Math. Proc. Camb. Phil. Soc. (1978), **83**, 19
Printed in Great Britain

Syntopogenous preordered spaces (II)

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(*Received* 2 *May* 1977)

1. *Introduction.* In (1) the authors introduced the idea of a syntopogenous preordered space (SPS), thereby generalizing simultaneously the concepts of 'uniform preordered space' and 'proximity preordered space'. In the case of a classical topology the SPS condition was seen to be equivalent to the condition, 'convex plus T_1-preordered'. In the present paper we show that, in the general case, the SPS condition may be characterized as the sum of two generalizations of the logically independent conditions of convexity and T_1-preorderedness.

In fact the notion of a convex syntopogenous space adopted herein turns out to be a generalization of Redfield's concept (5) of a 'nearly uniform ordered space'. As a consequence we are able to obtain – in a more general setting – analogues of several results in (5).

2. *Preliminaries.* It is assumed that the reader is familiar with the notation and results of Császár (2) and Burgess and Fitzpatrick (1). In addition note that if $\mathscr{S}$ is a syntopogenous structure on a non-empty set E, then by $\mathscr{U}(\mathscr{S})$ $(\delta(\mathscr{S}))$ we signify the classical quasi-uniformity (quasi-proximity) corresponding with $\mathscr{S}^b(\mathscr{S}^t)$. Moreover, if $\mathscr{U}(\delta)$ is a classical quasi-uniformity (quasi-proximity), then $\mathscr{U}^s(\delta^s)$ is used to denote the classical uniformity (proximity) generated by $\mathscr{U}(\delta)$.

Consider now the triple $(E, \mathscr{S}, \leqslant)$, where $\leqslant$ is a preordering on E. If $G(\leqslant) \subseteq G(\mathscr{S})$, then $\mathscr{S}$ is said to be $\leqslant$-inclusive; the same term is applied to a classical topology, quasi-uniformity or quasi-proximity if it corresponds to a $\leqslant$-inclusive syntopogenous structure. Note also that a pair of syntopogenous structures $(\mathscr{S}_1, \mathscr{S}_2)$ on E will be called a $\leqslant$-inclusive pair if $G(\leqslant) \subseteq G(\mathscr{S}_1) \cap G(\mathscr{S}_2^c)$.

3. *The $\mathscr{S}^u$ and $\mathscr{S}^1$ structures.* Given a space $(E, \mathscr{S}, \leqslant)$, let $\{\mathscr{S}_\alpha; \alpha \in \mathscr{A}\}$ $(\{\mathscr{S}_\beta; \beta \in \mathscr{B}\})$ denote the family of all $\leqslant$-$(\leqslant^d$-$)$ inclusive structures on E coarser than $\mathscr{S}$. These families are clearly non-empty since they each contain the trivial topogenous structure θ_E ((2); p. 86). Denoting $V\mathscr{S}_\alpha(V\mathscr{S}_\beta)$ by $\mathscr{S}^u(\mathscr{S}^1)$, we observe that $G(\mathscr{S}^u) = \cap G(\mathscr{S}_\alpha)$ $(G(\mathscr{S}^1) = \cap G(\mathscr{S}_\beta))$, so that $\mathscr{S}^u(\mathscr{S}^1)$ is the finest $\leqslant$-$(\leqslant^d$-$)$ inclusive structure on E coarser than $\mathscr{S}$ and $(\mathscr{S}^u, \mathscr{S}^1)$ is a $\leqslant$-inclusive pair. Moreover $\mathscr{S}$ is $\leqslant$-$(\leqslant^d$-$)$ inclusive if and only if $\mathscr{S} \sim \mathscr{S}^u(\mathscr{S} \sim \mathscr{S}^1)$.

An immediate consequence of (1); 3·3 (i)–(iv), is:

3·1. Theorem. *Given a space $(E, \mathscr{S}, \leqslant)$:*
(i) *if $\mathscr{S}_1 < \mathscr{S}$, then $\mathscr{S}_1^u < \mathscr{S}^u$ and $\mathscr{S}_1^1 < \mathscr{S}^1$;*
(ii) *if $\mathscr{S}_1 \sim \mathscr{S}$, then $\mathscr{S}_1^u \sim \mathscr{S}^u$ and $\mathscr{S}_1^1 \sim \mathscr{S}^1$;*

https://doi.org/10.1017/S0305004100054244 Published online by Cambridge University Press

(iii) *if* $a_1, \ldots, a_n \in \{i, b, t, p\}$, *then*

$$\mathscr{S}^{ua_1 \cdots a_n} < \mathscr{S}^{a_1 \cdots a_n u} \quad \text{and} \quad \mathscr{S}^{1a_1 \cdots a_n} < \mathscr{S}^{a_1 \cdots a_n 1};$$

(iv) $\mathscr{S}^{1c} < \mathscr{S}^{cu}$ *and* $\mathscr{S}^{uc} < \mathscr{S}^{c1}$.

3·2. COROLLARY. *Given a space* $(E, \mathscr{S}, \leqslant)$:

(i) *if* $a_1, \ldots, a_n \in \{i, b, t, p\}$ *and* $\mathscr{S} \sim \mathscr{S}^{a_1 \cdots a_n}$, *then* $\mathscr{S}^u \sim \mathscr{S}^{ua_1 \cdots a_n}$ *and* $\mathscr{S}^1 \sim \mathscr{S}^{1a_1 \cdots a_n}$;

(ii) *if* $\mathscr{S} \sim \mathscr{S}^s$, *then* $\mathscr{S}^u \sim \mathscr{S}^{1c}$, $\mathscr{S}^{uc} \sim \mathscr{S}^1$ *and* $\mathscr{S}^{us} \sim \mathscr{S}^{1s}$.

Proof. (ii) It is clear from (iv) and (i) above that $\mathscr{S}^{1c} < \mathscr{S}^u$ and $\mathscr{S}^{uc} < \mathscr{S}^1$. Applying the '$c$' operation to both results (and noting that $cc = c$), we have $\mathscr{S}^1 < \mathscr{S}^{uc}$ and $\mathscr{S}^u < \mathscr{S}^{1c}$. Hence $\mathscr{S}^u \sim \mathscr{S}^{1c}$ and $\mathscr{S}^{uc} \sim \mathscr{S}^1$. In addition $\mathscr{S}^{us} \sim \mathscr{S}^u \vee \mathscr{S}^{uc}$ ((2); 8·107) $\sim \mathscr{S}^{1c} \vee \mathscr{S}^1 \sim \mathscr{S}^{1s}$.

In the three results below we investigate the $\mathscr{S}^u$ and $\mathscr{S}^1$ concepts from the viewpoint of associated structures.

3·3. THEOREM. *Given a classical topology* τ *and its associated topology* $\mathscr{S}$ *on* E, *then* $\tau(\mathscr{S}^u) = \tau^u$ *and* $\tau(\mathscr{S}^1) = \tau^1$ (*where* $\tau^u(\tau^1)$ *denotes the classical topology consisting of all increasing* (*decreasing*) τ*-open sets* (1)).

If $\mathscr{U}$ is a classical quasi-uniformity on $(E, \leqslant)$, let $\mathscr{U}^u$ denote the family $\{V \in \mathscr{U}; \exists V_1, V_2, \ldots \in \mathscr{U}$ such that $V_1 = V$ and, for $n \geqslant 1$, $G(\leqslant) \subseteq V_n$ and $V_{n+1} \circ V_{n+1} \subseteq V_n\}$. By comparing the proof of proposition 5·1 (5) we deduce that $\mathscr{U}^u$ is a classical quasi-uniformity and is $\leqslant$-inclusive. $\mathscr{U}^1$ is defined analogously using $\leqslant^d$.

3·4. THEOREM. *Given a classical quasi-uniformity* $\mathscr{U}$ *and its associated biperfect syntopology* $\mathscr{S}$ *on* $(E, \leqslant)$, $\mathscr{U}(\mathscr{S}^u) = \mathscr{U}^u$ *and* $\mathscr{U}(\mathscr{S}^1) = \mathscr{U}^1$.

Proof. Since $\mathscr{U}^u \subseteq \mathscr{U}$ and $\mathscr{U}^u$ is $\leqslant$-inclusive, then $\mathscr{U}^u \subseteq \mathscr{U}(\mathscr{S}^u)$.

On the other hand, since $\mathscr{S}^{ub} < \mathscr{S}^b = \mathscr{S}$ and $G(\mathscr{S}^{ub}) = G(\mathscr{S}^u) \supseteq G(\leqslant)$, then each member of $\mathscr{U}(\mathscr{S}^u)$ is a member of $\mathscr{U}$ and a superset of $G(\leqslant)$. Hence $\mathscr{U}(\mathscr{S}^u) \subseteq \mathscr{U}^u$.

3·5. THEOREM. *Given a quasi-proximity* δ *and its associated topogenous structure* $\mathscr{S}$ *on* $(E, \leqslant)$, *let* $\delta^u(\delta^1)$ *denote* $\delta(\mathscr{S}^u)$ $(\delta(\mathscr{S}^1))$. *Then* $A\delta^u B(A\delta^1 B)$ *if and only if, for any finite coverings* $\{A_1, \ldots, A_m\}$ *and* $\{B_1, \ldots, B_n\}$ *of* A *and* B *respectively,* $\exists\, i$ *and* j *such that* $A_i \delta_\alpha B_j$, *for every* $\leqslant$*-*($\leqslant^d$*-*) *inclusive quasi-proximity* δ_α *coarser than* δ.

Proof. By (2); 7·23, $A\delta^u B$ if and only if $A \not< E - B$, where $\{<\} = \mathscr{S}^{ut}$. Taking $a = t$ in (2); 8·99, it is easy to see that in the usual representation of $\mathscr{S}^u$ there will be no loss of generality in taking each structure $\mathscr{S}_\alpha$ to be topogenous, with the form $\{<_\alpha\}$ and associated quasi-proximity δ_α. Then, by (2); 8·90, 31, 14,

$$(\vee \mathscr{S}_\alpha)^t = (\cup \mathscr{S}_\alpha)^{ot} = (\cup \mathscr{S}_\alpha)^t = \{(\cup <_\alpha)^q\}.$$

Hence $A\delta^u B$ if and only if $A(\cup <_\alpha)^q E - B$ is false. The rest follows from (2); 3·7, 7·23.

3·6. THEOREM. *If* $(E, \mathscr{S}, \leqslant)$ *is a SPS, then*:

(i) $\mathscr{S} \sim (\mathscr{S}^u \vee \mathscr{S}^1)^a$ *for some ordinary operation* a *in* $\{i, p, b\}$;

(ii) $(\mathscr{S}^u, \mathscr{S}^1)$ *is a* $\leqslant$*-defining pair.*

Proof. Suppose $\mathscr{S} \sim (\mathscr{S}_1 \vee \mathscr{S}_2)^a$ for some $\leqslant$-defining pair $(\mathscr{S}_1, \mathscr{S}_2)$ and some ordinary operation a in $\{i, p, b\}$. Since $\mathscr{S}_1 < \mathscr{S}$ and, *a fortiori*, $\mathscr{S}_1$ is $\leqslant$-inclusive, then $\mathscr{S}_1 < \mathscr{S}^u$, implying $G(\mathscr{S}^u) \subseteq G(\mathscr{S}_1) = G(\leqslant)$. Hence $G(\mathscr{S}^u) = G(\leqslant)$.

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Similarly $\mathscr{S}_2 < \mathscr{S}^1$ and $G(\mathscr{S}^1) = G(\leqslant^d)$. Then $\mathscr{S} < (\mathscr{S}^u \vee \mathscr{S}^1)^a < \mathscr{S}^a \sim \mathscr{S}$.

Conditions (i) and (ii) above will be studied in more detail in sections 4 and 5 respectively.

4. *Convex spaces*

4·1. DEFINITION. *A space* $(E, \mathscr{S}, \leqslant)$ *is called* convex *if* $\mathscr{S} \sim (\mathscr{S}^u \vee \mathscr{S}^1)^a$ *for some ordinary operation* a *in* $\{i, p, b\}$.

From an inspection of the argument in 3·6 it is clear that $(E, \mathscr{S}, \leqslant)$ is convex if and only if $\mathscr{S} \sim (\mathscr{S}_1 \vee \mathscr{S}_2)^a$ for some ordinary operation a in $\{i, p, b\}$ and for some $\leqslant$-inclusive pair $(\mathscr{S}_1, \mathscr{S}_2)$ on E. Comparing this with the definition of a SPS we remark that 'convex' analogues exist for a majority of the results and proofs in sections 4–8 of (1). Here, however, we shall be more concerned with descriptions of various types of convex spaces (and later SPSs) in terms of the $\mathscr{S}^u$ and $\mathscr{S}^1$ structures.

In the case where $\mathscr{S}$ is a topology our first result below shows that the 'a' in 4·1 may be replaced by 'tp'.

4·2. THEOREM. $(E, \mathscr{S}, \leqslant)$ *is a convex topological space if and only if* $\mathscr{S} = (\mathscr{S}_1^u \vee \mathscr{S}_2^1)^{tp}$ $(= (\mathscr{S}^{utp} \vee \mathscr{S}^{1tp})^{tp})$.

Proof. ($\Rightarrow$) By comparing the proof of (1); 5·2(i) $\Rightarrow$ (iii), we see that $\mathscr{S} = (\mathscr{S}_1 \vee \mathscr{S}_2)^{tp}$ for some $\leqslant$-inclusive pair $(\mathscr{S}_1, \mathscr{S}_2)$ on E. The rest follows by noting that $tptp = tp$ ((2); 8·50) and arguing as in the last sentence in the proof of 3·6. For the equality in brackets refer to (2); 8·100.

($\Leftarrow$) It suffices to note that $(\mathscr{S}^u \vee \mathscr{S}^1)^{tp} \sim (\mathscr{S}^{ut} \vee \mathscr{S}^{1t})^p$ ((2); 8·101) and that $G(\mathscr{S}^{ut}) = G(\mathscr{S}^u)$ and $G(\mathscr{S}^{1t}) = G(\mathscr{S}^1)$.

As motivation for the term 'convex' in 4·1 we observe that the bracketed statement in 4·2 is equivalent to saying that $\tau(\mathscr{S}) = \tau(\mathscr{S})^u \vee \tau(\mathscr{S})^1$. It follows that our definition of a convex topological space corresponds to Nachbin's definition (4) of a convex classical topological space.

Although by comparing the proof of (1); 5·2(ii) $\Rightarrow$ (iii), it is easily verified that if $(E, \mathscr{S}, \leqslant)$ is convex, then so too is $(E, \mathscr{S}^{tp}, \leqslant)$, the converse nevertheless is false (see Example 4·5 below). Thus the concept of a convex syntopogenous space is not purely topological.

By an argument analogous to that used in (1); 6·2(i) $\Rightarrow$ (v), it is readily seen that, for a symmetrical syntopology $\mathscr{S}$ on E, $(E, \mathscr{S}, \leqslant)$ is convex if and only if $\mathscr{S} \sim \mathscr{S}_0^{sb}$ for some $\leqslant$-inclusive biperfect syntopology $\mathscr{S}_0$ on E. This latter condition is equivalent to saying that the (classical) uniformity $\mathscr{U}(\mathscr{S})$ is generated by the quasi-uniformity $\mathscr{U}(\mathscr{S}_0)$, which is $\leqslant$-inclusive. Thus, in the case where $\leqslant$ is an ordering and $\mathscr{S}$ is separated, our definition of a convex symmetrical syntopological space corresponds to Redfield's definition (5) of a nearly uniform ordered space.

4·3. THEOREM. *If* $\mathscr{S}$ *is a symmetrical syntopology on* E, *then* $(E, \mathscr{S}, \leqslant)$ *is convex if and only if* $\mathscr{S} < \mathscr{S}^{usb} (= \mathscr{S}^{ubsb})$.

Proof. ($\Rightarrow$) Suppose $\mathscr{S} \sim \mathscr{S}_0^{sb}$ in the notation above. Since clearly $\mathscr{S}_0 < \mathscr{S}^u$ then $\mathscr{S} < \mathscr{S}^{usb}$. The equality $sb = bsb$ comes from (2); 8·52.

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($\Leftarrow$) Observe that $\mathscr{S}^{ubsb} \sim (\mathscr{S}^{ub} \vee \mathscr{S}^{ubc})^b$ ((2); 8·107) and $(\mathscr{S}^{ub}, \mathscr{S}^{ubc})$ is a $\leqslant$-inclusive pair.

With the concept of a convex classical uniform space defined in harmony with the remarks preceding 4·3, we note:

4·4 COROLLARY. *A classical uniform space $(E, \mathscr{U}, \leqslant)$ is convex if and only if $\mathscr{U} \subseteq \mathscr{U}^{us}$.*

In the separated/ordered case the above result corresponds to proposition 5·2(5).

4·5. EXAMPLE. Let R be the real line with the usual ordering $\leqslant$. For each $n \geqslant 1$, define subsets V_n and U_n of $R \times R$ by:

$$V_n = \{(x, y);\ x \geqslant n, y \geqslant n \quad \text{and} \quad x, y \text{ are both rational}\}$$

$$U_n = V_n \cup \{(x, x);\ x \in R\}.$$

It is easily verified that $\{U_n;\ n \geqslant 1\}$ constitutes a base for a classical uniformity $\mathscr{U}$ on R and that $\tau(\mathscr{U})$ is discrete. We claim in addition that

(i) $(R, \mathscr{U}, \leqslant)$ is not convex;

(ii) $(R, \tau(\mathscr{U}), \leqslant)$ is convex.

Proof of (i). If A is any member of $\mathscr{U}^u$, then there exists $A_1 \in \mathscr{U}^u$ such that

$$G(\leqslant) \subseteq A_1 \subseteq A_1 \circ A_1 \subseteq A,$$

and since $\mathscr{U}^u \subseteq \mathscr{U}$, $U_n \subseteq A_1$ for some $n \geqslant 1$. If θ is any irrational in $]0, 1[$, then

$$(n, n+\theta) \in G(\leqslant) \subseteq A,$$

while $(n+\theta, n+1) \in G(\leqslant) \subseteq A_1$ and $(n+1, n) \in U_n \subseteq A_1$ so that $(n+\theta, n) \in A$. Hence $A \cap A^d \notin U_1$ so that $\mathscr{U} \notin \mathscr{U}^{us}$.

Proof of (ii). Immediate since $\tau(\mathscr{U})$ is discrete.

By methods analogous to those used in the proof of theorem 7·1(1), we may show that, for a symmetrical topogenous structure $\mathscr{S}$ on E, $(E, \mathscr{S}, \leqslant)$ is convex if and only if one of the following conditions is satisfied:

(i) $\mathscr{S} = \mathscr{S}_0^s$ for some $\leqslant$-inclusive topogenous structure $\mathscr{S}_0$ on E;

(ii) $\mathscr{S} = \mathscr{S}_0^{sb}$ for some $\leqslant$-inclusive topogenous structure $\mathscr{S}_0$ on E.

In view of the fact that the study of convex symmetrical syntopological spaces already includes those spaces satisfying (ii) (see the first sentence of the paragraph preceding 4·3) and the fact that the only separated structure on $(E, \leqslant)$ with this condition is $\mathscr{D}_E$, the term 'convex' as applied to a symmetrical topogenous space will be reserved for a space satisfying (i) above. Incidentally the condition in (i) is equivalent to saying that the proximity defined by $\mathscr{S}$ is generated by a quasi-proximity which is $\leqslant$-inclusive.

4·6. THEOREM. *A symmetrical topogenous space $(E, \mathscr{S}, \leqslant)$ is convex if and only if $\mathscr{S} < \mathscr{S}^{uts}$.*

Proof. ($\Rightarrow$) Suppose $\mathscr{S} = \mathscr{S}_0^s$ as in (i) above. Then $\mathscr{S}_0 < \mathscr{S}^u$, implying

$$\mathscr{S} = \mathscr{S}_0^s = \mathscr{S}_0^{ts} < \mathscr{S}^{uts}.$$

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Defining a convex proximity space on the basis of the last remark before 4·6, we have:

4·7. COROLLARY. *A proximity space $(E, \delta, \leqslant)$ is convex if and only if δ^{us} is finer than δ.*

5. *T_1-preordered spaces.*

5·1. DEFINITION. *A space $(E, \mathscr{S}, \leqslant)$ is called* lower (upper) T_1-preordered *if $\mathscr{S}^u(\mathscr{S}^1)$ is $\leqslant$-($\leqslant^d$-) compatible. When both conditions are satisfied $(E, \mathscr{S}, \leqslant)$ is termed* T_1-preordered.

In the case where $\leqslant$ is the discrete ordering on E it is easily seen that $(E, \mathscr{S}, \leqslant)$ is T_1-ordered if and only if it is T_1. As further motivation for the terminology introduced in 5·1, compare McCartan's definition(3) of a lower (upper) T_1-ordered classical topological space with condition (iii) below.

5·2. THEOREM. *Given a topological space $(E, \mathscr{S}, \leqslant)$, where $\tau = \tau(\mathscr{S})$, the following are equivalent*:

(i) *$\mathscr{S}^u(\mathscr{S}^1)$ is $\leqslant$-($\leqslant^d$-) compatible*;

(ii) *$[\leftarrow, x]$ is τ^u-closed ($[x, \rightarrow]$ is τ^1-closed), $\forall\, x \in E$*;

(iii) *$[\leftarrow, x]$ is τ-closed ($[x, \rightarrow]$ is τ-closed), $\forall\, x \in E$.*

The statement of 3·6 may now be reformulated as: $(E, \mathscr{S}, \leqslant)$ is a SPS if and only if it is convex and T_1-preordered. That the concepts of convexity and T_1-preorderedness are logically independent is shown by the following examples.

5·3. EXAMPLE. Let E denote the real line and let Q denote the set of rational numbers. McCartan(3) has observed that a topology τ on E may be defined by calling a subset G of E τ-open if and only if, for each $x \in G$, $\exists$ an open interval I_x in E such that $x \in I_x$ and $I_x \cap Q \subseteq G$; also, if $\leqslant$ denotes the usual ordering on E, then $(E, \tau, \leqslant)$ is T_1-ordered. The space is not, however, convex. For example, note that Q is a τ-open neighbourhood of the point 1 and suppose $\exists\, \tau$-open sets G_1 and G_2, respectively increasing and decreasing, such that $1 \in G_1 \cap G_2 \subseteq Q$. Then G_1 and G_2 have the forms $]a, \rightarrow]$ and $[\leftarrow, b[$ respectively, where $a < 1 < b$. Hence $]a, b[\subseteq Q$, which is impossible.

5·4 EXAMPLE. Let $(E, \leqslant)$ be as in 5·3, and let τ be the usual topology on E. Define an ordering $\leqslant^*$ on E by: $x \leqslant^* y$ if and only if $x \leqslant y \leqslant 0$ or $0 < x \leqslant y$. Then $(E, \tau, \leqslant^*)$ is convex but not T_1-ordered (since, for example, $\{x \in E;\ x \leqslant^* 1\} = \{x \in E;\ 0 < x \leqslant 1\}$ and is therefore not τ-closed).

On the other hand, given a space $(E, \mathscr{S}, \leqslant)$, where $\leqslant$ is a total ordering and $\mathscr{S}$ is T_1, the following result shows that T_1-preorderedness is implied by convexity.

5·5. THEOREM. *Given a space $(E, \mathscr{S}, \leqslant)$, where $\leqslant$ is a total ordering and $\mathscr{S}$ is T_1, then $(E, \mathscr{S}, \leqslant)$ is a SPS if it is convex.*

Proof. Suppose $\mathscr{S} \sim (\mathscr{S}^u \vee \mathscr{S}^1)^a$, where $a \in \{i, b, p\}$. If $(E, \mathscr{S}, \leqslant)$ is not T_1-ordered, then, without loss of generality, we may assume $x \not\leqslant y$ for some $(x, y) \in G(\mathscr{S}^u)$. Thus $y < x$ and so $(x, y) \in G(\mathscr{S}^1)$, implying $(x, y) \in G(\mathscr{S}^u) \cap G(\mathscr{S}^1) = G((\mathscr{S}^u \vee \mathscr{S}^1)^a) = G(\mathscr{S})$. Since $\mathscr{S}$ is T_1 then $x = y$ – contradiction.

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When $\mathscr{S}$ is a symmetrical syntopology the above result corresponds to Redfield's proposition 2·4 (5).

5·6. THEOREM. *Given a space* $(E, \mathscr{S}, \leqslant)$, *where* $\mathscr{S} \sim \mathscr{S}^s$, $(E, \mathscr{S}, \leqslant)$ *is lower* T_1-*preordered if and only if it is upper* T_1-*preordered.*

Proof. It suffices to note from 3·2(ii) that $\mathscr{S}^u \sim \mathscr{S}^{1c}$.

The above result, combined with 3·6, 4·3 and 4·6, enables us to deduce:

5·7. THEOREM. (i) *A symmetrical syntopological space* $(E, \mathscr{S}, \leqslant)$ *is a SBPS if and only if* $\mathscr{S} < \mathscr{S}^{usb}$ *and* $G(\mathscr{S}^u) \subseteq G(\leqslant)$;

(ii) *a symmetrical topogenous space* $(E, \mathscr{S}, \leqslant)$ *is a STPS if and only if* $\mathscr{S} < \mathscr{S}^{uts}$ *and* $G(\mathscr{S}^u) \subseteq G(\leqslant)$.

5·8. COROLLARY. (i) *A classical uniform space* $(E, \mathscr{U}, \leqslant)$ *is a uniform preordered space if and only if* $\mathscr{U} \subseteq \mathscr{U}^{us}$ *and* $\cap \{U;\ U \in \mathscr{U}^u\} \subseteq G(\leqslant)$;

(ii) *a proximity space* $(E, \delta, \leqslant)$ *is a proximity preordered space if and only if* δ^{us} *is finer than* δ *and* $x \delta^u y \Rightarrow x \leqslant y$, $\forall\, x, y \in E$.

In the ordered case 5·8(i) corresponds to Redfield's proposition 5·3(5).

In closing, the authors wish to record their gratitude to the referee for his revision and simplification of Example 4·5.

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