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Relations between arithmetical functions

By C. J. A. EVELYN
33 *Grosvenor Square, London, W.* 1

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In a recent note(1) I proved that if $\mu(n)$ denotes the usual Möbius function, N denotes a fixed positive integer and if

$$M(x) = \sum_{n \leqslant x} \mu(n)$$

then

$$M(x^{1/N}) = \sum_{T \leqslant x} M(x/T),$$

where T runs through all natural numbers $\leqslant x$ which are not divisible by an Nth power. In the present paper I shall establish some further relations of this character and, in particular, I shall prove that if

$$L(x) = \sum_{n \leqslant x} \lambda(n),$$

where

$$\lambda(n) = \sum_{m^N | n} \mu(n/m^N),\dagger$$

then

$$\sum_{T \leqslant x} L(x/T) = 1. \tag{1}$$

Thus, in some respects, $L(x)$ appears more regular than $M(x)$, the sum over $L(x/T)$ being multiplicative, whereas $M(x^{1/N})$ is not.

To prove (1) we note first that

$$\lambda(m) = \sum_{n | m} \mu(m/n)\, \nu(n),$$

where $\nu(n) = 1$ if n is an Nth power and 0 otherwise, and so

$$\sum_{m | n} \lambda(m) = \nu(n).$$

Hence we obtain

$$\begin{aligned} \sum_{n \leqslant x} L(x/n) &= \sum_{n \leqslant x} \sum_{m \leqslant x/n} \lambda(m) \\ &= \sum_{l \leqslant x} \sum_{m | l} \lambda(m) \\ &= \sum_{l \leqslant x} \nu(l) \\ &= [x^{1/N}]. \end{aligned}$$

It follows that

$$\begin{aligned} \sum_{T \leqslant x} L(x/T) &= \sum_{n \leqslant x} L(x/n) \sum_{m^N | n} \mu(m) \\ &= \sum_{m \leqslant x^{1/N}} \mu(m) \sum_{n \leqslant x/m^N} L\left(\frac{x}{nm^N}\right) \\ &= \sum_{m \leqslant x^{1/N}} \mu(m)\, [x^{1/N}/m], \end{aligned}$$

† The function $\lambda(n)$ can be defined alternatively by the equation $\zeta(Ns)/\zeta(s) = \Sigma\lambda(n)n^{-s}$ where $\zeta(s) = \Sigma n^{-s}$; it is often written as $\lambda_N(n)$ and the suffix omitted only when $N = 2$, but the distinction is of no special significance here.

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and the latter sum is 1, as is easily verified from the familiar inversion formulae

$$F(z) = \sum_{n \leqslant z} G(z/n), \quad G(z) = \sum_{n \leqslant z} \mu(n) F(z/n).$$

This establishes the desired relation (1). In particular we note that (1) holds if T runs through all the square-free numbers not exceeding x.

We now consider the sums

$$R(x) = \sum_{n \leqslant x} \mu(n) \log (x/n),$$

$$S(x) = \sum_{n \leqslant x} \lambda(n) \log (x/n),$$

familiar from elementary prime number theory: clearly, by the above inversion formulae, we have

$$\log x = \sum_{n \leqslant x} R(x/n)$$

and so

$$\begin{aligned} \sum_{T \leqslant x} R(x/T) &= \sum_{n \leqslant x} R(x/n) \sum_{m^N \mid n} \mu(m) \\ &= \sum_{m \leqslant x^{1/N}} \mu(m) \sum_{n \leqslant x/m^N} R\left(\frac{x}{nm^N}\right) \\ &= \sum_{m \leqslant x^{1/N}} \mu(m) \log (x/m^N) \\ &= NR(x^{1/N}). \end{aligned}$$

This represents the analogue, for the function R, of the relation established in (1) for the function M. We proceed now to prove a corresponding result for S.

We have

$$\begin{aligned} S(x) &= \sum_{n \leqslant x} \lambda(n) \log (x/n) \\ &= \sum_{n \leqslant x} \log (x/n) \sum_{m^N \mid n} \mu(n/m^N) \\ &= \sum_{lm^N \leqslant x} \mu(l) \log \left(\frac{x}{lm^N}\right) \\ &= \sum_{m \leqslant x^{1/N}} R(x/m^N). \end{aligned}$$

Thus noting that every integer $\leqslant x$ can be written uniquely in the form Tm^N, where T is divisible by no Nth power, we obtain

$$\begin{aligned} \sum_{T \leqslant x} S(x/T) &= \sum_{T \leqslant x} \sum_{m \leqslant (x/T)^{1/N}} R\left(\frac{x}{Tm^N}\right) \\ &= \sum_{n \leqslant x} R(x/n) \\ &= \log x. \end{aligned}$$

Finally we observe that

$$\begin{aligned} \sum_{T \leqslant x} S(x/T) &= \sum_{m \leqslant x^{1/N}} \sum_{T \leqslant x/m^N} R\left(\frac{x}{Tm^N}\right) \\ &= N \sum_{m \leqslant x^{1/N}} R(x^{1/N}/m), \end{aligned}$$

and so

$$\sum_{m \leqslant x^{1/N}} R(x^{1/N}/m) = (1/N) \log x.$$

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More generally, the method of this paper can be used to establish the following results:

If p is any real or complex number and if

$$R_p(x) = \sum_{n \leqslant x} \mu(n) (\log (x/n))^p,$$

$$S_p(x) = \sum_{n \leqslant x} \lambda(n) (\log (x/n))^p,$$

then

$$\sum_{n \leqslant x} R_p(x/n) = (\log x)^p,$$

$$\sum_{T \leqslant x} S_p(x/T) = (\log x)^p.$$

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REFERENCE

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