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Thermodynamic formalism for certain nonhyperbolic maps

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Abstract. We establish a generalized thermodynamic formalism for certain nonhyperbolic maps with countably many preimages. We study existence and uniqueness of conformal measures and statistical properties of the equilibrium states absolutely continuous with respect to the conformal measures. We will see that such measures are not Gibbs but satisfy a version of Gibbs property (*weak Gibbs measure*). We apply our results to a one-parameter family of one-dimensional maps and a two-dimensional nonconformal map related to number theory. Both of them admit indifferent periodic points.

1. *Introduction*

In this paper, we shall study thermodynamic formalism for noninvertible discontinuous maps T defined on a subset X of a compact metric space $(\bar{X}, d)$. We assume that there exists a countable (Borel) measurable partition $Q = \{X_a\}_{a\in I}$ such that T restricted on each atom X_a of Q is a homeomorphism from X_a to TX_a. Given a function $f : X \to \mathbf{R}$, we define the Perron–Frobenius operator $\mathcal{L}_f$ by

$$\mathcal{L}_f g(x) = \sum_{y\in T^{-1}x} \exp f(y)g(y) \quad (g \in C(\bar{X})).$$

In the case when T satisfies a regular condition corresponding to hyperbolicity, Walters [**15**] established the existence and uniqueness of the equilibrium state for $f \in C(\bigcup_{a\in I} \operatorname{int} X_a)$ under some condition (see §3). Further, the equilibrium state is absolutely continuous with respect to the $\exp(P(f) - f)$ conformal measure, where $P(f)$ is the measure theoretical pressure for f. On the other hand, in the case when T is a finite-to-one continuous map of a compact metric space, similar results were established in [**1**] and also in [**8**], without hyperbolicity but assuming $P(f) > \sup f$. In this paper, we study these problems for infinite-to-one maps T and piecewise Hölder potentials f satisfying $P(f) \geq \sup f$ in a nonhyperbolic situation. More precisely, in §2 we study the existence and uniqueness problem of conformal measures ν for intermittent Bernoulli maps, i.e.

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maps admitting indifferent periodic points such that $T(\operatorname{int} X_a) = \operatorname{int} X\ \forall X_a \in Q$. We remark that the property (iii) introduced by Walters (property (iii) in [**15**]) fails for such nonhyperbolic maps and for piecewise Hölder potentials (see §5). Further, we discuss a relation between the radius of convergence of Zeta functions and the pressure of f through a generalized Gibbs measure. In §3, we study the existence of a T-invariant ergodic measure μ which is equivalent to the conformal measure ν without both uniform boundedness and equi-continuity of iterated Perron–Frobenius operators. To do this, we first construct a derived system T^* which expands distances and we establish a weak Hölder-type property of the potential f^* associated to T^*. Then we show the existence of an ergodic T^*-invariant measure equivalent to ν. In §4, we establish the variational principle for the pressure $P(f)$ and we will see that the μ is an equilibrium state for f with $P(f) \geq \sup f$. Our method uses the variational principle for the derived systems (T^*, f^*) (see §4). We remark that the Bernoulli property of Q is needed only for the existence of conformal measures in §2 and for no other results in §3 and §4. Further if we consider the existence problem of conformal measures only for symbolic dynamics (with respect to Q), a weaker condition than the Bernoulli one can be found in [**9**]. Note that the thermodynamic formalism of symbolic dynamics associated to Hölder potentials may not apply to nonhyperbolic maps since conjugacy maps from shift spaces to X do not preserve Hölder continuity of potentials.

2. *On the existence of conformal measures for intermittent Bernoulli maps*

Let (X, d) be a bounded metric space, $\mathcal{F}$ the σ-algebra of Borel sets of X and $Q = \{X_a\}_{a\in I}$ a disjoint partition of X with $X_a \in \mathcal{F}$. We assume that there exists a compact metric space $(\bar{X}, d)$ with $\bar{X} \supseteq X$ such that $\operatorname{cl}(\operatorname{int} X) = X$ and $X_0 = \bigcup_{X_a\in Q} \operatorname{int} X_a$ is an open dense subset of $\bar{X}$. Let $T : X \to \bar{X}$ be a piecewise invertible map with the Bernoulli structure, i.e. Q is a generating partition, $T|_{X_a} : X_a \to TX_a$ is a homeomorphism and $T(\operatorname{int} X_a) = \operatorname{int} X\ \forall X_a \in Q$. We denote $(T|_{X_a})^{-1}$ by ψ_a and assume that ψ_a extends to a homeomorphism from $\bar{X}$ onto $\operatorname{cl} X_a$. For the existence of conformal measures, we further assume Lipshitz continuity of the local inverse $\{\psi_a\}_{a\in I}$.

(C1) $\exists 0 < L_a < \infty$ such that $d(\psi_a x, \psi_a y) \leq L_a d(x, y)$ $(\forall a \in I)$ and $\sup L_a < \infty$, $a \in I$.

Let $f : X \to \mathbf{R}$ be a piecewise Hölder continuous function with exponent θ, i.e.

(C2) $\exists 0 < L_f < \infty$ such that $|f(x) - f(y)| \leq L_f d(x, y)^\theta$ $(\forall x, y \in X_a, \forall X_a \in Q)$.

Let $UC(\operatorname{int} X)$ be the space of uniformly continuous functions on $\operatorname{int} X$. Let $\mathcal{L}_f$ be the generalized Perron–Frobenius operator which is defined by

$$\mathcal{L}_f(g)(x) = \sum_{y\in T^{-1}x} \exp f(y)g(y) = \sum_{a\in I} \exp f(\psi_a x)g(\psi_a x)$$
$$(\forall g \in UC(\operatorname{int} X), \forall x \in \operatorname{int} X).$$

LEMMA 2.1. (Cf. [**15**]) *Suppose that (C1), (C2) and*

(C3) $\exists 0 < K < \infty$ *such that* $\mathcal{L}_f 1(x) = \sum_{a\in I} \exp f(\psi_a x) \leq K$ $(\forall x \in X)$

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are satisfied. Then $\mathcal{L}_f : UC(\operatorname{int} X) \to UC(\operatorname{int} X)$ *and* $\mathcal{L}_f$ *can be extended to a positive continuous linear operator of* $C(\bar{X})$. *Furthermore,* $\mathcal{L}_f^*(\cdot)/\mathcal{L}_f^*(\cdot)1 : \mathcal{M}(\bar{X}) \to \mathcal{M}(\bar{X})$ *is continuous, where* $\mathcal{M}(\bar{X})$ *is the set of all probability measures on the* σ*-algebra of Borel sets* $\bar{\mathcal{F}}$ *of* $\bar{X}$ *and* $\mathcal{L}_f^*$ *is the dual of* $\mathcal{L}_f$.

Proof. First we note the following fact:

$$|\mathcal{L}_f 1(x) - \mathcal{L}_f 1(y)| \to 0 \text{ as } d(x, y) \to 0.$$

Indeed, we see that

$$\begin{aligned}
&|\mathcal{L}_f 1(x) - \mathcal{L}_f 1(y)| \\
&\quad \le \sum_{a\in I} |\exp f(\psi_a x) - \exp f(\psi_a y)| \le \eta L_f \sum_{a\in I} \exp f(\psi_a x) d(\psi_a x, \psi_a y)^\theta \\
&\quad \le \eta L_f K \left(\sup_{a\in I} L_a\right)^\theta d(x, y)^\theta \quad (\exists 0 < \eta < \infty).
\end{aligned}$$

The above observations and (C3) allow us to make the first assertion. Since $\mathcal{M}(\bar{X})$ comes equipped with the weak*-topology, the second assertion is obvious. □

The next result is immediate from Lemma 2 in [**15**].

LEMMA 2.2. *Suppose that (C1)–(C3) are satisfied. Then there exists a real number* $p > 0$ *and a probability measure* ν *on* $\bar{\mathcal{F}}$ *such that* $\mathcal{L}_f^*\nu = p\nu$ *and* $p = \mathcal{L}_f^*\nu(1)$. ν *is the* $\exp(\log p - f)$*-conformal measure for* T, *i.e.*

$$\frac{d(\nu T)|_{X_a}}{d\nu|_{X_a}} = \exp(\log p - f) \quad (\forall X_a \in Q).$$

We shall consider sufficient conditions for the uniqueness of p and ν. Let $\mathcal{A}_n$ denote the set of all admissible sequences of length n, i.e. $(a_1 \dots a_n) \in \mathcal{A}_n$ iff $\operatorname{int}(X_{a_1} \cap T^{-1}X_{a_2} \cap \dots \cap T^{-(n-1)}X_{a_n}) \neq \emptyset$. For $(a_1 \dots a_n) \in \mathcal{A}_n$, we denote $X_{a_1} \cap T^{-1}X_{a_2} \cap \dots \cap T^{-(n-1)}X_{a_n}$ by $X_{a_1\dots a_n}$ and $(T^n|_{X_{a_1\dots a_n}})^{-1}$ by $\psi_{a_1\dots a_n}$. We suppose that $\nu(\operatorname{int} X) = 1$, which will be proved in §3 (Corollary 3.2).

(C4) *Weak bounded variation (cf. [**21**]).* There exists a sequence of positive numbers $\{C_n\}_{n>0}$ satisfying $\lim_{n\to\infty}(1/n)\log C_n = 0$ and

$$\sup_{(a_1\dots a_n)\in\mathcal{A}_n} \sup_{x,y\in X_{a_1\dots a_n}} \frac{\exp\left(\sum_{i=0}^{n-1} f(T^i x)\right)}{\exp\left(\sum_{i=0}^{n-1} f(T^i y)\right)} \le C_n.$$

Condition (C4) is weaker than Walters' corresponding assumption (assumption (iii) in [**15**]) (i.e. $\exists 1 \le C < \infty$ such that $C_n \le C$ for $n \ge 1$). For piecewise C^1-maps admitting indifferent periodic points ($T^q x_0 = x_0$ and $|\det DT^q(x_0)| = 1$), (C4) is typically satisfied for $f = -\log|\det DT|$ though (iii) in [**15**] is not satisfied.

PROPOSITION 2.1. *Suppose that the conditions (C1)–(C4) are satisfied. If* $\mathcal{L}_f^*\nu_1 = p_1\nu_1$ *and* $\mathcal{L}_f^*\nu_2 = p_2\nu_2$, *then* $p_1 = p_2$.

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Proof. Since ν_i $(i = 1, 2)$ are $\exp(\log p_i - f)$-conformal measures, we have the following inequalities:

$$\nu_i(T^n X_{a_1\dots a_n}) = \int_{X_{a_1\dots a_n}} \exp\left((\log p_i)n - \sum_{i=0}^{n-1} fT^i(x)\right) d\nu_i(x)$$
$$\leq p_i^n C_n \nu_i(X_{a_1\dots a_n}) \exp\left(-\sum_{i=0}^{n-1} fT^i(x)\right) \quad (\forall x \in X_{a_1\dots a_n}).$$

$$\nu_i(T^n X_{a_1\dots a_n}) \geq p_i^n C_n^{-1} \nu_i(X_{a_1\dots a_n}) \exp\left(-\sum_{i=0}^{n-1} fT^i(x)\right) \quad (\forall x \in X_{a_1\dots a_n}).$$

Combining the above inequalities gives

$$0 < \frac{\nu_1(X_{a_1\dots a_n})}{\nu_2(X_{a_1\dots a_n})} \leq \exp n\left(\log\frac{p_1}{p_2} + \frac{2\log C_n}{n}\right).$$

Suppose that $p_1 < p_2$. Since $\nu_i(T^n X_{a_1\dots a_n}) = \nu_i(X) = 1$ and so $\nu_i(X_{a_1\dots a_n}) > 0$ we have a contradiction. This implies that $p_1 = p_2$. □

The next result gives a characterization of the $\exp(\log p - f)$-conformal measure.

PROPOSITION 2.2. *Suppose that (C1)–(C4) are satisfied. Any solution of the equation* $\mathcal{L}_f^* \nu = p\nu$ *is a weak Gibbs measure for* f *with a constant* $-\log p$, *i.e. there exists a sequence of positive numbers* $\{K_n\}_{n>0}$ *satisfying* $\lim_{n\to\infty} 1/n \log K_n = 0$ *and* ν*-a.e.* x

$$K_n^{-1} \leq \frac{\nu(X_{a_1\dots a_n}(x))}{\exp\left(-n(\log p) + \sum_{i=0}^{n-1} f(T^i x)\right)} \leq K_n,$$

where $X_{a_1\dots a_n}(x)$ *denotes the cylinder containing* x.

(Cf. [**21**].)

Proof. From Lemma 2.2 (conformality of ν), we can easily see that

$$\int_{X_{a_1\dots a_n}} d\nu(x) = \int_{T^n X_{a_1\dots a_n}} \exp\left(\sum_{i=0}^{n-1} (fT^i - n\log p)(\psi_{a_1\dots a_n}(x))\right) d\nu(x).$$

The weak bounded variation (WBV) property for f allows us to have

$$C_n^{-1} \leq \frac{\nu(X_{a_1\dots a_n}(x))}{\exp\left(\sum_{i=0}^{n-1} fT^i(x) - n\log p\right)} \leq C_n. \qquad \square$$

Let $Z_n(X, f) = \sum_{x\in \mathrm{Fix}(T^n, X)} \exp\left(\sum_{i=0}^{n-1} fT^i(x)\right)$. Then the formal power series $\zeta_{X,f}(t) = \exp\left(\sum_{n=1}^{\infty} (t^n/n) Z_n(X, f)\right)$ is called *Artin–Mazur–Ruelle* zeta function and the radius of convergence of $\zeta_{X,f}$ is $\rho_f = \exp(-\limsup_{n\to\infty}(1/n)\log Z_n(X, f))$. We shall associate with ρ_f the constant p.

THEOREM 2.1. *Suppose that (C1)–(C4) are satisfied. Then*

$$\lim_{n\to\infty} (1/n)\log Z_n(X, f) = \log p \quad \textit{and} \quad \rho_f = p^{-1}.$$

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Proof. In [**21**], a result similar to Theorem 2.1 was established for maps with finite range structure (FRS) in the case when the Lebesgue measure is a weak Gibbs measure for f with a constant $-p$ and $\log|\det DT|$ is of weak bounded variation (C4). Since all cylinders are full cylinders (i.e. $T(\operatorname{int} X_a) = \operatorname{int} X, \forall X_a \in Q$) and (C4) is valid, the result follows from Theorem 5.1 in [**21**] immediately. □

COROLLARY 2.1. *If there exists a weak Gibbs measure for f with a constant $-\log p$, then $\zeta_{X,f}(t)$ converges for $|t| < p^{-1}$.*

PROPOSITION 2.3. *Suppose that*

(C5) $\sigma(n) = \sup_{X_{a_1\dots a_n}} \operatorname{diam} X_{a_1\dots a_n} \to 0$ *as* $n \to \infty$.

Then for every nonempty open set O we have $\nu(O) > 0$.

Proof. Since

$$\nu(X_{a_1\dots a_n}) = \int_{T^n X_{a_1\dots a_n}} \exp\left(\sum_{i=0}^{n-1} (fT^i - n\log p)(\psi_{a_1\dots a_n} x)\right) d\nu(x),$$

we have $\nu(X_{a_1\dots a_n}) > 0\ \forall X_{a_1\dots a_n} \in \bigvee_{i=0}^{n-1} T^{-i}Q$. Then the result follows from (C5) which implies $\sigma\left(\bigvee_{i=0}^{\infty} T^{-i}Q\right) = \mathcal{F}$. □

PROPOSITION 2.4. (Uniqueness of ν) *Suppose Proposition 2.1. If ν_i $(i = 1, 2)$ are ergodic and $\nu_1 \le \nu_2$, then $\nu_1 = \nu_2$.*

Proof. Since $d\nu_1/d\nu_2 \in L^1(\nu_2)$, we have

$$\begin{aligned}\nu_1(T^n(X_{a_1\dots a_n})) &= \int_X 1_{X_{a_1\dots a_n}}(\psi_{a_1\dots a_n} x)(d\nu_1/d\nu_2)(T^n\psi_{a_1\dots a_n}(x))\, d(\nu_2 T^n)(\psi_{a_1\dots a_n} x)\\ &= \int_{X_{a_1\dots a_n}} (d\nu_1/d\nu_2)T^n(y) \exp\left(n\log p - \sum_{i=0}^{n-1} fT^i(y)\right) d\nu_2(y).\end{aligned}$$

On the other hand, we have

$$\nu_1(T^n(X_{a_1\dots a_n})) = \int_{X_{a_1\dots a_n}} \exp\left(n\log p - \sum_{i=0}^{n-1} fT^i(x)\right)(d\nu_1/d\nu_2)(x)\, d\nu_2(x).$$

Combining the above equalities and by ergodicity, we have $\nu_1 = \nu_2$. □

By applying Walters' idea in [**15**] we have the following.

THEOREM 2.2. *Let ν be a weak Gibbs measure for f with a constant $-\log p$. If there exists a T-invariant ergodic probability measure μ equivalent to ν with $H_\mu(Q) < \infty$ and $f \in L^1(\mu)$, then $\log p = h_\mu(T) + \int f\, d\mu$.*

(Cf. [**19, 20**].)

LEMMA 2.3. *Either of the following conditions (a) or (b) is sufficient for $H_\mu(Q) < +\infty$:*
(a) $\sum_{a\in I} \nu(X_a) \inf_{X_a} f > -\infty$;

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(b) $\exists 0 < W_1 < W_2 < \infty$ *such that* $d\mu/d\nu \geq W_1$ *and* $I_{W_2} = \{a \in I : \sup_{X_a} d\mu/d\nu > W_2\}$ *is a finite set.*

Proof. (a) From Proposition 2.2, we see that for $x \in X_a$,

$$K^{-1} \exp(f(x) - \log p) \leq \nu(X_a) \leq K_1 \exp(f(x) - \log p).$$

The above inequalities allow us to have that

$$H_\nu(Q) \leq \sum_{a \in I} \nu(X_a)(\log K_1 + \log p + \sup_{X_a}(-f)) \leq \log K_1 + \log p - \sum_{a \in I} \nu(X_a) \inf_{X_a} f.$$

(b) We see that

$$H_\mu(Q) \leq - \sum_{a \in I_{W_2}} \mu(X_a) \log \mu(X_a) + W_2(H_\nu(Q) - \log W_1).$$

The above inequality completes the proof. □

In the next section, we will give sufficient conditions for ergodicity of ν and the existence of an ergodic T invariant measure μ equivalent to ν.

3. *Invariant measures for derived systems which expand distances*

We suppose that (C1)–(C4) are satisfied. Assume further that

(C1)* $\forall a \in I, \exists L_a \leq 1$.

Given $L < 1$, we define $\mathcal{B}_1 = \{X_b : L_b \leq L\}, \mathcal{D}_1 = \{X_d : L_d > L\}$, and inductively define

$$\mathcal{B}_n = \{X_{b_1 \dots b_n} : X_{b_1 \dots b_{n-1}} \in \mathcal{D}_{n-1}, L_{b_n} \leq L\},$$

$$\mathcal{D}_n = \{X_{d_1 \dots d_n} : X_{d_1 \dots d_i} \in \mathcal{D}_i \ (i = 1 \dots n-1), L_{d_n} > L\}.$$

Put $B_n = \bigcup_{X_{b_1 \dots b_n} \in \mathcal{B}_n} X_{b_1 \dots b_n}$ and $D_n = \bigcup_{X_{d_1 \dots d_n} \in \mathcal{D}_n} X_{d_1 \dots d_n}$. (We put $D_0 = X$.) We define $\mathcal{R} = \{X_{a_1 \dots a_n} : X_{a_n} \in \mathcal{B}_1\}$. The following properties will be important later for establishing our theorems.

LEMMA 3.1.

(1) $\mathcal{R}$ *satisfies the strong playback property, i.e.* $X_{a(m)b(n)} \in \mathcal{R}$ *if* $X_{b(n)} \in \mathcal{R}$ *(cf. [**10**]).*

(2) $B_{n+1} = D_n \cap T^{-n} B_1$.

Define $I^* = \bigcup_{n=1}^\infty \{(a_1 \dots a_n) : X_{a_1 \dots a_n} \in \mathcal{B}_n\}$. Then $Q^* = \{X_\alpha\}_{\alpha \in I^*}$ is a countable disjoint partition of $\bigcup_{i=1}^\infty B_i$. Define a map $T^* : \bigcup_{i=1}^\infty B_i \to \bar{X}$ by $T^* x = T^i x$ for $x \in B_i$. Then $T^*|_{X_\alpha}$ is invertible and a local homeomorphism from X_α to $T^* X_\alpha$. We denote $\psi^*_\alpha = (T^*|_{X_\alpha})^{-1}$ and $\psi^*_{\alpha_1 \dots \alpha_n} = (T^{*n}|_{X_{\alpha_1 \dots \alpha_n}})^{-1}$. Since $X = \bigcup_{i=1}^\infty B_i \cup (\bigcap_{n=0}^\infty D_n)$, we assume that

(C6) $\lim_{n \to \infty} \nu(D_n) = 0$.

Then we have $\bigcup_{i=1}^\infty B_i = X (\nu \bmod 0)$.

Remark A. In §5, we will see that if $\nu(X) = 1$ then (C6) is satisfied for all examples when $\bigcap_{n=0}^\infty D_n$ consists of only indifferent periodic points belonging to ∂X.

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Put $X^* = X\setminus(\bigcup_{m=0}^{\infty} T^{*-m}(\bigcap_{n=0}^{\infty} D_n))$. Define $\phi(x) = \inf\{n \in \mathbf{N} : X_{a_1\ldots a_n}(x) \in \mathcal{R}\}$. Then we have the following.

LEMMA 3.2. *If ν is an $\exp(\log p - f)$ conformal measure with respect to T, then ν is an $\exp(\phi(x)\log p - f^*(x))$ conformal measure with respect to T^*, where $f^*(x) = \sum_{i=0}^{\phi(x)-1} fT^i(x)$.*

From the above lemma we have the following fact.

LEMMA 3.3. $X = X^*$ (ν mod 0).

Proof. We have

$$\begin{aligned}
&\nu\left((T^*)^{-1}\left(\bigcap_{n=0}^{\infty} D_n\right)\right)\\
&\quad= \sum_{\alpha\in I^*} \nu\left(\psi_\alpha^*\left(\bigcap_{n=0}^{\infty} D_n \cap T^* X_\alpha\right)\right)\\
&\quad= \sum_{\alpha\in I^*} \int_{\bigcap_{n=0}^{\infty} D_n\cap T^* X_\alpha} \exp\left(\sum_{i=0}^{|\alpha|-1} (fT^i(x) - |\alpha|\log p)(\psi_\alpha^* x)\right) d\nu(x)\\
&\quad= 0.
\end{aligned}$$

Inductively we can show that $\nu(T^{*-i}(\bigcap_{n=0}^{\infty} D_n)) = 0$ which completes the proof. □

The next result is immediate from assumption (C1)*.

LEMMA 3.4. $d(\psi^*_{\alpha_1\ldots\alpha_n}x, \psi^*_{\alpha_1\ldots\alpha_n}y) \le L^n d(x, y)$.

THEOREM 3.1. *Suppose that:*

(C7) $\gamma = \sup_{a\in I}\{\sup_{X_a} f - \inf_{X_a} f\} < \infty$;
(C8) $\forall X_{a_1\ldots a_n} \in \mathcal{B}_n, \exists 0 < L_f(a_1\ldots a_n) < \infty$ *such that*

$$|f(\psi_{a_1\ldots a_n}x) - f(\psi_{a_1\ldots a_n}y)| \le L_f(a_1\ldots a_n)d(x, y)^\theta$$

$\forall x, y \in T^n X_{a_1\ldots a_n}$, *and*

$$L_f(\infty) = \sup_{n>0} \sup_{X_{a_1\ldots a_n}\in\mathcal{B}_n} \sum_{i=1}^{n} L_f(a_i\ldots a_n) < \infty.$$

Then $\{\mathcal{L}^n_{f^*-\phi\log p}1\}_{n\ge 0}$ is equicontinuous and uniformly bounded.

COROLLARY 3.1. (Renyi's condition) *There exists $C > 1$ such that $\forall x, y \in X^*$,*

$$\sup_{n\ge 1} \sup_{(\alpha_1\ldots\alpha_n)\in\mathcal{A}^*} \frac{\exp\left(\sum_{i=0}^{n-1} f^*T^{*i}(\psi^*_{\alpha_1\ldots\alpha_n}x)\right)}{\exp\left(\sum_{i=0}^{n-1} f^*T^{*i}(\psi^*_{\alpha_1\ldots\alpha_n}y)\right)} \le C.$$

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Proof of Theorem 3.1. We see easily that

$$\frac{\exp\left(\sum_{i=0}^{n-1} f^*T^{*i}(\psi^*_{\alpha_1\dots\alpha_n}x)\right)}{\exp\left(\sum_{i=0}^{n-1} f^*T^{*i}(\psi^*_{\alpha_1\dots\alpha_n}y)\right)} \tag{$*$}$$

$$\le \prod_{i=0}^{n-1}\prod_{j=0}^{|\alpha_{i+1}|-1}\left\{1+\exp\left(\sup_{x\in X_{\alpha_{i+1}^{j+1}\dots\alpha_{i+1}^{|\alpha_{i+1}|}\alpha_{i+2}\dots\alpha_n}} f(x) - \inf_{x\in X_{\alpha_{i+1}^{j+1}\dots\alpha_{i+1}^{|\alpha_{i+1}|}\alpha_{i+2}\dots\alpha_n}} f(x)\right)\right.$$

$$\left.\times\left|f\left(\psi_{\alpha_{i+1}^{j+1}\dots\alpha_{i+1}^{|\alpha_{i+1}|}}(\psi^*_{\alpha_{i+2}\dots\alpha_n}x)\right) - f\left(\psi_{\alpha_{i+1}^{j+1}\dots\alpha_{i+1}^{|\alpha_{i+1}|}}(\psi^*_{\alpha_{i+2}\dots\alpha_n}y)\right)\right|\right\}$$

$$\le \prod_{i=0}^{n-1}\prod_{j=0}^{|\alpha_{i+1}|-1}(1+\exp\gamma L_f(\alpha_{i+1}^{j+1}\dots\alpha_{i+1}^{|\alpha_{i+1}|})L^{n-(i+1)}d(x,y)^\theta),$$

where ${\alpha_{i+1}}^1\dots\alpha_{i+1}^{|\alpha_{i+1}|}$ denotes $\alpha_{i+1}\in I^*$. Then (2) and (C8) allow us to have an upper bound of $(*)$, $1+d(x,y)^\theta\times$ (constant). Similarly we have a lower bound; $1-d(x,y)^\theta\times$ (constant) so that $\exists C>1$, such that

$$\left|1-\frac{\exp\left(\sum_{i=0}^{n-1} f^*T^{*i}(\psi^*_{\alpha_1\dots\alpha_n}x)\right)}{\exp\left(\sum_{i=0}^{n-1} f^*T^{*i}(\psi^*_{\alpha_1\dots\alpha_n}y)\right)}\right| \le d(x,y)^\theta \times (\text{constant}) \le C$$

for $X_{\alpha_1\dots\alpha_n}, x, y \in T^{*n}X_{\alpha_1\dots\alpha_n}$, and for $n>0$. From Lemma 3.2 and the above observations, we see that $\exists C'>1$ such that

$$|\mathcal{L}^n_{f^*-\phi\log p}1(x) - \mathcal{L}^n_{f^*-\phi\log p}1(y)|$$

$$\le (\text{constant})\times d(x,y)^\theta \sum_{\alpha_1\dots\alpha_n}\prod_{i=0}^{n-1}\exp(f^*(\psi^*_{\alpha_{i+1}\dots\alpha_n}x))\,p^{-\sum_{i=0}^{n-1}\phi(\psi^*_{\alpha_{i+1}\dots\alpha_n}x)}$$

$$\le (\text{constant})\times d(x,y)^\theta C\sum_{\alpha_1\dots\alpha_n}\int_{T^{*n}X_{\alpha_1\dots\alpha_n}}\exp\left(\sum_{i=0}^{n-1} f^*T^{*i}(\psi^*_{\alpha_1\dots\alpha_n}x)\right)d\nu(x)\,p^{-\sum_{i=1}^{n}|\alpha_i|}$$

$$\le C'\sum_{\alpha_1\dots\alpha_n}\nu(X_{\alpha_1\dots\alpha_n})d(x,y)^\theta.$$

This completes the proof. □

COROLLARY 3.2. *There exists a T^*-invariant measure $\mu^*\sim\nu$ and $\nu(X^*)=\nu(X)=1$.*

Proof of Corollary 3.2. Since we have Renyi's condition and Bernoulli property of Q^*, the standard argument allows us to have a T^*-invariant measure $\mu^*\sim\nu$ with $d\mu^*/d\nu$ bounded away from zero and infinity. The last assertion follows from Lemma 9 in [**15**]. In fact, we have $\nu(X_0)=\nu(\text{int } X)=1$. □

THEOREM 3.2. *T^* is ergodic with respect to ν and exact with respect to μ^*.*

Proof. Condition (C5) is valid because of Lemma 3.4. Again, Renyi's condition, Bernoulli property of Q and conformality of ν lead us to apply the standard argument for establishing ergodicity [**10, 11, 16**]. Exactness of μ^* follows from [**17, 19**]. □

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We remark that the following formula [**5, 10, 11**] gives a T-invariant σ-finite measure:

$$\mu(E) = \sum_{n=0}^{\infty} \mu^*(D_n \cap T^{-n}E). \tag{F}$$

Then we see that Theorem 3.2 allows us to have ergodicity of T with respect to μ.

Remark B. We will see in §5 that property (C1)* fails for Example 2, but we can define $\mathcal{B}_n, \mathcal{D}_n$ so that Lemma 3.1 is valid and $\exists n_0 > 0$ and $\exists 0 < \delta < 1$ such that

$$d(\psi^*_{\alpha_1\ldots\alpha_n}x, \psi^*_{\alpha_1\ldots\alpha_n}y) \leq \text{ constant} \times \delta^n d(x, y) \quad (\forall n \geq n_0).$$

4. *Variational principle*

In this section, we suppose that

(C6)* $\sum_{n=0}^{\infty} \nu(D_n) < \infty$ which guarantees $\mu(X) < \infty$.

Let T_{B_1} be the induced transformation on B_1. Let μ, μ^* and μ_{B_1} be invariant probability measures with respect to T, T^* and T_{B_1} respectively. We assume (2) in Lemma 3.1. Then we have the following.

LEMMA 4.1. $T|_{B_1} \circ T_{B_1} = T^* \circ T|_{B_1}$. *Furthermore, we have*

$$\mu^*(F) = \mu_{B_1}((T|_{B_1})^{-1}F) \quad (\forall F \in \mathcal{F}^*),$$

and for $g : X^* \to \mathbf{R}$, $\mu^*(g) = \mu_{B_1}(g \circ T|_{B_1})$.

(Cf. [**10, 11**].) Let ϕ_{B_1} be the first return function to B_1. Put $f^*(x) = \sum_{i=0}^{\phi(x)-1} fT^i(x)$ and $f_{B_1}(x) = \sum_{i=1}^{\phi_{B_1}(x)} fT^i(x)$. Then it is easy to see that $\forall x \in B_1$, $f_{B_1}(x) = f^*T(x)$. Furthermore, we have the following fact.

LEMMA 4.2. $\int_{X^*} f^*(x)\, d\mu^*(x) = \int_{B_1} f_{B_1}\, d\mu_{B_1}(x)$.

Proof. We note that for $x \in B_1$, $\phi_{B_1}(x) = k$ implies $\phi(Tx) = k$. We also have that $f_{B_1}T_{B_1}(x) = f^*T(T_{B_1}x) = f^*T^*(T|_{B_1}x)$. It is enough to use one of these facts. □

We can easily verify the next formula.

LEMMA 4.3. $h_{\mu^*}(T^*) = h_{\mu_{B_1}}(T_{B_1})$.

We shall suppose that μ^* is ergodic so that both μ and μ_{B_1} are ergodic. Then combining the above lemmas we have the following.

THEOREM 4.1.

$$\begin{aligned} h_{\mu^*}(T^*) + \int_{X^*} f^*(x)\, d\mu^*(x) &= h_{\mu_{B_1}}(T_{B_1}) + \int_{B_1} f_{B_1}(x)\, d\mu_{B_1}(x) \\ &= \frac{h_\mu(T) + \int_X f(x)\, d\mu(x)}{\mu(B_1)}. \end{aligned}$$

LEMMA 4.4. *If a T-invariant probability measure m satisfies $m(B_1) = 0$, then* $m\big(\bigcap_{n=0}^{\infty} D_n\big) = 1$.

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Remark C. For all examples that we shall consider in §5, $\bigcap_{n=0}^{\infty} D_n$ consists of only indifferent fixed points. For such maps, m with $m(B_1) = 0$ is a T-invariant measure supported on the fixed points.

Proof of Lemma 4.4. Under assumption (2), we see that

$$m\left(\bigcup_{i=1}^{\infty} B_i\right) = \sum_{i=1}^{\infty} m(D_{i-1} \cap T^{-(i-1)} B_1) \leq \sum_{i=0}^{\infty} m(T^{-(i-1)} B_1).$$

If $m(B_1) = 0$, then from T-invariance of m we have

$$m\left(\bigcup_{i=1}^{\infty} B_i\right) = m\left(X \setminus \bigcap_{n=0}^{\infty} D_n\right) = 0. \qquad \square$$

We note that $h^* = d\mu^*/d\nu$ is bounded away from zero and infinity. Put $g^* = (h^*/h^* T^*) \exp(f^* - \phi \log p)$. Applying Theorem 10 in [**15**] gives the following variational principle for the Pressure. I_μ denotes the conditional information of $\mathcal{F}$ with respect to $T^{-1}\mathcal{F}$.

PROPOSITION 4.1. *For any T^*-invariant probability measure m^*:*

(3) $\mu^*(I_{\mu^*} + \log g^*) \geq m^*(I_{m^*} + \log g^*)$.

The left-hand side of (3) is zero and for $m^* \neq \mu^*$, $0 > m^*(I_{m^*} + \log g^*)$.

In particular, if

(C9) $H_\nu(Q^*) < \infty$,

then we have the follwing.

COROLLARY 4.1. *(3) is equivalent to the following inequality:*

(4) $h_{\mu^*} + \mu^*(f^*) - \mu(B_1)^{-1} \log p \geq m^*(I_{m^*} + f^*) - m(B_1)^{-1} \log p$,

where m is the T-invariant measure arising from m^* via the formula (F).

Proof. Since $m^*(\phi) = m_{B_1}(\phi_{B_1}) = 1/m(B_1) < \infty$, the assertion is obtained by noting that $d\mu^*/d\nu$ is bounded away from zero and infinity. (See Corollary 3.2.) $\square$

We have a sufficient condition for $H_\nu(Q^*) < \infty$ as follows.

LEMMA 4.5. *If $\int_{X^*} (\phi(x) \log p - f^*(x))\, d\nu(x) < \infty$, then $H_\nu(Q^*) < \infty$.*

Let $E(X, T)$ denote the set of T-invariant ergodic probability measures on $\mathcal{F}$. Applying Theorem 4.1 we have the following.

THEOREM 4.2. *Suppose that (C1)*, (2), (C6)* and (C7)–(C9) are satisfied. Then we have:*

(5) $\log p = h_\mu(T) + \mu(f)$.

(6) $\log p > m(B_1) m^*(I_{m^*} + f^*)$. *In particular, $\forall m \in E(X, T)$ satisfying $m^*(f^*) > -\infty$ or $m^*(I_{m^*}) < \infty$, we have* $\log p > h_m(T) + m(f)$.

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Proof of Theorem 4.2. (5) and the first assertion of (6) are immediately verified. Suppose that $m^*(f^*) > -\infty$. Then it follows from $I_m^* \geq 0$ and (4) that $f^* \in L^1(m^*)$ and we have that $m^*(I_{m^*} + f^*) = m^*(I_{m^*}) + m^*(f^*)$. This equality allows us to have the desired strict inequality for $m \in E(X, T)$ arising from T^*-invariant ergodic measure m^*. Suppose that $m^*(f^*) = -\infty$ and $m^*(I_{m^*}) < \infty$. Then we have $m^*(I_{m^*}) + m^*(f^*) = -\infty < \log p$. The second assertion of (6) has just been proved. □

Remark D. The collection of ergodic T-invariant measures arising from T^*-invariant ergodic measures does not contain the T-invariant probability measures supported on the periodic points belonging to $\bigcap_{n\geq\infty} D_n$. For such a measure m, we have $h_m(T) = 0$.

Definition. A periodic point x_0 with period q is said to be *indifferent* with respect to ν if

$$\frac{d(\nu T^q)}{d\nu}(x_0)\left(= \exp\left(q \log p - \sum_{i=0}^{q-1} f T^i(x_0)\right)\right) = 1.$$

In particular, for piecewise C^1 maps T and for the Lebesgue measure ν, the definition coincides with the usual one (i.e. $\det |DT^q(x_0)| = 1$).

COROLLARY 4.2. *Suppose that (C1)*, (2), (C6) and (C7)–(C9) are satisfied. Assume further that $\bigcap_{n\geq 0} D_n$ consists of periodic points. If $\sup_X f \leq \log p$, then μ is an equilibrium state for f in the following sense:*

$$h_\mu(T) + \mu(f) \geq \sup\{h_m(T) + m(f) : m \in E(X, T), m(f) > -\infty \text{ or } m(I_m) < \infty\}.$$

If $\sup_X f < \log p$, then μ is a unique equilibrium state for f. If $\bigcap_{n\geq 0} D_n$ consists of indifferent periodic points with respect to ν, then the equilibrium state for f is not unique.

Proof of Corollary 4.2. By Theorem 4.2, we have $\log p > h_m(T) + m(f)$ for $m \in E(X, T)$ which arises from some $m^* \in E(X^*, T^*)$. If m is a probability measure supported on a periodic orbit contained in $\bigcap_{n\geq\infty} D_n$, we have $h_m(T) + m(f) = m(f) \leq \sup_X f \leq \log p$. If $\sup_X f < \log p$, then the strict inequality is obtained. Let x_0 be an indifferent periodic point with period q. Then for $m = (1/q)\sum_{i=0}^{q-1} \delta_{T^i x_0}$ we have $h_m(T) + m(f) = m(f) = \log p$. □

5. *Examples*

Example 1. A one-parameter family of maps on the interval $[0, 1]$. For β with $0 < \beta$ we define a map T_β of $X = [0, 1]$ by

$$T_\beta(x) = \begin{cases} \dfrac{x}{(1 - x^\beta)^{1/\beta}} & \text{on } X_0 = [0, (1/2)^{1/\beta}) \\ \dfrac{x}{(1/2)^{1/\beta} - 1} + \dfrac{1}{1 - (1/2)^{1/\beta}} & \text{on } X_1 = [(1/2)^{1/\beta}, 1]. \end{cases}$$

The map T_β admits an indifferent fixed point zero. We can easily see that (C1)* and (C1) are satisfied. In fact, $L_0 = 1$ and $L_1 = 1 - (1/2)^{1/\beta}$. Since Q is a finite partition, (C3) and (C7) are automatically satisfied. (C5) and (C4) for $f = -\log |T'_\beta|$ were established in

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[**17**]. Combining these observations we can apply results in §2 to T_β, i.e. for f satisfying (C2) and (C4), we have Propositions 2.1–2.4 and Theorems 2.1 and 2.2. In particular for $f = -\log|T'_\beta|$ and the Lebesgue measure ν, we have Proposition 2.2 and Theorems 2.1 and 2.2. By using $L \equiv L_1$ we can define inductively,

$$\mathcal{B}_n = \{X_{\underbrace{0\ldots01}_{n}}\}, \quad \mathcal{D}_n = \{X_{\underbrace{0\ldots0}_{n}}\}.$$

The exact formula [**17, 18**]

$$\psi'_{\underbrace{0\ldots0}_{n}}(x) = \frac{1}{(1+nx^\beta)^{1+1/\beta}}$$

allows us to verify (C8) for f satisfying (C2). We can take

$$L_f(1-(1/2)^{1/\beta})2^{1+1/\beta}\sum_{i=1}^{\infty}(2+i)^{-(1+1/\beta)\theta}$$

as $L_f(\infty)$ and if $\theta > \beta/(\beta+1)$ then we have $L_f(\infty) < \infty$. Hence we can apply results in §3 to T_β with $0 < \beta$ and to f satisfying (C2) and (C4) with $\theta > \beta/(\beta+1)$. In particular, for $f = -\log|T'_\beta|$ and for the Lebesgue measure ν, (C8) is valid $\forall\beta > 0$. On the other hand, (C9) is valid for $0 < \beta < 1$ and it fails for $\beta \geq 1$. Further statistical properties of T_β were established in [**17, 18, 20, 22, 23**].

Example 2. Brun's map. (Cf. [**6, 11, 12, 18**].) Let $X = \{(x_1, x_2) \in \mathbf{R}^2 : 0 \leq x_2 \leq x_1 \leq 1\}$, and let

$$X_i = \{(x_1, x_2) \in X : x_i + x_1 \geq 1 \geq x_{i+1} + x_1\}$$

for $i = 0, 1, 2$ where we put $x_0 = 1$ and $x_3 = 0$. T is defined by

$$T(x_1, x_2) = \left(\frac{x_1}{1-x_1}, \frac{x_2}{1-x_1}\right) \quad \text{on } X_0,$$

$$T(x_1, x_2) = \left(\frac{1}{x_1} - 1, \frac{x_2}{x_1}\right) \quad \text{on } X_1,$$

$$T(x_1, x_2) = \left(\frac{x_2}{x_1}, \frac{1}{x_1} - 1\right) \quad \text{on } X_2.$$

It is easy to see that $TX_i = X$ $(i = 0, 1, 2)$, i.e. $Q = \{X_i\}_{i=0}^2$ is a Bernoulli partition. $(0, 0)$ is an indifferent fixed point and T is a continuous and piecewise C^2-map. Since Q is a finite partition, (C1), (C3) and (C7) are automatically satisfied. (C5), (C6) and (C2), (C4) for $f = -\log|\det DT|$ were verified in [**18**] (cf. [**6, 12**]). From these observations we can apply Propositions 2.1–2.4 and Theorems 2.1 and 2.2 to T and f satisfying (C2), (C4) (e.g. $f = -\log|\det DT|$). The next facts are useful to verify (C1)*.

(7) for $i = 0, 1$ $\exists 0 < \delta < \frac{3}{160}(5+\sqrt{5})$ such that $\|\psi_i(x_1, x_2) - \psi_i(x'_1, x'_2)\| \leq (1+x_1)^{-\delta}\|(x_1, x_2) - (x'_1, x'_2)\|$ $(x_1 \leq x'_1)$.

(8) $\|\psi_2(x_1, x_2) - \psi_2(x'_1, x'_2)\| \leq \max\left(\sqrt{\frac{1+x_1^2}{(1+x_2)^2}}, \sqrt{\frac{1+x_1^2+(1+x'_2)^2}{(1+x_2)^4}}\right)\|(x_1, x_2) - (x'_1, x'_2)\|$ $(x_1 \geq x'_1)$.

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From the above facts, we see that $L_i \leq 1$ $(i = 0, 1)$ but $L_2 > 1$. To see this it is enough to take points $(1, 0)$, $(1, \epsilon)$ with $\epsilon < \sqrt{2} - 1$. Then we have

$$\frac{\|\psi_2(1, 0) - \psi_2(1, \epsilon)\|}{\|(1, 0) - (1, \epsilon)\|} > 1.$$

As we mentioned in Remark B, even if $L_2 > 1$ so that (C1)* fails, the definitions

$$\mathcal{D}_n = \{X_{\underbrace{0 \ldots 0}_{n}}\}, \quad \mathcal{B}_n = \{X_{\underbrace{0 \ldots 0i}_{n}}\} \quad (i = 1, 2)$$

still allow us to have Lemma 3.1 and the version of Lemma 3.4 which is enough to establish Theorem 3.1. In fact, we can show that $\exists 0 < \theta^* < 1$, $\exists 0 < \Delta < \infty$ and $\exists n_0 > 0$ such that $\forall (x_1, x_2), (x_1', x_2') \in X$

$$\|\psi^*_{\alpha_1 \ldots \alpha_n}(x_1, x_2) - \psi^*_{\alpha_1 \ldots \alpha_n}(x_1', x_2')\| \leq \Delta \theta^{*n} \|(x_1, x_2) - (x_1', x_2')\|$$

$\forall (\alpha_1 \ldots \alpha_n) \in I^{*n}$ and $\forall n \geq n_0$. For example, if $\alpha_i = 2$ $(i = 1, \ldots, n)$ then we can take $\theta^* = 15/16$. We shall omit the details for other cases (cf. [**18**]). For f satisfying (C2), it follows from (7) that for $i = 1, 2$,

$$\begin{aligned}
&|f\psi_{\underbrace{0 \ldots 0}_{n} i}(x_1, x_2) - f\psi_{\underbrace{0 \ldots 0}_{n} i}(x_1', x_2')| \\
&\quad \leq L_f (1 + 1/n)^{-\delta\theta} \|\psi_{\underbrace{0 \ldots 0}_{n-1} i}(x_1, x_2) - \psi_{\underbrace{0 \ldots 0}_{n-1} i}(x_1', x_2')\|^{\theta} \\
&\quad \ldots \leq L_f \left(\frac{n}{n+1}\right)^{\delta\theta} \left(\frac{n-1}{n}\right)^{\delta\theta} \ldots \left(\frac{1}{2}\right)^{\delta\theta} \|(x_1, x_2) - (x_1', x_2')\|^{\theta} \\
&\quad = L_f \left(\frac{1}{n+1}\right)^{\delta\theta} \|(x_1, x_2) - (x_1', x_2')\|^{\theta}.
\end{aligned}$$

We see that (C8) is automatically satisfied for $\delta\theta > 1$. Let p_1 be the natural projection $(x_1, x_2) \to x_1$ and let $\overline{f}$ be a piecewise Hölder continuous function on $p_1 X$ with respect to $\{p_1 X_i\}_{i=0,1,2}$ with exponent $\delta\theta > 1/2$. Then the function f defined by $f = \overline{f} p_1$ is also a piecewise Hölder continuous function with exponent $\theta > 1/2$ and satisfies (C8). We can apply results in §3 to T and to such an f satisfying (C4). In particular, for $f = -\log|\det DT|$ we can verify directly equi-Lipschitz continuity of $\{f^* \circ \psi^*_\alpha\}_{\alpha \in I^*}$ so that Theorem 3.1 holds. (C9) is easily checked in the case when ν is the Lebesgue measure. In fact, we have

$$H_\nu(Q^*) \leq \sum_{k=1}^{\infty} \sharp\mathcal{B}_k \, O(k^{-3}) \log O(k^{-3}) \leq 2 \sum_{k=1}^{\infty} O(k^{-3}) \log O(k^{-3})$$

(cf. [**18**]) and we can apply results in §4 to $f = -\log|\det DT|$.

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