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On Measures of Average Degree for Lattices

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The usual definition of average degree for a non-regular lattice has the disadvantage that it takes the same value for many lattices with clearly different connectivity. We introduce an alternative definition of average degree, which better separates different lattices.

These measures are compared on a class of lattices and are analysed using a Markov chain describing a random walk on the lattice. Using the new measure, we conjecture the order of both the critical probabilities for bond percolation and the connective constants for self-avoiding walks on these lattices.

1. Introduction

By a lattice we will understand a connected (doubly) periodic planar graph with a finite number, K, of vertex classes, with degrees $q_1, \ldots, q_K$. Two vertices are equivalent if they can be mapped onto each other through symmetry operations (translation, rotation, reflection) that conserve the lattice. The degree of a vertex is the number of edges from the vertex. We restrict our interest to non-directed graphs.

If $K = 1$, the lattice is regular and it is natural to define the degree of the lattice as $q = q_1$.

For lattices with $K > 1$, the *average degree* of the lattice is usually defined as a weighted average

$$\bar{q} = \sum_{i=1}^{K} w_i \cdot q_i, \tag{1.1}$$

where the weights $w = (w_1, \ldots w_K)$ are the proportions in which the vertex classes appear.

This definition of average degree, or *coordination number* as it is called in the physics literature, has some disadvantages, which are discussed in Section 2, and we will introduce

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an alternative definition of average degree, $\tilde{q}$, in Section 4. We show that $\tilde{q} \geqslant \bar{q}$, and give several examples of pairs of graphs (G, H), where $\bar{q}(G) = \bar{q}(H)$, but $\tilde{q}(G) \neq \tilde{q}(H)$. We have found no examples where the opposite holds.

A comparison of these measures of degree is performed on a class of lattices, the ALB lattices, which are defined in Section 3.

Much of the analysis is based on a study of a Markov chain describing a simple random walk on the lattice, with the vertex classes as states.

We conclude with some conjectures, including predicted order of the critical probabilities for bond percolation and the connective constants for self-avoiding walks on these lattices. We also discuss the possibility of extending these conjectures to a larger class of lattices.

2. Average degree

The measure $\bar{q}$ of (1.1) is frequently used by physicists attempting to give universal formulas for critical probabilities in percolation; for a review, see Wierman [6]. These formulas are usually based on the dimension and the average degree of the lattice. As our lattices all have dimension 2, these formulas will predict the same critical probability for all graphs with the same average degree.

To compute $\bar{q}$, we need to know the weights $w_1, \ldots, w_K$. These can be expressed by means of the stationary distribution of the finite Markov chain, $X = \{X_n, n \geqslant 0\}$, describing a simple symmetric random walk on the lattice, with the vertex classes as states. The random walk is symmetric in the sense that in each step, with equal probabilities, it moves to one of the neighbouring vertices.

Let $P_{K\times K}$ be the transition matrix of this finite Markov chain. As the graph is connected, the Markov chain is irreducible, and has a unique stationary distribution $\pi = (\pi_1, \ldots, \pi_K)$, where all $\pi_i > 0$. Further, let Q denote a $K \times K$ diagonal matrix with elements $q_1, \ldots, q_K$.

Theorem 2.1. *$w = C \cdot \pi \cdot Q^{-1}$, where the constant C is chosen so that $\sum w_i = 1$.*

Proof. The proportion of edges from vertices of vertex class i is $C' \cdot w_i \cdot q_i$, where $C' = 1/\sum w_i \cdot q_i$. This proportion must equal the proportion of edges to vertices of vertex class i, *i.e.*,

$$C' \cdot w_i \cdot q_i = \sum_{j=1}^{K} C' \cdot w_j \cdot q_j \cdot P_{ji},$$

or

$$w \cdot Q = w \cdot Q \cdot P,$$

so that, as π is the unique stationary distribution given by $\pi = \pi \cdot P$ with $\sum \pi_i = 1$,

$$w \cdot Q = C \cdot \pi,$$

where $C = 1/\sum(\pi_i/q_i)$, which proves the theorem. □

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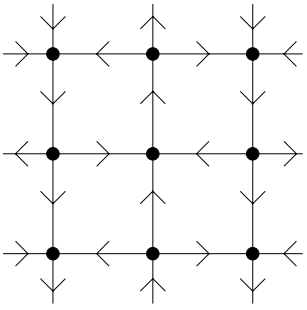


Figure 1. An unbalanced directed graph.

Corollary 2.2. *The average degree, $\bar{q}$, satisfies*

$$\bar{q} = \frac{1}{\sum_{i=1}^{K} \frac{\pi_i}{q_i}}.$$

Proof. By Theorem 2.1,

$$w_i = \frac{\pi_i/q_i}{\sum(\pi_j/q_j)},$$

so that

$$\bar{q} = \sum_i w_i \cdot q_i = \frac{\sum \pi_i}{\sum(\pi_j/q_j)} = \frac{1}{\sum(\pi_j/q_j)}. \qquad \square$$

Remark. $\bar{q}$ is the harmonic mean of $q_1, \ldots, q_K$ with weights π.

Remark. Theorem 2.1 also holds for balanced directed lattices, *i.e.*, where all the vertices have the same number of outgoing and incoming edges, although the above proof is not applicable.

However, it need not hold for unbalanced directed lattices, as can be seen from the following counterexample; see Figure 1. This lattice has two vertex classes, one with 3 outgoing edges and 1 incoming, and one with 1 outgoing edge and 3 incoming. These are equally common, so that $w = (\frac{1}{2}, \frac{1}{2})$. The lattice is bipartite so that

$$P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and $\pi = (\frac{1}{2}, \frac{1}{2})$. As

$$Q = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix},$$

we obviously have $w \neq C \cdot \pi \cdot Q^{-1}$.

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An easy consequence of Euler's formula for planar graphs is that for any periodic planar graph G,

$$\frac{2}{\bar{q}(G)} + \frac{2}{\bar{q}(D(G))} = 1, \tag{2.1}$$

where $D(G)$ denotes the dual of G; see for example Wierman [5].

One consequence of (2.1) is that $\bar{q} = 6$ for all fully triangulated lattices, as their duals all have $\bar{q} = 3$. Further, $\bar{q} = 4$ for all lattices where all faces have four sides and $\bar{q} = 10/3$ for all lattices where all faces have five sides.

This indicates that the average degree, $\bar{q}$, is not a very sharp characteristic of the lattice.

In Section 4, we will introduce an alternative average degree, $\tilde{q}$, which is a sharper graph characteristic.

We will compare $\bar{q}$ and $\tilde{q}$ on the class of ALB lattices, defined in Section 3.

3. ALB lattices

A *regular tiling* is a tiling of the plane which consists entirely of regular polygons. A vertex-transitive graph of such a regular tiling is called an *Archimedean lattice*. There are 11 such graphs, shown in Figure 2. They are denoted according to a notation given in Grünbaum and Shephard [2].

The Archimedean lattices are, by definition, all regular. The square lattice (4^4) is self-dual; the triangular (3^6) and hexagonal (6^3) lattices are each other's duals. The duals of the eight remaining Archimedean lattices constitute the class of *Laves lattices*, which are all non-regular. They are shown in Figure 3.

The Laves lattices serve well as test graphs when studying average degree. To get a slightly richer class, we will also include the bow-tie lattice and its dual, see Figure 4.

The class of Archimedean lattices, Laves lattices, the bow-tie lattice and its dual will be called the *ALB lattices* and will be used to compare different definitions of average degree.

4. Alternative definition of average degree

An alternative definition of average degree can be obtained by studying the number of walks on the graph. A *walk* of length n is a sequence $(v_0, e_1, v_1, \ldots, e_n, v_n)$ of vertices and edges such that the edge e_i connects the vertices v_{i-1} and v_i, for $i = 1, \ldots, n$.

Let $w_i(n)$ denote the number of walks of length n starting in a vertex of class i, $i = 1, \ldots, K$.

For a regular graph with degree q we obviously have

$$w(n) = w_1(n) = q^n,$$

so that

$$q = w(n)^{1/n}$$

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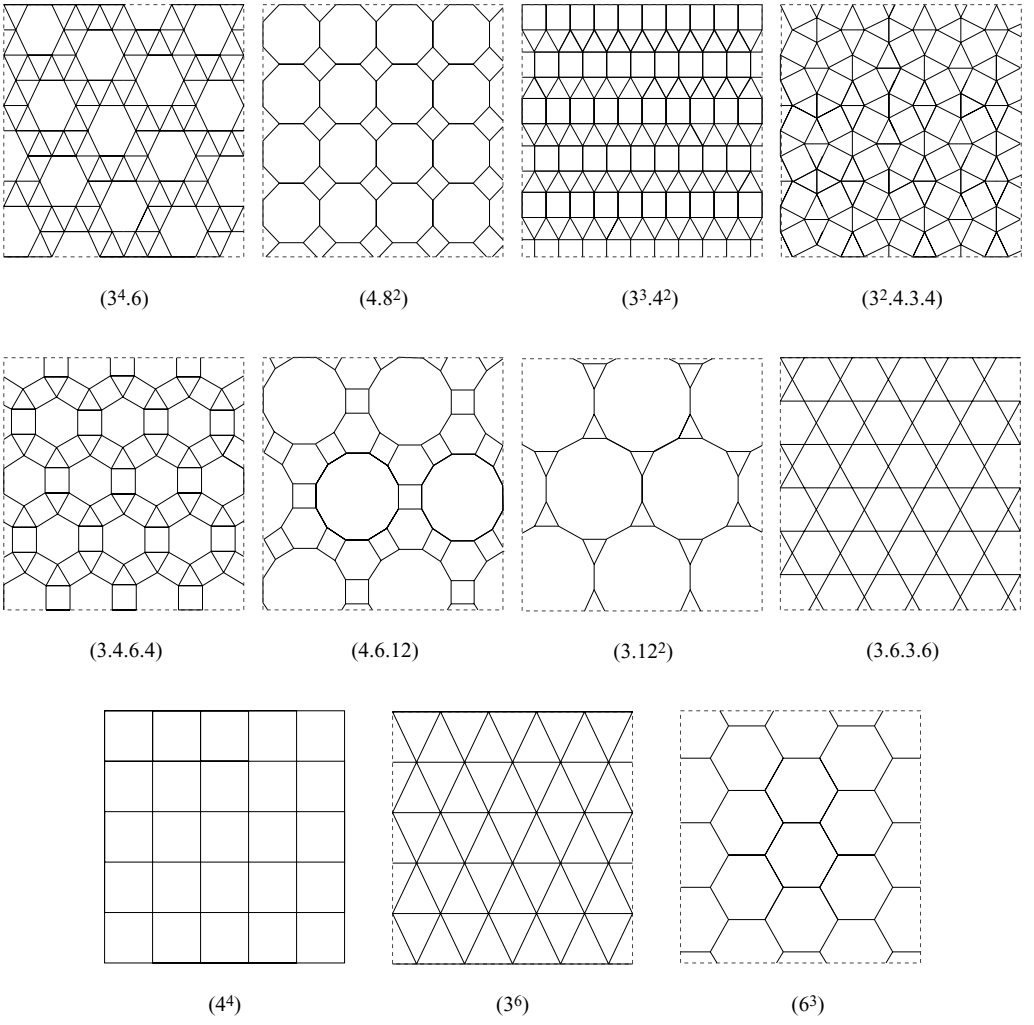


Figure 2. The Archimedean lattices.

for all n, and in particular

$$q = \lim_{n\to\infty} w(n)^{1/n}.$$

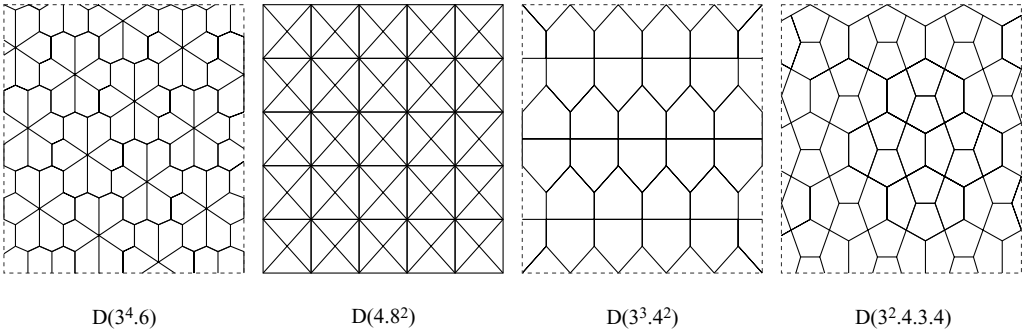
For a general lattice, define

$$\tilde{q}_i = \lim_{n\to\infty} w_i(n)^{1/n}, \quad i = 1,\ldots,K.$$

Theorem 4.1. *$\tilde{q}_i$ is independent of i for $i = 1,\ldots,K$.*

Proof. Follows immediately from the fact that the Markov chain $\{X_n\}$ is irreducible. □

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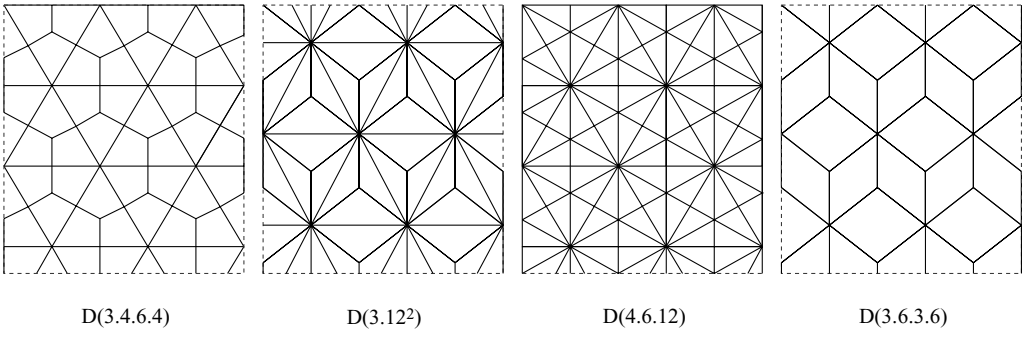


Figure 3. The Laves lattices.

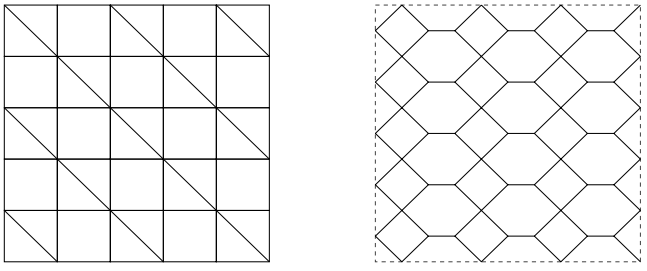


Figure 4. The bow-tie lattice and its dual.

Theorem 4.1 allows us to define a new average degree, $\tilde{q}$, as the common value of $\tilde{q}_1, \ldots, \tilde{q}_K$. The value of $\tilde{q}$ can be computed as the largest eigenvalue of the matrix $A = Q \cdot P$, where Q is the diagonal matrix with values $q_1, \ldots, q_K$ and P is the transition matrix of the Markov chain X, as defined in Section 2. Let $\lambda_1(A)$ denote the largest eigenvalue of the matrix A.

Theorem 4.2.

$$\tilde{q} = \lambda_1(A) = \lambda_1(Q \cdot P).$$

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Proof. We can write

$$w_1(n) = (1, 0, \ldots, 0) \cdot A^n \cdot \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix},$$

which proves the theorem. □

Example. There are four fully triangulated ALB lattices, (3^6), $D(4, 8^2)$, $D(4, 6, 12)$, and $D(3, 12^2)$, who all have $\bar{q} = 6$. As the triangular lattice, (3^6), is regular, obviously, $\tilde{q}(3^6) = 6$.

For $D(4, 8^2)$ we have

$$Q = \begin{pmatrix} 8 & 0 \\ 0 & 4 \end{pmatrix}, \quad P = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ 1 & 0 \end{pmatrix}, \quad \text{so that } A = Q \cdot P = \begin{pmatrix} 4 & 4 \\ 4 & 0 \end{pmatrix},$$

with $\tilde{q} = \lambda_1(A) = 2 + \sqrt{20} \approx 6.472$.

For $D(4, 6, 12)$ we have

$$Q = \begin{pmatrix} 12 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 4 \end{pmatrix}, \quad P = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}, \quad \text{so that } A = \begin{pmatrix} 0 & 6 & 6 \\ 3 & 0 & 3 \\ 2 & 2 & 0 \end{pmatrix},$$

with $\tilde{q} = \lambda_1(A) = 4 \cdot \sqrt{3} \cdot \cos(\pi/18) \approx 6.823$.

Finally, for $D(3, 12^2)$ we have

$$Q = \begin{pmatrix} 12 & 0 \\ 0 & 3 \end{pmatrix}, \quad P = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ 1 & 0 \end{pmatrix}, \quad \text{so that } A = \begin{pmatrix} 6 & 6 \\ 3 & 0 \end{pmatrix},$$

with $\tilde{q} = \lambda_1(A) = 3 + \sqrt{27} \approx 8.196$.

Thus, $\tilde{q}$ separates these four lattices.

Example. There are four ALB lattices with $\bar{q} = 10/3$, namely $D(\text{Bow-tie})$, $D(3^2, 4, 3, 4)$, $D(3^3, 4^2)$, and $D(3^4, 6)$.

Both $D(\text{Bow-tie})$ and $D(3^2, 4, 3, 4)$ have

$$Q = \begin{pmatrix} 4 & 0 \\ 0 & 3 \end{pmatrix}, \quad P = \begin{pmatrix} 0 & 1 \\ \frac{2}{3} & \frac{1}{3} \end{pmatrix}, \quad \text{so that } A = \begin{pmatrix} 0 & 4 \\ 2 & 1 \end{pmatrix},$$

with $\tilde{q} = \lambda_1(A) = \frac{1+\sqrt{33}}{2} \approx 3.372$.

For $D(3^3, 4^2)$ we have

$$Q = \begin{pmatrix} 4 & 0 \\ 0 & 3 \end{pmatrix}, \quad P = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{3} & \frac{2}{3} \end{pmatrix}, \quad \text{so that } A = \begin{pmatrix} 2 & 2 \\ 1 & 2 \end{pmatrix},$$

with $\tilde{q} = \lambda_1(A) = 2 + \sqrt{2} \approx 3.414$.

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Finally, for $D(3^4, 6)$ we have

$$Q = \begin{pmatrix} 6 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}, \quad P = \begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 1 & 0 \end{pmatrix}, \quad \text{so that } A = \begin{pmatrix} 0 & 6 & 0 \\ 1 & 1 & 1 \\ 0 & 3 & 0 \end{pmatrix},$$

with $\tilde{q} = \lambda_1(A) = \frac{1+\sqrt{37}}{2} \approx 3.541$.

Thus $\tilde{q}$ separates these lattices better than $\bar{q}$, although not completely.

Example. Among the four ALB lattices with $\bar{q} = 5$, three of these, $(3^4, 6)$, $(3^3, 4^2)$, and $(3^2, 4, 3, 4)$, are regular with $q = \bar{q} = \tilde{q} = 5$.

The fourth, Bow-tie, has

$$Q = \begin{pmatrix} 6 & 0 \\ 0 & 4 \end{pmatrix}, \quad P = \begin{pmatrix} \frac{1}{3} & \frac{2}{3} \\ 1 & 0 \end{pmatrix}, \quad \text{so that } A = \begin{pmatrix} 2 & 4 \\ 4 & 0 \end{pmatrix},$$

with $\tilde{q} = \lambda_1(A) = 1 + \sqrt{17} \approx 5.123 > 5 = \bar{q}$.

Example. There are five ALB lattices with $\bar{q} = 4$. Three of these, (4^4), $(3, 4, 6, 4)$, and $(3, 6, 3, 6)$, are regular with $q = \bar{q} = \tilde{q} = 4$.

For $D(3, 4, 6, 4)$ we have

$$Q = \begin{pmatrix} 6 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 3 \end{pmatrix}, \quad P = \begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \end{pmatrix}, \quad \text{so that } A = \begin{pmatrix} 0 & 6 & 0 \\ 2 & 0 & 2 \\ 0 & 3 & 0 \end{pmatrix},$$

with $\tilde{q} = \lambda_1(A) = \sqrt{18} \approx 4.243$, and for $D(3, 6, 3, 6)$ we have

$$Q = \begin{pmatrix} 6 & 0 \\ 0 & 3 \end{pmatrix}, \quad P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \text{so that } A = \begin{pmatrix} 0 & 6 \\ 3 & 0 \end{pmatrix},$$

with $\tilde{q} = \lambda_1(A) = \sqrt{18} \approx 4.243$, so that the two non-regular lattices have a higher average degree, $\tilde{q}$, than the regular ones.

Example. The remaining four ALB lattices, $(3, 12^2)$, $(4, 6, 12)$, $(4, 8^2)$, and (6^3), are all regular with $q = \bar{q} = \tilde{q} = 3$.

From the examples we note that $\tilde{q} \geqslant \bar{q}$ for all ALB lattices. This inequality holds for general lattices.

Theorem 4.3. *For all lattices*

$$\tilde{q} \geqslant \prod_{i=1}^{K} q_i^{\pi_i} \geqslant \bar{q},$$

with strict inequality in the right inequality unless all q_i, $i = 1, \ldots, K$, are equal.

Proof. Recall that π is the unique stationary distribution of the Markov chain X, and that by Corollary 2.2, $\bar{q}$ is the harmonic mean of $q_1, \ldots, q_K$ with weights π. Thus the

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right inequality follows from the relation between geometric and harmonic means, with equality if and only if all q_i, $i = 1, \ldots, K$, are equal (as $\pi_i > 0$ for all i).

To prove the first inequality, let X_0 have the stationary distribution π and let $Y(n) = (Y_1(n), \ldots, Y_K(n))$ be the number of times the Markov chain $\{X_r\}_{r=0}^{n-1}$ visits the states $1, \ldots, K$ respectively.

Then

$$\tilde{q} = \lambda_1(A) = \lim_{n\to\infty} \left(\sum_{i=1}^{K} \pi_i \cdot w_i(n) \right)^{1/n} = \lim_{n\to\infty} \left(E_\pi(w_{X_0}(n)) \right)^{1/n}.$$

Here,

$$E_\pi(w_{X_0}(n)) = E_\pi \left(\prod_{i=0}^{n-1} q_{X_i} \right) = E_\pi \left(\prod_{i=1}^{K} q_i^{Y_i(n)} \right).$$

By Jensen's inequality,

$$\left(E_\pi \left(\prod_{i=1}^{K} q_i^{Y_i(n)} \right) \right)^{1/n} \geqslant E_\pi \left(\prod_{i=1}^{K} q_i^{Y_i(n)/n} \right).$$

Finally, by bounded convergence, as $Y(n)/n$ converges almost surely to π as $n \to \infty$,

$$\lim_{n\to\infty} E_\pi \left(\prod_{i=1}^{K} q_i^{Y_i(n)/n} \right) = \prod_{i=1}^{K} q_i^{\pi_i}. \qquad \square$$

Remark. For most ALB lattices the arithmetic mean, $\bar{q}_a = \sum \pi_i \cdot q_i$, exceeds $\tilde{q}$. This does not hold generally though; for $D(3^3, 4^2)$ we get $\bar{q}_a = \frac{3}{5} \cdot 3 + \frac{2}{5} \cdot 4 = \frac{17}{5} < 2 + \sqrt{2} = \tilde{q}$.

Remark. As an alternative to the harmonic mean, $\bar{q}_h = \bar{q}$, or $\tilde{q}$ as defined above, we could use the geometric mean

$$\bar{q}_g = \prod_{i=1}^{K} q_i^{\pi_i}$$

as a measure of average degree.

For the ALB lattices we get the same separation using $\bar{q}_g$ as with $\tilde{q}$, with two exceptions, namely $\bar{q}_g(D(3^3, 4^2)) = \bar{q}_g(D(3^2, 4, 3, 4)) = \bar{q}_g(D(\text{Bow-tie})) = 3^{3/5} \cdot 4^{2/5} \approx 3.366$, whereas $\tilde{q}(D(3^3, 4^2)) > \tilde{q}(D(3^2, 4, 3, 4)) = \tilde{q}(D(\text{Bow-tie}))$, and $\bar{q}_g(D(3, 6, 3, 6)) = 3^{1/2} \cdot 6^{1/2} = \sqrt{18} > (16 \cdot 18)^{1/4} = 6^{1/4} \cdot 4^{1/2} \cdot 3^{1/4} = \bar{q}_g(D(3, 4, 6, 4))$, whereas $\tilde{q}(D(3, 6, 3, 6)) = \tilde{q}(D(3, 4, 6, 4)) = \sqrt{18}$.

There are nontrivial lattices for which $\tilde{q} = \bar{q}_g$.

Theorem 4.4. *For a bipartite lattice with $K = 2$,*

$$\tilde{q} = \bar{q}_g = \sqrt{q_1 \cdot q_2}.$$

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Table 1. Average degrees for the ALB lattices.

Graph	Alias	$\tilde{q}$	$\bar{q}_a$	$\bar{q}_g$	$\bar{q}_h$
$D(3,12^2)$	Asanoha	8.196	9	7.560	6
$D(4,6,12)$		6.823	7.333	6.604	6
$D(4,8^2)$	Octagonal	6.472	6.667	6.350	6
(3^6)	Triangular	6	6	6	6
$((3^2,4,3^2,4),(4^4))$	Bow-tie	5.123	5.2	5.102	5
$(3^2,4,3,4)$		5	5	5	5
$(3^3,4^2)$		5	5	5	5
$(3^4,6)$		5	5	5	5
$D(3,6,3,6)$	Dice	4.243	4.5	4.243	4
$D(3,4,6,4)$	D(Ruby)	4.243	4.25	4.120	4
(4^4)	Square	4	4	4	4
$(3,4,6,4)$	Ruby	4	4	4	4
$(3,6,3,6)$	Kagomé	4	4	4	4
$D(3^4,6)$		3.541	3.6	3.446	10/3
$D(3^3,4^2)$	Pentagonal	3.414	3.4	3.366	10/3
$D(3^2,4,3,4)$		3.372	3.4	3.366	10/3
$D((3^2,4,3^2,4),(4^4))$	D(Bow-tie)	3.372	3.4	3.366	10/3
(6^3)	Hexagonal	3	3	3	3
$(4,8^2)$	Bathroom tile, Briarwood	3	3	3	3
$(4,6,12)$	Cross	3	3	3	3
$(3,12^2)$	Star, Extended Kagomé	3	3	3	3

Proof. For a bipartite lattice with $K = 2$,

$$P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} 0 & q_1 \\ q_2 & 0 \end{pmatrix},$$

so that $\pi = (\frac{1}{2}, \frac{1}{2})$, $\tilde{q} = (q_1 \cdot q_2)^{1/2}$ and $\bar{q}_g = q_1^{\pi_1} \cdot q_2^{\pi_2} = (q_1 \cdot q_2)^{1/2}$. □

Conjecture 4.5. *$\tilde{q}(G) = \bar{q}_g(G)$ if and only if $K = 1$ or $K = 2$ and G is bipartite.*

Remark. For $K = 3$, the matrix

$$P = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

would give $\pi = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, so that $\tilde{q} = \bar{q}_g = (q_1 \cdot q_2 \cdot q_3)^{1/3}$. However, this matrix does not correspond to an undirected lattice, as these must satisfy $p_{ij} > 0 \Rightarrow p_{ji} > 0$.

A summary of average degrees for the ALB lattices is given in Table 1. In the table we also give some aliases for the lattices that are more or less common in the literature.

From Table 1 we observe a number of properties that hold for the average degrees of the ALB lattices.

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Property 4.6. *For lattices G and H in ALB,*

$$\text{(i)} \qquad \tilde{q}(G) > \tilde{q}(H) \quad \Rightarrow \quad \begin{cases} \bar{q}_g(G) \geqslant \bar{q}_g(H) \\ \bar{q}_h(G) \geqslant \bar{q}_h(H), \end{cases}$$

$$\text{(ii)} \qquad \bar{q}_g(G) > \bar{q}_g(H) \quad \Rightarrow \quad \begin{cases} \tilde{q}(G) \geqslant \tilde{q}(H) \\ \bar{q}_h(G) \geqslant \bar{q}_h(H), \end{cases}$$

$$\text{(iii)} \qquad \bar{q}_h(G) > \bar{q}_h(H) \quad \Rightarrow \quad \begin{cases} \tilde{q}(G) > \tilde{q}(H) \\ \bar{q}_g(G) > \bar{q}_q(H). \end{cases}$$

Property 4.7. *For lattices G and H in ALB,*

$$\text{(i)} \qquad \tilde{q}(G) > \tilde{q}(H) \quad \Rightarrow \quad \tilde{q}(D(G)) \leqslant \tilde{q}(D(H)),$$

$$\text{(ii)} \qquad \bar{q}_g(G) > \bar{q}_g(H) \quad \Rightarrow \quad \bar{q}_g(D(G)) \leqslant \bar{q}_g(D(H)).$$

Unfortunately, we can not expect Properties 4.6 and 4.7 to hold in general, although it is likely that they hold for a much larger class of lattices than ALB.

Remark. By the relation (2.1) between the average degree of a lattice and its dual, we get $\bar{q}_h(G) > \bar{q}_h(H) \Leftrightarrow \bar{q}_h(D(G)) < \bar{q}_h(D(H))$.

Let $\mu(G)$ be the connective constant for self-avoiding walks on G. It is known (see [1]) that, for any graphs G and H in ALB, $\bar{q}(G) > \bar{q}(H) \Rightarrow \mu(G) > \mu(H)$. There are also strong indications that the following conjecture holds.

Conjecture 4.8. *For any graphs G and H in ALB,* $\tilde{q}(G) > \tilde{q}(H) \Rightarrow \mu(G) > \mu(H)$.

Less is known about the ordering of critical probabilities, p_c, for bond percolations on the ALB lattices (see [7]), but what is known supports the following conjectures.

Conjecture 4.9. *For any graphs G and H in ALB,* $\bar{q}(G) > \bar{q}(H) \Rightarrow p_c(G) < p_c(H)$.

Conjecture 4.10. *For any graphs G and H in ALB,* $\tilde{q}(G) > \tilde{q}(H) \Rightarrow p_c(G) < p_c(H)$.

It seems likely that the dependence on average degree for μ and p_c is more general than stated in Conjectures 4.8 and 4.10.

It may be tempting to conjecture that, for general lattices, G and H,

(i) $\tilde{q}(G) > \tilde{q}(H) \Rightarrow p_c(G) < p_c(H)$,
(ii) $\tilde{q}(G) > \tilde{q}(H) \Rightarrow \mu(G) > \mu(H)$.

This is, however, not true, as is shown by Parviainen [4], who gives examples of pairs of lattices with connective constants and critical probabilities in the same order.

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Remark. If we introduce the ordering relation

$$G \succ H \quad \text{if} \quad \begin{cases} \tilde{q}(G) > \tilde{q}(H), & \text{or} \\ \bar{q}_g(G) > \bar{q}_g(H), & \text{or} \\ \tilde{q}(D(G)) < \tilde{q}(D(H)), & \text{or} \\ \bar{q}_g(D(G)) < \bar{q}_g(D(H)), \end{cases}$$

we get a complete ordering of the ALB lattices as shown in Table 1. This order is in full agreement with what is known of the partial order of these lattices that can be obtained by inclusions and non-inclusions (see Parviainen and Wierman [3]). It is also in agreement with the order of critical probabilities for bond percolation on these lattices (see Wierman and Parviainen [7]), and the order of the connective constants of self-avoiding walks on these lattices (see Alm [1]), with the possible exception of $D(3,6,3,6)$ and $D(3,4,6,4)$, which may have connective constants in reversed order. This indicates that $\tilde{q}$ may be a better measure of connectivity than $\bar{q}_g$, at least for predicting connective constants.

Conjecture 4.11. *The critical probabilities for bond percolation on the ALB lattices are ordered, in increasing order, as in Table* 1.

Conjecture 4.12. *The connective constants for self-avoiding walks on the ALB lattices are ordered, in decreasing order, as in Table* 1.

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