Bull. Aust. Math. Soc. (First published online 2026), page 1 of 6*
doi:10.1017/S0004972726101737
*Provisional—final page numbers to be inserted when paper edition is published

A NOTE ON TOPOLOGICAL ASPECTS OF TRANSCENDENTAL SELF-POWERS

SIDNEY A. MORRIS ✉ and MARCELO O. RIBEIRO

(Received 2 June 2026 UTC; accepted 18 June 2026 UTC)

Abstract

We study the self-power map $x \mapsto x^x$ on $(0, \infty)$ from a topological perspective. In particular, we show that the set of $x > 0$, for which x^x is a Liouville number, is topologically large, but metrically small.

2020 *Mathematics subject classification*: primary 11J81.

Keywords and phrases: Liouville numbers, self-powers, transcendence, G_δ sets.

1. Introduction

A real number is called *transcendental* if it is not a root of any nonzero polynomial with integer coefficients. Since the pioneering work of Liouville in 1844, transcendental numbers have occupied a central position in number theory. Liouville not only constructed the first explicit transcendental numbers, but also introduced a remarkable class of numbers that admit exceptionally accurate rational approximations. These numbers, now known as *Liouville numbers*, provided the first examples of transcendental real numbers and remain among the most important objects in transcendence theory.

The subject developed dramatically during the twentieth century. Among the most celebrated achievements are the theorems of Gel'fond [3] and Schneider [6], which independently solved Hilbert's seventh problem. They proved that if α and β are algebraic numbers with $\alpha \neq 0, 1$ and β irrational, then α^β is transcendental. These results established deep connections between algebraic and transcendental numbers and demonstrated the power of analytic methods in number theory. Since then, transcendence theory has expanded into a rich discipline concerned with algebraic independence, Diophantine approximation, special values of analytic functions and the arithmetic properties of exponential expressions.

© The Author(s), 2026. Published by Cambridge University Press on behalf of Australian Mathematical Publishing Association Inc. This is an Open Access article, distributed under the terms of the Creative Commons Attribution licence (https://creativecommons.org/licenses/by/4.0/), which permits unrestricted re-use, distribution, and reproduction in any medium, provided the original work is properly cited.

Downloaded from https://www.cambridge.org/core. UNIVERSIDAD AUTONOMA METROPOLITANA, on 17 Nov 2026 at 17:58:40, subject to the Cambridge Core terms of use, available at https://www.cambridge.org/core/terms. https://doi.org/10.1017/S0004972726101737

A real irrational number x is called a *Liouville number* if, for every $N \in \mathbb{N}$, there exist integers p and q with $q \geq 2$ such that

$$\left| x - \frac{p}{q} \right| < q^{-N}.$$

We denote by $\mathcal{L}$ the set of all Liouville numbers.

It is classical that every Liouville number is transcendental. From the viewpoint of measure theory, however, the set $\mathcal{L}$ is extremely small: it has Lebesgue measure zero. From the viewpoint of topology, the situation is very different. A standard application of Baire's theorem shows that $\mathcal{L}$ is a dense G_δ subset of $\mathbb{R}$. Consequently, $\mathcal{L}$ is topologically large despite being measure-theoretically negligible.

This contrast between measure and category has become a recurring theme in modern analysis and number theory. A classical account of such phenomena can be found in Oxtoby's monograph *Measure and Category* [5], which systematically develops the striking differences between measure-theoretic and category-theoretic notions of largeness. In transcendence theory, Baire category methods have repeatedly proved useful for establishing the abundance of exceptional classes of transcendental numbers. A particularly attractive example is provided by Steuding [7], who demonstrated how functional-analytic and Baire category techniques can be used to obtain results concerning Liouville numbers and related transcendence phenomena. Such methods complement more classical arithmetic approaches and reveal that many transcendence questions possess a natural topological dimension.

The arithmetic structure of $\mathcal{L}$ is surprisingly rich. In a celebrated theorem, Erdős [2] proved that every real number can be expressed as a sum and, if nonzero, as a product of two Liouville numbers. Later, Burger [1] extended these ideas to a broad class of two-variable functions, showing in particular that every positive real number $r \neq 1$ can be written in the form $r = \sigma^\tau$ for suitable Liouville numbers σ and τ. These results illustrate the remarkable flexibility of Liouville numbers and suggest that transcendental values frequently arise in exponential constructions.

Despite this abundance of representations, the behaviour of the self-power map $f(x) = x^x$ remains poorly understood. In particular, it is not known whether x^x must be transcendental whenever x is a Liouville number. A partial answer was obtained by Marques and Ribeiro [4], who proved transcendence for a dense G_δ subclass of Liouville numbers, namely the class of ε-strong Liouville numbers.

The purpose of the present paper is to investigate the self-power map from a topological perspective. Rather than studying individual Liouville numbers, we examine the size and structure of the sets

$$\mathcal{T} = \{x > 0 : x^x \text{ is transcendental}\} \quad \text{and} \quad \mathcal{G} = \{x > 0 : x^x \in \mathcal{L}\}.$$

Our main results show that $\mathcal{T}$ is comeagre and has full Lebesgue measure in $(0, \infty)$, while $\mathcal{G}$ is a dense G_δ subset of $(0, \infty)$ and has Lebesgue measure zero. Thus, the self-power map exhibits a striking category–measure dichotomy: transcendental values occur on a set that is both topologically and measure-theoretically large, whereas

Downloaded from https://www.cambridge.org/core. UNIVERSIDAD AUTONOMA METROPOLITANA, on 17 Nov 2026 at 17:58:40, subject to the Cambridge Core terms of use, available at https://www.cambridge.org/core/terms. https://doi.org/10.1017/S0004972726101737

Liouville values occur on a set that is topologically large but measure-theoretically negligible.

Throughout the paper, we write $\overline{\mathbb{Q}} \cap \mathbb{R}$ for the set of real algebraic numbers. This avoids possible confusion with the standard notation $\overline{\mathbb{Q}}$, which usually denotes the full algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$.

2. Results and proofs

The following result, although classical, is included for completeness.

LEMMA 2.1. *The set $\mathcal{L}$ of Liouville numbers is a dense G_δ subset of $\mathbb{R}$. Moreover, $\mathcal{L}$ has Lebesgue measure zero.*

PROOF. For $N, Q \in \mathbb{N}$, with $Q \geq 2$, define

$$U_{N,Q} = \bigcup_{q \geq Q} \bigcup_{p \in \mathbb{Z}} \left(\frac{p}{q} - \frac{1}{q^N}, \frac{p}{q} + \frac{1}{q^N} \right).$$

Each $U_{N,Q}$ is open. It is also dense in $\mathbb{R}$, since it contains all rational numbers p/q with $q \geq Q$, and these are dense in $\mathbb{R}$.

We claim that

$$\mathcal{L} = \left(\bigcap_{N=1}^{\infty} \bigcap_{Q=2}^{\infty} U_{N,Q} \right) \cap (\mathbb{R} \setminus \mathbb{Q}).$$

Indeed, the inclusion from right to left is immediate. Conversely, if $x \in \mathcal{L}$, then x is irrational. Given N and Q, choose $M > N$. By the Liouville property, there exists p/q such that

$$\left| x - \frac{p}{q} \right| < q^{-M}.$$

For fixed Q, the denominators q occurring in such approximations cannot remain bounded as $M \to \infty$, since x is irrational. Hence, for some such approximation, $q \geq Q$, and then

$$\left| x - \frac{p}{q} \right| < q^{-M} \leq q^{-N}.$$

Thus, $x \in U_{N,Q}$. The first factor is a countable intersection of open dense sets. Also, $\mathbb{R} \setminus \mathbb{Q}$ is a dense G_δ, since

$$\mathbb{R} \setminus \mathbb{Q} = \bigcap_{r \in \mathbb{Q}} (\mathbb{R} \setminus \{r\}).$$

Therefore, $\mathcal{L}$ is a dense G_δ subset of $\mathbb{R}$.

Finally, the classical Liouville theorem implies that every Liouville number is transcendental. Since the set of algebraic real numbers is countable, this does not by itself give measure zero. The standard Borel–Cantelli argument gives measure zero:

Downloaded from https://www.cambridge.org/core. UNIVERSIDAD AUTONOMA METROPOLITANA, on 17 Nov 2026 at 17:58:40, subject to the Cambridge Core terms of use, available at https://www.cambridge.org/core/terms. https://doi.org/10.1017/S0004972726101737

for fixed $N > 2$, the total length of the intervals with $q \geq Q$ over any bounded interval is bounded by a constant multiple of

$$\sum_{q \geq Q} q \cdot q^{-N} = \sum_{q \geq Q} q^{1-N},$$

which tends to 0 as $Q \to \infty$. Hence, the set of real numbers satisfying the Liouville inequalities for this fixed $N > 2$ has measure zero on every bounded interval. Since $\mathcal{L}$ is contained in that set, $\mathcal{L}$ has Lebesgue measure zero. □

THEOREM 2.2. *The set*

$$\mathcal{T} = \{x > 0 : x^x \text{ is transcendental}\}$$

is comeagre in $(0, \infty)$ *(in particular, it is a dense* G_δ*) and it has full Lebesgue measure in* $(0, \infty)$.

PROOF. Let $f\colon (0, \infty) \to (0, \infty)$ be the function $f(x) = x^x = \exp(x \log x)$. The derivative of f is $f'(x) = x^x(\log x + 1)$, and thus f is strictly decreasing on $(0, e^{-1})$ and strictly increasing on (e^{-1}, ∞), with a unique global minimum $f(e^{-1}) = e^{-1/e}$. Consequently, for each $\alpha \in \mathbb{R}$, the level set $F_\alpha = f^{-1}(\{\alpha\}) \subset (0, \infty)$ has cardinality $|F_\alpha| \leq 2$.

Consider the set

$$A := \bigcup_{\alpha \in \overline{\mathbb{Q}} \cap \mathbb{R}} F_\alpha = \{x > 0 : x^x \text{ is algebraic}\}.$$

Since each F_α is finite and $\overline{\mathbb{Q}} \cap \mathbb{R}$ is countable, A is countable. Hence, A is meagre and has Lebesgue measure 0. Therefore, its complement $\mathcal{T} = (0, \infty) \setminus A$ is comeagre (equivalently, a dense G_δ) and has full Lebesgue measure. □

COROLLARY 2.3. *The set*

$$\{x > 0 : x \in \mathcal{L},\ x^x \text{ is transcendental}\}$$

is a dense G_δ *subset of* $(0, \infty)$.

PROOF. By Theorem 2.2, $\mathcal{T}$ is a dense G_δ subset of $(0, \infty)$. By Lemma 2.1, $\mathcal{L} \cap (0, \infty)$ is also a dense G_δ. Since the intersection of two dense G_δ subsets of a Baire space is again a dense G_δ, the result follows. □

THEOREM 2.4. *The set*

$$\mathcal{G} = \{x > 0 : x^x \in \mathcal{L}\} = f^{-1}(\mathcal{L})$$

is a dense G_δ *subset of* $(0, \infty)$ *and has Lebesgue measure zero.*

PROOF. Let $f : (0, \infty) \to (0, \infty)$ be given by $f(x) = x^x$. The function f is continuous, and $f'(x) = x^x(\log x + 1)$. Hence, $f'(x) = 0$ only at $x = e^{-1}$. Since $\mathcal{L}$ is a G_δ subset of $\mathbb{R}$ and f is continuous, the preimage $\mathcal{G} = f^{-1}(\mathcal{L})$ is a G_δ subset of $(0, \infty)$.

We next prove density. Let $I \subset (0, \infty)$ be a nonempty open interval. Since f is continuous and nonconstant on I, the image $f(I)$ contains a nonempty open interval J.

Downloaded from https://www.cambridge.org/core. UNIVERSIDAD AUTONOMA METROPOLITANA, on 17 Nov 2026 at 17:58:40, subject to the Cambridge Core terms of use, available at https://www.cambridge.org/core/terms. https://doi.org/10.1017/S0004972726101737

Since $\mathcal{L}$ is dense in $\mathbb{R}$, we have $J \cap \mathcal{L} \neq \emptyset$. Choose $y \in J \cap \mathcal{L}$. Since $J \subset f(I)$, there exists $x \in I$ such that $f(x) = y$. Therefore $x \in I \cap \mathcal{G}$. Hence, $I \cap \mathcal{G} \neq \emptyset$, so $\mathcal{G}$ is dense in $(0, \infty)$.

It remains to prove that $\mathcal{G}$ has Lebesgue measure zero. Let $c = e^{-1}$. Since $f'(x) \neq 0$ for every $x \neq c$, every compact interval $K \subset (0, \infty) \setminus \{c\}$ has the property that $|f'(x)| \geq m_K > 0$ for all $x \in K$. Since K is an interval not containing c, the derivative f' has constant sign on K. Moreover, by the mean value theorem,

$$|f(x) - f(y)| \geq m_K |x - y| \quad (x, y \in K).$$

Therefore, $f|_K$ is injective and its inverse branch is Lipschitz. In particular, the inverse branch $(f|_K)^{-1}$ is Lipschitz on $f(K)$.

Since $\mathcal{L}$ has Lebesgue measure zero, the set $\mathcal{L} \cap f(K)$ has measure zero. Since $f|_K$ is injective and its inverse branch is Lipschitz, the image of every null set under $(f|_K)^{-1}$ is a null set. Since $\mathcal{L} \cap f(K)$ has measure zero,

$$(f|_K)^{-1}(\mathcal{L} \cap f(K)) = K \cap f^{-1}(\mathcal{L})$$

also has measure zero.

Now, $(0, \infty) \setminus \{c\}$ is a countable union of compact intervals of the above type. For example,

$$(0, \infty) \setminus \{c\} = \bigcup_{n=1}^{\infty} \left[\frac{1}{n}, c - \frac{1}{n}\right] \cup \bigcup_{n=1}^{\infty} \left[c + \frac{1}{n}, n\right],$$

where empty intervals are ignored. Therefore, $f^{-1}(\mathcal{L}) \setminus \{c\}$ has measure zero. Adding the single point c does not change the measure, and hence

$$\mu(\mathcal{G}) = \mu(f^{-1}(\mathcal{L})) = 0.$$

Thus, $\mathcal{G}$ is a dense G_δ subset of $(0, \infty)$ and has Lebesgue measure zero. □

COROLLARY 2.5. *The set*

$$\{x > 0 : x \in \mathcal{L},\ x^x \in \mathcal{L}\} = \mathcal{G} \cap \mathcal{L}$$

is a dense G_δ subset of $(0, \infty)$.

PROOF. By Theorem 2.4, $\mathcal{G}$ is a dense G_δ subset of $(0, \infty)$. By Lemma 2.1, $\mathcal{L} \cap (0, \infty)$ is also a dense G_δ subset of $(0, \infty)$. Hence, $\mathcal{G} \cap \mathcal{L}$ is the intersection of two dense G_δ subsets of the Baire space $(0, \infty)$ and is therefore again a dense G_δ. □

Acknowledgement

The authors thank the referee for valuable comments that improved the presentation of the paper.

Downloaded from https://www.cambridge.org/core. UNIVERSIDAD AUTONOMA METROPOLITANA, on 17 Nov 2026 at 17:58:40, subject to the Cambridge Core terms of use, available at https://www.cambridge.org/core/terms. https://doi.org/10.1017/S0004972726101737

References

[1] E. Burger, 'On Liouville decompositions in local fields', *Proc. Amer. Math. Soc.* **124** (1996), 3305–3310.
[2] P. Erdős, 'Representations of real numbers as sums and products of Liouville numbers', *Michigan Math. J.* **9** (1962), 59–60.
[3] A. O. Gel'fond, *Transcendental and Algebraic Numbers* (Dover Publications Inc., New York, 1960).
[4] D. Marques and M. O. Ribeiro, 'On the transcendence of some powers related to U-numbers', *Bull. Braz. Math. Soc. (N.S.)* **54** (2023), Article no. 25.
[5] J. C. Oxtoby, *Measure and Category*, 2nd edn (Springer, New York, 1980).
[6] T. Schneider, *Einführung in die transzendenten Zahlen*, Grundlehren der mathematischen Wissenschaften, 81 (Springer-Verlag, Berlin, 1957).
[7] J. Steuding, 'On Liouville numbers: yet another application of functional analysis to number theory', in: *From Arithmetic to Zeta Functions* (eds. J. Sander, J. Steuding and R. Steuding) (Springer, Cham, 2016), 279–288.

SIDNEY A. MORRIS, School of Engineering, IT and Physical Sciences,
Federation University Australia, PO Box 663, Ballarat, Victoria 3353, Australia
and
Department of Mathematical and Physical Sciences,
La Trobe University, Melbourne, Victoria 3086, Australia
e-mail: morris.sidney@gmail.com

MARCELO O. RIBEIRO, São Bernardo Science Center,
Federal University of Maranhão, São Bernardo, Brazil
e-mail: marcelo.or@ufma.br

Downloaded from https://www.cambridge.org/core. UNIVERSIDAD AUTONOMA METROPOLITANA, on 17 Nov 2026 at 17:58:40, subject to the Cambridge Core terms of use, available at https://www.cambridge.org/core/terms. https://doi.org/10.1017/S0004972726101737