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Wave propagation in a diffusive SAIV epidemic model with time delays

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Based on the fact that the incubation periods of epidemic disease in asymptomatically infected and infected individuals are inevitable and different, we propose a diffusive susceptible, asymptomatically infected, symptomatically infected and vaccinated (SAIV) epidemic model with delays in this paper. To see whether epidemic disease can propagate spatially with a constant speed, we focus on the travelling wave solution for this model. When the basic reproduction number of the corresponding spatial-homogenous delayed differential system is greater than one and the wave speed is greater than or equal to the critical speed, we prove that this model admits nontrivial positive travelling wave solutions. Our theoretical results are of benefit to the prevention and control of epidemic.

Key words: Reaction–diffusion equation, epidemic, travelling wave, time delay

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1 Introduction

Since epidemic transmission in space is associated with the travelling wave solutions with constant speed, in the last three decades, there has been a lot of work on travelling wave solutions for diffusive epidemic models [1, 2, 6–18, 20–52]. In 2014, Zhang and Wang [43] studied a diffusive influenza model with treatment

$$\begin{cases} \dfrac{\partial S(t,x)}{\partial t} = d_s \dfrac{\partial^2 S(t,x)}{\partial x^2} - \beta(I_u(t,x) + \delta I_h(t,x))S(t,x), \\ \dfrac{\partial I_u(t,x)}{\partial t} = d_u \dfrac{\partial^2 I_u(t,x)}{\partial x^2} + (1-\mu)\beta(I_u(t,x) + \delta I_h(t,x))S(t,x) - k_u I_u(t,x), \\ \dfrac{\partial I_h(t,x)}{\partial t} = d_h \dfrac{\partial^2 I_h(t,x)}{\partial x^2} + \mu\beta(I_u(t,x) + \delta I_h(t,x))S(t,x) - k_h I_h(t,x), \\ \dfrac{\partial R(t,x)}{\partial t} = d_r \dfrac{\partial^2 R(t,x)}{\partial x^2} + k_u I_u(t,x) + k_h I_h(t,x), \end{cases} \tag{1.1}$$

where $S(t,x)$, $I_u(t,x)$, $I_h(t,x)$ and $R(t,x)$ are the densities of susceptible, infected-untreated, treated and recovered individuals at time t and location x, respectively. For the biological meanings of the positive constants d_s, d_u, d_h, d_r, β, δ, μ, k_u and k_h, we refer to [43]. They proved that there exist three positive constants c^*, c_1^* and c_2^* such that if the basic reproduction

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number $R_0 > 1$, the wave speed $c > c^*$ and $d_u = d_h$ or $R_0 > 1$ and $c > c_1^*$, model (1.1) has a nonnegative and nontrivial travelling wave solution satisfying $(S, I_u, I_h, R)(-\infty) = (S_1, 0, 0, 0)$ and $(S, I_u, I_h, R)(+\infty) = (S_2, 0, 0, S_1 - S_2)$, where the constants $S_1 > S_2 \geqslant 0$; if $R_0 > 1$, $d_u = d_h$, $0 < c < c^*$ or $R_0 > 1$, $d_u \neq d_h$, $0 < c < c_2^* \leqslant c_1^*$ or $R_0 < 1$, $c > 0$, model (1.1) admits no nonnegative and nontrivial travelling wave solutions. Very recently, Deng and Zhang [5] derived the existence of travelling wave solutions for model (1.1) when $R_0 > 1$, $d_u = d_h$ and $c = c^*$. In 2016, Xu and Ai [33] considered a diffusive influenza epidemic model with vaccination

$$\begin{cases} \dfrac{\partial S(t,x)}{\partial t} = d_1 \dfrac{\partial^2 S(t,x)}{\partial x^2} - \beta[\delta_A A(t,x) + I(t,x)]S(t,x), \\ \dfrac{\partial A(t,x)}{\partial t} = d_2 \dfrac{\partial^2 A(t,x)}{\partial x^2} + \beta[(1-p)S(t,x) + (1-p_V)\delta_V V(t,x)] \\ \qquad\qquad \times [\delta_A A(t,x) + I(t,x)] - \mu_A A(t,x), \\ \dfrac{\partial I(t,x)}{\partial t} = d_3 \dfrac{\partial^2 I(t,x)}{\partial x^2} + \beta[pS(t,x) + p_V \delta_V V(t,x)] \\ \qquad\qquad \times [\delta_A A(t,x) + I(t,x)] - \mu I(t,x), \\ \dfrac{\partial V(t,x)}{\partial t} = d_4 \dfrac{\partial^2 V(t,x)}{\partial x^2} - \delta_V \beta[\delta_A A(t,x) + I(t,x)]V(t,x), \end{cases} \tag{1.2}$$

where $S(t,x)$, $A(t,x)$, $I(t,x)$ and $V(t,x)$ denote the densities of susceptible, asymptomatically infected, symptomatically infected and vaccinated individuals at position x and time t, respectively. The coefficients $d_i > 0$ $(i = 1, 2, 3, 4)$ are the diffusion rates of each class; $\beta > 0$ refers to transmission rate; p and p_V denote, respectively, the probabilities that susceptible and vaccinated population have symptoms after infection; δ_A represents the reduced transmissibility of asymptomatic infection; δ_V is the reduced transmission factors due to dose vaccination; and μ and μ_A are the recovery rates of symptomatic infected and asymptomatic infected individuals. They showed that when the basic reproduction number $R_0 > 1$, there is a positive constant c^* such that for each $c > c^*$, model (1.2) admits a nontrivial travelling wave solution satisfying $(S, A, I, V)(-\infty) = (S^0, 0, 0, V^0)$ and $(S, A, I, V)(+\infty) = (S^1, 0, 0, V^1)$, where S^0, V^0, S^1, V^1 are nonnegative constants; when $R_0 < 1$ or $c < c^*$, model (1.2) has no nontrivial travelling wave solutions. *When $R_0 > 1$ and $c = c^*$, the existence problem of (critical) travelling wave solutions for model (1.2) is still unsolved.*

In the real word, the incubation periods of asymptomatically infected and infected individuals seem to be inevitable and may be different. Based on this concern, we introduce two delays into model (1.2) and investigate the following delayed epidemic model:

$$\begin{cases} \dfrac{\partial S(t,x)}{\partial t} = d_1 \dfrac{\partial^2 S(t,x)}{\partial x^2} - \beta[\delta_A A(t-\tau_1,x) + I(t-\tau_2,x)]S(t,x), \\ \dfrac{\partial A(t,x)}{\partial t} = d_2 \dfrac{\partial^2 A(t,x)}{\partial x^2} + \beta[(1-p)S(t,x) + (1-p_V)\delta_V V(t,x)] \\ \qquad\qquad \times [\delta_A A(t-\tau_1,x) + I(t-\tau_2,x)] - \mu_A A(t,x), \\ \dfrac{\partial I(t,x)}{\partial t} = d_3 \dfrac{\partial^2 I(t,x)}{\partial x^2} + \beta[pS(t,x) + p_V \delta_V V(t,x)] \\ \qquad\qquad \times [\delta_A A(t-\tau_1,x) + I(t-\tau_2,x)] - \mu I(t,x), \\ \dfrac{\partial V(t,x)}{\partial t} = d_4 \dfrac{\partial^2 V(t,x)}{\partial x^2} - \delta_V \beta[\delta_A A(t-\tau_1,x) + I(t-\tau_2,x)]V(t,x), \end{cases} \tag{1.3}$$

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where $\tau_1 \geqslant 0$ and $\tau_2 \geqslant 0$ denote the incubation periods of the disease in asymptomatically infected and infected individuals, respectively. We will prove the existence of *super-critical and critical* travelling wave solutions with constant speed c for (1.3) in the form of

$$(S, A, I, V)(t, x) = (S, A, I, V)(\xi), \quad \xi = x + ct, \tag{1.4}$$

which satisfies the wave system

$$\begin{cases} cS'(\xi) = d_1 S''(\xi) - \beta[\delta_A A(\xi - c\tau_1) + I(\xi - c\tau_2)]S(\xi), \\ cA'(\xi) = d_2 A''(\xi) + \beta[(1-p)S(\xi) + (1-p_V)\delta_V V(\xi)] \\ \qquad \times [\delta_A A(\xi - c\tau_1) + I(\xi - c\tau_2)] - \mu_A A(\xi), \\ cI'(\xi) = d_3 I''(\xi) + \beta[pS(\xi) + p_V \delta_V V(\xi)][\delta_A A(\xi - c\tau_1) + I(\xi - c\tau_2)] - \mu I(\xi), \\ cV'(\xi) = d_4 V''(\xi) - \delta_V \beta[\delta_A A(\xi - c\tau_1) + I(\xi - c\tau_2)]V(\xi) \end{cases} \tag{1.5}$$

and the asymptotic boundary conditions

$$(S, A, I, V)(-\infty) = (S_1, 0, 0, V_1) \quad \text{and} \quad (S, A, I, V)(+\infty) = (S_2, 0, 0, V_2), \tag{1.6}$$

where S_1, V_1 are given two positive constants and S_2, V_2 are nonnegative constants which will be proved to exist.

Due to the appearance of time delays in model (1.3), the eigenvalue problem of the linearisation equation of the second equation and the third equation in (1.5) is completely different from that in [33]. Motivated by Xu and Ai [33] and Wu and Hsu [30], we introduce a perturbed system of (1.5) and derive some information of our eigenvalue problem. Using these information, we construct a pair of upper and lower solutions for this perturbed system and define a convex cone Γ whose elements sandwich in this pair of upper and lower solutions. Then we define an operator G on Γ and apply Schauder's fixed point theorem to obtain the existence of a fixed point for the operator G. This yields that this perturbed system admits a solution on Γ. With the aid of approximation method, we derive the existence of a super-critical travelling wave solution for (1.3). Moreover, by virtue of squeezing method and some important properties of the solution, we obtain the asymptotic boundaries of the super-critical travelling wave solution. Finally, we derive the *positiveness* of the super-critical travelling wave solution using the expression of the operator G and the lower solutions. Here, we note that the positiveness of the super-critical travelling wave solution for (1.2) was not obtained in [33] and this can be deduced by the strategy used in the present paper.

In the literature, the method of upper and lower solutions together with Schauder's fixed point theorem has been employed to obtain the existence of critical travelling wave solutions for some diffusive epidemic models consisting of two or three equations [11, 14, 27, 28, 48, 49]. It seems to be difficult to apply this method to (1.3) since it consists of four equations. On the other hand, the limiting argument has been used to derive the existence of nontrivial critical travelling wave solutions for some epidemic models whose reaction system has one disease-free equilibrium and one endemic equilibrium [17, 18, 36, 42]. Note that the reaction system of model (1.3) has infinitely many equilibria, the limiting argument used in [17, 18, 36, 42] cannot be carried over to obtain the existence of nontrivial positive critical travelling wave solutions for (1.3). We utilise a limiting approach together with some new ingredients to derive the existence of a *nontrivial*

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critical travelling wave solution for (1.3). Moreover, we derive the *positiveness* of critical travelling waves by the strong maximum principle. Our result on the existence of the critical travelling wave solutions for (1.3) with $\tau_1 = \tau_2 = 0$ supplements the result in [33]. We also mention that Deng and Zhang [5] derived the existence of critical travelling wave solutions for (1.1) under an unnatural condition, d_u=d_h, while our result on the existence of critical travelling wave solutions for (1.3) has no restrictions on the diffusion rates.

The remainder of this paper is organised as follows. In Section 2, we introduce a perturbed system of (1.5), construct a pair of upper and lower solutions and apply Schauder's fixed point theorem to derive the existence of a solution for this system. In Section 3, we obtain the existence of a nontrivial positive super-critical travelling wave solution for (1.3). In Section 4, we deduce the existence of a nontrivial positive critical travelling wave solution for (1.3). In Section 5, we present some concluding remarks.

2 A perturbed system

Following [3, 33], we define the basic reproduction number R_0 of (1.5) by

$$R_0 = \beta S_1\left[\frac{(1-p)\delta_A}{\mu_A} + \frac{p}{\mu}\right] + \beta\delta_V V_1\left[\frac{(1-p_V)\delta_A}{\mu_A} + \frac{p_V}{\mu}\right]. \tag{2.1}$$

For our purpose, we introduce a perturbed system of (1.5) as follows:

$$\begin{cases} cS'(\xi) = d_1 S''(\xi) - \beta[\delta_A A(\xi - c\tau_1) + I(\xi - c\tau_2)]S(\xi), \\ cA'(\xi) = d_2 A''(\xi) + \beta[(1-p)S(\xi) + (1-p_V)\delta_V V(\xi)] \\ \qquad\quad \times [\delta_A A(\xi - c\tau_1) + I(\xi - c\tau_2)] - \mu_A A(\xi) - \varepsilon A^2(\xi), \\ cI'(\xi) = d_3 I''(\xi) + \beta[pS(\xi) + p_V\delta_V V(\xi)][\delta_A A(\xi - c\tau_1) + I(\xi - c\tau_2)] \\ \qquad\quad - \mu I(\xi) - \varepsilon I^2(\xi), \\ cV'(\xi) = d_4 V''(\xi) - \delta_V\beta[\delta_A A(\xi - c\tau_1) + I(\xi - c\tau_2)]V(\xi), \end{cases} \tag{2.2}$$

where $\varepsilon > 0$ is a small parameter.

2.1 Eigenvalue problem

Linearising the second and third equations in (2.2) at $(S_1, 0, 0, V_1)$ yields

$$\begin{cases} d_2 A''(\xi) - cA'(\xi) - \mu_A A(\xi) + \nu_1\delta_A A(\xi - c\tau_1) + \nu_1 I(\xi - c\tau_2) = 0, \\ d_3 I''(\xi) - cI'(\xi) - \mu I(\xi) + \nu_2\delta_A A(\xi - c\tau_1) + \nu_2 I(\xi - c\tau_2) = 0, \end{cases} \tag{2.3}$$

where

$$\nu_1 = \beta[(1-p)S_1 + (1-p_V)\delta_V V_1] \quad \text{and} \quad \nu_2 = \beta(pS_1 + p_V\delta_V V_1). \tag{2.4}$$

Inserting $A(\xi) = L_1 e^{\rho\xi}$ and $I(\xi) = L_2 e^{\rho\xi}$ into (2.3) gives

$$A(c, \rho)\begin{pmatrix} L_1 \\ L_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

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where

$$A(c,\rho)=\begin{pmatrix} h_1(c,\rho) & \nu_1 e^{-c\tau_2\rho} \\ \nu_2\delta_A e^{-c\tau_1\rho} & h_2(c,\rho)\end{pmatrix}$$

with

$$h_1(c,\rho)=d_2\rho^2-c\rho-\mu_A+\nu_1\delta_A e^{-c\tau_1\rho} \quad \text{and} \quad h_2(c,\rho)=d_3\rho^2-c\rho-\mu+\nu_2 e^{-c\tau_2\rho}.$$

For convenience, let

$$\Delta(c,\rho):=h_1(c,\rho)h_2(c,\rho)-\nu_1\nu_2\delta_A e^{-c(\tau_1+\tau_2)\rho}. \tag{2.5}$$

Lemma 2.1 *Suppose that $\mu_A>\nu_1\delta_A$ and $\mu>\nu_2$. Then for each $c>0$, $h_i(c,\rho)=0$ has a negative root ρ_i^- and a positive root ρ_i^+ $(i=1,2)$.*

Proof Since $\mu_A>\nu_1\delta_A$ and $\mu>\nu_2$, we have for each $c>0$ that

$$h_1(c,0)=-\mu_A+\nu_1\delta_A<0,\ h_2(c,0)=-\mu+\nu_2<0$$

and

$$\lim_{\rho\to\pm\infty} h_i(c,\rho)=+\infty \quad (i=1,2).$$

On the other hand, for each pair $(\rho,c)\in\mathbb{R}\times\mathbb{R}^+$, it follows that

$$\frac{\partial^2 h_1(c,\rho)}{\partial\rho^2}=2d_2+c^2\tau_1^2\nu_1\delta_A e^{-c\tau_1\rho}>0$$

and

$$\frac{\partial^2 h_2(c,\rho)}{\partial\rho^2}=2d_3+c^2\tau_2^2\nu_2 e^{-c\tau_2\rho}>0.$$

Then the assertion of this lemma follows. □

Lemma 2.2 *Assume that $\mu_A>\nu_1\delta_A$, $\mu>\nu_2$ and $R_0>1$. Then there exists a positive constant c^* such that*

(i) for each $c>c^*$, equation $\Delta(c,\rho)=0$ has at least three positive roots and a negative root. Let $\rho_1:=\rho_1(c)$ be the smallest positive root of $\Delta(c,\rho)=0$ and $\rho_2:=\rho_2(c)$ be the positive root which is closest to ρ_1. Then $\Delta(c,\rho)>0$ for $\rho\in(\rho_1,\rho_2)$;

(ii) when $c=c^*$, there exists a positive constant ρ^* satisfying that

$$\Delta(c^*,\rho^*)=0 \quad \text{and} \quad \frac{\partial}{\partial\rho}\Delta(c^*,\rho^*)=0.$$

Proof By elementary calculations, for each $c>0$, we obtain from (2.1), (2.4) and Lemma 2.1 that

$$\lim_{\rho\to\pm\infty}\Delta(c,\rho)=+\infty, \tag{2.6}$$

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$$\begin{aligned}\Delta(c,0) &= (\nu_1\delta_A - \mu_A)(\nu_2 - \mu) - \nu_1\nu_2\delta_A \\ &= \mu\mu_A - \mu_A\nu_2 - \nu_1\delta_A\mu \\ &= \mu\mu_A\left(1 - \frac{\nu_2}{\mu} - \frac{\nu_1\delta_A}{\mu_A}\right) \\ &= \mu\mu_A(1 - R_0) \\ &< 0\end{aligned} \tag{2.7}$$

and

$$\Delta(c,\rho_m^c) = -\nu_1\nu_2\delta_A e^{-c(\tau_1+\tau_2)\rho_m^c} < 0 \quad \text{for} \quad \rho_m^c = \min\{\rho_1^+, \rho_2^+\}, \tag{2.8}$$

where ρ_1^+ and ρ_2^+ are given in Lemma 2.1.

Define $\rho_0 := \min\left\{\sqrt{\frac{\mu_A-\nu_1\delta_A}{d_2}}, \sqrt{\frac{\mu-\nu_2}{d_3}}\right\}$, then we have

$$\mu_A - \nu_1\delta_A - d_2\rho^2 > 0 \quad \text{and} \quad \mu - \nu_2 - d_3\rho^2 > 0 \quad \text{for} \quad \rho \in (0,\rho_0). \tag{2.9}$$

By (2.9), we obtain

$$\begin{aligned}\Delta\,(0,\rho) &= \left(d_2\rho^2 - \mu_A + \nu_1\delta_A\right)\left(d_3\rho^2 - \mu + \nu_2\right) - \nu_1\nu_2\delta_A \\ &= \left(\mu_A - \nu_1\delta_A - d_2\rho^2\right)\left(\mu - \nu_2 - d_3\rho^2\right) - \nu_1\nu_2\delta_A \\ &< \left(\mu_A - \nu_1\delta_A\right)\left(\mu - \nu_2\right) - \nu_1\nu_2\delta_A \\ &= \mu\mu_A - \mu_A\nu_2 - \nu_1\delta_A\mu \\ &= \mu\mu_A\,(1 - R_0) \\ &< 0 \quad \text{for} \quad \rho \in (0,\rho_0)\,.\end{aligned} \tag{2.10}$$

Noting that

$$\begin{aligned}&\lim_{c\to+\infty}\Delta\left(c, \frac{1}{\sqrt{c}}\right) \\ &= \lim_{c\to+\infty}\left[\left(\frac{d_2}{c} - \sqrt{c} - \mu_A + \nu_1\delta_A e^{-\sqrt{c}\tau_1}\right)\left(\frac{d_3}{c} - \sqrt{c} - \mu + \nu_2 e^{-\sqrt{c}\tau_2}\right) - \nu_1\nu_2\delta_A e^{-\sqrt{c}(\tau_1+\tau_2)}\right] \\ &= +\infty,\end{aligned} \tag{2.11}$$

we infer that $\Delta(c,\rho)|_{\rho=\frac{1}{\sqrt{c}}} > 0$ and $\frac{1}{\sqrt{c}} \in (0,\rho_m^c)$ for sufficiently large $c > 0$. Then by (2.6)–(2.11) and the continuity of $\Delta(c,\rho)$ in both c and ρ, one can get the results of this lemma. □

Remark 2.1 *In [33], Xu and Ai obtained the existence of a travelling wave solution for non-delayed model (1.2) under the condition $R_0 > 1$. Here, we note that if $R_0 > 1$, then inequalities (2.7) and (2.10) hold whatever the characteristic equation $\Delta(c,\rho) = 0$ has incubation times or not. Therefore, one will see in Sections 3 and 4 that the incubation times do not influence the validity of the inequality $R_0 > 1$ as the condition for the epidemic wave propagation in delayed model (1.3).*

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2.2 Construction of upper and lower solutions

In the following, we always assume that $\mu_A > \nu_1\delta_A$, $\mu > \nu_2$, $R_0 > 1$ and $c > c^*$. For $\xi \in \mathbb{R}$, we define the following nonnegative continuous functions:

$$S_+(\xi) := S_1,$$

$$A_+(\xi) := \begin{cases} L_1 e^{\rho_1 \xi}, & \xi \leqslant \xi_1, \\ L_1 M, & \xi > \xi_1, \end{cases}$$

$$I_+(\xi) := \begin{cases} L_2 e^{\rho_1 \xi}, & \xi \leqslant \xi_1, \\ L_2 M, & \xi > \xi_1, \end{cases}$$

$$V_+(\xi) := V_1,$$

$$S_-(\xi) := \begin{cases} S_1 - \sigma^{-1} e^{\sigma \xi}, & \xi \leqslant \xi_2, \\ 0, & \xi > \xi_2, \end{cases}$$

$$A_-(\xi) := \begin{cases} L_1 e^{\rho_1 \xi} - q e^{(\rho_1+\eta)\xi}, & \xi \leqslant \xi_3, \\ 0, & \xi > \xi_3, \end{cases}$$

$$I_-(\xi) := \begin{cases} L_2 e^{\rho_1 \xi} - q\kappa e^{(\rho_1+\eta)\xi}, & \xi \leqslant \xi_4, \\ 0, & \xi > \xi_4, \end{cases}$$

$$V_-(\xi) := \begin{cases} V_1 - \sigma^{-1} e^{\sigma \xi}, & \xi \leqslant \xi_5, \\ 0, & \xi > \xi_5, \end{cases}$$

where

$$\xi_1 = \frac{1}{\rho_1} \ln M, \ \ \xi_2 = \frac{1}{\sigma} \ln(\sigma S_1), \ \ \xi_3 = \frac{1}{\eta} \ln \frac{L_1}{q}, \ \ \xi_4 = \frac{1}{\eta} \ln \frac{L_2}{q\kappa}, \ \ \xi_5 = \frac{1}{\sigma} \ln(\sigma V_1), \tag{2.12}$$

$L_1, L_2, \rho_1, M, \sigma, \eta, \kappa$ and q are positive constants to be determined in Lemma 2.3-Lemma 2.6.

Lemma 2.3 *The functions $S_+(\xi)$ and $V_+(\xi)$ satisfy*

$$d_1 S_+''(\xi) - c S_+'(\xi) - \beta[\delta_A A_-(\xi - c\tau_1) + I_-(\xi - c\tau_2)] S_+(\xi) \leqslant 0 \tag{2.13}$$

and

$$d_4 V_+''(\xi) - c V_+'(\xi) - \delta_V \beta[\delta_A A_-(\xi - c\tau_1) + I_-(\xi - c\tau_2)] V_+(\xi) \leqslant 0 \tag{2.14}$$

for $\xi \in \mathbb{R}$.

Proof Since $S_+(\xi) = S_1$, $V_+(\xi) = V_1$, $A_-(\xi - c\tau_1) \geqslant 0$ and $I_-(\xi - c\tau_2) \geqslant 0$ for $\xi \in \mathbb{R}$, inequalities (2.13) and (2.14) hold obviously. □

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Let ρ_1 be the smallest positive eigenvalue of equation $\Delta(c,\rho)=0$ (see Lemma 2.2) and $(L_1,L_2)>0$, its associating eigenvector with

$$h_1(c,\rho_1)L_1+\nu_1L_2e^{-c\tau_2\rho_1}=0 \quad \text{and} \quad \nu_2\delta_AL_1e^{-c\tau_1\rho_1}+h_2(c,\rho_1)L_2=0. \tag{2.15}$$

Lemma 2.4 *Assume that the constant*

$$M>\max\left\{0,\frac{\nu_1\delta_AL_1+\nu_1L_2-\mu_AL_1}{\varepsilon L_1^2},\frac{\nu_2\delta_AL_1+\nu_2L_2-\mu L_2}{\varepsilon L_2^2}\right\},$$

then the functions $A_+(\xi)$ and $I_+(\xi)$ satisfy

$$\begin{aligned}&d_2A_+''(\xi)-cA_+'(\xi)+\beta[(1-p)S_+(\xi)+(1-p_V)\delta_VV_+(\xi)]\\&\quad\times[\delta_AA_+(\xi-c\tau_1)+I_+(\xi-c\tau_2)]-\mu_AA_+(\xi)-\varepsilon A_+^2(\xi)\leqslant 0\end{aligned} \tag{2.16}$$

and

$$\begin{aligned}&d_3I_+''(\xi)-cI_+'(\xi)+\beta[pS_+(\xi)+p_V\delta_VV_+(\xi)][\delta_AA_+(\xi-c\tau_1)+I_+(\xi-c\tau_2)]\\&\quad-\mu I_+(\xi)-\varepsilon I_+^2(\xi)\leqslant 0\end{aligned} \tag{2.17}$$

for $\xi\neq\xi_1$.

Proof When $\xi<\xi_1$, $A_+(\xi)=L_1e^{\rho_1\xi}$, $I_+(\xi)=L_2e^{\rho_1\xi}$, $S_+(\xi)=S_1$ and $V_+(\xi)=V_1$. We then infer from (2.4) and (2.15) that

$$\begin{aligned}&d_2A_+''(\xi)-cA_+'(\xi)+\beta[(1-p)S_+(\xi)+(1-p_V)\delta_VV_+(\xi)]\\&\times[\delta_AA_+(\xi-c\tau_1)+I_+(\xi-c\tau_2)]-\mu_AA_+(\xi)-\varepsilon A_+^2(\xi)\\&=d_2L_1\rho_1^2e^{\rho_1\xi}-cL_1\rho_1e^{\rho_1\xi}+\beta[(1-p)S_1+(1-p_V)\delta_VV_1]\\&\times\left(\delta_AL_1e^{\rho_1\xi}e^{-c\tau_1\rho_1}+L_2e^{\rho_1\xi}e^{-c\tau_2\rho_1}\right)-\mu_AL_1e^{\rho_1\xi}-\varepsilon A_+^2(\xi)\\&=L_1e^{\rho_1\xi}\left(d_2\rho_1^2-c\rho_1-\mu_A+\nu_1\delta_Ae^{-c\tau_1\rho_1}\right)+\nu_1L_2e^{\rho_1\xi}e^{-c\tau_2\rho_1}-\varepsilon A_+^2(\xi)\\&=e^{\rho_1\xi}\left[h_1(c,\rho_1)L_1+\nu_1L_2e^{-c\tau_2\rho_1}\right]-\varepsilon A_+^2(\xi)\\&=-\varepsilon A_+^2(\xi)\\&\leqslant 0.\end{aligned}$$

When $\xi>\xi_1$, $A_+(\xi)=L_1M$, $I_+(\xi)=L_2M$, $S_+(\xi)=S_1$ and $V_+(\xi)=V_1$. Then a direct computation yields

$$\begin{aligned}&d_2A_+''(\xi)-cA_+'(\xi)+\beta[(1-p)S_+(\xi)+(1-p_V)\delta_VV_+(\xi)]\\&\quad\times[\delta_AA_+(\xi-c\tau_1)+I_+(\xi-c\tau_2)]-\mu_AA_+(\xi)-\varepsilon A_+^2(\xi)\\&\quad=\nu_1(\delta_AL_1M+L_2M)-\mu_AL_1M-\varepsilon L_1^2M^2\\&\quad=M(\nu_1\delta_AL_1+\nu_1L_2-\mu_AL_1-\varepsilon L_1^2M)\\&\quad\leqslant 0,\end{aligned}$$

where we have used the assumption $M>(\nu_1\delta_AL_1+\nu_1L_2-\mu_AL_1)/\varepsilon L_1^2$. Inequality (2.17) can be proved similarly. □

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Define

$$\hat{\sigma} = \min\left\{\frac{\rho_1}{2}, \frac{1}{S_1}, \frac{1}{V_1}, \frac{c}{d_1 + \beta(\delta_A L_1 + L_2)S_1^2}, \frac{c}{d_4 + \delta_V \beta(\delta_A L_1 + L_2)V_1^2}\right\}. \tag{2.18}$$

Lemma 2.5 *Suppose that the constant $\sigma \in (0, \hat{\sigma})$ and is sufficiently small. Then the functions $S_-(\xi)$ and $V_-(\xi)$ satisfy*

$$d_1 S_-''(\xi) - cS_-'(\xi) - \beta[\delta_A A_+(\xi - c\tau_1) + I_+(\xi - c\tau_2)]S_-(\xi) \geqslant 0 \text{ for } \xi \neq \xi_2 \tag{2.19}$$

and

$$d_4 V_-''(\xi) - cV_-'(\xi) - \delta_V \beta[\delta_A A_+(\xi - c\tau_1) + I_+(\xi - c\tau_2)]V_-(\xi) \geqslant 0 \text{ for } \xi \neq \xi_5. \tag{2.20}$$

Proof Choose sufficiently small $\sigma \in (0, \hat{\sigma})$ such that for $\xi < \xi_2$,

$$-d_1\sigma + c - \beta(\delta_A L_1 + L_2)S_1^2\sigma \geqslant 0 \tag{2.21}$$

and

$$e^{(\rho_1 - \sigma)\xi} < (\sigma S_1)^{\frac{\rho_1 - \sigma}{\sigma}} < \sigma S_1. \tag{2.22}$$

When $\xi < \xi_2$, $S_-(\xi) = S_1 - \sigma^{-1}e^{\sigma\xi}$, $A_+(\xi) \leqslant L_1 e^{\rho_1\xi}$ and $I_+(\xi) \leqslant L_2 e^{\rho_1\xi}$. By (2.21) and (2.22), we get

$$\begin{aligned}
&d_1 S_-''(\xi) - cS_-'(\xi) - \beta[\delta_A A_+(\xi - c\tau_1) + I_+(\xi - c\tau_2)]S_-(\xi) \\
&\geqslant -d_1\sigma e^{\sigma\xi} + ce^{\sigma\xi} - \beta\left(\delta_A L_1 e^{\rho_1\xi}e^{-c\tau_1\rho_1} + L_2 e^{\rho_1\xi}e^{-c\tau_2\rho_1}\right)\left(S_1 - \sigma^{-1}e^{\sigma\xi}\right) \\
&> e^{\sigma\xi}\left[-d_1\sigma + c - \beta\left(\delta_A L_1 e^{-c\tau_1\rho_1} + L_2 e^{-c\tau_2\rho_1}\right)S_1 e^{(\rho_1 - \sigma)\xi}\right] \\
&> e^{\sigma\xi}\left[-d_1\sigma + c - \beta(\delta_A L_1 + L_2)S_1^2\sigma\right] \\
&\geqslant 0.
\end{aligned}$$

When $\xi > \xi_2$, $S_-(\xi) = 0$ and inequality (2.19) holds. Analogously, one can show (2.20) is valid. □

By Lemma 2.1, we can choose a constant $\eta \in (0, \min\{\sigma, \rho_1, \rho_2 - \rho_1\})$ to be small enough such that

$$\begin{aligned}
&h_1(c, \rho_1 + \eta)h_2(c, \rho_1 + \eta) - \nu_1\nu_2\delta_A e^{-c(\tau_1 + \tau_2)(\rho_1 + \eta)} > 0, \\
&h_1(c, \rho_1 + \eta) < 0 \quad \text{and} \quad h_2(c, \rho_1 + \eta) < 0.
\end{aligned} \tag{2.23}$$

Then there exists a constant $\kappa > 0$ satisfying

$$-\frac{\nu_2\delta_A e^{-c\tau_1(\rho_1 + \eta)}}{h_2(c, \rho_1 + \eta)} < \kappa < -\frac{h_1(c, \rho_1 + \eta)}{\nu_1 e^{-c\tau_2(\rho_1 + \eta)}}. \tag{2.24}$$

Define

$$\begin{aligned}
\hat{q} := \max\Bigg\{&L_1, \frac{L_2}{\kappa}, -\frac{\beta[(1-p) + (1-p_V)\delta_V](\delta_A L_1 + L_2) + \sigma\varepsilon L_1^2}{\sigma[h_1(c, \rho_1 + \eta) + \nu_1\kappa e^{-c\tau_2(\rho_1 + \eta)}]}, \\
&-\frac{\beta(p + p_V\delta_V)(\delta_A L_1 + L_2) + \sigma\varepsilon L_2^2}{\sigma[\kappa h_2(c, \rho_1 + \eta) + \nu_2\delta_A e^{-c\tau_1(\rho_1 + \eta)}]}\Bigg\}.
\end{aligned}$$

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Lemma 2.6 *Suppose that the constant $q > \hat{q}$ and is large enough. Then the functions $A_-(\xi)$ and $I_-(\xi)$ satisfy*

$$\begin{aligned} &d_2A''_-(\xi) - cA'_-(\xi) + \beta[(1-p)S_-(\xi) + (1-p_V)\delta_V V_-(\xi)] \\ &\times [\delta_A A_-(\xi - c\tau_1) + I_-(\xi - c\tau_2)] - \mu_A A_-(\xi) - \varepsilon A^2_-(\xi) \\ &\geqslant 0 \text{ for } \xi \neq \xi_3 \end{aligned} \tag{2.25}$$

and

$$\begin{aligned} &d_3I''_-(\xi) - cI'_-(\xi) + \beta[pS_-(\xi) + p_V\delta_V V_-(\xi)] \\ &\times [\delta_A A_-(\xi - c\tau_1) + I_-(\xi - c\tau_2)] - \mu I_-(\xi) - \varepsilon I^2_-(\xi) \\ &\geqslant 0 \text{ for } \xi \neq \xi_4. \end{aligned} \tag{2.26}$$

Proof By (2.24), we have $h_1(c, \rho_1 + \eta) + \nu_1\kappa e^{-c\tau_2(\rho_1+\eta)} < 0$. Then it follows from $q > \hat{q}$ that

$$q[h_1(c, \rho_1 + \eta) + \nu_1\kappa e^{-c\tau_2(\rho_1+\eta)}] + \beta\sigma^{-1}[(1-p) + (1-p_V)\delta_V][\delta_A L_1 + L_2] + \varepsilon L_1^2 < 0. \tag{2.27}$$

When $\xi < \xi_3$, $A_-(\xi) = L_1e^{\rho_1\xi} - qe^{(\rho_1+\eta)\xi}$, $S_-(\xi) \geqslant S_1 - \sigma^{-1}e^{\sigma\xi}$, $V_-(\xi) \geqslant V_1 - \sigma^{-1}e^{\sigma\xi}$ and $I_-(\xi) \geqslant L_2e^{\rho_1\xi} - q\kappa e^{(\rho_1+\eta)\xi}$. Then we derive that

$$\begin{aligned} &d_2A''_-(\xi) - cA'_-(\xi) + \beta[(1-p)S_-(\xi) + (1-p_V)\delta_V V_-(\xi)] \\ &\quad\times [\delta_A A_-(\xi - c\tau_1) + I_-(\xi - c\tau_2)] - \mu_A A_-(\xi) - \varepsilon A^2_-(\xi) \\ &\geqslant d_2L_1\rho_1^2e^{\rho_1\xi} - d_2q(\rho_1+\eta)^2e^{(\rho_1+\eta)\xi} - cL_1\rho_1e^{\rho_1\xi} + cq(\rho_1+\eta)e^{(\rho_1+\eta)\xi} + \beta[(1-p) \\ &\quad\times (S_1 - \sigma^{-1}e^{\sigma\xi}) + (1-p_V)\delta_V(V_1 - \sigma^{-1}e^{\sigma\xi})] \times [\delta_A(L_1e^{\rho_1\xi}e^{-c\tau_1\rho_1} \\ &\quad- qe^{(\rho_1+\eta)\xi}e^{-c\tau_1(\rho_1+\eta)}) + L_2e^{\rho_1\xi}e^{-c\tau_2\rho_1} - q\kappa e^{(\rho_1+\eta)\xi}e^{-c\tau_2(\rho_1+\eta)}] \\ &\quad- \mu_AL_1e^{\rho_1\xi} + \mu_Aqe^{(\rho_1+\eta)\xi} - \varepsilon[L_1e^{\rho_1\xi} - qe^{(\rho_1+\eta)\xi}]^2 \\ &> d_2L_1\rho_1^2e^{\rho_1\xi} - cL_1\rho_1e^{\rho_1\xi} - \mu_AL_1e^{\rho_1\xi} + \nu_1\delta_AL_1e^{\rho_1\xi}e^{-c\tau_1\rho_1} + \nu_1L_2e^{\rho_1\xi}e^{-c\tau_2\rho_1} \\ &\quad- d_2q(\rho_1+\eta)^2e^{(\rho_1+\eta)\xi} + cq(\rho_1+\eta)e^{(\rho_1+\eta)\xi} + \mu_Aqe^{(\rho_1+\eta)\xi} - \nu_1\delta_Aqe^{(\rho_1+\eta)\xi}e^{-c\tau_1(\rho_1+\eta)} \\ &\quad- \nu_1q\kappa e^{(\rho_1+\eta)\xi}e^{-c\tau_2(\rho_1+\eta)} + \beta[(1-p)(-\sigma^{-1}e^{\sigma\xi}) + (1-p_V)\delta_V(-\sigma^{-1}e^{\sigma\xi})] \\ &\quad\times [\delta_A(L_1e^{\rho_1\xi}e^{-c\tau_1\rho_1} - qe^{(\rho_1+\eta)\xi}e^{-c\tau_1(\rho_1+\eta)}) + L_2e^{\rho_1\xi}e^{-c\tau_2\rho_1} \\ &\quad- q\kappa e^{(\rho_1+\eta)\xi}e^{-c\tau_2(\rho_1+\eta)}] - \varepsilon(L_1e^{\rho_1\xi})^2 \\ &= -e^{(\rho_1+\eta)\xi}\Big\{q[h_1(c, \rho_1+\eta) + \nu_1\kappa e^{-c\tau_2(\rho_1+\eta)}] + \beta\sigma^{-1}e^{(\sigma-\eta)\xi}[(1-p) + (1-p_V)\delta_V] \\ &\quad\times \left[\delta_A\left(L_1e^{-c\tau_1\rho_1} - qe^{\eta\xi}e^{-c\tau_1(\rho_1+\eta)}\right) + L_2e^{-c\tau_2\rho_1} - q\kappa e^{\eta\xi}e^{-c\tau_2(\rho_1+\eta)}\right] + \varepsilon L_1^2e^{(\rho_1-\eta)\xi}\Big\} \end{aligned}$$

(by (2.15))

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$$
\begin{aligned}
&> -e^{(\rho_1+\eta)\xi}\Big\{q\left[h_1(c,\rho_1+\eta)+\nu_1\kappa e^{-c\tau_2(\rho_1+\eta)}\right]\\
&\quad+\beta\sigma^{-1}e^{(\sigma-\eta)\xi}\left[(1-p)+(1-p_V)\delta_V\right]\left[\delta_A L_1 e^{-c\tau_1\rho_1}+L_2e^{-c\tau_2\rho_1}\right]+\varepsilon L_1^2 e^{(\rho_1-\eta)\xi}\Big\}\\
&> -e^{(\rho_1+\eta)\xi}\Big\{q\left[h_1(c,\rho_1+\eta)+\nu_1\kappa e^{-c\tau_2(\rho_1+\eta)}\right]\\
&\quad+\beta\sigma^{-1}[(1-p)+(1-p_V)\delta_V][\delta_A L_1 e^{-c\tau_1\rho_1}+L_2e^{-c\tau_2\rho_1}]+\varepsilon L_1^2\Big\}\\
&\text{(since } e^{(\rho_1-\eta)\xi}<1 \text{ and } e^{(\sigma-\eta)\xi}<1 \text{ for } \eta<\min\{\sigma,\rho_1,\rho_2-\rho_1\} \text{ and } \xi<\xi_3<0)\\
&> -e^{(\rho_1+\eta)\xi}\Big\{q\left[h_1(c,\rho_1+\eta)+\nu_1\kappa e^{-c\tau_2(\rho_1+\eta)}\right]+\beta\sigma^{-1}\left[(1-p)+(1-p_V)\delta_V\right]\\
&\quad\times(\delta_A L_1+L_2)+\varepsilon L_1^2\Big\}\\
&\geqslant 0 \quad \text{(by (2.27))}.
\end{aligned}
$$

When $\xi>\xi_3$, $A_-(\xi)=0$ and inequality (2.25) holds. The proof of (2.26) can be carried out similarly. □

2.3 Application of Schauder's fixed point theorem

Define $M^*=M\max\{L_1,L_2\}$ and choose

$$
\begin{aligned}
&\beta_1>\beta(\delta_A+1)M^*, \quad \beta_2>\mu_A+2\varepsilon M^*,\\
&\beta_3>\mu+2\varepsilon M^* \quad \text{and} \quad \beta_4>\delta_V\beta(\delta_A+1)M^*.
\end{aligned}
\tag{2.28}
$$

Then for $(S,A,I,V)(\xi)\in C(\mathbb{R},\Omega)$ with $\Omega=[0,S_1]\times[0,M^*]\times[0,M^*]\times[0,V_1]$, we deduce that

$$
H_1[S,A,I,V](\xi):=\beta_1 S(\xi)-\beta[\delta_A A(\xi-c\tau_1)+I(\xi-c\tau_2)]S(\xi)
$$

is monotonically increasing with respect to S and monotonically decreasing in both A and I,

$$
\begin{aligned}
H_2[S,A,I,V](\xi):=&\beta_2 A(\xi)+\beta[(1-p)S(\xi)+(1-p_V)\delta_V V(\xi)][\delta_A A(\xi-c\tau_1)+I(\xi-c\tau_2)]\\
&-\mu_A A(\xi)-\varepsilon A^2(\xi)
\end{aligned}
$$

and

$$
H_3[S,A,I,V](\xi):=\beta_3 I(\xi)+\beta[pS(\xi)+p_V\delta_V V(\xi)][\delta_A A(\xi-c\tau_1)+I(\xi-c\tau_2)]-\mu I(\xi)-\varepsilon I^2(\xi)
$$

are monotonically increasing in S, A, I, V, and

$$
H_4[S,A,I,V](\xi):=\beta_4 V(\xi)-\delta_V\beta[\delta_A A(\xi-c\tau_1)+I(\xi-c\tau_2)]V(\xi)
$$

is monotonically increasing with respect to V, and monotonically decreasing in both A and I.

Now we rewrite (2.2) as

$$
\begin{cases}
d_1S''(\xi)-cS'(\xi)-\beta_1S(\xi)+H_1[S,A,I,V](\xi)=0,\\
d_2A''(\xi)-cA'(\xi)-\beta_2A(\xi)+H_2[S,A,I,V](\xi)=0,\\
d_3I''(\xi)-cI'(\xi)-\beta_3I(\xi)+H_3[S,A,I,V](\xi)=0,\\
d_4V''(\xi)-cV'(\xi)-\beta_4V(\xi)+H_4[S,A,I,V](\xi)=0.
\end{cases}
\tag{2.29}
$$

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Define

$$\Gamma := \left\{ (S, A, I, V)(\xi) \in C(\mathbb{R}, \Omega) \;\middle|\; \begin{array}{l} S_-(\xi) \leqslant S(\xi) \leqslant S_+(\xi), \\ A_-(\xi) \leqslant A(\xi) \leqslant A_+(\xi), \\ I_-(\xi) \leqslant I(\xi) \leqslant I_+(\xi), \\ V_-(\xi) \leqslant V(\xi) \leqslant V_+(\xi), \\ \text{for any } \xi \in \mathbb{R} \end{array} \right\}.$$

Clearly, Γ is bounded, nonempty, closed and convex in $C(\mathbb{R}, \Omega)$. Moreover, we define an operator $G = (G_1, G_2, G_3, G_4)$: $\Gamma \mapsto C(\mathbb{R}, \Omega)$ by

$$\begin{aligned} G_i[S, A, I, V](\xi) = & \frac{1}{\Lambda_i} \int_{-\infty}^{\xi} e^{\rho_{i1}(\xi - s)} H_i[S, A, I, V](s) ds \\ & + \frac{1}{\Lambda_i} \int_{\xi}^{+\infty} e^{\rho_{i2}(\xi - s)} H_i[S, A, I, V](s) ds, \end{aligned} \tag{2.30}$$

where

$$\rho_{i1} = \frac{c - \sqrt{c^2 + 4 d_i \beta_i}}{2 d_i}, \quad \rho_{i2} = \frac{c + \sqrt{c^2 + 4 d_i \beta_i}}{2 d_i} \quad \text{and} \quad \Lambda_i = d_i(\rho_{i2} - \rho_{i1})$$

for $i = 1, 2, 3, 4$.

Lemma 2.7 $G = (G_1, G_2, G_3, G_4) : \Gamma \mapsto \Gamma$.

Proof By the monotonicity of $H_i[S, A, I, V](\xi)$ with respect to S, A, I and V, we only need to show that

$$\begin{aligned} S_-(\xi) \leqslant G_1[S_-, A_+, I_+, V](\xi) \leqslant G_1[S, A, I, V](\xi) \\ \leqslant G_1[S_+, A_-, I_-, V](\xi) \leqslant S_+(\xi), \end{aligned} \tag{2.31}$$

$$\begin{aligned} A_-(\xi) \leqslant G_2[S_-, A_-, I_-, V_-](\xi) \leqslant G_2[S, A, I, V](\xi) \\ \leqslant G_2[S_+, A_+, I_+, V_+](\xi) \leqslant A_+(\xi), \end{aligned} \tag{2.32}$$

$$\begin{aligned} I_-(\xi) \leqslant G_3[S_-, A_-, I_-, V_-](\xi) \leqslant G_3[S, A, I, V](\xi) \\ \leqslant G_3[S_+, A_+, I_+, V_+](\xi) \leqslant I_+(\xi), \end{aligned} \tag{2.33}$$

$$\begin{aligned} V_-(\xi) \leqslant G_4[S, A_+, I_+, V_-](\xi) \leqslant G_4[S, A, I, V](\xi) \\ \leqslant G_4[S, A_-, I_-, V_+](\xi) \leqslant V_+(\xi) \end{aligned} \tag{2.34}$$

for any $(S, A, I, V)(\xi) \in \Gamma$.

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When $\xi < \xi_2$, it follows from (2.19) that

$$\begin{aligned}
G_1[S_-, A_+, I_+, V](\xi) &= \frac{1}{\Lambda_1}\int_{-\infty}^{\xi} e^{\rho_{11}(\xi-s)} H_1[S_-, A_+, I_+, V](s)ds \\
&\quad + \frac{1}{\Lambda_1}\int_{\xi}^{+\infty} e^{\rho_{12}(\xi-s)} H_1[S_-, A_+, I_+, V](s)ds \\
&\geqslant \frac{1}{\Lambda_1}\int_{-\infty}^{\xi} e^{\rho_{11}(\xi-s)}[-d_1 S_-''(s) + cS_-'(s) + \beta_1 S_-(s)]ds \\
&\quad + \frac{1}{\Lambda_1}\int_{\xi}^{+\infty} e^{\rho_{12}(\xi-s)}[-d_1 S_-''(s) + cS_-'(s) + \beta_1 S_-(s)]ds \\
&= \frac{1}{\Lambda_1}\int_{-\infty}^{\xi} e^{\rho_{11}(\xi-s)}[-d_1 S_-''(s) + cS_-'(s) + \beta_1 S_-(s)]ds \\
&\quad + \frac{1}{\Lambda_1}\int_{\xi}^{\xi_2} e^{\rho_{12}(\xi-s)}[-d_1 S_-''(s) + cS_-'(s) + \beta_1 S_-(s)]ds \\
&\quad + \frac{1}{\Lambda_1}\int_{\xi_2}^{+\infty} e^{\rho_{12}(\xi-s)}[-d_1 S_-''(s) + cS_-'(s) + \beta_1 S_-(s)]ds \\
&= S_-(\xi) + \frac{d_1}{\Lambda_1} e^{\rho_{12}(\xi-\xi_2)}[S_-'(\xi_2+0) - S_-'(\xi_2-0)] \\
&\geqslant S_-(\xi).
\end{aligned}$$

When $\xi > \xi_2$, one can get

$$G_1[S_-, A_+, I_+, V](\xi) \geqslant 0 = S_-(\xi).$$

Then by the continuity of $G_1[S_-, A_+, I_+, V](\xi)$ and $S_-(\xi)$ at the point ξ_2, we have

$$G_1[S_-, A_+, I_+, V](\xi) \geqslant S_-(\xi) \quad \text{for} \quad \xi \in \mathbb{R}.$$

On the other hand, we infer from (2.13) that for all $\xi \in \mathbb{R}$,

$$H_1[S_+, A_-, I_-, V](\xi) \leqslant -d_1 S_+''(\xi) + cS_+'(\xi) + \beta_1 S_+(\xi) = \beta_1 S_1,$$

which leads to

$$\begin{aligned}
G_1[S_+, A_-, I_-, V](\xi) &= \frac{1}{\Lambda_1}\int_{-\infty}^{\xi} e^{\rho_{11}(\xi-s)} H_1[S_+, A_-, I_-, V](s)ds \\
&\quad + \frac{1}{\Lambda_1}\int_{\xi}^{+\infty} e^{\rho_{12}(\xi-s)} H_1[S_+, A_-, I_-, V](s)ds \\
&\leqslant \frac{\beta_1 S_1}{\Lambda_1}\int_{-\infty}^{\xi} e^{\rho_{11}(\xi-s)}ds + \frac{\beta_1 S_1}{\Lambda_1}\int_{\xi}^{+\infty} e^{\rho_{12}(\xi-s)}ds \\
&= S_1 \quad \text{for} \quad \xi \in \mathbb{R}.
\end{aligned}$$

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Therefore, (2.31) holds. The proofs of (2.32)–(2.34) are similar to that of (2.31), so we omit them here. □

Now we introduce a functional space

$$B_\omega(\mathbb{R},\Omega)=\left\{\Psi=(\Psi_1,\Psi_2,\Psi_3,\Psi_4)(\xi)\in C(\mathbb{R},\mathbb{R}):\|\Psi\|_\omega:=\max_{1\leqslant i\leqslant 4}\{\sup_{\xi\in\mathbb{R}}|\Psi_i(\xi)|e^{-\omega|\xi|}\}<+\infty\right\},$$

where $\omega\in(0,\min_{1\leqslant i\leqslant 4}\{-\rho_{i1},\rho_{i2}\})$.

Lemma 2.8 *The operator* $G=\big(G_1,G_2,G_3,G_4\big):\Gamma\mapsto\Gamma$ *is continuous with respect to the norm* $||\cdot||_\omega$ *in* $B_\omega(\mathbb{R},\Omega)$.

Proof For any $(S,A,I,V)(\xi)$ and $(\tilde S,\tilde A,\tilde I,\tilde V)(\xi)\in\Gamma$, it is not difficult to verify that there exist constants $K_i>0$ $(i=1,2,3,4)$ such that

$$\begin{aligned}&|H_i[S,A,I,V](\xi)-H_i[\tilde S,\tilde A,\tilde I,\tilde V](\xi)|e^{-\omega|\xi|}\\&\quad\leqslant K_i[||S-\tilde S||_\omega+||A-\tilde A||_\omega+||I-\tilde I||_\omega+||V-\tilde V||_\omega].\end{aligned}\tag{2.35}$$

Then using (2.35), we have

$$\begin{aligned}&|G_i[S,A,I,V](\xi)-G_i[\tilde S,\tilde A,\tilde I,\tilde V](\xi)|e^{-\omega|\xi|}\\&\leqslant\frac{K_i}{\Lambda_i}\left[\int_{-\infty}^{\xi}e^{\rho_{i1}(\xi-s)}e^{\omega|s|-\omega|\xi|}ds+\int_{\xi}^{+\infty}e^{\rho_{i2}(\xi-s)}e^{\omega|s|-\omega|\xi|}ds\right]\\&\quad\times[||S-\tilde S||_\omega+||A-\tilde A||_\omega+||I-\tilde I||_\omega+||V-\tilde V||_\omega]\\&\leqslant\frac{K_i}{\Lambda_i}\left[\int_{-\infty}^{\xi}e^{\rho_{i1}(\xi-s)}e^{\omega|s-\xi|}ds+\int_{\xi}^{+\infty}e^{\rho_{i2}(\xi-s)}e^{\omega|s-\xi|}ds\right]\\&\quad\times[||S-\tilde S||_\omega+||A-\tilde A||_\omega+||I-\tilde I||_\omega+||V-\tilde V||_\omega]\\&=\frac{K_i}{\Lambda_i}\left(\frac{1}{\rho_{i2}-\omega}-\frac{1}{\rho_{i1}+\omega}\right)[||S-\tilde S||_\omega+||A-\tilde A||_\omega+||I-\tilde I||_\omega+||V-\tilde V||_\omega],\end{aligned}$$

which implies that the operator G is continuous with respect to the norm $||\cdot||_\omega$ in $B_\omega(\mathbb{R},\Omega)$. □

Lemma 2.9 *The operator* $G=\big(G_1,G_2,G_3,G_4\big):\Gamma\mapsto\Gamma$ *is compact with respect to the norm* $||\cdot||_\omega$ *in* $B_\omega(\mathbb{R},\Omega)$.

Proof By the definition of $H_i[S,A,I,V](\xi)$, we have that

$$|H_i[S,A,I,V](\xi)|\leqslant L_0,\quad i=1,2,3,4,$$

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where $L_0 := \max\{\beta_1 S_1,\ \beta_2 M^*,\ \beta_3 M^*,\ \beta_4 V_1\}$. Then it follows that

$$\begin{aligned}\left|\frac{d}{d\xi}G_i[S,A,I,V](\xi)\right| &= \frac{1}{\Lambda_i}\left|\rho_{i1}\int_{-\infty}^{\xi} e^{\rho_{i1}(\xi-s)}H_i[S,A,I,V](s)ds\right.\\ &\quad\left.+\rho_{i2}\int_{\xi}^{+\infty} e^{\rho_{i2}(\xi-s)}H_i[S,A,I,V](s)ds\right|\\ &\leqslant \frac{L_0}{\Lambda_i}\left(|\rho_{i1}|\int_{-\infty}^{\xi} e^{\rho_{i1}(\xi-s)}ds + |\rho_{i2}|\int_{\xi}^{+\infty} e^{\rho_{i2}(\xi-s)}ds\right)\\ &\leqslant \frac{2L_0}{\Lambda_i}, \quad i=1,2,3,4.\end{aligned} \tag{2.36}$$

Using Lemma 2.7, we have

$$\sum_{i=1}^{4}|G_i[S,A,I,V](\xi)| \leqslant (S_1 + 2M^* + V_1). \tag{2.37}$$

Hence, for any $\epsilon > 0$, there exists a constant $N > 0$ such that

$$\left(\sum_{i=1}^{4}|G_i[S,A,I,V](\xi)|\right)e^{-\omega|\xi|} < (S_1 + 2M^* + V_1)e^{-\omega N} < \epsilon \quad \text{for } |\xi| > N. \tag{2.38}$$

Utilising (2.36), (2.37) and Arzelà–Ascoli's theorem, we can choose finite elements in $G(\Gamma)$ such that there are a finite ϵ-net of $G(\Gamma)(\xi)$ with the supremum norm if we restrict them on $\xi \in [-N, N]$, which is also a finite ϵ-net of $G(\Gamma)(\xi)$ for $\xi \in \mathbb{R}$ with the norm $||\cdot||_\omega$ (by (2.38)). Therefore, the operator G is compact with respect to the norm $||\cdot||_\omega$ in $B_\omega(\mathbb{R}, \Omega)$. The proof is completed. □

According to Lemma 2.7–Lemma 2.9 and Schauder's fixed point theorem, we can obtain the following result.

Proposition 2.1 *Assume that $\mu_A > \nu_1\delta_A$, $\mu > \nu_2$, $R_0 > 1$ and $c > c^*$, then system (2.2) admits a solution $(S, A, I, V)(\xi)$ satisfying*

$$\begin{aligned}&S_-(\xi) \leqslant S(\xi) \leqslant S_+(\xi), \quad A_-(\xi) \leqslant A(\xi) \leqslant A_+(\xi),\\ &I_-(\xi) \leqslant I(\xi) \leqslant I_+(\xi), \quad V_-(\xi) \leqslant V(\xi) \leqslant V_+(\xi), \quad \xi \in \mathbb{R}.\end{aligned} \tag{2.39}$$

In the following, we will derive some properties of the solution for (2.2).

Proposition 2.2 *Assume that $\mu_A > \nu_1\delta_A$, $\mu > \nu_2$, $R_0 > 1$ and $c > c^*$, then system (2.2) has a solution $(S, A, I, V)(\xi) \in \Gamma$ satisfying*

(i) $S(\xi) > 0$, $A(\xi) > 0$, $I(\xi) > 0$ *and* $V(\xi) > 0$ *for* $\xi \in \mathbb{R}$;
(ii) $S(\xi) \to S_1$, $A(\xi) \to 0$, $I(\xi) \to 0$, $V(\xi) \to V_1$, $A(\xi) = O(e^{\rho_1\xi})$ *and* $I(\xi) = O(e^{\rho_1\xi})$ *as* $\xi \to -\infty$;
(iii) $S(\xi)$ *and* $V(\xi)$ *are strictly decreasing for* $\xi \in \mathbb{R}$;
(iv) $S(\xi) \to S_2$, $A(\xi) \to 0$, $I(\xi) \to 0$, $V(\xi) \to V_2$ *as* $\xi \to +\infty$;
(v) $S'(\xi)$, $A'(\xi)$, $I'(\xi)$, $V'(\xi)$, $S''(\xi)$, $A''(\xi)$, $I''(\xi)$, $V''(\xi) \to 0$ *as* $\xi \to \pm\infty$;

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(vi)

$$\int_{-\infty}^{+\infty}[\mu_A A(\eta)+\varepsilon A^2(\eta)]d\eta = c(1-p)(S_1-S_2)+c(1-p_V)(V_1-V_2), \tag{2.40}$$

and

$$\int_{-\infty}^{+\infty}[\mu I(\eta)+\varepsilon I^2(\eta)]d\eta = cp(S_1-S_2)+cp_V(V_1-V_2); \tag{2.41}$$

(vii) $S(\xi)<S_1$, $A(\xi)<(1-p)(S_1-S_2)+(1-p_V)(V_1-V_2)$, $I(\xi)<p(S_1-S_2)+p_V(V_1-V_2)$ *and* $V(\xi)<V_1$ *for* $\xi\in\mathbb{R}$.

Proof (i) Noting that $H_1[S,A,I,V](\xi)$ is monotonically increasing with respect to S and monotonically decreasing in both A and I, we have from (2.28) that

$$\begin{aligned} S(\xi) &= \frac{1}{\Lambda_1}\int_{-\infty}^{\xi} e^{\rho_{11}(\xi-s)}H_1[S,A,I,V](s)ds \\ &\quad + \frac{1}{\Lambda_1}\int_{\xi}^{+\infty} e^{\rho_{12}(\xi-s)}H_1[S,A,I,V](s)ds \\ &\geqslant \frac{1}{\Lambda_1}\int_{-\infty}^{\xi} e^{\rho_{11}(\xi-s)}[\beta_1-\beta(\delta_A+1)M^*]S_-(s)ds \\ &\quad + \frac{1}{\Lambda_1}\int_{\xi}^{+\infty} e^{\rho_{12}(\xi-s)}[\beta_1-\beta(\delta_A+1)M^*]S_-(s)ds \\ &> 0. \end{aligned} \tag{2.42}$$

Similarly, one can obtain the positiveness of $A(\xi)$, $I(\xi)$ and $V(\xi)$ on $\mathbb{R}$.

(ii) Applying squeeze rule to (2.39), one can get

$$\begin{aligned} &S(\xi)\to S_1,\ A(\xi)\to 0,\ I(\xi)\to 0,\ V(\xi)\to V_1, \\ &A(\xi)=O(e^{\rho_1\xi}) \text{ and } I(\xi)=O(e^{\rho_1\xi}) \quad \text{as} \quad \xi\to-\infty. \end{aligned} \tag{2.43}$$

(iii) In view of (2.30), we have

$$\begin{aligned} S'(\xi) &= \frac{\rho_{11}}{\Lambda_1}\int_{-\infty}^{\xi} e^{\rho_{11}(\xi-s)}H_1[S,A,I,V](s)ds \\ &\quad + \frac{\rho_{12}}{\Lambda_1}\int_{\xi}^{+\infty} e^{\rho_{12}(\xi-s)}H_1[S,A,I,V](s)ds, \end{aligned} \tag{2.44}$$

which together with L'Hospital rule yields

$$\lim_{\xi\to\pm\infty} S'(\xi)=0. \tag{2.45}$$

Similarly, one can get

$$\lim_{\xi\to\pm\infty} A'(\xi)=\lim_{\xi\to\pm\infty} I'(\xi)=\lim_{\xi\to\pm\infty} V'(\xi)=0. \tag{2.46}$$

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Rewrite the first equation in system (2.2) as

$$-[e^{-\frac{c}{d_1}\xi}S'(\xi)]' = -\frac{\beta}{d_1}[\delta_A A(\xi - c\tau_1) + I(\xi - c\tau_2)]S(\xi)e^{-\frac{c}{d_1}\xi}. \tag{2.47}$$

Then using (2.45) together with the positiveness of A, I, S on $\mathbb{R}$ and integrating (2.47) from ξ to $+\infty$, we obtain

$$S'(\xi) = -\frac{\beta}{d_1}e^{\frac{c}{d_1}\xi}\int_{\xi}^{+\infty}[\delta_A A(\eta - c\tau_1) + I(\eta - c\tau_2)]S(\eta)e^{-\frac{c}{d_1}\eta}d\eta < 0 \quad \text{for } \xi \in \mathbb{R}, \tag{2.48}$$

which implies that $S(\xi)$ is strictly decreasing on $\mathbb{R}$. Analogously, one can deduce that $V(\xi)$ is strictly decreasing on $\mathbb{R}$.

(iv) Using (2.42) and (2.48), we get that $\lim_{\xi\to+\infty} S(\xi)$ exists, which is denoted by S_2. Then we infer that $S_1 > S_2 \geqslant 0$. Similarly, one can obtain that $\lim_{\xi\to+\infty} V(\xi) := V_2$ exists and $V_1 > V_2 \geqslant 0$.

Next we show that $\lim_{\xi\to+\infty} A(\xi) = \lim_{\xi\to+\infty} I(\xi) = 0$. From (2.36), we know that $S'(\xi)$, $A'(\xi)$, $I'(\xi)$ and $V'(\xi)$ are uniformly bounded on $\mathbb{R}$. Then integrating the first equation in system (2.2) over $(-\infty, \xi)$ and using (2.43) and (2.45) give that

$$\int_{-\infty}^{\xi}\beta[\delta_A A(\eta - c\tau_1) + I(\eta - c\tau_2)]S(\eta)d\eta = d_1S'(\xi) - c[S(\xi) - S_1] < +\infty, \tag{2.49}$$

for $\xi \in \mathbb{R}$, which together with the positiveness of the integrand implies that

$$\int_{-\infty}^{+\infty}[\delta_A A(\eta - c\tau_1) + I(\eta - c\tau_2)]S(\eta)d\eta < +\infty. \tag{2.50}$$

By the similar argument as above, one can obtain that

$$\int_{-\infty}^{+\infty}[\delta_A A(\eta - c\tau_1) + I(\eta - c\tau_2)]V(\eta)d\eta < +\infty. \tag{2.51}$$

Integrating the second equation in system (2.2) from $-\infty$ to ξ, we have

$$\begin{aligned}
&\mu_A\int_{-\infty}^{\xi}A(\eta)d\eta + \varepsilon\int_{-\infty}^{\xi}A^2(\eta)d\eta \\
&\quad = d_2A'(\xi) - cA(\xi) + \beta(1-p)\int_{-\infty}^{\xi}[\delta_A A(\eta - c\tau_1) + I(\eta - c\tau_2)]S(\eta)d\eta \\
&\qquad + \beta(1-p_V)\delta_V\int_{-\infty}^{\xi}[\delta_A A(\eta - c\tau_1) + I(\eta - c\tau_2)]V(\eta)d\eta \\
&\quad < +\infty \quad \text{for } \xi \in \mathbb{R},
\end{aligned} \tag{2.52}$$

where we have used (2.43), (2.46), (2.50), (2.51) and the uniform boundedness of $A(\xi)$ and $A'(\xi)$ on $\mathbb{R}$. Then we derive that $\int_{\mathbb{R}} A(\eta)d\eta < +\infty$. Consequently, we have $\lim_{\xi\to+\infty} A(\xi) = 0$. Similarly, one can prove that $\lim_{\xi\to+\infty} I(\xi) = 0$.

(v) In the proof of (iii), we have derived that $S'(\xi), A'(\xi), I'(\xi), V'(\xi) \to 0$ as $\xi \to \pm\infty$. Then we obtain from system (2.2) and the asymptotic boundaries of S, A, I, V at infinity that

$$S''(\xi),\ A''(\xi),\ I''(\xi),\ V''(\xi) \to 0 \ \text{ as } \xi \to \pm\infty. \tag{2.53}$$

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(vi) Using (2.45), (2.49) and $S(+\infty)=S_2$, we have

$$\beta\int_{-\infty}^{+\infty}[\delta_A A(\eta-c\tau_1)+I(\eta-c\tau_2)]S(\eta)d\eta=c(S_1-S_2). \tag{2.54}$$

Similarly, we get

$$\delta_V\beta\int_{-\infty}^{+\infty}[\delta_A A(\eta-c\tau_1)+I(\eta-c\tau_2)]V(\eta)d\eta=c(V_1-V_2). \tag{2.55}$$

Using $A(+\infty)=A'(+\infty)=0$, (2.52), (2.54) and (2.55), we obtain

$$\int_{-\infty}^{+\infty}[\mu_A A(\eta)+\varepsilon A^2(\eta)]d\eta=c(1-p)(S_1-S_2)+c(1-p_V)(V_1-V_2). \tag{2.56}$$

Analogously, one can deduce that

$$\int_{-\infty}^{+\infty}[\mu I(\eta)+\varepsilon I^2(\eta)]d\eta=cp(S_1-S_2)+cp_V(V_1-V_2). \tag{2.57}$$

(vii) By (2.39), we have that $S(\xi)\leqslant S_1$ for $\xi\in\mathbb{R}$. Suppose that there exists some $\acute{\xi}\in\mathbb{R}$ such that $S(\acute{\xi})=S_1$, then $S'(\acute{\xi})=0$ and $S''(\acute{\xi})\leqslant 0$. Using the positiveness of $S(\xi)$, $A(\xi)$ and $I(\xi)$ on $\mathbb{R}$, we deduce from the first equation in (2.2) that

$$\begin{aligned}0&=d_1S''(\acute{\xi})-cS'(\acute{\xi})-\beta[\delta_A A(\acute{\xi}-c\tau_1)+I(\acute{\xi}-c\tau_2)]S(\acute{\xi})\\&\leqslant-\beta[\delta_A A(\acute{\xi}-c\tau_1)+I(\acute{\xi}-c\tau_2)]S(\acute{\xi})\\&<0,\end{aligned}$$

which leads to a contradiction. Hence $S(\xi)<S_1$ for $\xi\in\mathbb{R}$. In a similar way, one can get that $V(\xi)<V_1$ for $\xi\in\mathbb{R}$.

Construct a function

$$F(\xi):=\frac{1}{c}\int_{-\infty}^{\xi}[\mu I(\eta)+\varepsilon I^2(\eta)]d\eta+\frac{1}{c}\int_{\xi}^{+\infty}e^{\frac{c}{d_3}(\xi-\eta)}[\mu I(\eta)+\varepsilon I^2(\eta)]d\eta. \tag{2.58}$$

By L'Hospital rule and (2.57), we obtain $F(-\infty)=0$, $F(+\infty)=p(S_1-S_2)+p_V(V_1-V_2)$ and $F'(+\infty)=0$. It is easy to see that $F(\xi)$ satisfies the following equation:

$$cF'(\xi)-d_3F''(\xi)=\mu I(\xi)+\varepsilon I^2(\xi). \tag{2.59}$$

Define $W(\xi):=F(\xi)+I(\xi)$. Then $W(+\infty)=p(S_1-S_2)+p_V(V_1-V_2)$ and $W'(+\infty)=0$. By (2.59) and the third equation in (2.2), we have

$$cW'(\xi)-d_3W''(\xi)=\beta[pS(\xi)+p_V\delta_V V(\xi)][\delta_A A(\xi-c\tau_1)+I(\xi-c\tau_2)], \tag{2.60}$$

that is,

$$[-d_3e^{-\frac{c}{d_3}\xi}W'(\xi)]'=e^{-\frac{c}{d_3}\xi}\beta[pS(\xi)+p_V\delta_V V(\xi)][\delta_A A(\xi-c\tau_1)+I(\xi-c\tau_2)]. \tag{2.61}$$

Integrating (2.61) from ξ to $+\infty$ yields

$$\begin{aligned}W'(\xi)&=\frac{1}{d_3}\int_{\xi}^{+\infty}e^{\frac{c}{d_3}(\xi-\eta)}\beta[pS(\eta)+p_V\delta_V V(\eta)][\delta_A A(\eta-c\tau_1)+I(\eta-c\tau_2)]d\eta\\&>0\quad\text{for}\quad\xi\in\mathbb{R}.\end{aligned} \tag{2.62}$$

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Hence, $W(\xi)$ is strictly increasing on $\mathbb{R}$, which implies that

$$I(\xi) < W(\xi) < W(+\infty) = p(S_1 - S_2) + p_V(V_1 - V_2) \quad \text{for} \quad \xi \in \mathbb{R}. \tag{2.63}$$

Similarly, one can obtain that $A(\xi) < (1-p)(S_1 - S_2) + (1-p_V)(V_1 - V_2)$ for $\xi \in \mathbb{R}$. The proof of this proposition is finished. □

3 Super-critical travelling waves

In this section, we will obtain the existence of super-critical travelling wave solutions for (1.3).

Theorem 3.1 *Suppose that $\mu_A > \nu_1\delta_A$, $\mu > \nu_2$, $R_0 > 1$ and $c > c^*$, then (1.3) admits a travelling wave solution $(S, A, I, V)(\xi)$ satisfying*

(i) $S(\xi) > 0$, $A(\xi) > 0$, $I(\xi) > 0$ *and* $V(\xi) > 0$ *for* $\xi \in \mathbb{R}$;

(ii) $S(\xi) \to S_1$, $A(\xi) \to 0$, $I(\xi) \to 0$, $V(\xi) \to V_1$, $A(\xi) = O(e^{\rho_1 \xi})$ *and* $I(\xi) = O(e^{\rho_1 \xi})$ *as* $\xi \to -\infty$;

(iii) $S(\xi)$ *and* $V(\xi)$ *are strictly decreasing for* $\xi \in \mathbb{R}$;

(iv) $S(\xi) \to S_2$, $A(\xi) \to 0$, $I(\xi) \to 0$, $V(\xi) \to V_2$ *as* $\xi \to +\infty$;

(v) $S'(\xi)$, $A'(\xi)$, $I'(\xi)$, $V'(\xi)$, $S''(\xi)$, $A''(\xi)$, $I''(\xi)$, $V''(\xi) \to 0$ *as* $\xi \to \pm\infty$;

(vi)

$$\int_{-\infty}^{+\infty} \mu_A A(\eta) d\eta = c(1-p)(S_1 - S_2) + c(1-p_V)(V_1 - V_2),$$

and

$$\int_{-\infty}^{+\infty} \mu I(\eta) d\eta = cp(S_1 - S_2) + cp_V(V_1 - V_2);$$

(vii) $S(\xi) < S_1$, $A(\xi) < (1-p)(S_1 - S_2) + (1-p_V)(V_1 - V_2)$, $I(\xi) < p(S_1 - S_2) + p_V(V_1 - V_2)$ *and* $V(\xi) < V_1$ *for* $\xi \in \mathbb{R}$.

Proof Let $\{\varepsilon_n\}_{n\in\mathbb{N}}$ be a decreasing sequence such that $0 < \varepsilon_n < 1$ and $\varepsilon_n \to 0$ as $n \to \infty$, then we have from Proposition 2.1 that for each $n \in \mathbb{N}$, there exists a solution $(S_n, A_n, I_n, V_n)(\xi)$ of the system

$$\begin{cases} cS_n'(\xi) = d_1 S_n''(\xi) - \beta[\delta_A A_n(\xi - c\tau_1) + I_n(\xi - c\tau_2)]S_n(\xi), \\ cA_n'(\xi) = d_2 A_n''(\xi) + \beta[(1-p)S_n(\xi) + (1-p_V)\delta_V V_n(\xi)][\delta_A A_n(\xi - c\tau_1) + I_n(\xi - c\tau_2)] \\ \qquad\qquad - \mu_A A_n(\xi) - \varepsilon_n A_n^2(\xi), \\ cI_n'(\xi) = d_3 I_n''(\xi) + \beta[pS_n(\xi) + p_V\delta_V V_n(\xi)][\delta_A A_n(\xi - c\tau_1) + I_n(\xi - c\tau_2)] \\ \qquad\qquad - \mu I_n(\xi) - \varepsilon_n I_n^2(\xi), \\ cV_n'(\xi) = d_4 V_n''(\xi) - \delta_V\beta[\delta_A A_n(\xi - c\tau_1) + I_n(\xi - c\tau_2)]V_n(\xi), \end{cases} \tag{3.1}$$

which satisfies the asymptotic boundary conditions

$$\begin{aligned} (S_n, A_n, I_n, V_n)(-\infty) &= (S_1, 0, 0, V_1), \\ (S_n, A_n, I_n, V_n)(+\infty) &= (S_2, 0, 0, V_2). \end{aligned} \tag{3.2}$$

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Using Proposition 2.2 (*vii*), we obtain that there exists a constant $M_1 > 0$ independent of ε_n such that

$$\left|[\delta_A A_n(\xi - c\tau_1) + I_n(\xi - c\tau_2)]S_n(\xi)\right| \leqslant M_1 \quad \text{for} \quad \xi \in \mathbb{R}.$$

Then it follows from the first equation in (3.1) and Proposition 2.2 (*v*) that

$$\begin{aligned} |S_n'(\xi)| &= \left| \frac{\beta}{d_1} e^{\frac{c}{d_1}\xi} \int_{\xi}^{+\infty} [\delta_A A_n(\eta - c\tau_1) + I_n(\eta - c\tau_2)]S_n(\eta) e^{-\frac{c}{d_1}\eta} d\eta \right| \\ &\leqslant \frac{\beta}{d_1} e^{\frac{c}{d_1}\xi} M_1 \int_{\xi}^{+\infty} e^{-\frac{c}{d_1}\eta} d\eta \\ &\leqslant \frac{\beta}{c} M_1, \quad \xi \in \mathbb{R}. \end{aligned} \tag{3.3}$$

Similarly, it can be shown that there exist constants $M_i > 0$ $(i = 2, 3, 4)$ independent of ε_n such that

$$|A'(\xi)| \leqslant M_2, \quad |I'(\xi)| \leqslant M_3, \quad |V'(\xi)| \leqslant M_4, \quad \xi \in \mathbb{R}. \tag{3.4}$$

By Proposition 2.2 (*vii*), (3.1), (3.3) and (3.4), we deduce that $S_n''(\xi)$, $A_n''(\xi)$, $I_n''(\xi)$ and $V_n''(\xi)$ are uniformly bounded on $\mathbb{R}$. Differentiating (3.1) with respect to ξ, we then conclude that $S_n'''(\xi)$, $A_n'''(\xi)$, $I_n'''(\xi)$ and $V_n'''(\xi)$ are uniformly bounded on $\mathbb{R}$. Now it follows from Arzelà–Ascoli's theorem that there exists a subsequence, which is still denoted by $\{\varepsilon_n\}_{n\in\mathbb{N}}$ such that

$$S_n^{(j)}(\xi) \to S^{(j)}(\xi), \quad A_n^{(j)}(\xi) \to A^{(j)}(\xi), \quad I_n^{(j)}(\xi) \to I^{(j)}(\xi) \quad \text{and} \quad V_n^{(j)}(\xi) \to V^{(j)}(\xi)$$

$(j = 0, 1, 2)$ uniformly in any bounded closed interval of $\mathbb{R}$ as $n \to \infty$. Passing to the limit in (3.1) as $n \to \infty$ yields that $(S, A, I, V)(\xi)$ is the solution of (1.5).

By Proposition 2.1, one can get that

$$\begin{aligned} &\max\{S_1 - \sigma^{-1}e^{\sigma\xi}, 0\} \leqslant S_n(\xi) \leqslant S_1, \\ &\max\{L_1 e^{\rho_1\xi} - qe^{(\rho_1+\eta)\xi}, 0\} \leqslant A_n(\xi) \leqslant \min\{L_1 e^{\rho_1\xi}, L_1 M\}, \\ &\max\{L_2 e^{\rho_1\xi} - q\kappa e^{(\rho_1+\eta)\xi}, 0\} \leqslant I_n(\xi) \leqslant \min\{L_2 e^{\rho_1\xi}, L_2 M\}, \\ &\max\{V_1 - \sigma^{-1}e^{\sigma\xi}, 0\} \leqslant V_n(\xi) \leqslant V_1 \end{aligned} \tag{3.5}$$

with

$$M > \max\left\{0, \frac{\nu_1\delta_A L_1 + \nu_1 L_2 - \mu_A L_1}{\varepsilon_n L_1^2}, \frac{\nu_2\delta_A L_1 + \nu_2 L_2 - \mu L_2}{\varepsilon_n L_2^2}\right\}.$$

Passing to the limit in (3.5) as $n \to \infty$ yields that

$$\begin{aligned} &\max\{S_1 - \sigma^{-1}e^{\sigma\xi}, 0\} \leqslant S(\xi) \leqslant S_1, \\ &\max\{L_1 e^{\rho_1\xi} - qe^{(\rho_1+\eta)\xi}, 0\} \leqslant A(\xi) \leqslant L_1 e^{\rho_1\xi}, \\ &\max\{L_2 e^{\rho_1\xi} - q\kappa e^{(\rho_1+\eta)\xi}, 0\} \leqslant I(\xi) \leqslant L_2 e^{\rho_1\xi}, \\ &\max\{V_1 - \sigma^{-1}e^{\sigma\xi}, 0\} \leqslant V(\xi) \leqslant V_1. \end{aligned} \tag{3.6}$$

Applying squeeze rule in (3.6) gives that $(S, A, I, V)(-\infty) = (S_1, 0, 0, V_1)$, $A(\xi) = O(e^{\rho_1\xi})$ and $I(\xi) = O(e^{\rho_1\xi})$. Similar to the proofs of Proposition 2.2 (*i*) and (*iii*) – (*vii*), one can derive the rest of the assertions in this theorem. □

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4 Critical travelling waves

This section is devoted to establishing the existence of critical travelling wave solutions for (1.3).

Theorem 4.1 *Suppose that $\mu_A > \nu_1\delta_A$, $\mu > \nu_2$, $R_0 > 1$ and $c = c^*$, then (1.3) admits a travelling wave solution $(S_*, A_*, I_*, V_*)(\xi)$ satisfying*

(i) $S_*(\xi) > 0$, $A_*(\xi) > 0$, $I_*(\xi) > 0$, $V_*(\xi) > 0$ *for* $\xi \in \mathbb{R}$;
(ii) $S_*(\xi) \to S_1$, $A_*(\xi) \to 0$, $I_*(\xi) \to 0$, $V_*(\xi) \to V_1$ *as* $\xi \to -\infty$;
(iii) $S_*(\xi)$ *and* $V_*(\xi)$ *are strictly decreasing for* $\xi \in \mathbb{R}$;
(iv) $S_*(\xi) \to S_2$, $A_*(\xi) \to 0$, $I_*(\xi) \to 0$, $V_*(\xi) \to V_2$ *as* $\xi \to +\infty$;
(v) $S_*'(\xi)$, $A_*'(\xi)$, $I_*'(\xi)$, $V_*'(\xi)$, $S_*''(\xi)$, $A_*''(\xi)$, $I_*''(\xi)$, $V_*''(\xi) \to 0$ *as* $\xi \to \pm\infty$;
(vi)

$$\int_{-\infty}^{+\infty} \mu_A A_*(\eta)d\eta = c^*(1-p)(S_1 - S_2) + c^*(1-p_V)(V_1 - V_2),$$

and

$$\int_{-\infty}^{+\infty} \mu I_*(\eta)d\eta = c^* p(S_1 - S_2) + c^* p_V(V_1 - V_2);$$

(vii) $S_*(\xi) < S_1$, $A_*(\xi) < (1-p)(S_1 - S_2) + (1-p_V)(V_1 - V_2)$, $I_*(\xi) < p(S_1 - S_2) + p_V(V_1 - V_2)$ *and* $V_*(\xi) < V_1$ *for* $\xi \in \mathbb{R}$.

Proof Let $\{c_n\}_{n\in\mathbb{N}} \subset (c^*, c^* + 1]$ be a decreasing sequence such that $c_n \to c^*$ as $n \to \infty$. By Theorem 3.1, for each $n \in \mathbb{N}$, we have that (1.5) with $c = c_n$ admits a positive solution $(S_n, A_n, I_n, V_n)(\xi)$, that is,

$$\begin{cases} c_n S_n'(\xi) = d_1 S_n''(\xi) - \beta[\delta_A A_n(\xi - c_n\tau_1) + I_n(\xi - c_n\tau_2)]S_n(\xi), \\ c_n A_n'(\xi) = d_2 A_n''(\xi) + \beta[(1-p)S_n(\xi) + (1-p_V)\delta_V V_n(\xi)][\delta_A A_n(\xi - c_n\tau_1) \\ \qquad + I_n(\xi - c_n\tau_2)] - \mu_A A_n(\xi), \\ c_n I_n'(\xi) = d_3 I_n''(\xi) + \beta[pS_n(\xi) + p_V\delta_V V_n(\xi)][\delta_A A_n(\xi - c_n\tau_1) \\ \qquad + I_n(\xi - c_n\tau_2)] - \mu I_n(\xi), \\ c_n V_n'(\xi) = d_4 V_n''(\xi) - \delta_V\beta[\delta_A A_n(\xi - c_n\tau_1) + I_n(\xi - c_n\tau_2)]V_n(\xi), \end{cases} \tag{4.1}$$

which satisfies

$$\begin{aligned} &\max\left\{0, S_1 - \sigma^{-1}e^{\sigma\xi}\right\} \leqslant S_n(\xi) \leqslant S_1, \\ &\max\left\{0, L_1 e^{\rho_1(c_n)\xi} - qe^{[\rho_1(c_n)+\eta]\xi}\right\} \leqslant A_n(\xi) \leqslant L_1 e^{\rho_1(c_n)\xi}, \\ &\max\left\{0, L_2 e^{\rho_1(c_n)\xi} - q\kappa e^{[\rho_1(c_n)+\eta]\xi}\right\} \leqslant I_n(\xi) \leqslant L_2 e^{\rho_1(c_n)\xi}, \\ &\max\left\{0, V_1 - \sigma^{-1}e^{\sigma\xi}\right\} \leqslant V_n(\xi) \leqslant V_1 \end{aligned} \tag{4.2}$$

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and

$$A_n(\xi) < (1-p)(S_1 - S_2) + (1-p_V)(V_1 - V_2),\ \ I_n(\xi) < p(S_1 - S_2) + p_V(V_1 - V_2), \xi \in \mathbb{R}, \tag{4.3}$$

where $\rho_1(c_n)$ is the smallest positive root of $\Delta(c_n, \rho) = 0$. From (4.3), we obtain that there exists a constant $C_1 > 0$ independent of c_n such that

$$\left|[\delta_A A_n(\xi - c_n\tau_1) + I_n(\xi - c_n\tau_2)]S_n(\xi)\right| \leqslant C_1, \quad \xi \in \mathbb{R}.$$

By the similar deductions as that in (3.3) and (3.4), we have

$$|S_n'(\xi)| + |A_n'(\xi)| + |I_n'(\xi)| + |V_n'(\xi)| < C_2, \quad \xi \in \mathbb{R}, \tag{4.4}$$

where $C_2 > 0$ is independent of c_n. By (4.1)–(4.4), we deduce that $S_n''(\xi)$, $A_n''(\xi)$, $I_n''(\xi)$, $V_n''(\xi)$, $S_n'''(\xi)$, $A_n'''(\xi)$, $I_n'''(\xi)$ and $V_n'''(\xi)$ are uniformly bounded on $\mathbb{R}$. Then we infer from Arzelà–Ascoli's theorem that there exists a subsequence, which is still labelled by $\{c_n\}_{n\in\mathbb{N}}$ such that

$$S_n^{(j)}(\xi) \to S_*^{(j)}(\xi), \quad A_n^{(j)}(\xi) \to A_*^{(j)}(\xi), \quad I_n^{(j)}(\xi) \to I_*^{(j)}(\xi) \quad \text{and} \quad V_n^{(j)}(\xi) \to V_*^{(j)}(\xi),$$

$j = 0, 1, 2$ uniformly in any bounded closed interval of $\mathbb{R}$ as $n \to \infty$. Taking the limit in (4.1) as $n \to \infty$ yields that $(S_*, A_*, I_*, V_*)(\xi)$ is the solution of (1.5), that is,

$$\begin{cases} c^* S_*'(\xi) = d_1 S_*''(\xi) - \beta[\delta_A A_*(\xi - c^*\tau_1) + I_*(\xi - c^*\tau_2)]S_*(\xi), \\ c^* A_*'(\xi) = d_2 A_*''(\xi) + \beta[(1-p)S_*(\xi) + (1-p_V)\delta_V V_*(\xi)][\delta_A A_*(\xi - c^*\tau_1) \\ \qquad\qquad + I_*(\xi - c^*\tau_2)] - \mu_A A_*(\xi), \\ c^* I_*'(\xi) = d_3 I_*''(\xi) + \beta[pS_*(\xi) + p_V\delta_V V_*(\xi)][\delta_A A_*(\xi - c^*\tau_1) \\ \qquad\qquad + I_*(\xi - c^*\tau_2)] - \mu I_*(\xi), \\ c^* V_*'(\xi) = d_4 V_*''(\xi) - \delta_V\beta[\delta_A A_*(\xi - c^*\tau_1) + I_*(\xi - c^*\tau_2)]V_*(\xi). \end{cases} \tag{4.5}$$

Moreover, by (4.2) one can get

$$(S_*, A_*, I_*, V_*)(-\infty) = (S_1, 0, 0, V_1). \tag{4.6}$$

Next, we investigate the positiveness of $S_*(\xi)$, $A_*(\xi)$, $I_*(\xi)$ and $V_*(\xi)$ on $\mathbb{R}$. By contradiction, we assume that $S_*(\tilde{\xi}) = 0$ for some $\tilde{\xi} \in \mathbb{R}$. Then there exist two constants l_1, $l_2 \in \mathbb{R}$ such that $\xi_2, \tilde{\xi} \in (l_1, l_2)$, where ξ_2 is defined in (2.12). This implies that $S_*(\xi)$ attains its minimum in (l_1, l_2). Using Theorem 3.1 (*vii*) and the first equation in (4.5), we get that $-d_1 S_*''(\xi) + c^* S_*'(\xi) + \beta_5 S_*(\xi) \geqslant 0$ for $\xi \in [l_1, l_2]$, where $\beta_5 = \delta_A[(1-p)(S_1 - S_2) + (1-p_V)(V_1 - V_2) + p(S_1 - S_2) + p_V(V_1 - V_2)]$. Then it follows from the strong maximum principle that

$$S_*(\xi) \equiv 0 \quad \text{for} \quad \xi \in [l_1, l_2],$$

which contradicts the fact that $S_*(\xi) \geqslant S_-(\xi) > 0$ for $\xi \in [l_1, \xi_2)$. Hence, we have

$$S_*(\xi) > 0 \quad \text{for} \quad \xi \in \mathbb{R}. \tag{4.7}$$

In an analogous manner, one can get that

$$V_*(\xi) > 0 \quad \text{for} \quad \xi \in \mathbb{R}. \tag{4.8}$$

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Now we prove $A_*(\xi) > 0$ and $I_*(\xi) > 0$ on $\mathbb{R}$. By (4.2), one can get

$$A_n(\xi) \leqslant L_1 e^{\rho_1(c_n)\xi} \quad \text{for} \quad \xi \leqslant \xi_3 < 0, \tag{4.9}$$

where ξ_3 is defined in (2.12). Note that

$$\int_{-\infty}^{\xi_3} L_1 e^{\rho_1(c_n)\xi} d\xi = \frac{L_1 e^{\rho_1(c_n)\xi_3}}{\rho_1(c_n)} \leqslant \frac{L_1}{\tilde{\rho}}, \tag{4.10}$$

where $\tilde{\rho} = \min_{c_n \in (c^*, c^*+1]} \rho_1(c_n)$. Then applying dominated convergence theorem and (4.2), we obtain

$$\begin{aligned}
\int_{-\infty}^{\xi_3} A_*(s)ds &= \int_{-\infty}^{\xi_3} \lim_{n\to\infty} A_n(s)ds \\
&= \lim_{n\to\infty} \int_{-\infty}^{\xi_3} A_n(s)ds \\
&\geqslant \lim_{n\to\infty} \int_{-\infty}^{\xi_3} [L_1 e^{\rho_1(c_n)s} - q e^{(\rho_1(c_n)+\eta)s}]ds \\
&= \lim_{n\to\infty} \left[L_1 \frac{e^{\rho_1(c_n)\xi_3}}{\rho_1(c_n)} - q \frac{e^{(\rho_1(c_n)+\eta)\xi_3}}{\rho_1(c_n)+\eta} \right] \\
&= \lim_{n\to\infty} e^{\rho_1(c_n)\xi_3} \frac{L_1[\rho_1(c_n)+\eta] - q e^{\eta\xi_3}\rho_1(c_n)}{\rho_1(c_n)[\rho_1(c_n)+\eta]} \\
&= \lim_{n\to\infty} \frac{e^{\rho_1(c_n)\xi_3} L_1 \eta}{\rho_1(c_n)[\rho_1(c_n)+\eta]} \quad \text{(by the definition of } \xi_3, \text{ i.e., } L_1 = q e^{\eta\xi_3}) \\
&= \frac{e^{\rho_1(c^*)\xi_3} L_1 \eta}{\rho_1(c^*)[\rho_1(c^*)+\eta]} \\
&> 0,
\end{aligned} \tag{4.11}$$

which implies that there exists a finite interval $[a, b] \subseteq (-\infty, \xi_3)$ such that

$$A_*(\xi) > 0 \quad \text{for} \quad \xi \in [a, b]. \tag{4.12}$$

Assume that $A_*(\hat{\xi}) = 0$ for some $\hat{\xi} \in \mathbb{R}$. Then there exist two constants $c, d \in \mathbb{R}$ such that $(a, b) \subsetneqq [c, d]$ and $\hat{\xi} \in [c, d]$. This yields that $A_*(\xi)$ attains its minimum in (c, d). From the second equation in (4.5), we obtain that $-d_2 A_*''(\xi) + c^* A_*'(\xi) + \mu_A A_*(\xi) \geqslant 0$ for $\xi \in [c, d]$. Then we infer from the strong maximum principle that $A_*(\xi) \equiv 0$ for $\xi \in [c, d]$, which contradicts (4.12). Therefore, $A_*(\xi) > 0$ for $\xi \in \mathbb{R}$. Similarly, one can show $I_*(\xi) > 0$ for $\xi \in \mathbb{R}$. The rest of the claims in this theorem can be proved by the similar arguments as in Theorem 3.1 $(iii) - (vii)$. □

5 Concluding remarks

In this paper, we proposed a diffusive SAIV epidemic model with time delays (i.e., (1.3)), which seems to be more realistic than the non-delayed model (1.2). Introducing a perturbed system and employing upper-lower solutions method, Schauder's fixed point theorem and limiting approach, we obtain the existence of super-critical and critical nontrivial positive travelling wave solutions

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for (1.3). Our work presents under what conditions can an epidemic disease spread. This may help people to take effective measures in advance to prevent the influenza epidemic or decrease the damage.

What is the influence of incubation times on the quantitative properties of the travelling wave solutions is an interesting problem in the field of epidemiology. But, it is a very challenging problem from the mathematical point of view. Since the incubation times are introduced in our model, the corresponding travelling wave system is a system of ordinary differential equations with time delays. At present, we have no good ideas to investigate this system analytically. Moreover, to the best of our knowledge, there have been almost no analytical results on this topic in the literature. Nevertheless, we can discuss the influence of incubation times on the value of the (critical) speed as below.

From the proof of Lemma 2.1, we have that $0 < \rho_1 \leqslant \rho^* \leqslant \rho_2 < \rho_m^c := \min\{\rho_1^+, \rho_2^+\}$ (where ρ_1^+ and ρ_2^+ are the positive roots of the equations $h_1(c, \rho) = 0$ and $h_2(c, \rho) = 0$, respectively). It follows from the expression of $\Delta(c, \rho)$ that

$$\frac{\partial \Delta(c,\rho)}{\partial c} = (-\rho - \nu_1\delta_A\tau_1\rho e^{-c\tau_1\rho})h_2(c,\rho) + (-\rho - \nu_2\tau_2\rho e^{-c\tau_2\rho})h_1(c,\rho) + \nu_1\nu_2\delta_A(\tau_1+\tau_2)\rho e^{-c(\tau_1+\tau_2)\rho} > 0,$$

$$\frac{\partial \Delta(c,\rho)}{\partial \tau_1} = -c\rho\nu_1\delta_A e^{-c\tau_1\rho}h_2(c,\rho) + \nu_1\nu_2\delta_A c\rho e^{-c(\tau_1+\tau_2)\rho} > 0$$

and

$$\frac{\partial \Delta(c,\rho)}{\partial \tau_2} = -c\rho\nu_2 e^{-c\tau_2\rho}h_1(c,\rho) + \nu_1\nu_2\delta_A c\rho e^{-c(\tau_1+\tau_2)\rho} > 0,$$

since $h_1(c, \rho) < 0$ and $h_2(c, \rho) < 0$ for $0 < \rho < \rho_m^c$. Then we conclude that

$$\frac{dc}{d\tau_1} = -\frac{\frac{\partial \Delta(c,\rho)}{\partial \tau_1}}{\frac{\partial \Delta(c,\rho)}{\partial c}} < 0 \quad \text{and} \quad \frac{dc}{d\tau_2} = -\frac{\frac{\partial \Delta(c,\rho)}{\partial \tau_2}}{\frac{\partial \Delta(c,\rho)}{\partial c}} < 0 \quad \text{for} \quad 0 < \rho < \rho_{\mathrm{m}}^{\mathrm{c}}.$$

Therefore, the decreasing of the incubation times can lead to the increasing of the value of the (critical) speed. From the biological point of view, this indicates that the more effective the infections are, the faster the disease spreads.

In addition, one can make numerical simulation of the travelling wave system to see the influence of incubation times on the quantitative properties of the travelling wave solutions. However, this is not the topic of the present paper, and we leave it as our future work.

Finally, we note that the nonexistence of travelling wave solutions for diffusive epidemic models has been investigated widely [2, 4, 8, 9, 12–14, 16–18, 21, 23, 24, 26, 32, 36, 37, 39, 40, 44, 45, 47, 50–52]. By the similar analysis as that in [33], we can deduce that (1.3) has no travelling wave solutions under the conditions $R_0 < 1$ and $c \geqslant 0$. On the other hand, since the introduction of time delays in (1.3), characteristic equation $\Delta(c, \rho) = 0$ is a transcendental equation and the detailed information of the eigenvalues for $\Delta(c, \rho) = 0$ is not worked out completely. However, the nonexistence of travelling wave solutions for (1.3) under the conditions $R_0 > 1$ and $0 < c < c^*$ depends heavily on the detailed information of the eigenvalues. In view of this, the problem is unsolved at present and needs to be investigated further.

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Conflicts of interest

The authors declare that they have no conflicts of interest.

Authors contributions

All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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