J. Aust. Math. Soc. **112** (2022), 145–169
doi:10.1017/S1446788721000082

CONDITION (K) FOR BOOLEAN DYNAMICAL SYSTEMS

TOKE MEIER CARLSEN and EUN JI KANG

(Received 22 January 2020; accepted 5 March 2021; first published online 27 May 2021)

Communicated by Aidan Sims

Abstract

We generalize Condition (K) from directed graphs to Boolean dynamical systems and show that a locally finite Boolean dynamical system $(\mathcal{B}, \mathcal{L}, \theta)$ with countable $\mathcal{B}$ and $\mathcal{L}$ satisfies Condition (K) if and only if every ideal of its C^*-algebra is gauge-invariant, if and only if its C^*-algebra has the (weak) ideal property, and if and only if its C^*-algebra has topological dimension zero. As a corollary we prove that if the C^*-algebra of a locally finite Boolean dynamical system with $\mathcal{B}$ and $\mathcal{L}$ countable either has real rank zero or is purely infinite, then $(\mathcal{B}, \mathcal{L}, \theta)$ satisfies Condition (K). We also generalize the notion of maximal tails from directed graph to Boolean dynamical systems and use this to give a complete description of the primitive ideal space of the C^*-algebra of a locally finite Boolean dynamical system that satisfies Condition (K) and has countable $\mathcal{B}$ and $\mathcal{L}$.

2020 Mathematics subject classification: primary 46L05; secondary 46L55.

Keywords and phrases: C^*-algebras of Boolean dynamical systems, Condition (K), gauge-invariant ideals, graph C^*-algebras, ideal property, maximal tail, primitive ideal space, real rank zero, topological dimension zero, ultrafilter cycle.

1. Introduction

1.1. Background. C^*-algebras associated to Boolean dynamical systems were introduced in [7] as a generalization of graph C^*-algebras. The class of C^*-algebras of Boolean dynamical systems also contains the class of ultragraph C^*-algebras, the class of C^*-crossed products of Cantor minimal systems, C^*-algebras of second countable zero-dimensional topological graphs, C^*-algebras of shift spaces, as well as many labeled graph C^*-algebras.

One of the merits of C^*-algebras of Boolean dynamical systems is that many of the results about graph C^*-algebras can be generalized to C^*-algebras of Boolean dynamical systems. In [7], the K-theory of the C^*-algebra of a Boolean dynamical system was computed, the set of gauge-invariant ideals of the C^*-algebra of a Boolean dynamical system was determined, Condition (L) for a Boolean dynamical system was

Research partially supported by NRF-2017R1D1A1B03030540.
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https://doi.org/10.1017/S1446788721000082 Published online by Cambridge University Press

introduced as a generalization of Condition (L) for directed graphs, and it was shown that a Boolean dynamical system $(\mathcal{B}, \mathcal{L}, \theta)$ with countable $\mathcal{B}$ and $\mathcal{L}$ satisfies Condition (L) if and only if its C^*-algebra satisfies the Cuntz–Krieger uniqueness theorem.

Condition (K) for directed graphs was introduced in [18]. A directed graph satisfies Condition (K) if and only if every ideal of its C^*-algebra is gauge-invariant [1, Corollary 3.8], and if and only if its C^*-algebra has real rank zero [12, Theorem 3.5]. If a C^*-algebra has real rank zero, then it has the ideal property [21, Remark 2.1] and is K_0-liftable [21, Definition 3.1]. It is proven in [21, Proposition 2.11 and Theorem 4.2] that the converse holds for a separable purely infinite C^*-algebra. The weak ideal property was introduced in [19], where it was also shown that the ideal property implies the weak ideal property. It was proven in [20, Theorem 2.8] that a C^*-algebra with the weak ideal property has topological dimension zero.

In [2], the notion of a maximal tail of a row-finite directed graph with no sinks was introduced, and it was shown that if a row-finite directed graph E with no sinks satisfies Condition (K), then there is a bijective correspondence between the primitive ideals of the C^*-algebra of E and the maximal tails of E. In addition, a topology on the set of maximal tails was introduced, and it was proved that the previously mentioned bijective correspondence becomes a homeomorphism if the set of maximal tails is equipped with this topology and the set of primitive ideals of the C^*-algebra of E is equipped with the hull-kernel topology. This was generalized to arbitrary directed graphs in [11].

1.2. The contents of this paper. In this paper we introduce Condition (K) for Boolean dynamical systems (Definition 5.1) and prove that a locally finite Boolean dynamical system $(\mathcal{B}, \mathcal{L}, \theta)$ with countable $\mathcal{B}$ and $\mathcal{L}$ satisfies Condition (K) if and only if the quotient Boolean dynamical system $(\mathcal{B}/\mathcal{H}, \mathcal{L}, \theta)$ satisfies Condition (L) for every hereditary saturated ideal $\mathcal{H}$ of $\mathcal{B}$, and if and only if every ideal of its C^*-algebra $C^*(\mathcal{B}, \mathcal{L}, \theta)$ is gauge-invariant (Theorem 6.3).

We also generalize the notion of maximal tails from directed graph to Boolean dynamical systems (Definition 4.1), and show that if a locally finite Boolean dynamical system $(\mathcal{B}, \mathcal{L}, \theta)$ with countable $\mathcal{B}$ and $\mathcal{L}$ satisfies Condition (K), then there is a bijective correspondence between the primitive ideals of the C^*-algebra of $(\mathcal{B}, \mathcal{L}, \theta)$ and the maximal tails of $(\mathcal{B}, \mathcal{L}, \theta)$ (Proposition 7.4); and we introduce a topology and the set of maximal tails of $(\mathcal{B}, \mathcal{L}, \theta)$ (Proposition 7.1) such that this correspondence becomes a homeomorphism when the set of maximal tails of $(\mathcal{B}, \mathcal{L}, \theta)$ is equipped with this topology and the set of primitive ideals of the C^*-algebra of $(\mathcal{B}, \mathcal{L}, \theta)$ is equipped with the hull-kernel topology (Theorem 7.5).

Using these results, we also prove that a locally finite Boolean dynamical system $(\mathcal{B}, \mathcal{L}, \theta)$ with countable $\mathcal{B}$ and $\mathcal{L}$ satisfies Condition (K) if and only if its C^*-algebra has the (weak) ideal property, if and only if its C^*-algebra has topological dimension zero, and if and only if its C^*-algebra has no quotient that contains a corner that is isomorphic to $M_n(C(\mathbb{T}))$ for some $n \in \mathbb{N}$ (Theorem 8.1). As a corollary we prove that if the C^*-algebra of a locally finite Boolean dynamical system with $\mathcal{B}$ and $\mathcal{L}$ countable

https://doi.org/10.1017/S1446788721000082 Published online by Cambridge University Press

either has real rank zero or is purely infinite, then $(\mathcal{B}, \mathcal{L}, \theta)$ satisfies Condition (K) (Corollary 8.2).

1.3. Further discussions. There are plenty of Boolean dynamical systems that satisfy Condition (K) without their C^*-algebras being purely infinite, for instance Boolean dynamical systems that give rise to approximately finite-dimensional (AF) algebras (see [13]) and Boolean dynamical systems that give rise to Cantor minimal systems (see [14]). In contrast to this, the authors do not know of any Boolean dynamical system that satisfies Condition (K) without its C^*-algebra having real rank zero.

The reason that we have to assume that $\mathcal{B}$ and $\mathcal{L}$ are countable in many of our results is that this is an assumption in the Cuntz–Krieger uniqueness theorem [7, Theorem 9.9]. The authors are currently working on a paper [6] to prove that the Cuntz–Krieger uniqueness theorem holds without the assumption that $\mathcal{B}$ and $\mathcal{L}$ are countable, and generalize most of the main results of this paper to Boolean dynamical systems $(\mathcal{B}, \mathcal{L}, \theta)$ where $\mathcal{B}$ and $\mathcal{L}$ are not necessarily countable (and to the generalized Boolean dynamical systems introduced in [5]). Even though the authors do not know of any interesting Boolean dynamical systems with $\mathcal{B}$ or $\mathcal{L}$ uncountable, this might be interesting for future applications.

The authors are convinced that if a (generalized) Boolean dynamical system satisfies Condition (K), then its C^*-algebra is K_0-liftable. However, to prove this one faces the problem that a gauge-invariant ideal of the C^*-algebra of a Boolean dynamical system is not necessarily Morita equivalent to a C^*-algebra of a Boolean dynamical system. The authors have in [5] enlarged the class of C^*-algebras of Boolean dynamical systems by introducing the notation of relative generalized Boolean dynamical systems and constructed C^*-algebras corresponding to these relative generalized Boolean dynamical systems. The authors are planning to show that any gauge-invariant ideal of a C^*-algebra of a (relative generalized) Boolean dynamical system is Morita equivalent to the C^*-algebra of a relative generalized Boolean dynamical system, and to use this to prove that the C^*-algebra of a (generalized) Boolean dynamical system that satisfies Condition (K) is K_0-liftable.

If it is correct that the C^*-algebra of a (generalized) Boolean dynamical system that satisfies Condition (K) is K_0-liftable, then it would follow that if the C^*-algebra of a (generalized) Boolean dynamical system is separable and purely infinite, then it has real rank zero.

1.4. The organization of this paper. The rest of this paper is organized in the following way. In Section 2 we recall some preliminary results about Boolean dynamical systems and their C^*-algebras. In Sections 3 and 4 we introduce the notions of ultrafilter cycles and maximal tails, respectively. In Section 5 we define Condition (K) for Boolean dynamical systems. In Section 6 we prove that a necessary and sufficient condition for Condition (K) of a locally finite Boolean dynamical system $(\mathcal{B}, \mathcal{L}, \theta)$ with countable $\mathcal{B}$ and $\mathcal{L}$ is that every ideal of $C^*(\mathcal{B}, \mathcal{L}, \theta)$ is gauge-invariant.

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In Section 7 we completely characterize the primitive ideal space of the C^*-algebras of Boolean dynamical systems. In Section 8 we show that a locally finite Boolean dynamical system $(\mathcal{B}, \mathcal{L}, \theta)$ with countable $\mathcal{B}$ and $\mathcal{L}$ satisfies Condition (K) if and only if its C^*-algebra has the (weak) ideal property, if and only if its C^*-algebra has topological dimension zero. We also illustrate some of the introduced concepts with a recurring example throughout the paper.

2. Preliminaries

For the convenience of the reader, in this section we briefly recall the definition of a Boolean dynamical system and the C^*-algebra of a Boolean dynamical system as well as some basic results about C^*-algebras of Boolean dynamical systems from [7].

We let $\mathbb{N}$ denote the set of positive integers.

2.1. Boolean algebras. A *Boolean algebra* [7, Definition 2.1] is a set $\mathcal{B}$ with a distinguished element $\emptyset$ and maps $\cap : \mathcal{B} \times \mathcal{B} \to \mathcal{B}$, $\cup : \mathcal{B} \times \mathcal{B} \to \mathcal{B}$ and $\setminus : \mathcal{B} \times \mathcal{B} \to \mathcal{B}$ such that $(\mathcal{B}, \cap, \cup)$ is a distributive lattice, $A \cap \emptyset = \emptyset$ for all $A \in \mathcal{B}$, and $(A \cap B) \cup (A \setminus B) = A$ and $(A \cap B) \cap (A \setminus B) = \emptyset$ for all $A, B \in \mathcal{B}$. The Boolean algebra $\mathcal{B}$ is called *unital* if there exists $1 \in \mathcal{B}$ such that $1 \cup A = 1$ and $1 \cap A = A$ for all $A \in \mathcal{B}$ (Boolean algebras are often assumed to be unital, and what we here call a Boolean algebra is often called a *generalized Boolean algebra*).

We call $A \cup B$ the *union* of A and B, $A \cap B$ the *intersection* of A and B, and $A \setminus B$ the *relative complement* of B with respect to A. A subset $\mathcal{B}' \subseteq \mathcal{B}$ is called a *Boolean subalgebra* if $\emptyset \in \mathcal{B}'$ and $\mathcal{B}'$ is closed under taking union, intersection and the relative complement. A Boolean subalgebra of a Boolean algebra is itself a Boolean algebra.

We define a partial order on $\mathcal{B}$ as follows: for $A, B \in \mathcal{B}$,

$$A \subseteq B \quad \text{if and only if} \quad A \cap B = A.$$

Then $(\mathcal{B}, \subseteq)$ is a partially ordered set, and $A \cup B$ and $A \cap B$ are respectively the least upper bound and the greatest lower bound of A and B with respect to the partial order $\subseteq$. If a family $\{A_\lambda\}_{\lambda \in \Lambda}$ of elements from $\mathcal{B}$ has a least upper bound, then we denote it by $\cup_{\lambda \in \Lambda} A_\lambda$. If $A \subseteq B$, then we say that A is a *subset of* B.

A nonempty subset $\mathcal{I}$ of $\mathcal{B}$ is called an *ideal* [7, Definition 2.4] if the following two conditions hold.

(i) If $A, B \in \mathcal{I}$, then $A \cup B \in \mathcal{I}$.
(ii) If $A \in \mathcal{I}$ and $B \in \mathcal{B}$, then $A \cap B \in \mathcal{I}$.

An ideal $\mathcal{I}$ of a Boolean algebra $\mathcal{B}$ is a Boolean subalgebra. For $A \in \mathcal{B}$, the ideal generated by A is defined by $\mathcal{I}_A := \{B \in \mathcal{B} : B \subseteq A\}$.

If $\mathcal{I}$ is an ideal of a Boolean algebra $\mathcal{B}$, then the relation

$$A \sim B \iff A \cup A' = B \cup B' \quad \text{for some } A', B' \in \mathcal{I} \tag{2-1}$$

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defines an equivalence relation on $\mathcal{B}$ (see [7, Definition 2.5]). We denote by $[A]$ the equivalence class of $A \in \mathcal{B}$ and by $\mathcal{B}/\mathcal{I}$ the set of all equivalent classes of $\mathcal{B}$. It is easy to check that $\mathcal{B}/\mathcal{I}$ becomes a Boolean algebra with operations defined by $[A] \cap [B] = [A \cap B]$, $[A] \cup [B] = [A \cup B]$, and $[A] \setminus [B] = [A \setminus B]$. The partial order $\subseteq$ on $\mathcal{B}/\mathcal{I}$ is characterized by

$$\begin{aligned} [A] \subseteq [B] &\iff A \subseteq B \cup W \quad \text{for some } W \in \mathcal{I} \\ &\iff [A] \cap [B] = [A]. \end{aligned}$$

A *filter* [7, Definition 2.6] ξ in a Boolean algebra $\mathcal{B}$ is a nonempty subset $\xi \subseteq \mathcal{B}$ such that:

F0 $\emptyset \notin \xi$;
F1 if $A \in \xi$ and $A \subseteq B$, then $B \in \xi$;
F2 if $A, B \in \xi$, then $A \cap B \in \xi$.

If in addition ξ satisfies

F3 if $A \in \xi$ and $B, B' \in \mathcal{B}$ with $A = B \cup B'$, then either $B \in \xi$ or $B' \in \xi$,

then it is called an *ultrafilter* [7, Definition 2.6] of $\mathcal{B}$. A filter is an ultrafilter if and only if it is a maximal element in the set of filters with respect to inclusion. We write $\widehat{\mathcal{B}}$ for the set of all ultrafilters of $\mathcal{B}$. Notice that if $A \in \mathcal{B} \setminus \{\emptyset\}$, then $\{B \in \mathcal{B} : A \subseteq B\}$ is a filter, and it then follows from Zorn's lemma that there is an ultrafilter $\eta \in \widehat{\mathcal{B}}$ that contains A. For $A \in \mathcal{B}$, we let $Z(A) := \{\xi \in \widehat{\mathcal{B}} : A \in \xi\}$ and we equip $\widehat{\mathcal{B}}$ with the topology generated by $\{Z(A) : A \in \mathcal{B}\}$. Then $\widehat{\mathcal{B}}$ is a totally disconnected locally compact Hausdorff space, $\{Z(A) : A \in \mathcal{B}\}$ is a basis for the topology, and each $Z(A)$ is compact and open.

We now give a simple example of a Boolean algebra and illustrate some of the concepts introduced above. We shall return to this example throughout the paper.

EXAMPLE 2.1. Let $\mathcal{A} := \{1, 2, 3\}$ and let $X := \mathcal{A}^{\mathbb{N}}$ be the full one-sided shift space on $\mathcal{A}$. Equip X with the product topology. Then X is a Cantor set (that is, it is a second countable compact Hausdorff space with no isolated points and with a basis of compact open sets). Let $\mathcal{B}$ be the set of compact open subsets of X. Then $\mathcal{B}$ is a Boolean algebra with unit X where $\cup$, $\cap$ and $\setminus$ are the usual set operations, and $\emptyset$ is the empty set.

The map $U \mapsto \mathcal{I}_U := \{A \in \mathcal{B} : A \subseteq U\}$ is a bijection between the set of open subsets of X and the set of ideals of $\mathcal{B}$. If $U \in \mathcal{B}$, then the map $A \mapsto [A]$ is a Boolean isomorphism from $\mathcal{I}_{X \setminus U}$ to the quotient $\mathcal{B}/\mathcal{I}_U$.

The map $C \mapsto \mathcal{F}_C := \{A \in \mathcal{B} : C \subseteq A\}$ is a bijection between the set of nonempty compact subsets of X and the set of filters in $\mathcal{B}$, and $x \mapsto \hat{x} := \{A \in \mathcal{B} : x \in A\}$ is a bijection from X to $\widehat{\mathcal{B}}$. If Λ is a subset of $\mathcal{B}$, then the family $\{A\}_{A \in \Lambda}$ has a least upper bound if and only if $\cup_{A \in \Lambda} A \in \mathcal{B}$ (i.e. if and only if $\cup_{A \in \Lambda} A$ is compact), in which case $\cup_{A \in \Lambda} A$ is the least upper bound of $\{A\}_{A \in \Lambda}$.

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2.2. Boolean dynamical systems. A map $\phi : \mathcal{B} \to \mathcal{B}'$ between two Boolean algebras is called a *Boolean homomorphism* if $\phi(A \cap B) = \phi(A) \cap \phi(B)$, $\phi(A \cup B) = \phi(A) \cup \phi(B)$, and $\phi(A \setminus B) = \phi(A) \setminus \phi(B)$ for all $A, B \in \mathcal{B}$.

If $\mathcal{B}$ is a Boolean algebra, then a Boolean homomorphism $\theta : \mathcal{B} \to \mathcal{B}$ is an *action* on $\mathcal{B}$ if $\theta(\emptyset) = \emptyset$. An action θ has *compact range* [7, Definition 3.1] if $\{\theta(A)\}_{A\in\mathcal{B}}$ has a least upper bound. We denote by $\mathcal{R}_\theta$ this least upper bound if it exists. An action θ has *closed domain* [7, Definition 3.1] if there exists $\mathcal{D}_\theta \in \mathcal{B}$ such that $\theta(\mathcal{D}_\theta) = \mathcal{R}_\theta$. Notice that if an action $\theta : \mathcal{B} \to \mathcal{B}$ has compact range and $\mathcal{B}$ has a unit, then θ has closed domain.

Given a set $\mathcal{L}$ and any $n \in \mathbb{N}$, we define $\mathcal{L}^n := \{(\alpha_1, \dots, \alpha_n) : \alpha_i \in \mathcal{L}\}$ and $\mathcal{L}^* := \cup_{n\geq 0}\mathcal{L}^n$, where $\mathcal{L}^0 := \{\emptyset\}$. We define $|\alpha|$ to be n if $\alpha \in \mathcal{L}^n$. For $\alpha = (\alpha_1, \dots, \alpha_n)$, $\beta = (\beta_1, \dots, \beta_m) \in \mathcal{L}^*$, we usually write $\alpha_1 \cdots \alpha_n$ instead of $(\alpha_1, \dots, \alpha_n)$ and use $\alpha\beta$ to denote the word $\alpha_1 \cdots \alpha_n\beta_1 \cdots \beta_m$ (if $\alpha = \emptyset$, then $\alpha\beta := \beta$; and if $\beta = \emptyset$, then $\alpha\beta := \alpha$). For $k \in \mathbb{N}$, we let $\alpha^k := \alpha\alpha \cdots \alpha$ where the concatenation on the right has k terms. Similarly, we let $\alpha^0 := \emptyset$. For $1 \leq i \leq j \leq |\alpha|$, we denote by $\alpha_{[i,j]}$ the subword $\alpha_i \cdots \alpha_j$ of $\alpha = \alpha_1\alpha_2 \cdots \alpha_{|\alpha|}$, where $\alpha_{[i,i]} = \alpha_i$.

A *Boolean dynamical system* [7, Definition 3.3] is a triple $(\mathcal{B}, \mathcal{L}, \theta)$ where $\mathcal{B}$ is a Boolean algebra, $\mathcal{L}$ is a set, and $\{\theta_\alpha\}_{\alpha\in\mathcal{L}}$ is a set of actions on $\mathcal{B}$ such that for $\alpha = \alpha_1 \cdots \alpha_n \in \mathcal{L}^* \setminus \{\emptyset\}$, the action $\theta_\alpha : \mathcal{B} \to \mathcal{B}$ defined as $\theta_\alpha := \theta_{\alpha_n} \circ \cdots \circ \theta_{\alpha_1}$ has compact range and closed domain. Given any $\alpha \in \mathcal{L}^* \setminus \{\emptyset\}$, we write $\mathcal{R}_\alpha := \mathcal{R}_{\theta_\alpha}$. We also define $\theta_\emptyset := \mathrm{Id}$.

For $B \in \mathcal{B}$, we define

$$\Delta_B := \{\alpha \in \mathcal{L} : \theta_\alpha(B) \neq \emptyset\} \quad \text{and} \quad \lambda_B := |\Delta_B|.$$

We say that $A \in \mathcal{B}$ is a *regular* [7, Definition 3.5] set if for any $\emptyset \neq B \in \mathcal{I}_A$, we have $0 < \lambda_B < \infty$. If $A \in \mathcal{B}$ is not regular, then it is called a *singular* set. We write $\mathcal{B}_{\mathrm{reg}}$ for the set of all regular sets. Notice that $\emptyset \in \mathcal{B}_{\mathrm{reg}}$.

A Boolean dynamical system $(\mathcal{B}, \mathcal{L}, \theta)$ is *locally finite* [7, Definition 3.6] if for every $\xi \in \widehat{\mathcal{B}}$ there exists $A \in \xi$ such that $\lambda_A < \infty$. Notice that if $\mathcal{L}$ is finite, then $(\mathcal{B}, \mathcal{L}, \theta)$ is locally finite.

EXAMPLE 2.2. Let $\mathcal{A}$, X and $\mathcal{B}$ be as in Example 2.1. If $x = (x_k)_{k\in\mathbb{N}} \in X$ and $a \in \mathcal{A}$, then we denote by ax the element of X with $(ax)_1 = a$ and $(ax)_k = x_{k-1}$ for $k > 1$. Let $\mathcal{L} := \{1, 2, i, j\}$. Define maps $\theta_a : \mathcal{B} \to \mathcal{B}$ for $a \in \mathcal{L}$ by

$$\begin{aligned}
\theta_1(A) &:= \{1x : x \in A,\ x_1 \in \{1, 2\}\},\\
\theta_2(A) &:= \{2x : x \in A,\ x_1 \in \{1, 2\}\},\\
\theta_i(A) &:= \{x : x \in A,\ x_1 = 3\},\\
\theta_j(A) &:= \{1x : x \in A,\ x_1 = 3\}.
\end{aligned}$$

Then $(\mathcal{B}, \mathcal{L}, \theta)$ is a Boolean dynamical system. Since $\mathcal{L}$ is finite, $(\mathcal{B}, \mathcal{L}, \theta)$ is locally finite. Moreover, $\lambda_A \in \{2, 4\}$ for any $\emptyset \neq A \in \mathcal{B}$, so $\mathcal{B}_{\mathrm{reg}} = \mathcal{B}$.

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2.3. C^*-algebras associated with Boolean dynamical systems. A *Cuntz–Krieger representation* [7, Definition 3.7] of a Boolean dynamical system $(\mathcal{B}, \mathcal{L}, \theta)$ is a family of projections $\{P_A : A \in \mathcal{B}\}$ and partial isometries $\{S_\alpha : \alpha \in \mathcal{L}\}$ in a C^*-algebra such that, for $A, B \in \mathcal{B}$ and $\alpha, \beta \in \mathcal{L}$,

(i) $P_\emptyset = 0$, $P_{A\cap B} = P_A P_B$, and $P_{A\cup B} = P_A + P_B - P_{A\cap B}$,
(ii) $P_A S_\alpha = S_\alpha P_{\theta_\alpha(A)}$,
(iii) $S_\alpha^* S_\beta = \delta_{\alpha,\beta} P_{\mathcal{R}_\alpha}$,
(iv) $P_A = \sum_{\alpha\in\Delta_A} S_\alpha P_{\theta_\alpha(A)} S_\alpha^*$ if $A \in \mathcal{B}_{\mathrm{reg}}$.

A representation is called *faithful* if $P_A \neq 0$ for all $A \in \mathcal{B}$.

It is shown in [7, Theorem 5.8] that if $(\mathcal{B}, \mathcal{L}, \theta)$ is a Boolean dynamical system, then there exists a universal Cuntz–Krieger representation $\{s_\alpha, p_A\}$ of $(\mathcal{B}, \mathcal{L}, \theta)$ (here universal means that if $\{S_\alpha, P_A\}$ is a Cuntz–Krieger representation, then there is a $*$-homomorphism ϕ from the C^*-algebra generated by $\{s_\alpha, p_A\}$ to the C^*-algebra generated by $\{S_\alpha, P_A\}$ such that $\phi(s_\alpha) = S_\alpha$ for $\alpha \in \mathcal{L}$ and $\phi(p_A) = P_A$ for $A \in \mathcal{B}$). We let $C^*(\mathcal{B}, \mathcal{L}, \theta)$ denote the C^*-algebra generated by a universal representation of $(\mathcal{B}, \mathcal{L}, \theta)$ and call it the *Cuntz–Krieger Boolean C^*-algebra*, or just the *C^*-algebra*, of the Boolean dynamical system $(\mathcal{B}, \mathcal{L}, \theta)$.

REMARK 2.3. Let $(\mathcal{B}, \mathcal{L}, \theta)$ be a Boolean dynamical system.

(1) The universal property of $C^*(\mathcal{B}, \mathcal{L}, \theta) = C^*(s_a, p_A)$ defines a strongly continuous action $\gamma : \mathbb{T} \to \mathrm{Aut}(C^*(\mathcal{B}, \mathcal{L}, \theta))$, called the *gauge action*, such that

$$\gamma_z(s_\alpha) = z s_\alpha \quad \text{and} \quad \gamma_z(p_A) = p_A$$

for $\alpha \in \mathcal{L}$ and $A \in \mathcal{B}$.
(2) If $\mathcal{B}$ and $\mathcal{L}$ are countable, then $C^*(\mathcal{B}, \mathcal{L}, \theta)$ is separable.

Let $(\mathcal{B}, \mathcal{L}, \theta)$ be a Boolean dynamical system and let $\alpha = \alpha_1 \cdots \alpha_{|\alpha|} \in \mathcal{L}^* \setminus \{\emptyset\}$ and $A \in \mathcal{B} \setminus \{\emptyset\}$.

(i) A (α, A) is called a *cycle* [7, Definition 9.5] if $B = \theta_\alpha(B)$ for all $B \subseteq A$.
(ii) A cycle (α, A) has an *exit* if there exist a $t \leq |\alpha|$ and a $B \in \mathcal{B}$ such that $\emptyset \neq B \subseteq \theta_{\alpha_1\cdots\alpha_t}(A)$ and $\Delta_B \neq \{\alpha_{t+1}\}$ (where $\alpha_{|\alpha|+1} := \alpha_1$).
(iii) A cycle (α, A) has no exits [7, Definition 9.5] if and only if for all $t \leq |\alpha|$ and all $\emptyset \neq B \subseteq \theta_{\alpha_1\cdots\alpha_t}(A)$, we have $B \in \mathcal{B}_{\mathrm{reg}}$ with $\Delta_B = \{\alpha_{t+1}\}$ (where $\alpha_{|\alpha|+1} := \alpha_1$).
(iv) We say that $(\mathcal{B}, \mathcal{L}, \theta)$ satisfies *Condition (L)* [7, Definition 9.5] if every cycle has an exit (in [7], this condition is called *Condition* $(L_{\mathcal{B}})$).

EXAMPLE 2.4. We continue Examples 2.1 and 2.2. A pair (α, A) is a cycle if and only if $\alpha = i^n$ for some $n \in \mathbb{N}$ and $A \subseteq \{x \in X : x_1 = 3\}$. Since $\Delta_A = \{i, j\}$ for any $A \subseteq \{x \in X : x_1 = 3\}$, every cycle in $\mathcal{B}$ has an exit, and $(\mathcal{B}, \mathcal{L}, \theta)$ satisfies Condition (L).

We need the following easy strengthening of [7, Theorem 9.9].

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THEOREM 2.5 [7, Cuntz–Krieger uniqueness theorem]. *Let $(\mathcal{B},\mathcal{L},\theta)$ be a Boolean dynamical system such that $\mathcal{B}$ and $\mathcal{L}$ are countable. If $(\mathcal{B},\mathcal{L},\theta)$ satisfies Condition (L), then for any $*$-homomorphism $\pi : C^*(\mathcal{B},\mathcal{L},\theta) := C^*(s_a, p_A) \to B$, the following are equivalent.*

(i) $\pi(p_A) \neq 0$ *for all* $\emptyset \neq A \in \mathcal{B}$.
(ii) π *is injective.*

PROOF. (ii) $\Longrightarrow$ (i). This follows from [7, Corollary 5.3].

(i) $\Longrightarrow$ (ii). It follows from [7, Theorem 9.9] that it suffices to show that $\pi(s_\alpha p_A s_\alpha^*) \neq 0$ for all $\alpha \in \mathcal{L}^*$ and all $\emptyset \neq A \in \mathcal{B}$ with $A \subseteq \mathcal{R}_\alpha$. Suppose for contradiction that $\pi(s_\alpha p_A s_\alpha^*) = 0$ for some $\alpha \in \mathcal{L}^*$ and some $\emptyset \neq A \in \mathcal{B}$ with $A \subseteq \mathcal{R}_\alpha$. Then

$$\pi(p_A) = \pi(s_\alpha^* s_\alpha p_A s_\alpha^* s_\alpha) = \pi(s_\alpha^*)\pi(s_\alpha p_A s_\alpha^*)\pi(s_\alpha) = 0.$$

This shows that (i) $\Longrightarrow$ (ii). □

2.4. Gauge-invariant ideals of $C^*(\mathcal{B},\mathcal{L},\theta)$. We now recall the characterization given in [7] of the gauge-invariant ideals of the C^*-algebra of a locally finite Boolean dynamical system.

Let $(\mathcal{B},\mathcal{L},\theta)$ be a Boolean dynamical system. An ideal $\mathcal{H}$ of $\mathcal{B}$ is said to be *hereditary* if $\theta_\alpha(A) \in \mathcal{H}$ for $A \in \mathcal{H}$ and $\alpha \in \mathcal{L}$, and *saturated* if $A \in \mathcal{H}$ whenever $A \in \mathcal{B}_{\mathrm{reg}}$ and $\theta_\alpha(A) \in \mathcal{H}$ for all $\alpha \in \Delta_A$.

Given a hereditary ideal $\mathcal{H}$ of $\mathcal{B}$, if we define $\theta_\alpha([A]) := [\theta_\alpha(A)]$ for all $[A] \in \mathcal{B}/\mathcal{H}$ and $\alpha \in \mathcal{L}$, then $(\mathcal{B}/\mathcal{H},\mathcal{L},\theta)$ is a Boolean dynamical system [7, Proposition 10.7]. We call it a *quotient Boolean dynamical system* of $(\mathcal{B},\mathcal{L},\theta)$.

EXAMPLE 2.6. We continue Examples 2.1, 2.2 and 2.4. Recall that the map $U \mapsto \mathcal{I}_U := \{A \in \mathcal{B} : A \subseteq U\}$ is a bijection between the set of open subsets of X and the set of ideals of $\mathcal{B}$. If U is a nonempty open subset of X, then $\mathcal{I}_U$ is hereditary if and only if $\{x \in X : x_1 \in \{1,2\}\} \subseteq U$, in which case $\mathcal{I}_U$ is also saturated.

For an open subset U of X with $\{x \in X : x_1 \in \{1,2\}\} \subseteq U$, if we let $\mathcal{B}_U$ be the set of compact and open subsets of $X \setminus U$ (where the latter is equipped with the subspace topology), then $\mathcal{B}_U$ equipped with the usual set operations is a Boolean algebra, and the map $A \setminus U \mapsto [A]$ is an isomorphism from $\mathcal{B}_U$ to $\mathcal{B}/\mathcal{I}_U$.

For an open subset U of X with $\{x \in X : x_1 \in \{1,2\}\} \subseteq U$, the action θ of $\mathcal{L}$ on $\mathcal{B}/\mathcal{I}_U$ is given by $\theta_1([A]) = [\theta_1(A)] = \emptyset$, $\theta_2([A]) = [\theta_2(A)] = \emptyset$, $\theta_j([A]) = [\theta_j(A)] = \emptyset$, and $\theta_i([A]) = [\theta_i(A)] = [A]$ for any $A \in \mathcal{B}$. It follows that if $A \in \mathcal{B}$ with $A \setminus U \neq \emptyset$, then $(i,[A])$ is a cycle with no exits. So if U is an open subset of X with $\{x \in X : x_1 \in \{1,2\}\} \subseteq U$, then $(\mathcal{B}/\mathcal{I}_U,\mathcal{L},\theta)$ does not satisfy Condition (L).

For a hereditary saturated ideal $\mathcal{H}$ of $\mathcal{B}$, we denote by $I_\mathcal{H}$ the ideal of $C^*(\mathcal{B},\mathcal{L},\theta)$ generated by the projections $\{p_A : A \in \mathcal{H}\}$. Given an ideal I of $C^*(\mathcal{B},\mathcal{L},\theta)$, we let $\mathcal{H}_I := \{A \in \mathcal{B} : p_A \in I\}$. Then $\mathcal{H}_I$ is an ideal of $\mathcal{B}$.

It is shown in [7, Proposition 10.11] that if $(\mathcal{B},\mathcal{L},\theta)$ is a locally finite Boolean dynamical system, then the maps $I \mapsto \mathcal{H}_I$ and $\mathcal{H} \mapsto I_\mathcal{H}$ define a one-to-one corre-

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spondence between the set of all nonzero gauge-invariant ideals of $C^*(\mathcal{B},\mathcal{L},\theta)$ and the set of all nonempty hereditary saturated ideals of $\mathcal{B}$.

3. Ultrafilter cycles

Our definition of Condition (K) for Boolean dynamical systems relies on the notion of *ultrafilter cycles* which we now introduce.

DEFINITION 3.1. Let $(\mathcal{B},\mathcal{L},\theta)$ be a Boolean dynamical system. We say that a pair (α,η), where $\alpha\in\mathcal{L}^*\setminus\{\emptyset\}$ and $\eta\in\widehat{\mathcal{B}}$, is an *ultrafilter cycle* if $\theta_\alpha(A)\in\eta$ for all $A\in\eta$.

If $\eta\in\widehat{\mathcal{B}}$ and $\alpha\in\mathcal{L}^*$, then $\widehat{\theta}_\alpha(\eta):=\{A\in\mathcal{B}:\theta_\alpha(A)\in\eta\}$ is either empty or an ultrafilter. In fact, $\widehat{\theta}_\alpha(\eta)\in\widehat{\mathcal{B}}$ if and only if $\mathcal{R}_\alpha\in\eta$ or $\alpha=\emptyset$. Let

$$\widehat{\mathcal{R}}_\alpha:=\{\eta\in\widehat{\mathcal{B}}:\mathcal{R}_\alpha\in\eta\}$$

if $\alpha\neq\emptyset$, and $\widehat{\mathcal{R}}_\alpha:=\widehat{\mathcal{B}}$ if $\alpha=\emptyset$. Then $\eta\mapsto\widehat{\theta}_\alpha(\eta)$ is a map from $\widehat{\mathcal{R}}_\alpha$ to $\widehat{\mathcal{B}}$ which we denote by $\widehat{\theta}_\alpha$. Moreover, $\widehat{\theta}_\alpha(\widehat{\mathcal{R}}_\alpha)=\{\eta\in\widehat{\mathcal{B}}:\theta_\alpha(A)\neq\emptyset \text{ for all } A\in\eta\}$.

LEMMA 3.2. *Let $(\mathcal{B},\mathcal{L},\theta)$ be a Boolean dynamical system.*

(1) *If (α,η) is an ultrafilter cycle, then (α^k,η) is an ultrafilter cycle for all $k\in\mathbb{N}$.*
(2) *A pair (α,η) is an ultrafilter cycle if and only if $\eta=\widehat{\theta}_\alpha(\eta)$, and if and only if $\eta=\widehat{\theta}_{\alpha^k}(\eta)$ for all $k\in\mathbb{N}$.*
(3) *Let (α,A) be a cycle and η be an ultrafilter of $\mathcal{B}$ such that $A\in\eta$. Then (α,η) is an ultrafilter cycle.*

PROOF. (1) This follows by induction on k.

(2) If $\eta=\widehat{\theta}_\alpha(\eta)$, then obviously (α,η) is an ultrafilter cycle. Conversely, if (α,η) is an ultrafilter cycle, then $\eta\subseteq\widehat{\theta}_\alpha(\eta)$, and hence $\eta=\widehat{\theta}_\alpha(\eta)$ since η is an ultrafilter. Statement (2) therefore follows from (1).

(3) Let $B\in\eta$. Then we have that

$$\eta\ni A\cap B=\theta_\alpha(A\cap B)=\theta_\alpha(A)\cap\theta_\alpha(B)\subseteq\theta_\alpha(B),$$

which implies that $\theta_\alpha(B)\in\eta$. Thus (α,η) is an ultrafilter cycle. □

EXAMPLE 3.3. We continue Examples 2.1, 2.2, 2.4 and 2.6. Recall that $x\mapsto\hat{x}$ is a bijection from X to $\widehat{\mathcal{B}}$. We have that

$$\begin{aligned}
\widehat{\mathcal{R}}_1&=\{\hat{x}:x_1=1,\ x_2\in\{1,2\}\},\\
\widehat{\mathcal{R}}_2&=\{\hat{x}:x_1=2,\ x_2\in\{1,2\}\},\\
\widehat{\mathcal{R}}_i&=\{\hat{x}:x_1=3\},\\
\widehat{\mathcal{R}}_j&=\{\hat{x}:x_1=1,\ x_2=3\},
\end{aligned}$$

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and that the maps $\widehat{\theta}_a : \widehat{\mathcal{R}}_a \to \widehat{\mathcal{B}}$ are given by

$$\widehat{\theta}_1(\widehat{1x}) = \hat{x}, \quad \widehat{\theta}_2(\widehat{2x}) = \hat{x}, \quad \widehat{\theta}_i(\widehat{x}) = \hat{x}, \quad \widehat{\theta}_j(\widehat{1x}) = \hat{x}.$$

Moreover, $(i, \hat{x})$ is an ultrafilter cycle for any $\hat{x} \in \widehat{\mathcal{R}}_i$, and $(\alpha, \hat{x})$ is an ultrafilter cycle if α is a finite word with letters from $\{1, 2\}$ and $x := \alpha\alpha\cdots$ is the infinite word we get by concatenating α with itself infinitely many times.

4. Maximal tails

We now generalize the notion of *maximal tails* from directed graphs (see [2, 11]) to our setting. We write $A \geq B$ for $A, B \in \mathcal{B}$ if there exists an $\alpha \in \mathcal{L}^*$ such that $B \subseteq \theta_\alpha(A)$.

DEFINITION 4.1. Let $(\mathcal{B}, \mathcal{L}, \theta)$ be a Boolean dynamical system. A nonempty subset $\mathcal{T}$ of $\mathcal{B}$ is called a *maximal tail* if the following statements hold:

(T0) $\emptyset \notin \mathcal{T}$;
(T1) if $A \in \mathcal{B}$ and $\theta_\alpha(A) \in \mathcal{T}$ for some $\alpha \in \mathcal{L}$, then $A \in \mathcal{T}$;
(T2) if $A \cup B \in \mathcal{T}$, then $A \in \mathcal{T}$ or $B \in \mathcal{T}$;
(T3) if $A \in \mathcal{T}$, $B \in \mathcal{B}$ and $A \subseteq B$, then $B \in \mathcal{T}$;
(T4) if $A \in \mathcal{T}$ is regular, then there is an $\alpha \in \mathcal{L}$ such that $\theta_\alpha(A) \in \mathcal{T}$;
(T5) if $A_1, A_2 \in \mathcal{T}$, then there exists $C \in \mathcal{T}$ such that $A_1 \geq C$ and $A_2 \geq C$.

REMARK 4.2. In the above definition, (T1) and (T4) are equivalent to (T1′) and (T4′), respectively:

(T1′) if $A \in \mathcal{B}$ and $\theta_\alpha(A) \in \mathcal{T}$ for some $\alpha \in \mathcal{L}^*$, then $A \in \mathcal{T}$;
(T4′) if $A \in \mathcal{T}$ is regular, then there is an $\alpha \in \mathcal{L}^* \setminus \{\emptyset\}$ such that $\theta_\alpha(A) \in \mathcal{T}$.

LEMMA 4.3. *If $\mathcal{T}$ is a maximal tail, then $\mathcal{B} \setminus \mathcal{T}$ is a hereditary and saturated ideal of $\mathcal{B}$.*

PROOF. Suppose $\mathcal{T}$ is a maximal tail. Then $\mathcal{B} \setminus \mathcal{T}$ is nonempty because $\emptyset \notin \mathcal{T}$. It follows from property (T2) that if $A, B \in \mathcal{B} \setminus \mathcal{T}$, then $A \cup B \in \mathcal{B} \setminus \mathcal{T}$; and it follows from property (T3) that if $A \in \mathcal{B} \setminus \mathcal{T}$ and $B \in \mathcal{B}$, then $A \cap B \in \mathcal{B} \setminus \mathcal{T}$. This shows that $\mathcal{B} \setminus \mathcal{T}$ is an ideal of $\mathcal{B}$. It follows from property (T1) that $\mathcal{B} \setminus \mathcal{T}$ is hereditary, and from property (T4) that $\mathcal{B} \setminus \mathcal{T}$ is saturated. □

EXAMPLE 4.4. We continue Examples 2.1, 2.2, 2.4, 2.6 and 3.3. Recall that $x \mapsto \hat{x}$ is a bijection from X to $\widehat{\mathcal{B}}$ and that each hereditary and saturated ideal of $\mathcal{B}$ has the form $\mathcal{I}_U := \{A \in \mathcal{B} : A \subseteq U\}$ for some open subset $U \subseteq X$ for which $\{x \in X : x_1 \in \{1, 2\}\}$. Using this and the lemma above, it is straightforward to check that $\{\hat{x} : x \in X,\ x_1 = 3\}$ is the set of all maximal tails of $(\mathcal{B}, \mathcal{L}, \theta)$.

In the example above, each maximal tail is an ultrafilter. In general, a maximal tail is not necessarily a filter (consider, for example, the case where $\mathcal{B}$ is the Boolean

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algebra of subsets of $\mathbb{Z}$, $\mathcal{L} = \{a\}$ and $\theta_a(A) = \{x + 1 : x \in A\}$ for $A \in \mathcal{B}$; then $\mathcal{B} \setminus \{\emptyset\}$ is a maximal tail, but not a filter).

PROPOSITION 4.5. *Let $(\mathcal{B}, \mathcal{L}, \theta)$ be a Boolean dynamical system. Suppose (α, η) is an ultrafilter cycle and $A \in \eta$ is such that if $\beta \in \mathcal{L}^* \setminus \{\emptyset\}$, $B \in \mathcal{I}_A$ and $\theta_\beta(B) \in \eta$, then $B \in \eta$ and $\beta = \alpha^k$ for some $k \in \mathbb{N}$. Then*

$$\mathcal{T} := \{B \in \mathcal{B} : \theta_\beta(B) \in \eta \text{ for some } \beta \in \mathcal{L}^*\}$$

is a maximal tail such that $(\mathcal{B}/(\mathcal{B} \setminus \mathcal{T}), \mathcal{L}, \theta)$ does not satisfy Condition (L).

PROOF. It is straightforward to check that $\mathcal{T}$ satisfies (T0)–(T4). To show that it satisfies (T5), we choose $B_1, B_2 \in \mathcal{T}$. Then there exist $\beta_1, \beta_2 \in \mathcal{L}^*$ such that $\theta_{\beta_1}(B_1), \theta_{\beta_2}(B_2) \in \eta$. Thus $\theta_{\beta_1}(B_1) \cap \theta_{\beta_2}(B_2) \in \eta$, and hence $\theta_\alpha(\theta_{\beta_1}(B_1) \cap \theta_{\beta_2}(B_2)) \in \eta$ since (α, η) is an ultrafilter cycle. It then follows that $\theta_{\beta_1}(B_1) \cap \theta_{\beta_2}(B_2) \in \mathcal{T}$ and $B_i \geq \theta_{\beta_1}(B_1) \cap \theta_{\beta_2}(B_2)$ for all $i = 1, 2$.

Let $\pi : \mathcal{B} \to \mathcal{B}/(\mathcal{B} \setminus \mathcal{T})$ be the quotient map given by $\pi(B) = [B]$. We claim that $(\alpha, [A])$ is a cycle with no exit in $(\mathcal{B}/(\mathcal{B} \setminus \mathcal{T}), \mathcal{L}, \theta)$. We first show that

$$B \setminus \theta_\alpha(B) \notin \mathcal{T} \quad \text{and} \quad \theta_\alpha(B) \setminus B \notin \mathcal{T}$$

for any $B \in \mathcal{I}_A$. For contradiction, suppose $B \setminus \theta_\alpha(B) \in \mathcal{T}$. Then $\theta_\beta(B) \setminus \theta_{\alpha\beta}(B) \in \eta$ for some $\beta \in \mathcal{L}^*$. It follows that $\theta_\beta(B) \in \eta$, and thus that $\beta = \alpha^k$ for some $k \in \mathbb{N}$. But then $\theta_{\alpha\beta}(B) = \theta_{\alpha^{k+1}}(B) = \theta_\alpha(\theta_\beta(B)) \in \eta$, and

$$\emptyset = (\theta_\beta(B) \setminus \theta_{\alpha\beta}(B)) \cap \theta_{\alpha\beta}(B) \in \eta,$$

which is not the case. Thus, we must have that $B \setminus \theta_\alpha(B) \notin \mathcal{T}$. Similarly, if $\theta_\alpha(B) \setminus B \in \mathcal{T}$, then $\theta_{\alpha\beta}(B) \setminus \theta_\beta(B) \in \eta$ for some $\beta \in \mathcal{L}^*$, and then $\theta_{\alpha\beta}(B) \in \eta$, which implies that $B \in \eta$, and thus that $\theta_\beta(B) \in \eta$ and

$$\emptyset = (\theta_{\alpha\beta}(B) \setminus \theta_\beta(B)) \cap \theta_\beta(B) \in \eta,$$

which is not the case. Thus, we must have that $\theta_\alpha(B) \setminus B \notin \mathcal{T}$.

Now for any $[B] \subseteq [A]$, we see that

$$\theta_\alpha([B]) = \theta_\alpha([A \cap B]) = [A \cap B] = [B].$$

This shows that $(\alpha, [A])$ is a cycle. To show that $(\alpha, [A])$ has no exit, assume to the contrary that there are $t \leq |\alpha|$ and $[\emptyset] \neq [B] \subseteq [\theta_{\alpha_1 \cdots \alpha_t}(A)]$ such that we have $\Delta_{[B]} := \{\beta \in \mathcal{L} : [\theta_\beta(B)] \neq [\emptyset]\} \neq \{\alpha_{t+1}\}$. Then $[\theta_{\alpha_1 \cdots \alpha_t \beta}(A)] \neq [\emptyset]$ for some $\beta \neq \alpha_{t+1}$, and then $\theta_{\alpha_1 \cdots \alpha_t \beta}(A) \in \mathcal{T}$. Thus there is a $\gamma \in \mathcal{L}^*$ such that $\theta_\gamma(\theta_{\alpha_1 \cdots \alpha_t \beta}(A)) = \theta_{\alpha_1 \cdots \alpha_t \beta\gamma}(A) \in \eta$, and hence $\alpha_1 \cdots \alpha_t \beta\gamma = \alpha^k$ for some $k \in \mathbb{N}$, which means that $\beta = \alpha_{t+1}$, a contradiction. □

DEFINITION 4.6. Let $(\mathcal{B}, \mathcal{L}, \theta)$ be a Boolean dynamical system. A maximal tail $\mathcal{T}$ of $(\mathcal{B}, \mathcal{L}, \theta)$ is *cyclic* (cf. [8, page 112]) if there exist an ultrafilter cycle (α, η) such that $\mathcal{T} := \{B \in \mathcal{B} : \theta_\beta(B) \in \eta \text{ for some } \beta \in \mathcal{L}^*\}$ and an $A \in \eta$ such that if $\beta \in \mathcal{L}^* \setminus \{\emptyset\}$, $B \in \mathcal{I}_A$ and $\theta_\beta(B) \in \eta$, then $B \in \eta$ and $\beta = \alpha^k$ for some $k \in \mathbb{N}$.

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EXAMPLE 4.7. We continue Examples 2.1, 2.2, 2.4, 2.6, 3.3 and 4.4. Recall that $x \mapsto \hat{x}$ is a bijection from X to $\widehat{\mathcal{B}}$ and that $\{\hat{x} : x \in X,\ x_1 = 3\}$ is the set of all maximal tails of $(\mathcal{B}, \mathcal{L}, \theta)$. We next show that these maximal tails are all cyclic.

To see this, let x be an element of X such that $x_1 = 3$. We saw in Example 3.3 that $(i, \hat{x})$ is then an ultrafilter cycle. Moreover,

$$\hat{x} = \{B \in \mathcal{B} : \theta_\beta(B) \in \hat{x} \text{ for some } \beta \in \mathcal{L}^*\}.$$

Let $A := \{y \in X : y_1 = 3\}$. Suppose $\beta \in \mathcal{L}^* \setminus \{\emptyset\}$, $B \in \mathcal{I}_A$ and $\theta_\beta(B) \in \hat{x}$. Since $\theta_\alpha(B) \notin \hat{x}$ for $\alpha \in \{1, 2, t\}$ (see Example 2.2), it follows that $\beta = i^k$ for some $k \in \mathbb{N}$, and thus that $B = \theta_\beta(B) \in \hat{x}$. We thus have that $\hat{x}$ is a cyclic maximal tail.

Next, we show a converse to Proposition 4.5.

PROPOSITION 4.8. *Let $(\mathcal{B}, \mathcal{L}, \theta)$ be a Boolean dynamical system. Suppose $\mathcal{H}$ is a hereditary and saturated ideal of $\mathcal{B}$ such that $(\mathcal{B}/\mathcal{H}, \mathcal{L}, \theta)$ does not satisfy Condition (L). Then there is a cyclic maximal tail $\mathcal{T}$ such that $\mathcal{T} \cap \mathcal{H} = \emptyset$.*

Before we give the proof of Proposition 4.8, we first introduce the following two lemmas which we use in the proof and again later in the paper.

LEMMA 4.9. *Let $(\mathcal{B}, \mathcal{L}, \theta)$ be a Boolean dynamical system and suppose (α, A) is a cycle with no exits. Fix $1 \le k < |\alpha|$ and let $\alpha' = \alpha_{[1,k]}$. Suppose that $B \cap \theta_{\alpha'}(B) \neq \emptyset$ for every nonempty $B \in \mathcal{B}$ with $B \subseteq A$, and let $A' := A \cap \theta_{\alpha'}(A)$. Then (α', A') is a cycle with no exit.*

PROOF. It suffices to show that $B = \theta_{\alpha'}(B)$ for all $B \subseteq A'$. Choose a nonempty $B \in \mathcal{B}$ such that $B \subseteq A'$. Then

$$(B \setminus \theta_{\alpha'}(B)) \cap \theta_{\alpha'}(B \setminus \theta_{\alpha'}(B)) = \emptyset.$$

From the assumption that $B' \cap \theta_{\alpha'}(B') \neq \emptyset$ for every nonempty $B' \subseteq A$, we thus get that $B \setminus \theta_{\alpha'}(B) = \emptyset$. It follows that $B \subseteq \theta_{\alpha'}(B)$.

Since (α, A') is a cycle with no exit and $B' \cap \theta_{\alpha'}(B') \neq \emptyset$ for every nonempty B' with $B' \subseteq A$, it follows that $\alpha_{k+i} = \alpha_i$ for every i, where the indices are computed modulo $|\alpha|$. We thus have that $(\alpha')^{|\alpha|} = \alpha^k$, and hence $\theta_{\alpha'}(B) \subseteq \theta_{(\alpha')^2}(B) \subseteq \cdots \subseteq \theta_{(\alpha')^{|\alpha|}}(B) = B$. So, $B = \theta_{\alpha'}(B)$. □

LEMMA 4.10. *Let $(\mathcal{B}, \mathcal{L}, \theta)$ be a Boolean dynamical system and suppose (α, A) is a cycle with no exits. Then there exist $1 \le j \le |\alpha|$ and $B \subseteq A$ such that (β, B) is a cycle with no exits and $B \cap \theta_{\beta_{[1,k]}}(B) = \emptyset$ for all $1 \le k < j$, where $\beta = \alpha_{[1,j]}$.*

PROOF. Let

$$j := \min\{k : 1 \le k \le |\alpha| \text{ and there exists } B \in \mathcal{I}_A \text{ such that } (\alpha_{[1,k]}, B) \text{ is a cycle}\}.$$

Choose $A_j \in \mathcal{I}_A$ such that $(\alpha_{[1,j]}, A_j)$ is a cycle. By applying Lemma 4.9 inductively on $(\alpha_{[1,j]}, A_{k+1})$, where $k = j-1, j-2, \ldots, 1$, we see that there are nonempty $A_j \supseteq A_{j-1} \supseteq$

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$\cdots \supseteq A_1$ in $\mathcal{B}$ such that $A_k \cap \theta_{\alpha_{[1,k]}}(A_j) = \emptyset$ for each $1 \leq k < j$. Then $B := A_1$ has the desired properties. □

PROOF OF PROPOSITION 4.8. It follows from Lemma 4.10 that there is a cycle $(\alpha, [A])$ with no exits in $(\mathcal{B}/\mathcal{H}, \mathcal{L}, \theta)$ such that $[A] \cap \theta_{\alpha_{[1,k]}}([A]) = \emptyset$ for $1 \leq k < |\alpha|$. Let η' be an ultrafilter in $\mathcal{B}/\mathcal{H}$ such that $[A] \in \eta'$. By Lemma 3.2(3), we then have that (α, η') is ultrafilter cycle in $(\mathcal{B}/\mathcal{H}, \mathcal{L}, \theta)$. Let

$$\eta := \{B \in \mathcal{B} : [B] \in \eta'\}.$$

Then η is an ultrafilter in $\mathcal{B}$ and $A \in \eta$.

If $B \in \eta$, then $[\theta_\alpha(B)] = \theta_\alpha([B]) \in \eta'$, and hence $\theta_\alpha(B) \in \eta$. This shows that (α, η) is an ultrafilter cycle. Suppose $\beta \in \mathcal{L}^* \setminus \{\emptyset\}$, $B \in \mathcal{I}_A$ and $\theta_\beta(B) \in \eta$. Then $\theta_\beta([B]) = [\theta_\beta(B)] \in \eta'$. Since also $[A] \in \eta'$, it follows that $[A] \cap \theta_\beta([B]) \neq \emptyset$. Since $[B] \subseteq [A]$, we thus have

$$\emptyset \neq [A] \cap \theta_\beta([B]) \subseteq [A] \cap \theta_\beta([A]).$$

Using that $(\alpha, [A])$ is a cycle with no exits and $[A] \cap \theta_{\alpha_{[1,k]}}([A]) = \emptyset$ for $1 \leq k < |\alpha|$, we see that $\beta = \alpha^k$ for some $k \in \mathbb{N}$. Thus

$$[\theta_\beta(B)] = \theta_\beta([B]) = [B] \in \eta'.$$

This shows that $B \in \eta$.

We thus have that $\mathcal{T} := \{B \in \mathcal{B} : \theta_\beta(B) \in \eta \text{ for some } \beta \in \mathcal{L}^*\}$ is a cyclic maximal tail. Suppose $B \in \mathcal{T} \cap \mathcal{H}$. Since $B \in \mathcal{T}$, we have that $\theta_\beta([B]) \in \eta'$ for some $\beta \in \mathcal{L}^*$. On the other hand, since $B \in \mathcal{H}$ we also have that $[B] = \emptyset$, a contradiction. Thus, $\mathcal{T} \cap \mathcal{H} = \emptyset$. □

5. Condition (K)

Motivated by Propositions 4.5 and 4.8 we make the following definition.

DEFINITION 5.1. We say that a Boolean dynamical system $(\mathcal{B}, \mathcal{L}, \theta)$ satisfies Condition (K) if there exists no pair $((\alpha, \eta), A)$ where (α, η) is an ultrafilter cycle and $A \in \eta$ such that if $\beta \in \mathcal{L}^* \setminus \{\emptyset\}$, $B \in \mathcal{I}_A$ and $\theta_\beta(B) \in \eta$, then $B \in \eta$ and $\beta = \alpha^k$ for some $k \in \mathbb{N}$.

REMARK 5.2. It immediately follows from Proposition 4.5 and Definition 4.6 that a Boolean dynamical system satisfies Condition (K) if and only if it has no cyclic maximal tails.

EXAMPLE 5.3. We continue Examples 2.1, 2.2, 2.4, 2.6, 3.3, 4.4 and 4.7. We saw in Example 4.7 that $(\mathcal{B}, \mathcal{L}, \theta)$ has cyclic maximal tail. It therefore does not satisfy Condition (K).

We see below in Theorem 6.3 that a locally finite Boolean dynamical system with countable $\mathcal{B}$ and $\mathcal{L}$ satisfies Condition (K) if and only if every ideal in $C^*(\mathcal{B}, \mathcal{L}, \theta)$ is gauge-invariant, but we shall first look at how our Condition (K) for Boolean

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dynamical systems is related to Condition (K) for directed graphs. This also provides us with examples of Boolean dynamical systems that satisfy Condition (K).

We refer the reader to [1, 2, 11, 17, 18, 22], among others, for the definitions of directed graphs and their C^*-algebras. Let $E = (E^0, E^1, r, s)$ be a directed graph. We shall define two Boolean dynamical systems $(\mathcal{B}_E, \mathcal{L}_E, \theta_E)$ and $(\mathcal{B}_{\partial E}, \mathcal{L}_{\partial E}, \theta_{\partial E})$, both of which have the same C^*-algebra as E.

We let $\mathcal{L}_E := E^1$, let $\mathcal{B}_E$ be the set of finite subsets of E^0, and define for each $e \in \mathcal{L}_E$ a map $(\theta_E)_e : \mathcal{B}_E \to \mathcal{B}_E$ by

$$(\theta_E)_e(A) = \begin{cases} \{r(e)\} & \text{if } s(e) \in A, \\ \emptyset & \text{if } s(e) \notin A. \end{cases}$$

Then $(\mathcal{B}_E, \mathcal{L}_E, \theta_E)$ is a Boolean dynamical system. Let $\{p_A,\ s_e : A \in \mathcal{B}_E,\ e \in \mathcal{L}_E\}$ be a universal Cuntz–Krieger representation of $(\mathcal{B}_E, \mathcal{L}_E, \theta_E)$. Then $\{s_e, p_{\{v\}} : e \in E^1, v \in E^0\}$ is a Cuntz–Krieger E-family. It therefore follows from the universal property of $C^*(E)$ that there is a $*$-homomorphism $\phi : C^*(E) \to C^*(\mathcal{B}_E, \mathcal{L}_E, \theta_E)$ that maps s_e to s_e and p_v to $p_{\{v\}}$. Since $p_A = \sum_{v \in A} p_{\{v\}}$ for $A \in \mathcal{B}_E$, it follows that ϕ is onto; and it is injective by the gauge-invariant uniqueness theorem [1, Theorem 2.1], Remark 2.3(1) and the fact that $p_A \neq 0$ for $A \neq \emptyset$ (the latter follows, for instance, by an application of Theorem 2.5 to the identity map of $C^*(\mathcal{B}_E, \mathcal{L}_E, \theta_E)$). Thus, $C^*(\mathcal{B}_E, \mathcal{L}_E, \theta_E)$ is isomorphic to $C^*(E)$.

Moreover, the map

$$v \mapsto \hat{v} := \{A \in \mathcal{B}_E : v \in A\}$$

is a bijection between E^0 and $\widehat{\mathcal{B}_E}$ such that

$$\hat{\theta}_e(\hat{v}) = \begin{cases} \widehat{s(e)} & \text{if } r(e) = v, \\ \emptyset & \text{if } r(e) \neq v, \end{cases}$$

for $v \in E^0$ and $e \in \mathcal{L}_E$.

Let $E^0_{\text{sing}} := \{v \in E^0 : s^{-1}(v) \text{ is empty or infinite}\}$ be the set of singular vertices. For $n \in \mathbb{N}$, let

$$E^n := \{x_1 x_2 \cdots x_n \in (E^1)^n : r(x_i) = s(x_{i+1}) \text{ for all } i\},$$

and define $E^* = \cup_{n \geq 0} E^n$ to be the set of all finite paths, where we regard a vertex in E^0 as a path of length 0. Similarly, we let

$$E^\infty := \{x_1 x_2 \cdots \in (E^1)^{\mathbb{N}} : r(x_i) = s(x_{i+1}) \text{ for all } i\}.$$

The range and source maps r and s extend to E^* in the obvious way, and s extends to E^∞. We write $|u| = n$ if $u \in E^n$. The boundary path space of E is the space

$$\partial E := E^\infty \cup \{u \in E^* : r(u) \in E^0_{\text{sing}}\}$$

equipped with the topology for which the generalized cylinder sets

$$Z(\alpha \setminus F) := \{x \in \partial E : |\alpha| \leq |x|,\ s(x) = \alpha \text{ if } |\alpha| = 0, \\ x_1 = \alpha_1, \ldots, x_{|\alpha|} = \alpha_{|\alpha|} \text{ if } |\alpha| > 0, \text{ and either } |x| = |\alpha| \text{ or } x_{|\alpha|+1} \notin F\}$$

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parametrized by pairs (α, F), where $\alpha \in E^*$ and F is a finite subset of $s^{-1}(r(\alpha))$, form a basis of compact open sets; so that ∂E is locally compact Hausdorff space [24, Theorems 2.1 and 2.2].

We let $\mathcal{L}_{\partial E} := E^1$, define $\mathcal{B}_{\partial E}$ the set of compact open subsets of the boundary path space ∂E, and define for each $e \in \mathcal{L}_{\partial E}$ a Boolean homomorphism $(\theta_{\partial E})_e : \mathcal{B}_{\partial E} \to \mathcal{B}_{\partial E}$ by setting

$$(\theta_{\partial E})_e(A) := \{x \in \partial E : ex \in A\}.$$

Then $(\mathcal{B}_{\partial E}, \mathcal{L}_{\partial E}, \theta_{\partial E})$ is a Boolean dynamical system.

Let $\{p_A,\ s_e : A \in \mathcal{B}_{\partial E},\ e \in \mathcal{L}_{\partial E}\}$ be a universal Cuntz–Krieger representation of $(\mathcal{B}_{\partial E}, \mathcal{L}_{\partial E}, \theta_{\partial E})$. Then $\{s_e, p_{Z(v\setminus\emptyset)} : e \in E^1, v \in E^0\}$ is a Cuntz–Krieger E-family. It therefore follows from the universal property of $C^*(E)$ that there is a $*$-homomorphism $\phi : C^*(E) \to C^*(\mathcal{B}_{\partial E}, \mathcal{L}_{\partial E}, \theta_{\partial E})$ that maps s_e to s_e and p_v to $p_{Z(v\setminus\emptyset)}$.

It follows from the properties of $\{p_A,\ s_e : A \in \mathcal{B}_{\partial E},\ e \in \mathcal{L}_{\partial E}\}$ that if $v \in E^0$ and F is a finite subset of $s^{-1}(v)$, then $p_{Z(v\setminus F)} = p_{Z(v\setminus\emptyset)} - \sum_{e\in F} s_e s_e^*$, and if $\alpha \in E^* \setminus E^0$ and F is a finite subset of $s^{-1}(r(\alpha))$, then $p_{Z(\alpha\setminus F)} = s_\alpha s_\alpha^* - \sum_{e\in F} s_{\alpha e} s_{\alpha e}^*$. Since $\mathcal{B}_{\partial E}$ is generated by elements of the form $Z(\alpha \setminus F)$ where $\alpha \in E^*$ and F is a finite subset of $s^{-1}(r(\alpha))$, it follows that $\{p_A,\ s_e : A \in \mathcal{B}_{\partial E},\ e \in \mathcal{L}_{\partial E}\}$ is in the image of ϕ, and thus that ϕ is surjective. It follows from the gauge-invariant uniqueness theorem [1, Theorem 2.1], Remark 2.3(1) and the fact that $p_A \neq 0$ for $A \neq \emptyset$ (the latter follows, for instance, by an application of Theorem 2.5 to the identity map of $C^*(\mathcal{B}_{\partial E}, \mathcal{L}_{\partial E}, \theta_{\partial E})$) that ϕ is injective. Thus, $C^*(\mathcal{B}_{\partial E}, \mathcal{L}_{\partial E}, \theta_{\partial E})$ is isomorphic to $C^*(E)$.

Moreover, the map

$$x \in \partial E \mapsto \hat{x} := \{A \in \mathcal{B}_{\partial E} : x \in A\}$$

is a bijection between ∂E and $\widehat{\mathcal{B}_{\partial E}}$. Note that if $\beta \in \mathcal{L}_{\partial E}^* \setminus \{\emptyset\} = E^* \setminus E^0$, then

$$(\hat{\theta}_{\partial E})_\beta(\hat{x}) = \begin{cases} \widehat{\beta x} & \text{if } s(x) = r(\beta), \\ \emptyset & \text{if } s(x) \neq r(\beta). \end{cases}$$

Recall that the graph E satisfies Condition (K) if and only if whenever $v \in E^0$, $\alpha \in E^* \setminus E^0$ and $s(\alpha) = r(\alpha) = v$, there is a $\beta \in E^* \setminus E^0$ such that $s(\beta) = r(\beta) = v$ and for all $k \in N$ $\beta \neq \alpha^k$.

REMARK 5.4. Let E, $(\mathcal{B}_E, \mathcal{L}_E, \theta_E)$, and $(\mathcal{B}_{\partial E}, \mathcal{L}_{\partial E}, \theta_{\partial E})$ be as above.

(a) If $v \in E^0$ and $\alpha \in \mathcal{L}_E^*$, then $(\alpha, \hat{v})$ is an ultrafilter cycle in $(\mathcal{B}_E, \mathcal{L}_E, \theta_E)$ if and only if $s(\alpha) = r(\alpha) = v$.
(b) If $x \in \partial E$ and $\alpha \in \mathcal{L}_{\partial E}^*$, then $(\alpha, \hat{x})$ is an ultrafilter cycle in $(\mathcal{B}_{\partial E}, \mathcal{L}_{\partial E}, \theta_{\partial E})$ if and only if $x = \alpha^\infty$.
(c) $(\mathcal{B}_E, \mathcal{L}_E, \theta_E)$ satisfies Condition (K) if and only if E satisfies Condition (K).
(d) $(\mathcal{B}_{\partial E}, \mathcal{L}_{\partial E}, \theta_{\partial E})$ satisfies Condition (K) if and only if E satisfies Condition (K).

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PROOF. (a) and (b) easily follow from Lemma 3.2(2).

(c) Suppose first that $(\mathcal{B}_E, \mathcal{L}_E, \theta_E)$ satisfies Condition (K), and assume that $v \in E^0$, $\alpha \in E^* \setminus E^0$ and $s(\alpha) = r(\alpha) = v$. Then $(\alpha, \hat{v})$ is an ultrafilter cycle and $\{v\} \in \hat{v}$. Since we are assuming that $(\mathcal{B}_E, \mathcal{L}_E, \theta_E)$ satisfies Condition (K), it therefore follows that there are a $B \in \mathcal{I}_{\{v\}}$ and a $\beta \in \mathcal{L}^* \setminus \{\emptyset\}$ such that $(\theta_E)_\beta(B) \in \hat{v}$, and either $B \notin \hat{v}$ or $\beta \neq \alpha^k$ for any $k \in \mathbb{N}$. The conditions $B \in \mathcal{I}_{\{v\}}$ and $(\theta_E)_\beta(B) \in \hat{v}$ together imply that $B = \{v\}$ and $s(\beta) = r(\beta) = v$. We therefore have that $\beta \neq \alpha^k$ for any $k \in \mathbb{N}$, which shows that E satisfies Condition (K).

Conversely, suppose E satisfies Condition (K), and assume that $(\alpha, \hat{v})$ is an ultrafilter cycle in $(\mathcal{B}_E, \mathcal{L}_E, \theta_E)$ and $A \in \hat{v}$. Then $s(\alpha) = r(\alpha) = v$. Since we are assuming that E satisfies Condition (K), it follows that there is a $\beta \in E^* \setminus E^0$ such that $s(\beta) = r(\beta) = v$ and $\beta \neq \alpha^k$ for any $k \in \mathbb{N}$. Then $\beta \in \mathcal{L}_E^* \setminus \{\emptyset\}$, $\{v\} \in \mathcal{I}_A$, $(\theta_E)_\beta(\{v\}) = \{v\} \in \hat{v}$ and $\beta \neq \alpha^k$ for any $k \in \mathbb{N}$, which shows that $(\mathcal{B}_E, \mathcal{L}_E, \theta_E)$ satisfies Condition (K).

(d) Suppose first that $(\mathcal{B}_{\partial E}, \mathcal{L}_{\partial E}, \theta_{\partial E})$ satisfies Condition (K), and assume that $v \in E^0$, $\alpha \in E^* \setminus E^0$ and $s(\alpha) = r(\alpha) = v$. Then $x := \alpha^\infty \in \partial E$ and $(\alpha, \hat{x})$ is an ultrafilter cycle in $(\mathcal{B}_{\partial E}, \mathcal{L}_{\partial E}, \theta_{\partial E})$. Let $A := Z(\alpha \setminus \emptyset)$. Then $A \in \hat{x}$. Since we are assuming that $(\mathcal{B}_{\partial E}, \mathcal{L}_{\partial E}, \theta_{\partial E})$ satisfies Condition (K), it therefore follows that there are a $B \in \mathcal{I}_A$ and a $\beta \in \mathcal{L}^* \setminus \{\emptyset\}$ such that $(\theta_{\partial E})_\beta(B) \in \hat{x}$, and either $B \notin \hat{x}$ or $\beta \neq \alpha^k$ for any $k \in \mathbb{N}$. It follows from $(\theta_{\partial E})_\beta(B) \in \hat{x}$ that $\beta x \in B$, and thus that $r(\beta) = s(x) = s(\alpha) = v$. Since $B \in \mathcal{I}_A$, we also have that $\beta x \in B \subseteq A = Z(\alpha \setminus \emptyset)$ and thus $s(\beta) = s(\alpha) = v$. If $\beta = \alpha^k$ for some $k \in \mathbb{N}$, then $x = \beta x \in B$, so that cannot be the case. Thus, $\beta \neq \alpha^k$ for any $k \in \mathbb{N}$. This shows that E satisfies Condition (K).

Conversely, suppose E satisfies Condition (K), and assume that $(\alpha, \hat{x})$ is an ultrafilter cycle in $(\mathcal{B}_{\partial E}, \mathcal{L}_{\partial E}, \theta_{\partial E})$ and $A \in \hat{x}$. Then $s(\alpha) = r(\alpha)$ and $x = \alpha^\infty$. Since we are assuming that E satisfies Condition (K), it follows that there is a $\beta \in E^* \setminus E^0$ such that $s(\beta) = r(\beta) = v$ and $\beta \neq \alpha^k$ for any $k \in \mathbb{N}$. Since $A \in \hat{x}$, we have that A is an open neighborhood of x in ∂E. There is therefore an $n \in \mathbb{N}$ such that $Z(\alpha^n \setminus \emptyset) \subseteq A$. Let $\gamma := \alpha^n \beta$ and $B := Z(\gamma \setminus \emptyset)$. Then $\gamma \in \mathcal{L}_{\partial E}^* \setminus \{\emptyset\}$, $B \in \mathcal{I}_A$, $(\theta_{\partial E})_\gamma(B) \in \hat{x}$ and $\gamma \neq \alpha^k$ for any $k \in \mathbb{N}$. This shows that $(\mathcal{B}_{\partial E}, \mathcal{L}_{\partial E}, \theta_{\partial E})$ satisfies Condition (K). □

Remark 5.4 might make one think that Condition (K) for Boolean dynamical systems would be equivalent to one of the two conditions in the next lemma. However, we shall see in Proposition 5.6 and Remark 5.7 that the two conditions in Lemma 5.5 are, in general, strictly stronger than Condition (K).

LEMMA 5.5. *Let $(\mathcal{B}, \mathcal{L}, \theta)$ be a Boolean dynamical system. The following assertions are equivalent.*

(1) *If (α, η) is an ultrafilter cycle, then there exists $\beta \in \mathcal{L}^*$ such that (β, η) is an ultrafilter cycle and $\beta \neq \alpha^k$ for all $k \in \mathbb{N}$.*
(2) *If $\alpha \in \mathcal{L}^* \setminus \{\emptyset\}$, $\eta \in \widehat{\mathcal{B}}$ and $\eta = \widehat{\theta_\alpha}(\eta)$, then there exists $\beta \in \mathcal{L}^* \setminus \{\emptyset\}$ such that $\eta = \widehat{\theta_\beta}(\eta)$ and $\beta \neq \alpha^k$ for all $k \in \mathbb{N}$.*

PROOF. This follows from Lemma 3.2(2). □

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PROPOSITION 5.6. *Let $(\mathcal{B},\mathcal{L},\theta)$ be a Boolean dynamical system. If $(\mathcal{B},\mathcal{L},\theta)$ satisfies condition (1) in Lemma 5.5, then it satisfies Condition (K).*

PROOF. Assume that (1) in Lemma 5.5 holds, and let (α,η) be an ultrafilter cycle and $A\in\eta$. Then there is a $\beta\in\mathcal{L}^*\setminus\{\emptyset\}$ such that (β,η) is an ultrafilter cycle and $\beta\neq\alpha^k$ for all $k\in\mathbb{N}$. Since (β,η) is an ultrafilter cycle and $A\in\eta$, it follows that $\theta_\beta(A)\in\eta$. This shows that $(\mathcal{B},\mathcal{L},\theta)$ satisfies Condition (K). □

REMARK 5.7. If a directed graph E satisfies Condition (K), then the Boolean dynamical system $(\mathcal{B}_{\partial E},\mathcal{L}_{\partial E},\theta_{\partial E})$ satisfies Condition (K) by Remark 5.4(d). But it follows from Remark 5.4(b) that condition (1) in Lemma 5.5 cannot hold for $(\mathcal{B}_{\partial E},\mathcal{L}_{\partial E},\theta_{\partial E})$. Thus, Condition (K) does not imply condition (1) in Lemma 5.5 in general.

6. Boolean dynamical systems for which all ideals are gauge-invariant

In this section we show that a Boolean dynamical system $(\mathcal{B},\mathcal{L},\theta)$ satisfies Condition (K) if and only if the quotient Boolean dynamical system $(\mathcal{B}/\mathcal{H},\mathcal{L},\theta)$ satisfies Condition (L) for every hereditary saturated ideal $\mathcal{H}$ of $\mathcal{B}$. We also show that each of them is a necessary condition for every ideal of $C^*(\mathcal{B},\mathcal{L},\theta)$ to be gauge-invariant, and that if, moreover, $(\mathcal{B},\mathcal{L},\theta)$ is locally finite and has countable $\mathcal{B}$ and $\mathcal{L}$, each of them is also a sufficient condition for every ideal of $C^*(\mathcal{B},\mathcal{L},\theta)$ to be gauge-invariant.

We start with two technical results.

LEMMA 6.1. *Let $(\mathcal{B},\mathcal{L},\theta)$ be a Boolean dynamical system, and suppose (α,A) is a cycle with no exits. Then we have $\theta_{\alpha_{[1,i]}}(A)\cap\theta_{\alpha_{[1,j]}}(A)=\emptyset$ for all $1\le i<j\le|\alpha|$.*

PROOF. Let (α,A) be a cycle with no exits. By Lemma 4.10, we may assume that $A\cap\theta_{\alpha_{[1,j]}}(A)=\emptyset$ for $1\le j<|\alpha|$. Let $n:=|\alpha|$. We claim that then

$$\theta_{\alpha_{[1,i]}}(A)\cap\theta_{\alpha_{[1,j]}}(A)=\emptyset\quad\text{for all } 1\le i<j\le n.$$

To see that the claim holds, suppose that $\theta_{\alpha_{[1,i]}}(A)\cap\theta_{\alpha_{[1,j]}}(A)\neq\emptyset$ for some $1\le i<n$. Since (α,A) has no exit, we then have that $\alpha_{i+k}=\alpha_{j+k}$ for any k (where the indices are computed modulo n). Thus,

$$A\cap\theta_{\alpha_{[1,j-i]}}(A)=\theta_{\alpha_{i+1}\alpha_{i+2}\cdots\alpha_n}(\theta_{\alpha_{[1,i]}}(A)\cap\theta_{\alpha_{[1,j]}}(A))\neq\emptyset$$

because $\alpha_{i+1}\in\Delta_{\theta_{\alpha_{[1,i]}}(A)\cap\theta_{\alpha_{[1,j]}}(A)}$ and $\alpha_{i+k+1}\in\Delta_{\theta_{\alpha_{[i+1,i+k]}}(\theta_{\alpha_{[1,i]}}(A)\cap\theta_{\alpha_{[1,j]}}(A))}$ for $1\le k<n-i$. But this contradicts the assumption that $A\cap\theta_{\alpha_{[1,j]}}(A)=\emptyset$. □

PROPOSITION 6.2. *Let $(\mathcal{B},\mathcal{L},\theta)$ be a Boolean dynamical system. Suppose that $(\mathcal{B},\mathcal{L},\theta)$ has a cyclic maximal tail $\mathcal{T}$. Then $C^*(\mathcal{B}/(\mathcal{B}\setminus\mathcal{T}),\mathcal{L},\theta)$ contains an ideal that is not gauge-invariant, and there is a $B\in\mathcal{T}$ such that $p_{[B]}C^*(\mathcal{B}/(\mathcal{B}\setminus\mathcal{T}),\mathcal{L},\theta)p_{[B]}$ is isomorphic to $M_n(C(\mathbb{T}))$ for some $n\in\mathbb{N}$.*

PROOF. Choose a cyclic maximal tail $\mathcal{T}$ in $(\mathcal{B},\mathcal{L},\theta)$. Then there exist an ultrafilter cycle (α,η) such that $\mathcal{T}=\{B\in\mathcal{B}:\theta_\beta(B)\in\eta\text{ for some }\beta\in\mathcal{L}^*\}$ and an $A\in\eta$ such that

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if $\beta \in \mathcal{L}^* \setminus \{\emptyset\}$, $B \in \mathcal{I}_A$ and $\theta_\beta(B) \in \eta$, then $B \in \eta$ and $\beta = \alpha^k$ for some $k \in \mathbb{N}$. By Remark 4.3, $\mathcal{B} \setminus \mathcal{T}$ is a hereditary saturated ideal of $\mathcal{B}$. We first show that for any $B \in \mathcal{I}_A$, we have

$$\text{either} \quad A \setminus B \notin \mathcal{T} \quad \text{or} \quad B \notin \mathcal{T} \tag{6-1}$$

(or equivalently, either $A \setminus B \in \mathcal{B} \setminus \mathcal{T}$ or $B \in \mathcal{B} \setminus \mathcal{T}$). Since $A = B \cup (A \setminus B) \in \eta$, either $B \in \eta$ or $A \setminus B \in \eta$. First, if $B \in \eta$, then $A \setminus B \notin \eta$. We then have $\theta_\beta(A \setminus B) \notin \eta$ for all $\beta \in \mathcal{L}^*$. If not, then $\theta_\beta(A \setminus B) \in \eta$ for some $\beta \in \mathcal{L}^*$, and thus $A \setminus B \in \eta$. This is not the case. Thus $A \setminus B \notin \mathcal{T}$. Second, if $A \setminus B \in \eta$, it follows that $\theta_\beta(B) \notin \eta$ for all $\beta \in \mathcal{L}^*$ by the same reasoning. Thus, $B \notin \mathcal{T}$.

Put $B := \cup_{k=1}^n \theta_{\alpha_{[1,k]}}(A)$ where $n = |\alpha|$. We now claim that

$$p_{[B]} C^*(\mathcal{B}/(\mathcal{B} \setminus \mathcal{T}), \mathcal{L}, \theta) p_{[B]} \cong C(\mathbb{T}) \otimes M_n.$$

We proved that $(\alpha, [A])$ is a cycle with no exit in $(\mathcal{B}/(\mathcal{B} \setminus \mathcal{T}), \mathcal{L}, \theta)$ in the proof of Proposition 4.5. By (6-1), we see that $[A]$ is minimal in the sense that for any nonempty $[C] \in \mathcal{B}/(\mathcal{B} \setminus \mathcal{T})$, either $[A \cap C] = [A]$ or $[A \cap C] = [\emptyset]$. We also have by Lemma 6.1 that

$$[\theta_{\alpha_{[1,i]}}(A)] \cap [\theta_{\alpha_{[1,j]}}(A)] = \emptyset \quad \text{for all } \ 1 \le i < j \le n. \tag{6-2}$$

Then, for $s_\mu p_{[C]} s_\nu^* \in C^*(\mathcal{B}/(\mathcal{B} \setminus \mathcal{T}), \mathcal{L}, \theta)$, if

$$p_{[B]}(s_\mu p_{[C]} s_\nu^*) p_{[B]} = s_\mu p_{[\theta_\mu(B)] \cap [C] \cap [\theta_\nu(B)]} s_\nu^* \neq 0,$$

then $[\theta_\mu(B)] \cap [\theta_\nu(B)] \neq \emptyset$. Thus $[\theta_\mu(B)] \neq \emptyset$ and $[\theta_\nu(B)] \neq \emptyset$, and hence we see that the paths μ, ν are of the form

$$\mu = \alpha_{[i,n]} \alpha^l \alpha_{[1,k]}, \ \ \nu = \alpha_{[j,n]} \alpha^m \alpha_{[1,k']}$$

for some $i, j, l, m \ge 0$ and $1 \le k, k' \le n$ since $(\alpha, [A])$ is a cycle with no exit. Then $\emptyset \neq [\theta_\mu(B)] \cap [\theta_\nu(B)] = [\theta_{\alpha_{[1,i-1]}\mu}(A)] \cap [\theta_{\alpha_{[1,j-1]}\nu}(A)] = [\theta_{\alpha_{[1,k]}}(A)] \cap [\theta_{\alpha_{[1,k']}}(A)]$. Thus by (6-2), $k = k'$. It then follows that

$$\begin{aligned}
s_\mu p_{[\theta_\mu(B)] \cap [C] \cap [\theta_\nu(B)]} s_\nu^* &= s_{\alpha_{[i,n]} \alpha^l \alpha_{[1,k]}} p_{[\theta_{\alpha_{[1,k]}}(A) \cap C]} s^*_{\alpha_{[j,n]} \alpha^m \alpha_{[1,k]}} \\
&= s_{\alpha_{[i,n]} \alpha^l \alpha_{[1,k]}} s_{k+1} p_{[\theta_{\alpha_{[1,k+1]}}(A) \cap \theta_{\alpha_{k+1}}(C)]} s^*_{\alpha_{k+1}} s^*_{\alpha_{[j,n]} \alpha^m \alpha_{[1,k]}} \\
&\ \ \vdots \\
&= s_{\alpha_{[i,n]} \alpha^l \alpha_{[1,n]}} p_{[\theta_{\alpha_{[1,n]}}(A) \cap \theta_{\alpha_{[k+1,n]}}(C)]} s^*_{\alpha_{[j,n]} \alpha^m \alpha_{[1,n]}} \\
&= s_{\alpha_{[i,n]} \alpha^{l+1}} p_{[A \cap \theta_{\alpha_{[k+1,n]}}(C)]} s^*_{\alpha_{[j,n]} \alpha^{m+1}} \\
&= s_{\alpha_{[i,n]} \alpha^{l+1}} p_{[A]} s^*_{\alpha_{[j,n]} \alpha^{m+1}}.
\end{aligned}$$

This means that the hereditary subalgebra $p_{[B]} C^*(\mathcal{B}/(\mathcal{B} \setminus \mathcal{T}), \mathcal{L}, \theta) p_{[B]}$ is generated by the elements $s_{\alpha_i} p_{[\theta_{\alpha_{[1,i]}}(A)]}$ for $1 \le i \le n$. Let γ be the restriction of the gauge action on $C^*(\mathcal{B}/(\mathcal{B} \setminus \mathcal{T}), \mathcal{L}, \theta)$ to the hereditary subalgebra $p_{[B]} C^*(\mathcal{B}/(\mathcal{B} \setminus \mathcal{T}), \mathcal{L}, \theta) p_{[B]}$ which is obviously gauge-invariant, and let β be the gauge action of the universal (graph)

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C^*-algebra $C(\mathbb{T}) \otimes M_n$ generated by the partial isometries $t_1, \ldots, t_n$ satisfying the relations

$$t_i^* t_i = t_{i+1} t_{i+1}^*,\ t_n^* t_n = t_1 t_1^* \quad \text{and} \quad \sum_{j=1}^{n} t_j^* t_j = 1$$

for $1 \leq i \leq n-1$. But the partial elements $s_{\alpha_i} p_{[\theta_{\alpha_{[1,i]}}(A)]}$, $1 \leq i \leq n$, satisfy the above relations with $p_{[B]}$ in place of 1; hence there exists a homomorphism

$$\pi : C(\mathbb{T}) \otimes M_n \to p_{[B]} C^*(\mathcal{B}/(\mathcal{B} \setminus \mathcal{T}), \mathcal{L}, \theta) p_{[B]}$$

such that $\pi(t_i) = s_{\alpha_i} p_{[\theta_{\alpha_{[1,i]}}(A)]}$ for $1 \leq i \leq n$. It is then immediate that $\pi(\beta_z(t_i)) = \gamma_z(\pi(t_i))$ for all i and thus the gauge-invariant uniqueness theorem [7, Theorem 5.10] proves that π is an isomorphism. It then follows that $C^*(\mathcal{B}/(\mathcal{B} \setminus \mathcal{T}), \mathcal{L}, \theta)$ contains an ideal that is not gauge-invariant. □

We can now prove our main theorem.

THEOREM 6.3. *Let $(\mathcal{B}, \mathcal{L}, \theta)$ be a Boolean dynamical system. Consider the following conditions.*

(1) *$(\mathcal{B}, \mathcal{L}, \theta)$ satisfies Condition (K).*
(2) *$(\mathcal{B}, \mathcal{L}, \theta)$ has no cyclic maximal tails.*
(3) *For every hereditary saturated ideal $\mathcal{H}$ of $\mathcal{B}$, the Boolean dynamical system $(\mathcal{B}/\mathcal{H}, \mathcal{L}, \theta)$ satisfies Condition (L).*
(4) *Every ideal in $C^*(\mathcal{B}, \mathcal{L}, \theta)$ is gauge-invariant.*

We have that (1) $\iff$ (2) $\iff$ (3) and (4) implies each of conditions (1)–(3). If, moreover, $(\mathcal{B}, \mathcal{L}, \theta)$ is locally finite and $\mathcal{B}$ and $\mathcal{L}$ are countable, then all four conditions are equivalent.

PROOF. (1) $\implies$ (2) follows from the definition of a cyclic maximal tail. (2) $\implies$ (3) follows from Proposition 4.8, and (3) $\implies$ (1) follows from Proposition 4.5.

We show that (4) $\implies$ (2). Assume to the contrary that $(\mathcal{B}, \mathcal{L}, \theta)$ has a cyclic maximal tail $\mathcal{T}$. It then follows by Proposition 6.2 that there is a $B \in \mathcal{T}$ such that $p_{[B]} C^*(\mathcal{B}/(\mathcal{B} \setminus \mathcal{T}), \mathcal{L}, \theta) p_{[B]}$ is isomorphic to $M_n(C(\mathbb{T}))$ for some $n \in \mathbb{N}$. Thus $C^*(\mathcal{B}, \mathcal{L}, \theta)$ has a quotient containing a corner that is isomorphic to $M_n(C(\mathbb{T}))$ for some $n \in \mathbb{N}$ and contains an ideal that is not gauge-invariant, a contradiction.

Now assume that $(\mathcal{B}, \mathcal{L}, \theta)$ is locally finite and $\mathcal{B}$ and $\mathcal{L}$ are countable and prove (3) $\implies$ (4). Let I be an ideal of $C^*(\mathcal{B}, \mathcal{L}, \theta)$. Then $\mathcal{H}_I = \{A \in \mathcal{B} : p_A \in I\}$ is a hereditary saturated ideal of $\mathcal{B}$ and the ideal $I_{\mathcal{H}_I}$ generated by the projections $\{p_A : A \in \mathcal{H}_I\}$ is gauge-invariant. Since $I_{\mathcal{H}_I} \subseteq I$, the quotient map

$$q : C^*(\mathcal{B}, \mathcal{L}, \theta)/I_{\mathcal{H}_I} \to C^*(\mathcal{B}, \mathcal{L}, \theta)/I$$

given by $q(s + I_{\mathcal{H}_I}) := s + I$ for $s \in C^*(\mathcal{B}, \mathcal{L}, \theta)$, is well defined. From [7, Proposition 10.11], we have an isomorphism $\pi : C^*(\mathcal{B}/\mathcal{H}_I, \mathcal{L}, \theta) \to C^*(\mathcal{B}, \mathcal{L}, \theta)/I_{\mathcal{H}_I}$ which maps the canonical generators to the canonical generators. Then the composition map $q \circ \pi$:

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$C^*(\mathcal{B}/\mathcal{H}_I, \mathcal{L}, \theta) \to C^*(\mathcal{B}, \mathcal{L}, \theta)/I$ satisfies

$$q \circ \pi(p_{[A]}) = q(p_A + I_{\mathcal{H}_I}) = p_A + I,$$
$$q \circ \pi(s_\alpha) = q(s_\alpha + I_{\mathcal{H}_I}) = s_\alpha + I$$

for $[A] \in \mathcal{B}/\mathcal{H}_I$ and $\alpha \in \mathcal{L}$. If $p_{[A]} \neq 0$, then $[A] \neq [\emptyset]$ in $\mathcal{B}/\mathcal{H}_I$, hence $A \notin \mathcal{H}_I$. Thus $p_A + I \in C^*(\mathcal{B}, \mathcal{L}, \theta)/I$ is a nonzero projection. Since the quotient Boolean dynamical system $(\mathcal{B}/\mathcal{H}_I, \mathcal{L}, \theta)$ satisfies Condition (L), we see that the map $q \circ \pi$ is injective by the Cuntz–Krieger uniqueness theorem. Thus q is injective, so that I must coincide with the gauge-invariant ideal $I_{\mathcal{H}_I}$.

Note that if I is an ideal such that $\mathcal{H}_I = \{\emptyset\}$, then $I_{\mathcal{H}_I} = \{0\}$, and $q \circ \pi$ is the quotient map $q : C^*(\mathcal{B}, \mathcal{L}, \theta) \to C^*(\mathcal{B}, \mathcal{L}, \theta)/I$. Then the family $\{p_A + I, s_\alpha + I\}$ is a representation of the Boolean dynamical system $(\mathcal{B}, \mathcal{L}, \theta)$ in the C^*-algebra $C^*(\mathcal{B}, \mathcal{L}, \theta)/I$ such that $p_A + I \neq 0$ and $s_\alpha + I \neq 0$ for each $A \in \mathcal{B}$ and $\alpha \in \mathcal{L}$. Since $(\mathcal{B}, \mathcal{L}, \theta)$ satisfies Condition (L), the Cuntz–Krieger uniqueness theorem again says that q is injective. Thus we have $I = \{0\}$. □

7. The primitive ideal space of $C^*(\mathcal{B}, \mathcal{L}, \theta)$

In this section we describe the primitive ideal space of the C^*-algebra of a locally finite Boolean dynamical system that satisfies Condition (K) and has countable $\mathcal{B}$ and $\mathcal{L}$. The notion of a maximal tail plays a crucial role in characterizing primitive gauge-invariant ideals of $C^*(\mathcal{B}, \mathcal{L}, \theta)$. We analyze the topology on the set of all maximal tails M in $\mathcal{B}$ and then show that M is homeomorphic to the primitive ideal space of $C^*(\mathcal{B}, \mathcal{L}, \theta)$.

7.1. The space of maximal tails. Let $(\mathcal{B}, \mathcal{L}, \theta)$ be a Boolean dynamical system. Denote by M the set of all maximal tails in $\mathcal{B}$.

PROPOSITION 7.1. *Let $(\mathcal{B}, \mathcal{L}, \theta)$ be a Boolean dynamical system. For $A \in \mathcal{B}$, let*

$$\mathrm{U}_A := \{\mathcal{T} \in \mathrm{M} : A \in \mathcal{T}\}.$$

Then $\{\mathrm{U}_A : A \in \mathcal{B}\}$ is a basis of compact open sets for a topology on M.

PROOF. It is obvious that $\cup_{A\in\mathcal{B}} \mathrm{U}_A = \mathrm{M}$. For $A_1, A_2 \in \mathcal{B}$, choose $\mathcal{T} \in \mathrm{U}_{A_1} \cap \mathrm{U}_{A_2}$. Then $A_1, A_2 \in \mathcal{T}$, and thus there exists $C \in \mathcal{T}$ such that $A_1 \geq C$ and $A_2 \geq C$ by (T5). So there are $\alpha_i \in \mathcal{L}^*$ such that $C \subseteq \theta_{\alpha_i}(A_i)$ for $i = 1, 2$. Hence by (T3), we see that $\theta_{\alpha_i}(A) \in \mathcal{T}$ for $i = 1, 2$. Therefore, $A_1, A_2 \in \mathcal{T}$ by (T1). Thus it follows that

$$\mathcal{T} \in \mathrm{U}_C \subseteq \mathrm{U}_{A_1} \cap \mathrm{U}_{A_2}.$$

Thus, $\{\mathrm{U}_A : A \in \mathcal{B}\}$ forms a basis of a topology of M. To show that U_A is compact, consider the injective map $\iota : \mathrm{M} \to \{0, 1\}^{\mathcal{B}}$ given by

$$\iota(\mathcal{T})(A) = \begin{cases} 1 & \text{if } A \in \mathcal{T}, \\ 0 & \text{if } A \notin \mathcal{T}, \end{cases}$$

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for $A \in \mathcal{B}$. Equip $\{0,1\}^{\mathcal{B}}$ with the product topology. Then $\{0,1\}^{\mathcal{B}}$ is compact. Since

$$\iota(\mathrm{U}_A) = \{\eta \in \{0,1\}^{\mathcal{B}} : \eta(A) = 1\} = \pi_A^{-1}(\{1\}),$$

where $\pi_A : \{0,1\}^{\mathcal{B}} \to \{0,1\}$ is defined by $\pi_A(f) = f(A)$, the map $\iota : \mathrm{M} \to \{0,1\}^{\mathcal{B}}$ is an open map. Thus, $\iota : \mathrm{M} \to \iota(\mathrm{M})$ is a bijective open map, and hence $\iota^{-1} : \iota(\mathrm{M}) \to \mathrm{M}$ is continuous. So, $\mathrm{U}_A = \iota^{-1}(\iota(\mathrm{U}_A))$ is compact since $\iota(\mathrm{U}_A)$ is compact in $\{0,1\}^{\mathcal{B}}$. □

We now characterize the closed subsets of M. For a subset S of M, denote by $\overline{\mathrm{S}}$ the closure of S in M.

LEMMA 7.2. *Let $(\mathcal{B}, \mathcal{L}, \theta)$ be a Boolean dynamical system and* S *a subset of* M. *Then* $\overline{\mathrm{S}} = \{\mathcal{T} \in \mathrm{M} : \mathcal{T} \subseteq \cup_{\mathcal{S} \in \mathrm{S}} \mathcal{S}\}$.

PROOF. We have

$$\begin{aligned} \mathcal{T} \in \overline{\mathrm{S}} &\iff \mathrm{U}_A \cap \mathrm{S} \neq \emptyset \quad \text{for all } A \in \mathcal{T} \\ &\iff \mathcal{T} \subseteq \cup_{\mathcal{S} \in \mathrm{S}} \mathcal{S}. \end{aligned}$$

□

7.2. The primitive ideal spaces. If a locally finite Boolean dynamical system $(\mathcal{B}, \mathcal{L}, \theta)$ with countable $\mathcal{B}$ and $\mathcal{L}$ satisfies Condition (K), then every ideal has the form $I_{\mathcal{H}}$ for some hereditary saturated ideal $\mathcal{H}$ of $\mathcal{B}$ by Theorem 6.3. So, we only need to determine when the gauge-invariant ideal $I_{\mathcal{H}}$ is primitive. We start with the following lemma that holds true without assuming locally finiteness of $(\mathcal{B}, \mathcal{L}, \theta)$ and countability of $\mathcal{B}$ and $\mathcal{L}$.

LEMMA 7.3. *Let $(\mathcal{B}, \mathcal{L}, \theta)$ be a Boolean dynamical system. If I is a primitive ideal of $C^*(\mathcal{B}, \mathcal{L}, \theta)$, then $\mathcal{T} := \{A \in \mathcal{B} : p_A \notin I\}$ is a maximal tail of $\mathcal{B}$.*

PROOF. The set $\mathcal{H} := \{A \in \mathcal{B} : p_A \in I\}$ is a proper hereditary saturated ideal of $\mathcal{B}$ (see for example, [15, Lemma 3.5]). So, $\mathcal{T} = \mathcal{B} \setminus \mathcal{H}$ satisfies (T0)–(T4). To show (T5), choose $A_1, A_2 \in \mathcal{T}$ and take an irreducible representation $\pi : C^*(\mathcal{B}, \mathcal{L}, \theta) \to \mathfrak{B}(H_\pi)$ such that $\ker(\pi) = I$. Since

$$\mathcal{T} = \{A \in \mathcal{B} : p_A \notin I\},$$

we have $p_{A_1} \notin I$, and hence $\pi(p_{A_1})H_\pi \neq \{0\}$. Similarly, the space $\pi(p_{A_2})H_\pi$ is also a nontrivial subspace of H_π. Fix $h \in \pi(p_{A_1})H_\pi$ so that $\|h\| = 1$. Since π is irreducible, h is cyclic for π, so that there exists $a \in C^*(\mathcal{B}, \mathcal{L}, \theta)$ such that $\pi(p_{A_2})\pi(a)h = \pi(p_{A_2} a p_{A_1})h \neq 0$. In particular, we have $\pi(p_{A_2} a p_{A_1}) \neq 0$. Since

$$p_{A_1}(s_\mu p_B s_\nu^*)p_{A_2} = s_\mu p_{\theta_\mu(A_1) \cap B \cap \theta_\nu(A_2)} s_\nu^* \neq 0$$

only if $\theta_\mu(A_1) \cap B \cap \theta_\nu(A_2) \neq \emptyset$, we see that

$$\pi(p_{A_2} a p_{A_1}) \in \overline{\mathrm{span}}\{\pi(s_\mu p_C s_\nu^*) : \mu, \nu \in \mathcal{L}^*, C \in \mathcal{B}, \emptyset \neq C \subseteq \theta_\mu(A_1) \cap \theta_\nu(A_2)\} \setminus \{0\}.$$

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Thus there exist $\mu, \nu \in \mathcal{L}^*$ and $C \subseteq \theta_\mu(A_1) \cap \theta_\nu(A_2)$ such that $\pi(s_\mu p_C s_\nu^*) \neq 0$. One can also show that $\pi(p_C) \neq 0$, giving $p_C \notin I$. So $C \in \mathcal{T}$ satisfies $A_1 \geq C$ and $A_2 \geq C$. Therefore, $\mathcal{T}$ is a maximal tail. □

PROPOSITION 7.4. *Let $(\mathcal{B}, \mathcal{L}, \theta)$ be a locally finite Boolean dynamical system that satisfies Condition (K) and has countable $\mathcal{B}$ and $\mathcal{L}$. Then $\mathcal{H}$ is a hereditary saturated ideal of $\mathcal{B}$ such that $I_\mathcal{H}$ is primitive if and only if $\mathcal{T} := \mathcal{B} \setminus \mathcal{H}$ is a maximal tail.*

PROOF. ($\Rightarrow$) Suppose that $\mathcal{H}$ is a hereditary saturated ideal of $\mathcal{B}$ and $I_\mathcal{H}$ is primitive. Then by Lemma 7.3, the set $\mathcal{T} := \mathcal{B} \setminus \mathcal{H} = \{A \in \mathcal{B} : p_A \notin I_\mathcal{H}\}$ is a maximal tail.

($\Leftarrow$) Let $\mathcal{T}$ be a maximal tail. Then $\mathcal{H} := \mathcal{B} \setminus \mathcal{T}$ is a proper hereditary and saturated ideal in $\mathcal{B}$ by Remark 4.3. We show that $I_\mathcal{H}$ is a prime ideal. Suppose that I_1, I_2 are ideals in $C^*(\mathcal{B}, \mathcal{L}, \theta)$ such that $I_1 \cap I_2 \subseteq I_\mathcal{H}$. Since every ideal of $C^*(\mathcal{B}, \mathcal{L}, \theta)$ is gauge-invariant by Theorem 6.3, it follows from [7, Proposition 10.11] that there are hereditary saturated subsets $\mathcal{H}_i$ such that $I_i = I_{\mathcal{H}_i}$ for $i = 1, 2$. Then $\mathcal{H}_1 \cap \mathcal{H}_2 \subseteq \mathcal{H}$. If $\mathcal{H}_1 \nsubseteq \mathcal{H}$ and $\mathcal{H}_2 \nsubseteq \mathcal{H}$, then there are $A_i \in \mathcal{H}_i \setminus \mathcal{H}$ for $i = 1, 2$. By (T5), there exists $C \in \mathcal{T}$ such that $A_1 \geq C$ and $A_2 \geq C$. So there are $\alpha_i \in \mathcal{L}^*$ such that $C \subseteq \theta_{\alpha_i}(A_i)$ for $i = 1, 2$. Then $C \in \mathcal{H}_1 \cap \mathcal{H}_2 \subseteq \mathcal{H}$ since $\mathcal{H}_1$ and $\mathcal{H}_2$ are hereditary. This contradicts $C \notin \mathcal{H}$. Hence, either $\mathcal{H}_1 \subseteq \mathcal{H}$ or $\mathcal{H}_2 \subseteq \mathcal{H}$, which means either $I_1 = I_{\mathcal{H}_1} \subseteq I_\mathcal{H}$ or $I_2 = I_{\mathcal{H}_2} \subseteq I_\mathcal{H}$. This shows that $I_\mathcal{H}$ is prime. Thus, it is primitive since $C^*(\mathcal{B}, \mathcal{L}, \theta)$ is separable [23, Proposition A.17 and Remark A.18]. □

By $\mathrm{Prim}(C^*(\mathcal{B}, \mathcal{L}, \theta))$ we mean the set of primitive ideals in $C^*(\mathcal{B}, \mathcal{L}, \theta)$. We now obtain a complete list of primitive ideals of $C^*(\mathcal{B}, \mathcal{L}, \theta)$ and a description of the hull-kernel topology of $\mathrm{Prim}(C^*(\mathcal{B}, \mathcal{L}, \theta))$.

THEOREM 7.5. *Let $(\mathcal{B}, \mathcal{L}, \theta)$ be a locally finite Boolean dynamical system such that $\mathcal{B}$ and $\mathcal{L}$ are countable. Suppose that $(\mathcal{B}, \mathcal{L}, \theta)$ satisfies Condition (K). Then the map*

$$\phi : \mathrm{M} \to \mathrm{Prim}(C^*(\mathcal{B}, \mathcal{L}, \theta))$$

defined by $\phi(\mathcal{T}) = I_{H_\mathcal{T}}$ is a homeomorphism, where $H_\mathcal{T} = \mathcal{B} \setminus \mathcal{T}$.

PROOF. By Proposition 7.4, we see that $\mathcal{T} \mapsto I_{\mathcal{B}\setminus\mathcal{T}}$ is a surjective map from M to $\mathrm{Prim}(C^*(\mathcal{B}, \mathcal{L}, \theta))$. It follows from Lemma 7.3 that $\mathcal{T} = \{A \in \mathcal{B} : p_A \notin I_{\mathcal{B}\setminus\mathcal{T}}\}$, so ϕ is injective. If S a subset of M, then it follows from Lemma 7.2 that

$$\phi(\overline{\mathrm{S}}) = \{I_{\mathcal{B}\setminus\mathcal{T}} : \mathcal{T} \subseteq \cup_{\mathcal{S}\in\mathrm{S}}\mathrm{S}\} = \{I \in \mathrm{Prim}(C^*(\mathcal{B}, \mathcal{L}, \theta)) : \cap_{\mathcal{S}\in\mathrm{S}} I_{\mathcal{B}\setminus\mathcal{S}} \subseteq I\} = \overline{\phi(\mathrm{S})}.$$

This shows that ϕ is a homeomorphism. □

8. Topological dimension zero

A C^*-algebra A is said to have *topological dimension zero* if the primitive ideal space of A endowed with the hull-kernel topology has a basis of compact open sets [4]. Proposition 7.4 and Theorem 7.5 say that if a locally finite Boolean dynamical system $(\mathcal{B}, \mathcal{L}, \theta)$ satisfies Condition (K) and has countable $\mathcal{B}$ and $\mathcal{L}$, then the topological

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dimension of $C^*(\mathcal{B}, \mathcal{L}, \theta)$ is 0. We show in Theorem 8.1 that the converse is also true. We also show that this is equivalent to $C^*(\mathcal{B}, \mathcal{L}, \theta)$ having the (weak) ideal property.

To begin with, we recall that a C^*-algebra A is said to have *the ideal property* [21, Remark 2.1] if whenever I, J are ideals in A such that I is not contained in J, there is a projection in $I \setminus J$. A C^*-algebra A is said to have *the weak ideal property* [19, Definition 8.1] if whenever $I \subsetneq J \subset \mathcal{K} \otimes A$ are ideals in $\mathcal{K} \otimes A$, where $\mathcal{K}$ denotes the C^*-algebra of compact operators on a separable infinite-dimensional Hilbert space, then J/I contains a nonzero projection.

THEOREM 8.1. *Let $(\mathcal{B}, \mathcal{L}, \theta)$ be a locally finite Boolean dynamical system such that $\mathcal{B}$ and $\mathcal{L}$ are countable. Then the following assertions are equivalent.*

(1) *$(\mathcal{B}, \mathcal{L}, \theta)$ satisfies Condition (K).*
(2) *$C^*(\mathcal{B}, \mathcal{L}, \theta)$ has the ideal property.*
(3) *$C^*(\mathcal{B}, \mathcal{L}, \theta)$ has the weak ideal property.*
(4) *The topological dimension of $C^*(\mathcal{B}, \mathcal{L}, \theta)$ is 0.*
(5) *$C^*(\mathcal{B}, \mathcal{L}, \theta)$ has no quotients containing a corner that is isomorphic to $M_n(C(\mathbb{T}))$.*

PROOF. (1) $\Longrightarrow$ (2). Let I and J be ideals of $C^*(\mathcal{B}, \mathcal{L}, \theta)$ such that $I \not\subseteq J$. Then I and J are gauge-invariant by Theorem 6.3. Therefore it follows from [7, Theorem 10.12] that $\mathcal{H}_I := \{A \in \mathcal{B} : p_A \in I\} \not\subseteq \{A \in \mathcal{B} : p_A \in J\} =: \mathcal{H}_J$. Thus, J/I contains a projection.

(2) $\Longrightarrow$ (3). This follows from [19, Proposition 8.2].

(3) $\Longrightarrow$ (4). This follows from [20, Theorem 1.8].

(4) $\Longrightarrow$ (5). It follows from [10, Proposition 3.2.1] and [4, Proposition 2.6] that the property of having topological dimension zero passes to ideals and quotients. Since Morita equivalent C^*-algebras have homeomorphic primitive ideal spaces (see, for instance, [9, page 156]), topological dimension zero passes to full corners, and thus to corners. Since $M_n(C(\mathbb{T}))$ does not have topological dimension zero, it follows that (4) $\Longrightarrow$ (5).

(5) $\Longrightarrow$ (1). Suppose that $(\mathcal{B}, \mathcal{L}, \theta)$ does not satisfy Condition (K). Then once again $(\mathcal{B}, \mathcal{L}, \theta)$ has a cyclic maximal tail $\mathcal{T}$ and there is a $B \in \mathcal{T}$ such that $p_{[B]}C^*(\mathcal{B}/(\mathcal{B} \setminus \mathcal{T}), \mathcal{L}, \theta)p_{[B]}$ is isomorphic to $M_n(C(\mathbb{T}))$ for some $n \in \mathbb{N}$ by Proposition 6.2. Thus $C^*(\mathcal{B}, \mathcal{L}, \theta)$ has a quotient containing a corner that is isomorphic to $M_n(C(\mathbb{T}))$ for some $n \in \mathbb{N}$, a contradiction. □

COROLLARY 8.2. *Let $(\mathcal{B}, \mathcal{L}, \theta)$ be a locally finite Boolean dynamical system such that $\mathcal{B}$ and $\mathcal{L}$ are countable. If $C^*(\mathcal{B}, \mathcal{L}, \theta)$ has real rank zero or is purely infinite, then $(\mathcal{B}, \mathcal{L}, \theta)$ satisfies Condition (K).*

PROOF. Suppose first that $C^*(\mathcal{B}, \mathcal{L}, \theta)$ has real rank zero. It then follows from [3, Theorem 2.6] that $C^*(\mathcal{B}, \mathcal{L}, \theta)$ has the ideal property, and thus from Theorem 8.1 that $(\mathcal{B}, \mathcal{L}, \theta)$ satisfies Condition (K).

Suppose then that $C^*(\mathcal{B}, \mathcal{L}, \theta)$ is purely infinite and that A is a corner of a quotient of $C^*(\mathcal{B}, \mathcal{L}, \theta)$. Since the property of being purely infinite passes to quotients and corners [16, Propositions 4.3 and 4.17], A is purely infinite and thus cannot be

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isomorphic to $M_n(C(\mathbb{T}))$. It therefore follows from Theorem 8.1 that $(\mathcal{B}, \mathcal{L}, \theta)$ satisfies Condition (K). □

Acknowledgments

The discussion set out in this paper started when the second author visited the first author in the Faroe Islands. The second author is grateful for the hospitality of the first author and his family.

The authors would like to thank the referee for careful reading and valuable comments.

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TOKE MEIER CARLSEN, Department of Sciences and Technology,
University of the Faroe Islands,
Vestara Bryggja 15, FO-100 Tórshavn,
Faroe Islands
e-mail: toke.carlsen@gmail.com

EUN JI KANG, Research Institute of Mathematics,
Seoul National University,
Seoul 08826, Korea
e-mail: kkang33@snu.ac.kr

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