99.36 Stable tables

Introduction

When can we place a table (with four legs of equal length) on an uneven floor in such a way that the feet of all four legs are in contact with the floor? While browsing through some old copies of the *Gazette* I found the article [1] on this topic but by now, nearly forty years later, this has been considered by many authors. For example, [2] contains 28 references to the problem. However, the purpose of this note is to inject some caution into the discussion. Generally speaking, the authors use plausible, and very appealing, arguments which involve rotating or sliding the table over the floor, without considering carefully whether this is possible. Later we shall show that it is not always possible.

This is often regarded as a problem in 'real life', and in such cases the authors include physical constraints such as (i) '*not allowing the legs to penetrate the floor*', (ii) '*it is assumed that the table may be turned if it is not too near the walls ...*', and (iii) '*we assume, realistically, that the floor is not so mountainous that such a rotation could be obstructed ...*'. However, all such physical assumptions can be swept aside if we construct, and use, a mathematical model of a 'virtual' table and floor as might be displayed on a computer screen.

We shall work in $\mathbb{R}^3$, and we denote the distance between (x, y, z) and (a, b, c), defined in the usual way, by $\|(x, y, z) - (a, b, c)\|$. We now discuss the following problem:

> given a surface S of the form $z = \phi(x, y)$ in $\mathbb{R}^3$, and a plane quadrilateral Q whose vertices $v_1, \ldots, v_4$ represent the feet of the four legs of the virtual table, labelled in this order around Q, when can Q be placed in $\mathbb{R}^3$ so that each v_j is on S?

In some papers, one finds the following statement (or something equivalent to it):

> given that S is sufficiently smooth, Q can always be placed so that each v_j is on S.

Of course it can! We only have to interpret 'sufficiently smooth' to mean 'planar', and then the conclusion is trivially true. The point here is that such assertions can only have a non-trivial content if the phrase 'sufficiently smooth' is quantified in a precise way which will include some cases and exclude others. This is done in [2], but this is a rare exception.

Let us consider the conventional 'solution' to this problem when Q is a square. Place the table so that v_1 and v_3 are on S. Then move Q (in some unspecified way – except for vague references to sliding or rotating) such that at all times either v_1 and v_3 are on S, or v_2 and v_4 are on S, and such that in the final position, the positions originally occupied by v_1, v_2, v_3, v_4 are now occupied by v_2, v_3, v_4, v_1, respectively. In this final position, the vertices v_2 and v_4 are on S. Now, by some (often unspecified) use of the Intermediate Value Theorem, when the change from v_1 and v_3 being on S to

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v_2 and v_4 being on S occurs, all four vertices must be on S. This argument, which appears in several very well-known sources of articles on 'popular' mathematics, is neat and appealing, but is it really valid? It seems that the authors of such assertions implicitly assume that such a movement of Q is possible whereas, in fact, it is not (at least not without further assumptions). In Section 2 we shall construct a surface which contains two points X_0 and Y_0 such that, for X near X_0, and Y near Y_0, the function $\|X - Y\|$ attains its minimum if, and only if, $X = X_0$ and $Y = Y_0$. Clearly if $\|X_0 - Y_0\| = \|v_1 - v_3\|$, and we place the table on this surface with v_1 at X_0 and v_3 at Y_0, then it will be impossible to slide the table away from this position while always requiring that v_1 and v_3 remain on the surface.

An immobile table on a smooth floor

We shall now construct the surface referred to in the previous paragraph. First, for each positive integer n, and each non-negative number t, $t^{1/n}$ denotes the non-negative nth root of t. As usual $\sqrt{t} = t^{1/2} \geqslant 0$.

Consider the parabola given by $z^2 = 4x$ lying in the plane $y = 0$. The arc of this parabola which lies in the region $\{z > 0\}$ is given by $z = 2\sqrt{x}$, where $x \geqslant 0$, and if we rotate this about the z-axis we obtain the surface S_0 given by $z = 2(x^2 + y^2)^{1/4}$ (and which has a 'cusp' at $(0, 0, 0)$). Next, we construct a surface S_1 by translating S_0 by the vector $(1, 0, 0)$, and then reflecting it across the plane $z = 0$; thus S_1 is given by $z = -2((x - 1)^2 + y^2)^{1/4}$. We have illustrated the surfaces S_0 and S_1 in the vicinity of $(0, 0, 0)$ and $(1, 0, 0)$, respectively, in Figure 1. Finally, we construct our surface S by insisting that S is the same as S_0 for $x^2 + y^2 < 1/16$, the same as S_1 for $(x - 1)^2 + y^2 < 1/16$, and that S is chosen as the reader wishes in the remaining area. This latter choice is of no consequence for the following analysis, but obviously it can be chosen so that S is continuous throughout $\mathbb{R}^2$, and 'smooth' except at the points $(0, 0, 0)$ and $(1, 0, 0)$.

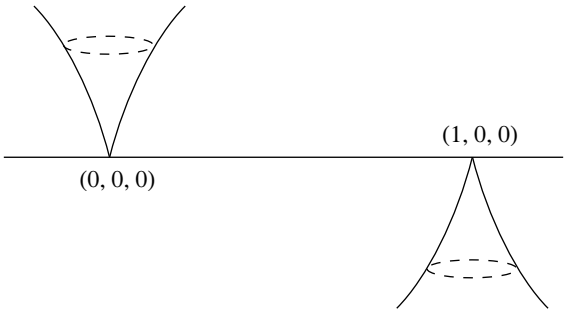


FIGURE 1: The surfaces S_0 and S_1

We claim that if $X \in S_0$, $Y \in S_1$ and $\|X - Y\| \leqslant 1$, then $X = (0, 0, 0)$ and $Y = (1, 0, 0)$. The significance of this is that if Q is

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such that $\|v_1 - v_3\| = 1$, and is placed so that $v_1 = (0, 0, 0)$ and $v_3 = (1, 0, 0)$, then *Q cannot be moved away from this position in such a way that these two vertices stay on S*. Hence, in this case, the Intermediate Value Theorem is not applicable.

We shall now verify the claim made above. We let $X = (r, s, t)$ and $Y = (1 + u, v, w)$; then

$$t = 2\left(r^2 + s^2\right)^{1/4}, \qquad w = -2\left(u^2 + v^2\right)^{1/4},$$

and

$$(t - w)^2 = 4\left(\left(r^2 + s^2\right)^{1/4} + \left(u^2 + v^2\right)^{1/4}\right)^2 \geqslant 4\left(\sqrt{|r|} + \sqrt{|u|}\right)^2 \geqslant 4\left(|r| + |u|\right).$$

Now suppose that $X \in S_0$, $Y \in S_1$ and $\|X - Y\| \leqslant 1$. Then

$$\begin{aligned} 1 &\geqslant \|X - Y\|^2 \\ &= (1 + u - r)^2 + (s - v)^2 + (t - w)^2 \\ &= 1 - (r - u)(2 + u - r) + (s - v)^2 + 4\left(\left(r^2 + s^2\right)^{1/4} + \left(u^2 + v^2\right)^{1/4}\right)^2 \quad (2.1) \\ &\geqslant 1 - (r - u)(2 + u - r) + (s - v)^2 + 4\left(|r| + |u|\right). \quad (2.2) \end{aligned}$$

Recall that since $X \in S_0$ and $Y \in S_1$, $|u| < 1/4$ and $|r| < 1/4$. As $2 + u - r \geqslant 2 - |u| - |r| > 3/2 > 0$, (2.2) shows that $r \geqslant u$ (else the right-hand side of (2.1) exceeds 1). Hence $2 + u - r \leqslant 2$ so that, from (2.2) again,

$$\begin{aligned} 1 &\geqslant 1 - 2(r - u) + 4\left(|r| + |u|\right) + (s - v)^2 \\ &= 1 + \left(4|r| - 2r\right) + \left(4|u| + 2u\right) + (s - v)^2. \end{aligned}$$

This shows that $r = u = 0$ and $s = v$, and then, from (2.1), which now reads $1 \geqslant 1 + 4\left(\sqrt{|s|} + \sqrt{|v|}\right)^2$, we find that $s = v = 0$. Thus $(r, s, t) = (u, v, w) = (0, 0, 0)$ and the proof is complete.

References

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2. W. Baritompa, R. Löwen, B. Polster and M. Ross, Mathematical table turning revisited, *Math. Intelligencer* **29** (2007) pp. 49-58.

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