma,0}) \subset \mathrm{int}(\Pi^\Gamma(U)) \cap H^0$. The inclusion of the latter in $\Pi^\Gamma(U) \cap H^0$ is obvious and, in turn, any element w of $\Pi^\Gamma(U) \cap H^0$ admits an antecedent element Λ in U, necessarily within $\Xi^{\Gamma,0}$ as the latter is saturated over H^0, which gives $\Pi^\Gamma(U) \cap H^0 \subset \Pi^\Gamma(U \cap \Xi^{\Gamma,0})$.

Let then K be a compact and topologically regular subset of Ξ^Γ: $\mathrm{int}(K) \cap \Xi^{\Gamma,0}$ is dense in K, so the image $\Pi^\Gamma(\mathrm{int}(K)) \cap H^0 \subset \mathrm{int}(\Pi(K)) \cap H^0 \subset \mathrm{int}(\Pi(K))$ is dense in $\Pi(K)$. In particular, $\mathrm{int}(\Pi(K))$ is dense in $\Pi(K)$, so the latter is the closure of its interior. This is in particular true for $K = \Xi_i$ and Ξ, as they are both open and closed in Ξ^Γ, and it follows that W_i and W are compact subsets which are the closure of their interior in H.

Now let us show that H^0 lies in NS^Γ: if w lies in the complementary set of NS^Γ in H, then it exactly means that $w - \gamma^*$ lies within $-\partial W_i$ for some $\gamma \in \Gamma$ and some $i \in I$. Then there exists Λ, with $\Pi(\Lambda) = w$, such that $\Lambda - \gamma \in \Xi_i$. But Λ cannot be in $\Xi^{\Gamma,0}$, because otherwise $\Lambda - \gamma$ would also be in $\Xi^{\Gamma,0}$ and, because $\Lambda - \gamma \in \Xi_i$, we would have $w - \gamma^* \in \mathrm{int}(-W_i) = -\mathrm{int}(W_i)$, which contradict the choice of $w - \gamma^* \in -\partial W_i$. Therefore, the complementary set of NS^Γ lies in the complementary set of H^0 in H^Γ, giving the desired inclusion. □

Now Theorem 6.3 will be proven once we show the last following lemma. The key argument here, added to the fact that W is topologically regular, is the fact that Π is a proper map (Proposition 4.9).

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LEMMA 6.8. $\Sigma := \{(\gamma^*, \gamma) \in H \times \mathbb{R}^d \mid \gamma \in \Gamma\}$ *is a discrete subgroup of* $H \times \mathbb{R}^d$.

Proof. First observe that the Γ-translates of the canonical transversal Ξ form an open cover of the subsystem Ξ^Γ. Hence, the preimage $(\Pi^\Gamma)^{-1}(-W)$, with $-W$ a compact subset of H and Π^Γ being proper, is covered by a finite number of translates of Ξ, say $\Xi + F$, where $F \subset \Gamma$ is finite. Let now $\Lambda \in \Xi^{\Gamma,0}$ be chosen, which can be selected in the open subset $\Xi \subset \Xi^\Gamma$ and for which we denote $w := \Pi^\Gamma(\Lambda) \in H^0 \subseteq H$ for simplicity. Since Λ is a Delone set and F is finite, the difference subset $\underline{\Lambda} - F$ of $\mathbb{R}^d$ admits a radius of uniform discreteness $\delta > 0$, and on the other hand $w \in -\mathring{W}$, so that we can form the open neighborhood of $(0, 0)$ in $H \times \mathbb{R}^d$ given by $(\mathring{W} + w) \times B(0, \delta)$.

Now suppose that there is another $(\gamma^*, \gamma) \in \Sigma$ in $(\mathring{W} + w) \times B(0, \delta)$: then $\gamma^* \in \mathring{W} + w$ and thus $w - \gamma^*$ lies in $-W$, yielding that $\Lambda - \gamma$ lies in $(\Pi^\Gamma)^{-1}(-W)$. Thus, there is some $f \in F$ such that $\Lambda - \gamma \in \Xi + f$, so that $\Lambda - (\gamma + f) \in \Xi$, which means that $\gamma + f$ lies in the support of Λ. This in turn means that γ remains in $\underline{\Lambda} - F$, and from the choice of δ we must get $\gamma = 0$, so that $(\gamma^*, \gamma) = (0, 0)$. This shows that $(0, 0)$ is isolated in Σ, which is consequently discrete in $H \times \mathbb{R}^d$. □

7. *Embedding of repetitive Meyer multiple sets into model multiple sets*

We shall now consider the family of repetitive model multiple sets arising from the cut and project scheme $(H, \Sigma, \mathbb{R}^d)$ and window $\{W_i\}_{i \in I}$ given in Theorem 6.3, as well as the connection with the point pattern Λ_0 we started with. Let us denote $\mathbb{X}_{\mathrm{MS}}$ for the hull of repetitive model multiple sets coming from the data provided by Theorem 6.3, which will be referred to as *the hull of repetitive model multiple sets associated to* Λ_0. The window $\{W_i\}_{i \in I}$ in H, which is provided in Theorem 6.3, is not necessarily irredundant, that is, the compact subgroup

$$\mathcal{R} := \{w \in H \mid W_i + w = W_i \ \forall i \in I\}$$

of H may not be trivial. This does not enable us to parameterize the hull $\mathbb{X}_{\mathrm{MS}}$ by the compact Abelian group $H \times_\Sigma \mathbb{R}^d$, but, as discussed in §2, one equally obtains the same hull $\mathbb{X}_{\mathrm{MS}}$ of point patterns by considering the cut and project scheme where the internal space arises as the quotient group $H_\mathcal{R}$ of H by $\mathcal{R}$. In this way the window becomes irredundant and, by an application of Theorem 2.8, one has a parameterization map

$$\pi_{\mathrm{MS}} : \mathbb{X}_{\mathrm{MS}} \longrightarrow\!\!\!\!\!\rightarrow H_\mathcal{R} \times_\Sigma \mathbb{R}^d$$

which is injective on the subset of non-singular point patterns of $\mathbb{X}_{\mathrm{MS}}$.

THEOREM 7.1. *Let* Λ_0 *be a repetitive Meyer multiple set of* $\mathbb{R}^d$ *with hull* $\mathbb{X}_{\Lambda_0}$, *and let* $\mathbb{X}_{\mathrm{MS}}$ *be its associated hull of repetitive model multiple sets. Then:*

(i) *any point pattern* Λ *of* $\mathbb{X}_{\Lambda_0}$ *admits a* Δ *in* $\mathbb{X}_{\mathrm{MS}}$ *such that* $\Lambda_i \subseteq \Delta_i$ *for each* $i \in I$;

(ii) *for residually many model multiple sets* Δ *of* $\mathbb{X}_{\mathrm{MS}}$, *there exists a strong regional proximality class* C_Δ *included in the saturated subset* $\mathbb{X}^0_{\Lambda_0} \subseteq \mathbb{X}_{\Lambda_0}$ *such that*

$$\Delta_i = \bigcup_{\Lambda \in C_\Delta} \Lambda_i;$$

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(iii) *the group of topological eigenvalues* $\mathcal{E}(\mathbb{X}_{\mathrm{MS}}, \mathbb{R}^d)$ *is a subgroup of* $\mathcal{E}(\mathbb{X}_{\Lambda_0}, \mathbb{R}^d)$ *and the quotient of the latter by the former is given by*

$$\widehat{\mathcal{R}} = \mathcal{E}(\mathbb{X}_{\Lambda_0}, \mathbb{R}^d) \diagup \mathcal{E}(\mathbb{X}_{\mathrm{MS}}, \mathbb{R}^d).$$

Proof. Consider first a point pattern $\Lambda \in \Xi^{\Gamma,0}$. For such a Λ, the element $w_\Lambda := \Pi^\Gamma(\Lambda)$ lies in H^0, and is in turn included in $NS^H := H \backslash [\Gamma^* - \bigcup_{i\in I} \partial W_i]$ by Proposition 6.7. Thus, w_Λ is a non-singular position in H and gives rise to a unique model multiple set in $\mathbb{X}_{\mathrm{MS}}$, given for each $i \in I$ by

$$\mathfrak{P}(\mathring{W}_i + w_\Lambda) = \mathfrak{P}(W_i + w_\Lambda) = \{\gamma \in \Gamma \mid \gamma^* \in W_i + w_\Lambda\}.$$

Now any $\gamma \in \Lambda_i$ yields $\Lambda - \gamma \in \Xi_i$, so that $w_\Lambda - \gamma^* \in -W_i$ and consequently $\gamma^* \in W_i + w_\Lambda$. This means that $\Lambda_i \subseteq \mathfrak{P}(W_i + w_\Lambda)$ for each $i \in I$.

Let now Λ be any point pattern in $\mathbb{X}_{\Lambda_0}$. By repetitivity of Λ_0 and thus by minimality of $(\mathbf{X}_{\Lambda_0}, \mathbb{R}^d)$, we can select some $\Lambda' \in \Xi^{\Gamma,0}$ as well as a sequence of vectors $t_n \in \mathbb{R}^d$ such that $\Lambda' - t_n$ converges to Λ with respect to the combinatorial topology. We showed that there exists some model multiple set $\Delta \in \mathbb{X}_{\mathrm{MS}}$ containing Λ': so, if γ is taken into Λ_i, then it lies within $\Lambda'_i - t_n$, and thus within $\Delta_i - t_n$ upon some rank, which shows that any accumulation point of $\Delta - t_n$ in the compact space $\mathbb{X}_{\mathrm{MS}}$, which does exist, must contain Λ index by index of I.

Let us rapidly prove (iii): the parameterization map of the hull $\mathbb{X}_{\mathrm{MS}}$ is over $H_{\mathcal{R}} \times_\Sigma \mathbb{R}^d$, for which there is the short exact sequence of compact Abelian groups

$$0 \longrightarrow \mathcal{R} \longrightarrow H \times_\Sigma \mathbb{R}^d \longrightarrow H_{\mathcal{R}} \times_\Sigma \mathbb{R}^d \longrightarrow 0$$

which dualizes in a short exact sequence of discrete groups

$$0 \longrightarrow \mathcal{E}(\mathbb{X}_{\mathrm{MS}}, \mathbb{R}^d) \longrightarrow \mathcal{E}(\mathbb{X}_{\Lambda_0}, \mathbb{R}^d) \longrightarrow \widehat{\mathcal{R}} \longrightarrow 0.$$

Now, to prove (ii), recall the existence of a dense residual subset $\mathbb{X}^0_{\mathrm{eq}}$ such that π is open at any point pattern over this subset. One has $\mathbb{X}_{\mathrm{eq}} = H \times_\Sigma \mathbb{R}^d$ by Theorem 6.3 and the image of $\mathbb{X}^0_{\mathrm{eq}}$ in $H_{\mathcal{R}} \times_\Sigma \mathbb{R}^d$ under the quotient map by $\mathcal{R}$, which is an open map, is then a dense residual subset. On the other hand, the subset of non-singular positions in $H_{\mathcal{R}} \times_\Sigma \mathbb{R}^d$ is also dense residual, and thus intersects the former subset in a dense residual subset of $H_{\mathcal{R}} \times_\Sigma \mathbb{R}^d$. This corresponds to a dense residual subcollection of non-singular model multiple sets of $\mathbb{X}_{\mathrm{MS}}$. Pick Δ as one of these: its parameter $\pi_{\mathrm{MS}}(\Delta) = [[w]_{\mathcal{R}}, t]_{\Sigma_{\mathcal{R}}}$ in $H_{\mathcal{R}} \times_{\Sigma_{\mathcal{R}}} \mathbb{R}^d$ is with $\mathcal{R}$-coset $[w]_{\mathcal{R}}$ included in H^0. Let $\Delta' := \Delta + t$. Then the latter is given symbol-wise by

$$\Delta'_i = \mathfrak{P}([W_i]_{\mathcal{R}} + [w]_{\mathcal{R}}) = \mathfrak{P}(W_i + w + \mathcal{R}) = \mathfrak{P}(W_i + w),$$

where w is any representative of the coset $[w]_{\mathcal{R}}$ in $H^0 \subseteq H$. Now the argument at the beginning of this proof shows that any $\Lambda \in \mathbb{X}_{\Lambda_0}$ with $\Pi^\Gamma(\Lambda) = w$ is symbol-wise contained in this model multiple set. On the other hand, the condition $\gamma^* \in W_i + w$ is equivalent to $w - \gamma^* \in -W_i$ and, since $-W_i = \Pi^\Gamma(\Xi_i)$, there must exist a Λ with $\Pi^\Gamma(\Lambda) = w$ such that $\Lambda - \gamma \in \Xi_i$, that is, with $\gamma \in \Lambda_i$. From this, we deduce that

$$\Delta'_i = \bigcup_{\Pi^\Gamma(\Lambda)=[w,0]_\Sigma} \Lambda_i \quad \text{and thus } \Delta_i = \bigcup_{\pi(\Lambda)=[w,t]_\Sigma} \Lambda_i.$$

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Now the set $\pi(\Lambda) = [w, t]_\Sigma$ is a strongly regionally proximal class in $\mathbb{X}^0_{\Lambda_0}$, which finishes the proof. □

The point (ii) shows that point patterns in $\mathbb{X}_{\mathrm{MS}}$ are independent of the choice of the group Γ used to construct the cut and project scheme $(H, \Sigma, \mathbb{R}^d)$ and window $\{W_i\}_{i\in I}$. Moreover, if one starts with a hull of repetitive model multiple sets, then the resulting hull $\mathbb{X}_{\mathrm{MS}}$ is exactly the original one.

8. *Characterization of model multiple sets by almost automorphic dynamical systems*

We apply here the result of Theorem 7.1 in the particular case of a hull with almost automorphic $\mathbb{R}^d$-action. We show in this context that the hull $\mathbb{X}_{\Lambda_0}$ precisely arises as the hull of repetitive model multiple sets formed from the cut and project scheme and window given in Theorem 6.3. It leads to the following dynamical characterization of repetitive model multiple sets.

THEOREM 8.1. *Let Λ_0 be a repetitive Meyer multiple set of $\mathbb{R}^d$. Then Λ is a model multiple set if and only if the dynamical system $(\mathbb{X}_{\Lambda_0}, \mathbb{R}^d)$ is almost automorphic.*

Proof. It is proven in [**26**] that, for an almost automorphic system, the subset of points where the factor map π onto its maximal equicontinuous factor is one-to-one is the set $\mathbb{X}^0_{\Lambda_0}$ where π is open. Let us pick a $\Delta \in \mathbb{X}_{\mathrm{MS}}$ such that point (ii) of Theorem 7.1 holds, that is, for which there is a strong regional proximality class C_Δ in $\mathbb{X}^0_{\Lambda_0}$ such that

$$\Delta_i = \bigcup_{\Lambda\in C_\Delta} \Lambda_i.$$

As π is injective on $\mathbb{X}^0_{\Lambda_0}$, the class C_Δ must consist of a single point pattern $\Lambda \in \mathbb{X}_{\Lambda_0}$, yielding $\Lambda = \Delta$. Minimality of $(\mathbb{X}_{\Lambda_0}, \mathbb{R}^d)$ and $(\mathbb{X}_{\mathrm{MS}}, \mathbb{R}^d)$ ensures that $\mathbb{X}_{\Lambda_0} = \mathbb{X}_{\mathrm{MS}}$, that is, Λ_0 is a repetitive model multiple set of $\mathbb{R}^d$. □

We wish to end this article with a comment on the Meyer property in the case of uncolored point patterns. In [**17**], Kellendonk and Sadun investigated whether a point pattern with hull conjugated with the hull of a Meyer set is itself a Meyer set. It turned out to be false, but still they showed the following theorem.

THEOREM 8.2. [**17**] *Let Λ_0 be a repetitive Delone set of finite local complexity of $\mathbb{R}^d$ with hull $\mathbb{X}_{\Lambda_0}$. Then the dynamical system $(\mathbb{X}_{\Lambda_0}, \mathbb{R}^d)$ admits d independent topological eigenvalues in $\mathbb{R}^d$ if and only if for each $\varepsilon > 0$ there exists a repetitive Meyer set Λ_ε of $\mathbb{R}^d$ such that:*

(i) *the dynamical systems $(\mathbb{X}_{\Lambda_0}, \mathbb{R}^d)$ and $(\mathbb{X}_{\Lambda_\varepsilon}, \mathbb{R}^d)$ are conjugated;*

(ii) *the Hausdorff distance between Λ_0 and Λ_ε is less than ε.*

If a repetitive Delone set with finite local complexity admits an almost automorphic system $(\mathbb{X}_{\Lambda_0}, \mathbb{R}^d)$, then it admits d independent topological eigenvalues in $\mathbb{R}^d$: for, the $\mathbb{R}^d$-action on $\mathbb{X}_{\Lambda_0}$ is locally free by uniform discreteness of any point pattern of this hull,

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and thus must be locally free on the maximal equicontinuous factor in case of almost automorphy. From [**8**, Lemma 2.22], one has by dualization an injective morphism

$$\mathcal{E}(\mathbb{X}_{\Lambda_0}, \mathbb{R}^d) = \widehat{\mathbb{X}_{\text{eq}}} \longrightarrow \widehat{\mathbb{R}^d} = \mathbb{R}^d$$

with relatively dense range in $\mathbb{R}^d$. Thus, $\mathcal{E}(\mathbb{X}_{\Lambda_0}, \mathbb{R}^d)$ as a relatively dense subgroup of $\mathbb{R}^d$ contains d linearly independent eigenfunctions. Consequently, we can provide a statement for repetitive Delone sets with finite local complexity.

THEOREM 8.3. *Let Λ_0 be a repetitive Delone set of finite local complexity of $\mathbb{R}^d$ with hull $\mathbb{X}_{\Lambda_0}$. Then the dynamical system $(\mathbb{X}_{\Lambda_0}, \mathbb{R}^d)$ is almost automorphic if and only if for each $\varepsilon > 0$ there exists a repetitive model set Λ_ε of $\mathbb{R}^d$ such that:*

(i) *the dynamical systems $(\mathbb{X}_{\Lambda_0}, \mathbb{R}^d)$ and $(\mathbb{X}_{\Lambda_\varepsilon}, \mathbb{R}^d)$ are conjugated;*

(ii) *the Hausdorff distance between Λ_0 and Λ_ε is less than ε.*

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