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An almost everywhere central limit theorem

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1. *Introduction and summary*

The purpose of this paper is the proof of an almost everywhere version of the classical central limit theorem (CLT). As is well known, the latter states that for IID random variables $Y_1, Y_2, \ldots$ on a probability space $(\Omega, \mathfrak{A}, P)$ with $EY_1 = 0$, $EY_1^2 = 1$, we have weak convergence of the distributions of $(1/\sqrt{n})\sum_{i=1}^n Y_i$ to the standard normal distribution on $\mathbb{R}$. We recall that weak convergence of finite measures μ_n on a metric space S to a finite measure μ on S is defined to mean that

$$\int_S f d\mu_n \to \int_S f d\mu \tag{1·1}$$

for all bounded, continuous real functions on S. Equivalently, one may require the validity of (1·1) only for bounded, uniformly continuous real functions, or even for all bounded measurable real functions which are μ-a.e. continuous.

If for $x \in \mathbb{R}$ we denote by δ_x the probability measure on $\mathbb{R}$ which assigns its total mass to x, we note that the distribution of $n^{-\frac{1}{2}}S_n$, where $S_n = \sum_{i=1}^n Y_i$, is just $\int_\Omega dP(\omega)\,\delta_{S_n(\omega)/\sqrt{n}}$ i.e. the average of the random measures $\delta_{S_n(\omega)/\sqrt{n}}$ with respect to P.

In this paper we shall form 'time averages' with respect to a logarithmic scale rather than 'space averages' and prove a.e. convergence for the resulting random measures. To be precise, we have the following

THEOREM 1·2. *Let* $Y_1, Y_2, \ldots$ *be IID random variables on* $(\Omega, \mathfrak{A}, P)$ *with* $EY_1 = 0$, $EY_1^2 = 1$, $E|Y_1|^{2+2\delta} < \infty$ *for some* $\delta > 0$. *Let* $S_n = \sum_{k=1}^n Y_k$, *for* $n \geqslant 1$. *Then P-a.e.*

$$\lim_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^{n} k^{-1} \delta_{S_k(\omega)/\sqrt{k}} = N(0,1),$$

where $N(0,1)$ *is the standard normal distribution on* $\mathbb{R}$, *and the convergence is weak convergence of measures on* $\mathbb{R}$.

Rephrasing this theorem, we have outside one single exceptional set of P-probability 0,

$$\lim_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^{n} k^{-1} f\left(\frac{S_k}{\sqrt{k}}\right) = \frac{1}{\sqrt{(2\pi)}} \int_{-\infty}^{+\infty} f(\xi)\, e^{-\frac{1}{2}\xi^2} d\xi, \tag{1·3}$$

for all bounded measurable functions $f: \mathbb{R} \to \mathbb{R}$, which are continuous a.e., or equivalently

$$\lim_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^{n} k^{-1} \chi_A\left(\frac{S_k}{\sqrt{k}}\right) = \frac{1}{\sqrt{(2\pi)}} \int_A e^{-\frac{1}{2}\xi^2} d\xi \tag{1·3′}$$

for all Borel sets $A \subseteq \mathbb{R}$ with $\mathrm{Leb}\,(\partial A) = 0$.

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To compare Theorem 1·2 with the classical CLT one might consider using (1·3) to test a random number generator. One would only have to check one (typical) sequence, rather than many sequences as in tests based on the classical CLT. In [**4**], we shall use a version of such an a.e. CLT to recover the spectrum of a compact Riemannian manifold from the development over time of the energy of one single diffusion path.

In order to get results like (1·3) for $f(S_1/\sqrt{k}, \ldots, S_k/\sqrt{k})$ rather than just for $f(S_k/\sqrt{k})$, we shall actually prove a $C[0,1]$-version of Theorem 1·2, which corresponds to the Donsker version of the classical CLT and immediately implies Theorem 1·2. For background material on the Donsker Theorem we refer the reader to [**8**], pp. 68–84. Following the notation there, we define for a sequence of real numbers $\{s_n, n \geqslant 1\}$ the sequence $\{s_{(n)}, n \geqslant 1\}$ of elements in $C[0,1]$ by setting $s_0 = 0$,

$$s_{(n)}\left(\frac{i}{n}\right) = n^{-\frac{1}{2}} s_i \quad \text{for} \quad i = 0, \ldots, n,$$

and by letting $s_{(n)}$ be linear on $[(i-1)/n, i/n]$ for $i = 1, \ldots, n$. We shall denote by μ_W the Wiener measure on $C[0,1]$, and for $x \in C[0,1]$ let δ_x be the probability measure on $C[0,1]$ which assigns its total mass to x. We have

THEOREM 1·4. *Under the assumptions of Theorem* 1·2, *P-a.e.*

$$\lim_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^{n} k^{-1} \delta_{S_{(k)}(\omega)} = \mu_W,$$

where the convergence is weak convergence of measures on $C[0,1]$.

This theorem can be rephrased thus: outside a single set of P-measure 0, we have

$$\lim_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^{n} k^{-1} \delta_{\phi \circ S_{(k)}(\omega)} = \phi(\mu_W),$$

for all measurable $\phi: C[0,1] \to \mathbb{R}$ which are continuous μ_W-a.e. Here $\phi(\mu_W)$ is the image measure of μ_W under ϕ. Letting in particular $\phi_1(x) = x(1)$, we obtain Theorem 1·2. Letting $\phi_2(x) = \max_{u \in [0,1]} |x(u)|^\alpha$, where $\alpha > 0$, we obtain that P-a.e.

$$\lim_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^{n} k^{-1} \delta_{k^{-\alpha/2} \max\{|S_1|^\alpha, \ldots, |S_k|^\alpha\}} = \phi_2(\mu_W).$$

We may also obtain pointwise asymptotic results for functionals such as the relative frequency of positive S_k's or the last change of sign in $\{S_k, k \leqslant n\}$, functionals which will have to be approximated by functions on $C[0,1]$. To be precise, let $S_0 = 0$,

$$L_n = (\#\{1 \leqslant k \leqslant n; S_k > 0\})/n,$$

$$C_n = \begin{cases} \max\{0 \leqslant j < n; S_j \leqslant 0\} & \text{if} \quad S_n > 0 \\ \max\{0 \leqslant j < n; S_j > 0\} & \text{if} \quad S_n \leqslant 0, \{\} \neq \varnothing \\ 0 & \text{if} \quad S_n \leqslant 0, \{\} = \varnothing. \end{cases}$$

We have the following

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COROLLARY 1·5. *Under the assumptions of Theorem* 1·2, *P-a.e.*

$$\lim_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^{n} k^{-1} \delta_{L_k(\omega)} = AS, \tag{1}$$

$$\lim_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^{n} k^{-1} \delta_{C_k(\omega)/k} = AS, \tag{2}$$

where AS is the arc sine distribution on $[0, 1]$, *and the convergence is weak convergence of measures.*

In order to prove Theorem 1·4, we shall first prove a pointwise CLT for one-dimensional Brownian motion and then use Skorokhod's representation theorem to deal with the case of IID random variables.

So let W be one-dimensional Brownian motion starting at 0, on some probability space $(\Omega, \mathfrak{A}, P)$, and define the $C[0, 1]$-valued random variables $W^{(s)}$, for $s > 0$, by

$$W_u^{(s)} = s^{-\frac{1}{2}} W_{us}, \quad (u \in [0, 1]).$$

By averaging the random point measures $\delta_{W^{(s)}(\omega)}$ on $C[0, 1]$ on a logarithmic time scale, we get the random measures

$$\mu_t(\omega) = \frac{1}{\log t} \int_1^t ds/s\, \delta_{W^{(s)}(\omega)}, \quad (t > 1),$$

and their discrete analogues

$$\nu_n(\omega) = \frac{1}{\log n} \sum_{k=1}^{n} k^{-1} \delta_{W^{(k)}(\omega)}, \quad (n \geqslant 2).$$

We shall prove

THEOREM 1·6. (*a*) $\lim_{t\to\infty} \mu_t = \mu_W$ P-a.e.

(*b*) $\lim_{n\to\infty} \nu_n = \mu_W$ P-a.e.

The convergence in Theorem 1·6 is again weak convergence of measures on $C[0, 1]$.

Again it follows that outside a single set of P-measure 0, for all measurable $\phi: C[0, 1] \to \mathbb{R}$ which are continuous μ_W-a.e.,

$$\lim_{t\to\infty} \frac{1}{\log t} \int_1^t ds/s\, \delta_{\phi \circ W^{(s)}} = \phi(\mu_W), \tag{1}$$

$$\lim_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^{n} k^{-1} \delta_{\phi \circ W^{(k)}} = \phi(\mu_W). \tag{2}$$

Functions ϕ one might want to consider in this context, include the functions

$$x \mapsto x(1), \ \max_{u\in[0,1]} |x(u)|^\alpha, \ \text{Leb}\{u \in [0, 1]; x(u) \geqslant 0\}, \ \max\{u \in [0, 1]; x(u) = 0\}.$$

The proof of Theorem 1·6 relies heavily on the process $X_s = W_s/\sqrt{s}$, $s > 0$. This process is stationary with respect to multiplication on the time scale, which means

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that for $h > 0$, the processes $\{X_s, s > 0\}$ and $\{X_{sh}, s > 0\}$ have the same laws. Stationarity allows extensive use of ergodic theory. We remark that the process $\{X_{e^s}, s \in \mathbb{R}\}$ is the Ornstein–Uhlenbeck process.

In a last section we shall discuss the relationship between our results and results obtained recently by A. Fisher.

2. *The Brownian motion case*

We shall first prove that the measures $\{\mu_t, t \geqslant e\}$ of Theorem 1·6 (a) are tight, P-a.e. Defining $\phi : C[0,1] \to \mathbb{R}$ by $\phi(x) = x(0)$, we see that the μ_t-distribution of ϕ is δ_0 for all $t > 1$, P-a.e., since $W_0^{(s)} = 0$ for all $s > 0$, P-a.e. Therefore, in order to prove tightness of the family $\{\mu_t, t \geqslant e\}$, P-a.e., it is, by theorem 12·3 in [1], sufficient to show that P-a.e., there exists a finite $C(\omega)$ such that for all $u_1, u_2 \in [0,1]$

$$\sup_{t \geqslant e} \int_{C[0,1]} |x(u_1) - x(u_2)|^4 \mu_t(\omega)(dx) \leqslant C(\omega)|u_1 - u_2|^{\frac{3}{2}},$$

or equivalently,

$$\sup_{t \geqslant e} \frac{1}{\log t} \int_1^t ds/s |W_{u_1 s}/\sqrt{s} - W_{u_2 s}/\sqrt{s}|^4 \leqslant C(\omega)|u_1 - u_2|^{\frac{3}{2}}. \tag{2·1}$$

We shall start with the following Lemma 2·2, which we shall prove, for the reader's convenience. This lemma follows also from the general theory of Gaussian processes.

LEMMA 2·2.

$$\sup_{\substack{u \in [0,1] \\ 0 < h < e^{-2}}} |W_{u+h} - W_u| / \sqrt{(h|\log h|)} \in L^p, \quad \textit{for all } p \geqslant 1.$$

Proof. Let

$$Z = \sup_{\substack{u \in [0,1] \\ 0 < h < e^{-2}}} |W_{u+h} - W_u| / \sqrt{(h|\log h|)}.$$

We have for $\lambda \geqslant 0$,

$$\begin{aligned}
P\{Z \geqslant \lambda\} &\leqslant \sum_{k=2}^{\infty} P\left\{ \sup_{\substack{0 \leqslant u \leqslant 1 \\ e^{-(k+1)} < h \leqslant e^{-k}}} |W_{u+h} - W_u| \geqslant \lambda \sqrt{(k+1)}\, e^{-\frac{1}{2}(k+1)} \right\} \\
&\leqslant \sum_{k=2}^{\infty} P\left\{ e^{-\frac{1}{2}k} \sup_{\substack{0 \leqslant u \leqslant 1 \\ e^{-(k+1)} < h \leqslant e^{-k}}} |W_{e^k(u+h)} - W_{e^k u}| \geqslant \lambda \sqrt{(k+1)}\, e^{-\frac{1}{2}(k+1)} \right\} \\
&\leqslant \sum_{k=2}^{\infty} P\left\{ \sup_{\substack{0 \leqslant v \leqslant e^k \\ 0 \leqslant w \leqslant 1}} |W_{v+w} - W_v| \geqslant \lambda\, e^{-\frac{1}{2}} \sqrt{(k+1)} \right\} \\
&\leqslant \sum_{k=2}^{\infty} \sum_{l=0}^{[e^k]} P\left\{ \sup_{\substack{l \leqslant v \leqslant l+1 \\ 0 \leqslant w \leqslant 1}} |W_{v+w} - W_v| \geqslant \lambda\, e^{-\frac{1}{2}} \sqrt{(k+1)} \right\} \\
&\leqslant \sum_{k=2}^{\infty} (e^k + 1) P\left\{ \sup_{\substack{0 \leqslant v \leqslant 1 \\ 0 \leqslant w \leqslant 1}} |W_{v+w} - W_v| \geqslant \lambda\, e^{-\frac{1}{2}} \sqrt{(k+1)} \right\} \\
&\leqslant \sum_{k=2}^{\infty} (e^k + 1) P\left\{ 2 \sup_{0 \leqslant v \leqslant 2} |W_v| \geqslant \lambda\, e^{-\frac{1}{2}} \sqrt{(k+1)} \right\}
\end{aligned}$$

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$$\leqslant 4 \sum_{k=2}^{\infty} (e^k+1) P\left\{W_1 \geqslant \tfrac{1}{4}\sqrt{2}\, e^{-\frac{1}{2}}\lambda \sqrt{(k+1)}\right\}$$

$$\leqslant c_1 \sum_{k=2}^{\infty} e^k \exp\left\{-c_2 \lambda^2 k\right\}.$$

Letting $\lambda_0 = \sqrt{(2/c_2)}$, we have for $\lambda \geqslant \lambda_0$

$$c_2 \lambda^2 k - k \geqslant \tfrac{1}{2} c_2 \lambda^2 k.$$

It follows that for $\lambda \geqslant \lambda_0$

$$P\{Z \geqslant \lambda\} \leqslant c_1 \sum_{k=2}^{\infty} \exp\{-\tfrac{1}{2} c_2 \lambda^2 k\}$$

$$\leqslant c_1 \int_1^{\infty} dt \exp\{-\tfrac{1}{2} c_2 \lambda^2 t\} \leqslant c_1\, e^{-\frac{1}{2} c_2 \lambda^2}.$$

This implies $\int_0^{\infty} \lambda^{p-1} P\{Z \geqslant \lambda\} d\lambda < \infty$ for all $p \geqslant 1$.

Remark 2·3. The preceding lemma implies immediately that for all $\epsilon < \frac{1}{2}$ and all $p \geqslant 1$,

$$\sup_{0 \leqslant u_1 < u_2 \leqslant 1} (u_2 - u_1)^{-\epsilon} |W_{u_2} - W_{u_1}| \in L^p.$$

The inequality (2·1), hence tightness of the family $\{\mu_t, t \geqslant e\}$, P-a.e., now follows immediately from the following lemma.

LEMMA 2·4. *For all* $p \geqslant 1$,

$$\sup_{t \geqslant e} \sup_{0 \leqslant u_1 < u_2 \leqslant 1} \frac{1}{\log t} \frac{1}{(u_2 - u_1)^{\frac{3}{2}}} \int_1^t ds/s\, |W_{u_1 s} - W_{u_2 s}|^4 / s^2 \in L^p.$$

Proof. The random variable in the lemma is majorized by

$$Z^* = \sup_{t \geqslant e} \frac{1}{\log t} \int_1^t ds/s\, Z_s = \sup_{t \geqslant 1} \frac{1}{t} \int_1^{e^t} ds/s\, Z_s,$$

where

$$Z_s = \sup_{0 \leqslant u_1 < u_2 \leqslant 1} \frac{|W_{u_1 s} - W_{u_2 s}|^4}{s^2 (u_2 - u_1)^{\frac{3}{2}}}.$$

Moreover,

$$Z_s = \sup_{0 \leqslant u_1 < u_2 \leqslant 1} (u_2 - u_1)^{-\frac{3}{2}} |\sqrt{u_1}\, X_{u_1 s} - \sqrt{u_2}\, X_{u_2 s}|^4,$$

where $X_s = W_s/\sqrt{s}$ for $s > 0$. As $\{X_s, s > 0\}$ and $\{X_{sh}, s > 0\}$ have the same laws for $h > 0$, $\{Z_s, s > 0\}$ and $\{Z_{sh}, s > 0\}$ have the same laws for $h > 0$. By Remark 2·3, $Z_1 \in L^p$ for $p \geqslant 1$. It follows by the dominated ergodic theorem that

$$\sup_{n \geqslant 1} \frac{1}{n} \int_1^{e^n} ds/s\, Z_s = \sup_{n \geqslant 1} \frac{1}{n} \sum_{k=0}^{n-1} \int_k^{k+1} du\, Z_{e^u} \in L^p,$$

for $p \geqslant 1$. Now we have for $t \geqslant 1$,

$$\frac{1}{t} \int_1^{e^t} ds/s\, Z_s \leqslant \frac{[t]+1}{t} \cdot \frac{1}{[t]+1} \int_1^{e^{[t]+1}} ds/s\, Z_s,$$

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which implies $$Z^* \leqslant 2 \sup_{n \geqslant 1} \frac{1}{n} \int_1^{e^n} ds/s\, Z_s,$$

hence $Z^* \in L^p$ for $p \geqslant 1$.

Proof of Theorem 1·6(*a*). Let $k \in \mathbb{N}$, $0 < u_1 < \ldots < u_k \leqslant 1$, $\alpha_1, \ldots, \alpha_k \in \mathbb{R}$. Let

$$\phi(x) = e^{i[\alpha_1 x(u_1) + \ldots + \alpha_k x(u_k)]}. \tag{2·5}$$

We want to show that P-a.e.

$$\lim_{t \to \infty} \int_{C[0,1]} \phi(x)\, \mu_t(\omega)(dx) = \int_{C[0,1]} \phi(x)\, \mu_W(dx). \tag{2·6}$$

Letting

$$\Psi_s = \exp\left(i[\alpha_1 \sqrt{u_1}\, X_{u_1 s} + \ldots + \alpha_k \sqrt{u_k}\, X_{u_k s}]\right), \quad (X_s = W_s/\sqrt{s}),$$

one can easily check that

$$\int_{C[0,1]} \phi(x)\, \mu_t(\omega)(dx) = \frac{1}{\log t} \int_1^t ds/s\, \Psi_s.$$

Since the laws of $\{X_s, s > 0\}$ and $\{X_{sh}, s > 0\}$ coincide for $h > 0$, it follows that the laws of $\{\Psi_s, s > 0\}$ and $\{\Psi_{sh}, s > 0\}$ coincide for $h > 0$. The ergodic theorem and the fact that Brownian motion has a trivial tail-field imply that P-a.e.

$$\lim_{n \to \infty} \frac{1}{n} \int_1^{e^n} ds/s\, \Psi_s = E\Psi_1.$$

Moreover

$$\left| \frac{1}{t} \int_1^{e^t} ds/s\, \Psi_s - \frac{[t]}{t} \cdot \frac{1}{[t]} \int_1^{e^{[t]}} ds/s\, \Psi_s \right| = \frac{1}{t} \left| \int_{e^{[t]}}^{e^t} ds/s\, \Psi_s \right| \leqslant \frac{1}{t} \int_{e^{[t]}}^{e^{[t]+1}} ds/s\, |\Psi_s| \leqslant \frac{1}{t},$$

so that P-a.e. $$\lim_{t \to \infty} \frac{1}{t} \int_1^{e^t} ds/s\, \Psi_s = E\Psi_1.$$

Obviously $E\Psi_1 = \int_{C[0,1]} \phi(x)\, \mu_W(dx)$. This proves (2·6). It follows that, P-a.e., (2·6) is valid for all ϕ, defined by (2·5), if k runs through $\mathbb{N}$, $0 < u_1 < \ldots < u_k \leqslant 1$ run through the rationals in $[0,1]$ and $\alpha_1, \ldots, \alpha_k$ run through the rationals in $\mathbb{R}$. In view of the tightness of the family $\{\mu_t, t \geqslant e\}$, proved earlier, this completes the proof of Theorem 1·6(*a*).

We now turn to the proof of Theorem 1·6(*b*) and define the $C[0,1]$-valued process $\{Z_s, s \geqslant 1\}$ by

$$Z_s = W^{(k)} \quad \text{for} \quad k \leqslant s < k+1. \tag{2·7}$$

LEMMA 2·8. *We have* P-a.e.,

$$\lim_{s \to \infty} \|Z_s - W^{(s)}\|_{C[0,1]} = 0. \tag{2·9}$$

Proof. We have to prove that P-a.e.

$$\lim_{n \to \infty} \sup_{0 \leqslant h \leqslant 1} \|W^{(n)} - W^{(n+h)}\|_{C[0,1]} = 0.$$

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Now

$$\|W^{(n)}-W^{(n+h)}\|_{C[0,1]} = \sup_{u\in[0,1]}\left|\frac{W_{un}}{\sqrt{n}}-\frac{W_{u(n+h)}}{\sqrt{(n+h)}}\right|$$

$$\leqslant \frac{1}{\sqrt{n}}\sup_{u\in[0,1]}|W_{u(n+h)}-W_{un}|+\sup_{u\in[0,1]}|W_{u(n+h)}|\left\{\frac{1}{\sqrt{n}}-\frac{1}{\sqrt{(n+1)}}\right\}$$

$$\leqslant \frac{1}{\sqrt{n}}\sup_{u\in[0,1]}|W_{u(n+h)}-W_{un}|+\frac{1}{2}\frac{1}{n^{\frac{3}{2}}}\sup_{u\in[0,1]}|W_{u(n+h)}|.$$

It remains to show that P-a.e.

$$\lim_{n\to\infty}\frac{1}{\sqrt{n}}\sup_{u\in[0,1]}\sup_{h\in[0,1]}|W_{u(n+h)}-W_{un}|=0 \tag{2·10}$$

and

$$\lim_{n\to\infty}\frac{1}{n^{\frac{3}{2}}}\sup_{u\in[0,1]}\sup_{h\in[0,1]}|W_{u(n+h)}|=0. \tag{2·11}$$

But (2·10) follows from the Borel–Cantelli Lemma and the estimate

$$P\left\{\sup_{u\in[0,1]}\sup_{h\in[0,1]}|W_{u(n+h)}-W_{un}|\geqslant \epsilon n^{\frac{1}{2}}\right\}$$

$$\leqslant P\left\{\sup_{v\in[0,n]}\sup_{\delta\in[0,1]}|W_{v+\delta}-W_v|\geqslant \epsilon n^{\frac{1}{2}}\right\}$$

$$\leqslant \sum_{l=0}^{n-1}P\left\{\sup_{v\in[l,l+1]}\sup_{\delta\in[0,1]}|W_{v+\delta}-W_v|\geqslant \epsilon n^{\frac{1}{2}}\right\}$$

$$= nP\left\{\sup_{v\in[0,1]}\sup_{\delta\in[0,1]}|W_{v+\delta}-W_v|\geqslant \epsilon n^{\frac{1}{2}}\right\}$$

$$\leqslant nP\left\{2\sup_{v\in[0,2]}|W_v|\geqslant \epsilon n^{\frac{1}{2}}\right\}$$

$$\leqslant 4nP\{W_1\geqslant \tfrac{1}{4}\sqrt{2}\,\epsilon n^{\frac{1}{2}}\}$$

$$\leqslant cn\,e^{-\frac{1}{16}\epsilon^2 n}.$$

As for (2·11), we use the estimate

$$\sup_{u\in[0,1]}\sup_{h\in[0,1]}|W_{u(n+h)}|\leqslant \sup_{0\leqslant s\leqslant n+1}|W_s|.$$

Moreover, P-a.e., for $n\geqslant e^2$

$$\frac{1}{\sqrt{(2n\log\log n)}}\sup_{e^2\leqslant s\leqslant n}|W_s|\leqslant \sup_{e^2\leqslant s}\frac{|W_s|}{\sqrt{(2s\log\log s)}}<\infty.$$

This implies (2·11).

The following lemma is the analogue of lemma 1·129 in [8].

LEMMA 2·12. (*a*) *If* $X, Z:\mathbb{R}^+\to C[0,1]$ *are measurable and are such that* $\lim_{s\to\infty}\|X_s-Z_s\|_{C[0,1]}=0$, *then for all bounded, uniformly continuous functions* $f: C[0,1]\to\mathbb{R}$,

$$\lim_{t\to\infty}\left\{\frac{1}{\log t}\int_1^t ds/s f(X_s)-\frac{1}{\log t}\int_1^t ds/s f(Z_s)\right\}=0.$$

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(*b*) *If* $X, Z: \mathbb{N} \to C[0,1]$ *are measurable and are such that* $\lim_{n\to\infty} \|X_n - Z_n\|_{C[0,1]} = 0$, *then for all bounded, uniformly continuous functions* $f: C[0,1] \to \mathbb{R}$,

$$\lim_{n\to\infty} \left\{ \frac{1}{\log n} \sum_{k=1}^{n} k^{-1} f(X_k) - \frac{1}{\log n} \sum_{k=1}^{n} k^{-1} f(Z_k) \right\} = 0.$$

The proof is simple and left to the reader.

Proof of Theorem 1·6(*b*). According to Theorem 1·6(*a*), we have P-a.e.

$$\lim_{n\to\infty} \frac{1}{\log n} \int_1^n ds/s f(W^{(s)}) = \int_{C[0,1]} f(x)\, d\mu_W(x), \tag{2·13}$$

for all bounded, continuous $f: C[0,1] \to \mathbb{R}$. From Lemmas 2·8 and 2·12(*a*) we conclude for the process $\{Z_s, s \geqslant 1\}$, defined by (2·7), that outside the two exceptional sets for (2·9) and (2·13), both of P-measure 0,

$$\lim_{n\to\infty} \frac{1}{\log n} \int_1^n ds/s f(Z_s) = \int_{C[0,1]} f(x)\, d\mu_W(x), \tag{2·14}$$

for all bounded and uniformly continuous functions $f: C[0,1] \to \mathbb{R}$, hence for all bounded and continuous functions $f: C[0,1] \to \mathbb{R}$. This however implies that P-a.e.

$$\lim_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^{n} k^{-1} f(W^{(k)}) = \int_{C[0,1]} f(x)\, d\mu_W(x) \tag{2·15}$$

for all bounded and continuous functions $f: C[0,1] \to \mathbb{R}$, since

$$\int_1^n ds/s f(Z_s) = \sum_{k=1}^{n-1} f(W^{(k)}) \log(1 + 1/k).$$

3. *The IID variables case*

We recall a theorem of Marcinkiewicz–Zygmund, as formulated e.g. as theorem 2 in [**5**], §5·2. This theorem states that for IID random variables $X_1, X_2, \ldots$ on some probability space $(\Omega, \mathfrak{A}, P)$ for which $E|X_1|^{1+\delta} < \infty$ for some $\delta \in (0,1)$, we have P-a.e.

$$\lim_{n\to\infty} \left(\sum_{i=1}^{n} X_i - nEX_1 \right) \Big/ n^{1-\rho} = 0, \tag{3·1}$$

where

$$\rho = \frac{\delta}{1+\delta}. \tag{3·2}$$

Equation (3·1) implies in particular that P-a.e.

$$\sup_{n\geqslant 1} \left| \sum_{i=1}^{n} X_i - nEX_1 \right| \Big/ n^{1-\rho} < \infty. \tag{3·3}$$

Remark. Under the above assumptions, one can also prove that for all $\rho \in (0, \delta/1+\delta)$ there exists C such that for $\lambda > 0$

$$P\left\{ \sup_{n\geqslant 1} \left| \sum_{i=1}^{n} X_i - nEX_1 \right| \Big/ n^{1-\rho} \geqslant \lambda \right\} \leqslant \frac{C}{\lambda^{1+\delta}} E|X_1|^{1+\delta}.$$

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The proof is based on the estimate

$$E\left|\sum_{i=1}^{n} X_i - nEX_1\right|^{1+\delta} \leqslant CnE|X_1 - EX_1|^{1+\delta}.$$

The corresponding result for certain mixing random variables is used in [**4**].

Assume now that on $(\Omega, \mathfrak{A}, P)$ there are defined a one-dimensional Brownian motion W and random variables $0 = \tau_0 < \tau_1 < \tau_2 < \dots$ such that

(3·4) (1), $\tau_k - \tau_{k-1}$, $k \geqslant 1$, are IID random variables;

(3·4) (2), $E\tau_1 = 1$;

(3·4) (3), $E\tau_1^{1+\delta} < \infty$ for some $\delta \in (0, 1)$.

We shall prove

LEMMA 3·5. $$\lim_{k\to\infty} |W_{\tau_k} - W_k|/\sqrt{k} = 0, \quad P\text{-a.e.}$$

Proof. Let ρ be as in (3·2). It follows from (3·3) that, for all $\epsilon > 0$, we may choose $\lambda > 0$ such that

$$P\left\{\sup_{k\geqslant 1} \frac{|\tau_k - k|}{k^{1-\rho}} \geqslant \lambda\right\} < \epsilon.$$

As $\epsilon > 0$ is arbitrary, it suffices to prove that P-a.e.

$$\lim_{n\to\infty} |W_{\tau_n} - W_n|/\sqrt{n} = 0,$$

on the set $$\Omega_\lambda = \left\{\sup_{k\geqslant 1} |\tau_k - k|/k^{1-\rho} < \lambda\right\}.$$

Now, for $\delta > 0$ and $n \in \mathbb{N}$, we have

$$\begin{aligned} a_n(\lambda) &\underset{\text{def}}{=} P\left\{|W_{\tau_n} - W_n| \geqslant \delta\sqrt{n}, \Omega_\lambda\right\} \\ &\leqslant P\left\{\sup_{|s-n|<\lambda n^{1-\rho}} |W_s - W_n| \geqslant \delta\sqrt{n}\right\}. \end{aligned}$$

Distinguishing the cases $n < s$ and $s < n$, we see that

$$a_n(\lambda) \leqslant 2P\left\{\sup_{0<s<\lambda n^{1-\rho}} |W_s| \geqslant \delta\sqrt{n}\right\}.$$

To be more precise, in the first case we use the fact that $\{W_s, s \geqslant 0\}$ and $\{W_{s+n} - W_n, s \geqslant 0\}$ have the same laws, and in the second case we use the fact that $\{W_s, 0 \leqslant s \leqslant n\}$ and $\{W_{n-s} - W_n, 0 \leqslant s \leqslant n\}$ have the same laws.

As $\{\sqrt{(\lambda n^{1-\rho})}\, W_t, t \geqslant 0\}$ and $\{W_{\lambda n^{1-\rho} t}, t \geqslant 0\}$ have the same laws, it follows that

$$a_n(\lambda) \leqslant 2P\left\{\sup_{0<t<1} |W_t| \geqslant \delta\lambda^{-\frac{1}{2}} n^{\rho/2}\right\} \leqslant 8P\{W_1 \geqslant \delta\lambda^{-\frac{1}{2}} n^{\rho/2}\} \leqslant C\exp\{-\tfrac{1}{2}\delta^2\lambda^{-1}n^{\rho}\}.$$

It follows that $\Sigma_{n=1}^{\infty} a_n(\lambda) < \infty$, and the proof is finished by the Borel–Cantelli Lemma.

Proof of Theorem 1·4. By a theorem of Skorokhod (see e.g. [**8**]), there exists a probability space which supports a Brownian motion W and random variables $0 = \tau_0 < \tau_1 < \tau_2, \dots$, satisfying (3·4) (1) and (3·4) (2), such that the sequence

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$\{W_{\tau_1}, W_{\tau_2}, \ldots\}$ has the same law as the sequence $\{S_1, S_2, \ldots\}$. Here $S_k = \Sigma_{i=1}^{k} Y_i$. We shall henceforth identify $S_1, S_2, \ldots$ and $W_{\tau_1}, W_{\tau_2}, \ldots$. Moreover τ_1 can be constructed as the first exit time of W from a bounded random interval, independent of W. (See [8].) Applying a Burkholder–Gundy inequality (see e.g. [6], theorem VII, 92) and the maximal inequality to the martingale $W_{t\wedge\tau_1}$, we have

$$E(t\wedge\tau_1)^{1+\delta} \leqslant C_1 E \sup_{s\leqslant t} |W_{s\wedge\tau_1}|^{2+2\delta} \leqslant C_2 E|W_{t\wedge\tau_1}|^{2+2\delta}.$$

For the special construction of τ_1 mentioned above,

$$E|W_{t\wedge\tau_1}|^{2+2\delta} \leqslant E|W_{\tau_1}|^{2+2\delta} = E|Y_1|^{2+2\delta},$$

which is easily verified if τ_1 is the first exit time of W from a fixed bounded interval. Thus

$$E(t\wedge\tau_1)^{1+\delta} \leqslant CE|Y_1|^{2+2\delta}.$$

Letting $t\to\infty$, we conclude that τ_1 satisfies (3·4) (3). Hence Lemma 3·5 is valid for W and τ_k of the Skorokhod representation above. Let $W^{(n)}, S_{(n)}$ be as in §1, and define $Z^n : \Omega \to C[0,1]$ by setting $Z^n(i/n) = W_i/\sqrt{n}$ for $i = 0, \ldots, n$, and letting Z^n be linear on each interval $[(i-1)/n, i/n]$, $i = 1, \ldots, n$. In view of Theorem 1·6(*b*) and Lemma 2·12(*b*), it is sufficient to prove that P-a.e.

$$\lim_{n\to\infty} \|W^{(n)} - S_{(n)}\|_{C[0,1]} = 0. \tag{3·6}$$

In order to prove (3·6), it is sufficient to show that P-a.e.

$$\lim_{n\to\infty} \|W^{(n)} - Z^n\|_{C[0,1]} = 0 \tag{3·7}$$

and

$$\lim_{n\to\infty} \|Z^n - S_{(n)}\|_{C[0,1]} = 0. \tag{3·8}$$

Now (3·7) follows by a minor modification of the proof of (1·131) in [8]: for all $\epsilon > 0$,

$$P\{\|W^{(n)} - Z^n\|_{C[0,1]} \geqslant \epsilon\} \leqslant \sum_{k=0}^{n-1} P\left\{\sup_{k\leqslant t\leqslant k+1} |W_t - W_k| \geqslant \tfrac{1}{2}\epsilon\sqrt{n}\right\}$$
$$= nP\left\{\sup_{0\leqslant t\leqslant 1} |W_t| \geqslant \tfrac{1}{2}\epsilon\sqrt{n}\right\} \leqslant 4nP\{W_1 \geqslant \tfrac{1}{2}\epsilon\sqrt{n}\} \leqslant Cne^{-\frac{1}{8}\epsilon^2 n}.$$

By the Borel–Cantelli Lemma we have (3·7).

Equation (3·8) follows from (3·5). Indeed,

$$\|Z^n - S_{(n)}\|_{C[0,1]} = \sup_{1\leqslant k\leqslant n} (1/\sqrt{n})|W_{\tau_k} - W_k|,$$

and the right side converges to 0 if and only if $(1/\sqrt{k})|W_{\tau_k} - W_k|$ converges to 0.

Proof of Corollary 1·5. Define $\phi_3, \phi_4 : C[0,1] \to \mathbb{R}$ by

$$\phi_3(x) = \text{Lebesgue measure } \{u\in[0,1];\ x(u) > 0\}$$

and

$$\phi_4(x) = \begin{cases} \sup\{u\in[0,1];\ x(u)\leqslant 0\} & \text{if } x(1) > 0 \\ \sup\{u\in[0,1];\ x(u) > 0\} & \text{if } x(1)\leqslant 0. \end{cases}$$

Here $\sup\{\,\} = 0$, if $\{\,\} = \varnothing$.

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The functionals ϕ_3, ϕ_4 are μ_W-a.e. continuous. (See e.g. pp. 78–79 in [**8**].) It follows from Theorem 1·4 that P-a.e.

$$\lim_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^{n} k^{-1} \delta_{\phi_i \circ S_{(k)}(\omega)} = \phi_i(\mu_W) \quad (i = 3, 4).$$

Now P-a.e.,

$$\lim_{n\to\infty} |\phi_3 \circ S_{(n)} - L_n| = 0, \quad \lim_{n\to\infty} |\phi_4 \circ S_{(n)} - n^{-1} C_n| = 0 \quad \text{and} \quad \phi_3(\mu_W) = \phi_4(\mu_W) = AS$$

(see pp. 78–79 in [**8**]). The proof is completed by applying the analogue of Lemma 2·12(*b*) for $X, Z: \mathbb{N} \to \mathbb{R}$.

Remark 3·9. According to a theorem of Prohorov, a family of probability measures on a separable complete metric space is relatively compact if and only if it is tight (cf. theorems 6·1 and 6·2 in [**1**]). It follows that not only the family $\{\mu_t, t \geqslant e\}$ in Theorem 1·6(*a*) is tight P-a.e., but also the family $\{\nu_n, n \geqslant 2\}$ in Theorem 1·6(*b*), as well as the families of random measures in Theorems 1·2 and 1·4.

Remark 3·10. It is possible to give a direct proof of Theorem 1·2, avoiding the tightness argument which appears to be indispensable in the proof of Theorem 1·4. In Theorem 1·2 we are considering finite measures μ_n, μ on $S = \mathbb{R}$, and in this case μ_n converges weakly to μ if and only if $\mu_n(\mathbb{R}) \to \mu(\mathbb{R})$ and (1·1) holds for all continuous $f: S \to \mathbb{R}$ with compact support. (This equivalence is not valid for finite measures μ_n, μ on $S = C[0, 1]$, where any continuous $f: S \to \mathbb{R}$ with compact support vanishes identically.) Thus, for the proof of Theorem 1·2, it suffices to prove that (1·3) holds outside a set of probability 0, for all f in the class of continuous real functions on $\mathbb{R}$ with compact support. This class, endowed with the sup-norm, is separable. It therefore suffices to prove that, for a fixed function $f: \mathbb{R} \to \mathbb{R}$ which is uniformly continuous and bounded, (1·3) holds P-a.e. Now by the ergodic theorem we have

$$\lim_{n\to\infty} 1/n \int_0^n ds f(W_{e^s}/\sqrt{(e^s)}) = Ef(W_1) \quad (P\text{-a.e.})$$

and hence

$$\lim_{t\to\infty} \frac{1}{\log t} \int_1^t ds/s f(W_s/\sqrt{s}) = Ef(W_1) \quad (P\text{-a.e.}).$$

We then conclude by Lemmas 2·8 and 3·5 that the left side can be replaced by the left side of (1·3).

Remark 3·11. It is possible to prove the validity of (1·3) P-a.e. for certain unbounded continuous functions $f: \mathbb{R} \to \mathbb{R}$, such as $f(x) = |x|^{2p}$, where $p > 0$. The case $p = 1$ gives for IID random variables $Y_1, Y_2, \ldots$ such that $E|Y_1|^{2+2\delta} < \infty$ for some $\delta > 0$ the variance formula

$$\lim_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^{n} k^{-1} |S_k - kEY_1|^2/k = \operatorname{Var} Y_1 \quad (P\text{-a.e.}). \tag{3·12}$$

It appears to be an interesting problem to determine the rate of convergence in the a.e. results which we have proved.

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4. *Comments on previous* a.e. CLT *results*

Weak versions of a.e. CLT's, at least for one-dimensional Brownian motion, are not new. For example, in [3], p. 314, l. 7, we concluded from the stationarity of the Ornstein–Uhlenbeck process $\{W_{e^s}/\sqrt{(e^s)}, s\in\mathbb{R}\}$ and the ergodic theorem, that for $\epsilon > 0$ and $c\in\mathbb{R}$, P-a.e.

$$\lim_{r\to 0} -\frac{1}{\log r}\int_r^\epsilon d\rho/\rho^2 \operatorname{Leb}\{s\in(0,\rho);\, W_s/\sqrt{s}\geqslant c\} = \frac{1}{\sqrt{(2\pi)}}\int_c^\infty e^{-\frac{1}{2}u^2}du, \tag{4·1}$$

which is equivalent (by integration by parts) to

$$\lim_{r\to 0} -\frac{1}{\log r}\int_r^\epsilon d\rho/\rho\, \chi_{(c,\infty)}(W_\rho/\sqrt{\rho}) = \frac{1}{\sqrt{(2\pi)}}\int_c^\infty e^{-\frac{1}{2}u^2}du. \tag{4·1′}$$

This law is an a.e. CLT for W at 0.

Stationarity of $\{W_{e^s}/\sqrt{(e^s)}, s\in\mathbb{R}\}$ as well as logarithmic (and $\log_2$-) averaging had been used previously in [2] to obtain a.e. results which are not available without averaging.

In [7] Fisher gives two a.e. CLT results for one-dimensional Brownian motion, which we shall now discuss.

(*a*) *Fisher's first CLT result*

This result ([7], p. 215, ll. 4–5) is in terms of averages. It states (without proof) that for $f\in L^\infty(\mathbb{R})$, P-a.e. the so-called order two (and higher) averaging methods, applied to the function $F_\omega\colon \mathbb{R}^+\to\mathbb{R}$, $s\mapsto f(W_s(\omega)/\sqrt{s})$, converge to $N(0,1)(f)$. As

$$F\mapsto \lim_{t\to\infty}\frac{1}{\log t}\int_1^t ds/s\, F(s)$$

is a particular averaging method of order two (see below), this includes the following result: for $f\in L^\infty(\mathbb{R})$, P-a.e.

$$\lim_{t\to\infty}\frac{1}{\log t}\int_1^t ds/s\, f(W_s/\sqrt{s}) = N(0,1)(f). \tag{4·2}$$

This latter result is the dual of (4·1′) (if $f=\chi_{(c,\infty)}$), and like (4·1′) it follows immediately from the stationarity of the process $\{W_{e^s}/\sqrt{(e^s)}, s\in\mathbb{R}\}$ and the ergodic theorem.

We shall show that Fisher's apparently more general result on general order two (and higher) averaging methods can be deduced from (4·2). The non-probabilistic Theorem 4·3 below will show that, for any $F\in L^\infty(\mathbb{R})$ for which

$$\lim_{t\to\infty}\frac{1}{\log t}\int_1^t ds/s\, F(s) = c,$$

we have convergence to c, of *all* order two (and higher) averaging methods. Thus the general result on p. 215, ll.4–5 in [7] follows from (4·2) in a deterministic way.

In order to formulate Theorem 4·3 we recall the definition of the averaging operators used in [7]. Following [7], we let

$$P_1(\mathbb{R}) = \left\{\Phi\in L^1(\mathbb{R});\, \Phi\geqslant 0, \int\Phi\, ds = 1\right\}.$$

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The operators $E, E^{-1}: L^\infty(\mathbb{R}) \to L^\infty(\mathbb{R})$ are defined by $(EF)(s) = F(e^s)$ for $s \in \mathbb{R}$, $(E^{-1}F)(s) = F(\log s)$ for $s > 0$ and $= 0$ for $s \leqslant 0$. For $\Phi \in P_1(\mathbb{R})$, the operators $A_\Phi^n: L^\infty(\mathbb{R}) \to L^\infty(\mathbb{R})$, $n \geqslant 0$, are defined by $A_\Phi^0 F = \Phi * F$, $A_\Phi^{n+1} = E^{-1} A_\Phi^n E$. The operator A_Φ^n is called an averaging operator of order n. For $n \geqslant 1$ and $F \in L^\infty(\mathbb{R})$, the function $A_\Phi^n F$ depends only on the values of F on $\mathbb{R}^+$. So $A_\Phi^n F$ makes sense for $n \geqslant 1$ and $F \in L^\infty(\mathbb{R}^+)$. Letting $\Phi_0(s) = e^{-s}\chi_{(0,\infty)}(s)$, we have for sufficiently large t,

$$(A_{\Phi_0}^1 F)(t) = \frac{1}{t}\int_0^t F(s)\,ds, \quad (A_{\Phi_0}^2 F)(t) = \frac{1}{\log t}\int_1^t ds/s\, F(s), \quad \text{etc.}$$

In the following theorem $\hat{}: L^1(\mathbb{R}) \to L^\infty(\mathbb{R})$ denotes the Fourier transform

$$\hat{\Phi}(\alpha) = \int \Phi(s)\, e^{2\pi i \alpha s}\, ds \quad (\alpha \in \mathbb{R}).$$

THEOREM 4·3. *Let $F \in L^\infty(\mathbb{R}^+)$, $c \in \mathbb{R}$. If $\lim_{t\to\infty}(A_\Phi^n F)(t) = c$ for some $n \in \mathbb{N}$ and some $\Phi \in P_1(\mathbb{R})$ with $\hat{\Phi}(\alpha) \neq 0$ for all $\alpha \in \mathbb{R}$, then $\lim_{t\to\infty}(A_\Psi^{n+k}F)(t) = c$ for all integers $k \geqslant 0$ and all $\Psi \in P_1(\mathbb{R})$.*

Remark. For Φ_0 defined above we have $\hat{\Phi}_0(\alpha) = (1 - 2\pi i\alpha)^{-1}$ for $\alpha \in \mathbb{R}$, so that $\hat{\Phi}_0(\alpha) \neq 0$ for all $\alpha \in \mathbb{R}$.

Theorem 4·3 is probably well known, but for the reader's convenience we recall that for $k = 0$, the assertion follows from Wiener's Tauberian Theorem. (For this theorem see e.g. p. 10 in [9], and keep in mind that we use a slightly different definition of the Fourier transform.) For $k \geqslant 1$, it follows from the following

LEMMA 4·4. *Let $F \in L^\infty(\mathbb{R})$ and $c \in \mathbb{R}$. If $\lim_{t\to\infty}(\Phi * F)(t) = c$ for some $\Phi \in P_1(\mathbb{R})$, then $\lim_{t\to\infty}(\Psi * (EF))(t) = c$ for all $\Psi \in P_1(\mathbb{R})$.*

Proof. As $\lim_{t\to\infty}(\Phi * F)(t) = c$, we conclude that $\lim_{t\to\infty}(A_\Psi^1(\Phi * F))(t) = c$. Now $\lim_{t\to\infty}[A_\Psi^1((\Phi * F) - F)](t) = 0$ (see p. 221 of [7]), and therefore $\lim_{t\to\infty}(A_\Psi^1 F)(t) = c$.

(*b*) *Fisher's second CLT result*

This result, theorem 4·1 in [7], is stated in terms of means on $L^\infty(\mathbb{R})$.

We recall that a mean on $L^\infty(\mathbb{R})$ is a positive, normalized, (continuous) linear functional $\lambda: L^\infty(\mathbb{R}) \to \mathbb{R}$. The mean λ is called weighted at ∞ if $\lambda(f)$ only depends on the values of f arbitrarily far on the right. In particular, λ is weighted at ∞ if it is exponential-invariant. Fisher's paper is to a large extent concerned with the existence of means on $L^\infty(\mathbb{R})$ with certain properties. In particular he proves the existence of a mean λ on $L^\infty(\mathbb{R})$ which is convolution-invariant and exponential-invariant.

Theorem 4·1 of [7] may be stated as follows:

Let λ be a mean on $L^\infty(\mathbb{R})$. If (*a*) λ is convolution-invariant and exponential-invariant or if (*b*) λ is weighted at ∞ and 'measure-linear', then for any Lebesgue set $A \subseteq \mathbb{R}$, P-a.e.

$$\lambda(t \mapsto \chi_A(W_t/\sqrt{t})) = N(0,1)(\chi_A). \tag{4·5}$$

Under assumption (*a*) this result is weaker than the averaging result (4·2), in the

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sense that it can be deduced deterministically from (4·2), but not vice versa. Indeed, if λ satisfies (a), we have $\lambda(F) = \lambda(A^2_{\Phi_0} F)$ for $F \in L^\infty(\mathbb{R}^+)$. Recalling that

$$(A^2_{\Phi_0} F)(t) = \frac{1}{\log t} \int_1^t ds/s\, F(s),$$

for $t > 1$, we conclude by the remark on p. 219 in [7] (with $c(x) = e^{e^x}$) that

$$\underline{\lim}_{t\to\infty} \frac{1}{\log t} \int_1^t ds/s\, F(s) \leqslant \lambda(F) \leqslant \overline{\lim}_{t\to\infty} \frac{1}{\log t} \int_1^t ds/s\, F(s), \tag{4·6}$$

which together with (4·2) implies the result.

The result (4·5) under assumption (b) has no apparent relationship with the averaging result (4·2). It may be proved as follows. If λ is weighted at ∞, the triviality of the tail-field of W implies that $\lambda(t \mapsto \chi_A(W_t/\sqrt{t}))$ is constant P-a.e., and the definition of 'measure-linearity' allows the direct verification that $E\,\lambda(t \mapsto \chi_A(W_t/\sqrt{t})) = N(0,1)(\chi_A)$.

Notice that in [7] the exceptional ω-sets depend on f, A. Thus the results in [7] are not universal results for large classes of f or A. In this respect the formulation of theorem 4·1 in [7] is perhaps somewhat misleading, as the proof there clearly indicates that only one fixed A is considered. Together with our Theorem 1·6(a) however, (4·6) implies in a deterministic way that, outside one single set of probability 0, (4·5) holds simultaneously for all λ satisfying (a) and for all Borel sets $A \subseteq \mathbb{R}$ with $\text{Leb}(\partial A) = 0$.

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