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ZERO-ONE LAWS IN SIMULTANEOUS AND MULTIPLICATIVE DIOPHANTINE APPROXIMATION

LIANGPAN LI

Abstract. Answering two questions of Beresnevich and Velani, we develop zero-one laws in both simultaneous and multiplicative Diophantine approximation. Our proofs mainly rely on a versatile Cassels–Gallagher type theorem and the cross fibering principle of Beresnevich, Haynes and Velani.

§1. *Introduction.* Diophantine approximation is the quantitative study of rational number approximation to real numbers, originating from the celebrated theorem of Dirichlet that for any irrational number α, there exist infinitely many $(q,\, p) \in \mathbb{N} \times \mathbb{Z}$ satisfying $|\alpha - p/q| < 1/q^2$. Note also for each algebraic number ξ of degree $d \geqslant 2$, an inequality of Liouville produces a constant $c(\xi) > 0$ such that for all $(q,\, p) \in \mathbb{N} \times \mathbb{Z}$, $|\xi - p/q| > c(\xi)/q^d$, and the Thue–Siegel–Roth theorem [17] further gives for any real number $\gamma > 2$, a constant $c = c(\xi,\, \gamma) > 0$ such that for all $(q,\, p) \in \mathbb{N} \times \mathbb{Z}$, $|\xi - p/q| > c/q^\gamma$.

There is in flavour another type of question in Diophantine approximation, that is, to prove statements which are "almost always" or "almost never" true. For example, given $\psi : \mathbb{N} \to [0,\, \frac{1}{2})$, often referred to as an approximation function, a long-standing conjecture of Duffin and Schaeffer [10] says that

$$\mathcal{M}_1\left(\limsup_{q\to\infty} \mathcal{E}_q(\psi)\right) = 1 \Longleftrightarrow \sum_{q\in\mathbb{N}} \mathcal{M}_1(\mathcal{E}_q(\psi)) = \infty;$$

here and throughout, $\mathcal{M}_k(\cdot)$ $(k \in \mathbb{N})$ denotes the k-dimensional Lebesgue measure

$$\mathcal{E}_q(\psi) \triangleq \bigcup_{\substack{p=1\\(p,q)=1}}^{q} \left(\frac{p-\psi(q)}{q},\, \frac{p+\psi(q)}{q}\right).$$

In this paper we will study a specific question of such flavour, that is, the so-called zero-one law in Diophantine approximation. Given an approximation function ψ, Cassels [9] proved that the Lebesgue measure of the set of real numbers x in $\mathbb{I} \triangleq [0,\, 1)$ for which $|qx - p| < \psi(q)$ holds for infinitely many pairs of positive integers $(q,\, p)$ equals either 0 or 1, while Gallagher [11] showed $\mathcal{M}_1(\limsup_{q\to\infty} \mathcal{E}_q(\psi)) \in \{0,\, 1\}$. The Cassels–Gallagher theorems, also called zero-one laws, have played a fundamental role in the study of metric Diophantine approximation (see e.g. [14, 19]).

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One may naturally consider developing zero-one laws in simultaneous Diophantine approximation. Given non-negative functions $\{\Psi_j\}_{j=1}^m$ on $\mathbb{Z}^n\backslash\{\mathbf{0}\}$, let $\mathcal{A}_{n,m}(\Psi_1, \ldots, \Psi_m)$ denote the set of $\mathbf{X} = (\mathbf{X}^{(1)}, \ldots, \mathbf{X}^{(m)}) \in \mathbb{I}^{nm}$ for which

$$|\mathbf{q}\mathbf{X}^{(j)} + p_j| < \Psi_j(\mathbf{q}) \quad (j = 1, \ldots, m) \tag{1.1}$$

holds for infinitely many (i.m.) pairs $\mathbf{q} \in \mathbb{Z}^n\backslash\{\mathbf{0}\}$, $\mathbf{p} = (p_1, \ldots, p_m) \in \mathbb{Z}^m$, where $\mathbf{q}$ is regarded as a row while $\mathbf{X}^{(j)} \in \mathbb{R}^n$ a column, thus $\mathbf{q}\mathbf{X}^{(j)}$ is the standard inner product between $\mathbf{q}$ and $\mathbf{X}^{(j)}$ in $\mathbb{R}^n$. By imposing certain partial coprimality conditions, we can also introduce

$$\begin{aligned}\mathcal{B}_{n,m}(\Psi_1, \ldots, \Psi_m) &\triangleq \{\mathbf{X} \in \mathbb{I}^{nm} : (1.1) \text{ holds for i.m. } (\mathbf{p}, \mathbf{q}) \\ &\qquad \text{with } \gcd(p_j, \mathbf{q}) = 1 (\forall j)\}, \\ \mathcal{C}_{n,m}(\Psi_1, \ldots, \Psi_m) &\triangleq \{\mathbf{X} \in \mathbb{I}^{nm} : (1.1) \text{ holds for i.m. } (\mathbf{p}, \mathbf{q}) \\ &\qquad \text{with } \gcd(\mathbf{p}, \mathbf{q}) = 1\},\end{aligned}$$

where $\gcd(\mathbf{r})$ denotes the greatest common divisor of all the components of any prescribed element $\mathbf{r}$ of $\mathbb{Z}^k\backslash\{\mathbf{0}\}$ $(k \in \mathbb{N})$, and as usual we write $\gcd(\mathbf{r}, \mathbf{q})$ for $\gcd((\mathbf{r}, \mathbf{q}))$. Throughout the paper, let us agree on writing $\mathcal{F}$ for any of $\mathcal{A}, \mathcal{B}, \mathcal{C}$. In the case $\Psi_1 = \cdots = \Psi_m = \Psi$, Beresnevich and Velani ([**6**, Theorem 1]; see also [**20**]) proved that $\mathcal{M}_{nm}(\mathcal{F}_{n,m}(\Psi_1, \ldots, \Psi_m)) \in \{0, 1\}$. They [**6**, Theorem 4] also showed in general that

$$\mathcal{M}_{nm}\left(\bigcup_{k\in\mathbb{N}} \mathcal{F}_{n,m}(k\Psi_1, \ldots, k\Psi_m)\right) \in \{0, 1\}, \tag{1.2}$$

and asked among several others [**6**, Question 2] to prove that

$$\mathcal{M}_{nm}\left(\bigcup_{k\in\mathbb{N}} \mathcal{F}_{n,m}(k\Psi_1, \ldots, k\Psi_m)\right) = \mathcal{M}_{nm}\left(\bigcap_{k\in\mathbb{N}} \mathcal{F}_{n,m}\left(\frac{\Psi_1}{k}, \ldots, \frac{\Psi_m}{k}\right)\right). \tag{1.3}$$

We can also study Diophantine approximation of the following multiplicative form:

$$\prod_{j=1}^m |\mathbf{q}\mathbf{X}^{(j)} + p_j| < \Psi(\mathbf{q}), \tag{1.4}$$

where Ψ is a non-negative function on $\mathbb{Z}^n\backslash\{\mathbf{0}\}$, and similarly define

$$\begin{aligned}&\mathcal{A}^{\times}_{n,m}(\Psi) \triangleq \{\mathbf{X} \in \mathbb{I}^{nm} : (1.4) \text{ holds for i.m. } (\mathbf{p}, \mathbf{q})\}, \\ &\mathcal{B}^{\times}_{n,m}(\Psi) \triangleq \{\mathbf{X} \in \mathbb{I}^{nm} : (1.4) \text{ holds for i.m. } (\mathbf{p}, \mathbf{q}) \text{ with } \gcd(p_j, \mathbf{q}) = 1 \ (\forall j)\}, \\ &\mathcal{C}^{\times}_{n,m}(\Psi) \triangleq \{\mathbf{X} \in \mathbb{I}^{nm} : (1.4) \text{ holds for i.m. } (\mathbf{p}, \mathbf{q}) \text{ with } \gcd(\mathbf{p}, \mathbf{q}) = 1\}.\end{aligned}$$

In the case $\mathcal{G} \in \{\mathcal{A}, \mathcal{B}\}$, Beresnevich *et al* [**4**, Theorem 4] obtained $\mathcal{M}_{nm}(\mathcal{G}^{\times}_{n,m}(\Psi)) \in \{0, 1\}$. Beresnevich and Velani [**6**, Question 5] also asked to prove that

$$\mathcal{M}_{nm}\left(\mathcal{A}^{\times}_{n,m}(\Psi) \bigg\backslash \bigcap_{k\in\mathbb{N}} \mathcal{A}^{\times}_{n,m}\left(\frac{\Psi}{k}\right)\right) = 0. \tag{1.5}$$

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The purpose of this paper is to show that

$$\mathcal{M}_{nm}(\mathcal{F}_{n,m}(k_1\Psi_1, \ldots, k_m\Psi_m)) \in \{0, 1\} \text{ is independent of } \{k_j > 0\}_{j=1}^m, \quad (1.6)$$

and

$$\mathcal{M}_{nm}(\mathcal{F}^{\times}_{n,m}(k\Psi)) \in \{0, 1\} \text{ is independent of } k > 0, \quad (1.7)$$

which result in (1.3), (1.5), and all aforementioned zero-one laws as special cases.

We will prepare several preliminary facts in §2, and study zero-one laws in simultaneous and multiplicative Diophantine approximation respectively in §§3 and 4. A few remarks will be addressed in the last section.

§2. *Preliminaries.*

2.1. *Cassels–Gallagher theorem.* For each $\mathbf{q} \in \mathbb{Z}^n\backslash\{\mathbf{0}\}$, let $\omega(\mathbf{q})$ be a fixed subset of positive divisors of $\gcd(\mathbf{q})$, and let $M_{\mathbf{q}} : \omega(\mathbf{q}) \to [0, \infty)$ be a fixed function. Let $\mathcal{H}_n(\omega, M)$ denote the set of $\mathbf{X} \in \mathbb{I}^n$ for which $|\mathbf{q}\mathbf{X} + p| < M_{\mathbf{q}}(\gamma)$ holds for infinitely many $(\mathbf{q}, p, \gamma) \in (\mathbb{Z}^n\backslash\{\mathbf{0}\}) \times \mathbb{Z} \times \mathbb{Z}$ with $\gamma \in \omega(\mathbf{q})$ and $(p, \gamma) = 1$.

LEMMA 2.1. *$\mathcal{M}_n(\mathcal{H}_n(\omega, kM)) \in \{0, 1\}$ is independent of $k > 0$.*

The proof of Lemma 2.1 needs the following two lemmas, whose proofs and one-dimensional prototypes can be found in [**6, 9, 11**].

LEMMA 2.2. *Let $\{B_i\}_{i\in\mathbb{N}}$ be a sequence of balls in $\mathbb{R}^n$ with $\mathcal{M}_n(B_i) \to 0$ as $i \to \infty$. Let $\{U_i\}_{i\in\mathbb{N}}$ be a sequence of Lebesgue measurable sets such that $U_i \subset B_i$ for all $i \in \mathbb{N}$. Assume that for some constant $c > 0$, $\mathcal{M}_n(U_i) \geqslant c\mathcal{M}_n(B_i)$ for all $i \in \mathbb{N}$. Then the upper limits of $\{B_i\}_{i\in\mathbb{N}}$ and $\{U_i\}_{i\in\mathbb{N}}$ have the same Lebesgue measure.*

LEMMA 2.3. *For any integer $s \geqslant 2$ and $\mathbf{e} \in \mathbb{Z}^n$ consider the transformation of $\mathbb{I}^n$ into itself given by*

$$T : \mathbf{X} \mapsto s\mathbf{X} + \frac{\mathbf{e}}{s} \pmod 1.$$

Suppose $A \subset \mathbb{I}^n$ with $T(A) \subset A$. Then $\mathcal{M}_n(A) = 0$ or 1.

Proof of the independence part of Lemma 2.1. By letting $|\cdot|$ denote the supremum norm in $\mathbb{R}^n$, we have two cases to consider.

Case 1. Suppose

$$\limsup_{|\mathbf{q}|\to\infty} \frac{\max_{\gamma\in\omega(\mathbf{q})} M_{\mathbf{q}}(\gamma)}{|\mathbf{q}|} > 0.$$

In this case there exist a $\delta > 0$ and $\{\gamma^{(i)} \in \omega(\mathbf{q}^{(i)})\}_{i\in\mathbb{N}}$ with $|\mathbf{q}^{(i)}| \to \infty$ as $i \to \infty$ such that

$$\frac{M_{\mathbf{q}^{(i)}}(\gamma^{(i)})}{|\mathbf{q}^{(i)}|} \geqslant \delta \quad (\forall i \in \mathbb{N}).$$

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Write $\mathbf{q}^{(i)} = (q_1^{(i)}, \dots, q_n^{(i)})$ and assume without loss of generality that $|\mathbf{q}^{(i)}| = |q_1^{(i)}|$ for all $i \in \mathbb{N}$. From Gallagher's proof [**11**, Lemma 1] we know that if $|q|$ is large enough, then for any $\alpha \in \mathbb{R}$,

$$\left\{x_1 \in \mathbb{I} : \left|x_1 + \frac{\alpha + p}{q}\right| < k\delta \text{ for some } p \in \mathbb{Z} \text{ with } (p, q) = 1\right\} = \mathbb{I}.$$

Thus for any $x_2, \dots, x_n \in \mathbb{I}$,

$$\left\{x_1 \in \mathbb{I} : \left|x_1 + \frac{\sum_{j=2}^n q_j^{(i)} x_j + p}{q_1^{(i)}}\right| < \frac{kM_{\mathbf{q}^{(i)}}(\gamma^{(i)})}{|q_1^{(i)}|} \text{ for some } p \in \mathbb{Z} \text{ with } (p, \gamma^{(i)}) = 1\right\} = \mathbb{I}$$

provided $|q_1^{(i)}|$ is large enough. By definition, $\mathcal{H}_n(\omega, kM) = \mathbb{I}^n$.

Case 2. Suppose

$$\lim_{|\mathbf{q}|\to\infty} \frac{\max_{\gamma\in\omega(\mathbf{q})} M_{\mathbf{q}}(\gamma)}{|\mathbf{q}|} = 0.$$

For any $(\mathbf{q}, p, \gamma) \in (\mathbb{Z}^n \backslash \{\mathbf{0}\}) \times \mathbb{Z} \times \mathbb{Z}$ with $N(\mathbf{q}, p) \triangleq \{\mathbf{X} \in \mathbb{I}^n : \mathbf{q}\mathbf{X} + p = 0\}$ non-empty, $\gamma \in \omega(\mathbf{q})$, $(p, \gamma) = 1$ and $M_{\mathbf{q}}(\gamma) > 0$, we fix a finite subset $N(\mathbf{q}, p, \gamma)$ of $N(\mathbf{q}, p)$ such that for any $\mathbf{X} \in N(\mathbf{q}, p)$ there is a $\mathbf{Y} \in N(\mathbf{q}, p, \gamma)$ satisfying $|\mathbf{X} - \mathbf{Y}| < M_{\mathbf{q}}(\gamma)/|\mathbf{q}|$. All the other remaining $(\mathbf{q}, p, \gamma)$ will be completely ignored since they have nothing to do with our study. For any $k > 0$ we define

$$\mathcal{H}_{\mathbf{q},p,\gamma}(k) = \bigcup_{\mathbf{Y}\in N(\mathbf{q},p,\gamma)} B\left(\mathbf{Y}, k\frac{M_{\mathbf{q}}(\gamma)}{|\mathbf{q}|}\right)$$

and

$$\mathcal{H}_{\mathbf{q}}(k) = \bigcup_p \bigcup_\gamma \mathcal{H}_{\mathbf{q},p,\gamma}(k),$$

where $B(\mathbf{Y}, r)$ denotes the ball in $\mathbb{R}^n$ with center $\mathbf{Y}$ and radius r. Then for all $\epsilon \in (0, \frac{1}{2})$, it is easy to verify that for any sufficient large $|\mathbf{q}|$,

$$[\epsilon, 1-\epsilon]^n \cap \mathcal{H}_{\mathbf{q},p,\gamma}\left(\frac{k}{n}\right) \subset \{\mathbf{X} \in \mathbb{I}^n : |\mathbf{q}\mathbf{X} + p| < kM_{\mathbf{q}}(\gamma)\},$$

$$[\epsilon, 1-\epsilon]^n \cap \{\mathbf{X} \in \mathbb{I}^n : |\mathbf{q}\mathbf{X} + p| < kM_{\mathbf{q}}(\gamma)\} \subset \mathcal{H}_{\mathbf{q},p,\gamma}(k + \sqrt{n}).$$

These facts naturally imply that

$$\mathcal{M}_n\left(\limsup_{|\mathbf{q}|\to\infty} \mathcal{H}_{\mathbf{q}}\left(\frac{k}{n}\right)\right) \leqslant \mathcal{M}_n(\mathcal{H}_n(\omega, kM)) \leqslant \mathcal{M}_n\left(\limsup_{|\mathbf{q}|\to\infty} \mathcal{H}_{\mathbf{q}}(k + \sqrt{n})\right),$$

and consequently by Lemma 2.2, $\mathcal{M}_n(\mathcal{H}_n(\omega, kM))$ is independent of $k > 0$. □

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Proof of the zero-one part of Lemma 2.1. For each prime number s and each non-negative integer ν, we consider the approximation

$$|\mathbf{q}\mathbf{X} + p| < s^{\nu} M_{\mathbf{q}}(\gamma) \quad \text{for some pair } p \in \mathbb{Z},\ \gamma \in \omega(\mathbf{q}) \text{ with } (p, \gamma) = 1, \tag{2.1}$$

and define three non-decreasing sequences of sets $\mathcal{R}(s^{\nu})$, $\mathcal{S}(s^{\nu})$ and $\mathcal{T}(s^{\nu})$ as follows:

$\mathbf{X} \in \mathcal{R}(s^{\nu})$ if $\mathbf{X} \in \mathbb{I}^n$ satisfies (2.1) for i.m. $\mathbf{q}$ with $s \nmid \gcd(\mathbf{q})$,

$\mathbf{X} \in \mathcal{S}(s^{\nu})$ if $\mathbf{X} \in \mathbb{I}^n$ satisfies (2.1) for i.m. $\mathbf{q}$ with $s | \gcd(\mathbf{q})$ and $s^2 \nmid \gcd(\mathbf{q})$,

$\mathbf{X} \in \mathcal{T}(s^{\nu})$ if $\mathbf{X} \in \mathbb{I}^n$ satisfies (2.1) for i.m. $\mathbf{q}$ with $s^2 | \gcd(\mathbf{q})$.

If $\mathbf{X} \in \mathcal{R}(s^{\nu})$, then it is easy to verify, a little lengthily but quite elementarily (see also [**14**, p. 33]), that

$$|\mathbf{q}\langle s\mathbf{X}\rangle + \mathbf{q}\lfloor s\mathbf{X}\rfloor + sp| < s^{\nu+1} M_{\mathbf{q}}(\gamma) \quad (\gamma, \mathbf{q}\lfloor s\mathbf{X}\rfloor + sp) = 1,$$

where as usual $\langle (x_1, \ldots, x_n)\rangle \triangleq (\langle x_1\rangle, \ldots, \langle x_n\rangle)$, $\lfloor (x_1, \ldots, x_n)\rfloor \triangleq (\lfloor x_1\rfloor, \ldots, \lfloor x_n\rfloor)$, $\langle \alpha\rangle$ and $\lfloor \alpha\rfloor$ denote respectively the fractional and integer parts of $\alpha \in \mathbb{R}$. The details of demonstrating $(\gamma, \mathbf{q}\lfloor s\mathbf{X}\rfloor + sp) = 1$ are left as an exercise for the interested reader. This shows the map $\mathbf{X} \mapsto \langle s\mathbf{X}\rangle$ on $\mathbb{I}^n$ sends $\bigcup_{\nu \geqslant 0} \mathcal{R}(s^{\nu})$ into itself. By Lemma 2.3, $\mathcal{M}_n(\bigcup_{\nu \geqslant 0} \mathcal{R}(s^{\nu})) \in \{0, 1\}$.

If $\mathbf{X} \in \mathbb{I}^n$ satisfies (2.1) with $s | \gcd(\mathbf{q})$ and $s^2 \nmid q_i$ for some $i \in \{1, \ldots, n\}$, then

$$\left|\mathbf{q}\left\langle s\mathbf{X} + \frac{\mathbf{1}_i}{s}\right\rangle + \mathbf{q}\left\lfloor s\mathbf{X} + \frac{\mathbf{1}_i}{s}\right\rfloor - \frac{q_i}{s} + sp\right| < s^{\nu+1} M_{\mathbf{q}}(\gamma),$$
$$\left(\gamma, \mathbf{q}\left\lfloor s\mathbf{X} + \frac{\mathbf{1}_i}{s}\right\rfloor - \frac{q_i}{s} + sp\right) = 1,$$

where $\mathbf{q} = (q_1, \ldots, q_n)$, $\mathbf{1}_i$ denotes the element of $\mathbb{Z}^n$ with zero entries everywhere except in the ith position where the entry is 1. Once again, the verification work of the coprimality condition is omitted as an exercise. This shows the map $\mathbf{X} \mapsto \langle s\mathbf{X} + \mathbf{1}_i/s\rangle$ on $\mathbb{I}^n$ sends $\bigcup_{\nu \geqslant 0} \mathcal{S}^{(i)}(s^{\nu})$ into itself, where we define

$\mathbf{X} \in \mathcal{S}^{(i)}(s^{\nu})$ if $\mathbf{X} \in \mathbb{I}^n$ satisfies (2.1) for i.m. $\mathbf{q}$ with $s | \gcd(\mathbf{q})$ *and* $s^2 \nmid q_i$.

By Lemma 2.3,

$$\mathcal{M}_n\left(\bigcup_{\nu \geqslant 0} \mathcal{S}^{(i)}(s^{\nu})\right) \in \{0, 1\}.$$

Note $\mathcal{S}(s^{\nu}) = \bigcup_{i=1}^{n} \mathcal{S}^{(i)}(s^{\nu})$, which easily implies that $\mathcal{M}_n(\bigcup_{\nu \geqslant 0} \mathcal{S}(s^{\nu})) \in \{0, 1\}$.

If $\mathbf{X} \in \mathcal{T}(s^{\nu})$, then for each $i \in \{1, \ldots, n\}$,

$$\left|\mathbf{q}\left\langle \mathbf{X} + \frac{\mathbf{1}_i}{s}\right\rangle + \mathbf{q}\left\lfloor \mathbf{X} + \frac{\mathbf{1}_i}{s}\right\rfloor - \frac{q_i}{s} + p\right| < s^{\nu} M_{\mathbf{q}}(\gamma),$$
$$\left(\gamma, \mathbf{q}\left\lfloor \mathbf{X} + \frac{\mathbf{1}_i}{s}\right\rfloor - \frac{q_i}{s} + p\right) = 1.$$

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For the third time, the verification work of the coprimality condition is omitted as an exercise. This means $\mathcal{T}(s^\nu)$ remains invariant under the map $\mathbf{X} \mapsto \langle \mathbf{X} + \mathbf{1}_i/s \rangle$ on $\mathbb{I}^n$ for each $i \in \{1, \ldots, n\}$, so obviously is $\bigcup_{\nu \geqslant 0} \mathcal{T}(s^\nu)$. Thus for any n-dimensional cube $U(\mathbf{Y}, 1/s)$ in $\mathbb{I}^n$ with center $\mathbf{Y}$ and side length $1/s$,

$$\frac{\mathcal{M}_n(\bigcup_{\nu \geqslant 0} \mathcal{T}(s^\nu) \cap U(\mathbf{Y}, 1/s))}{\mathcal{M}_n(U(\mathbf{Y}, 1/s))} = \frac{\mathcal{M}_n(\bigcup_{\nu \geqslant 0} \mathcal{T}(s^\nu) \cap \mathbb{I}^n)}{\mathcal{M}_n(\mathbb{I}^n)} = \mathcal{M}_n\left(\bigcup_{\nu \geqslant 0} \mathcal{T}(s^\nu)\right). \tag{2.2}$$

If either

$$\mathcal{M}_n\left(\bigcup_{\nu \geqslant 0} \mathcal{R}(s^\nu)\right) = 1 \quad \text{or} \quad \mathcal{M}_n\left(\bigcup_{\nu \geqslant 0} \mathcal{S}(s^\nu)\right) = 1,$$

then it is easy to see that $\mathcal{M}_n(\bigcup_{k \in \mathbb{N}} \mathcal{H}_n(\omega, kM)) = 1$. So we can assume

$$\mathcal{M}_n\left(\bigcup_{\nu \geqslant 0} \mathcal{R}(s^\nu)\right) = \mathcal{M}_n\left(\bigcup_{\nu \geqslant 0} \mathcal{S}(s^\nu)\right) = 0.$$

In this case we observe that for any prime number s, $\bigcup_{k \in \mathbb{N}} \mathcal{H}_n(\omega, kM)$ differs from $\bigcup_{\nu \geqslant 0} \mathcal{T}(s^\nu)$ only by a null set. If $\bigcup_{k \in \mathbb{N}} \mathcal{H}_n(\omega, kM)$ itself is null, then we have nothing to prove. So we can assume that $\bigcup_{k \in \mathbb{N}} \mathcal{H}_n(\omega, kM)$ is of positive Lebesgue measure, from which we apply Lebesgue's density theorem to get a point $\mathbf{Y} \in (0, 1)^n$ so that

$$\lim_{r \to 0} \frac{\mathcal{M}_n(\bigcup_{k \in \mathbb{N}} \mathcal{H}_n(\omega, kM) \cap U(\mathbf{Y}, r))}{\mathcal{M}_n(U(\mathbf{Y}, r))} = 1. \tag{2.3}$$

Combining (2.2), (2.3) as well as the fact that $\bigcup_{k \in \mathbb{N}} \mathcal{H}_n(\omega, kM)$ differs from $\bigcup_{\nu \geqslant 0} \mathcal{T}(s^\nu)$ by a null set, we get $\mathcal{M}_n(\bigcup_{k \in \mathbb{N}} \mathcal{H}_n(\omega, kM)) = 1$. Since in the previous part of the proof we have shown that the Lebesgue measures of the non-decreasing sequence of sets $\{\mathcal{M}_n(\mathcal{H}_n(\omega, kM))\}_{k \in \mathbb{N}}$ are the same, their common value must be 1. This finishes the proof. □

Remark 2.4. When $n = 1$, Lemma 2.1 reduces to Cassels' and Gallagher's theorems (see the introduction) if choosing $\omega(q) = \{1\}$ and $\{q\}$ respectively.

2.2. *Cross fibering principle.* The cross fibering principle of Beresnevich, Haynes and Velani [**4**, Theorem 3] provides an elegant way to verify that a set, or its complement in a product measure space (PMS) of finitely† many σ-finite ones, is null. With all the coordinate subspaces chosen to be $\mathbb{I}^n$, their principle to our purpose basically is the following lemma.

LEMMA 2.5. *View $\mathbb{I}^{nm}$ as the product of m coordinate planes $\mathbb{I}^n$. Let $A \subset \mathbb{I}^{nm}$ be a Lebesgue measurable set such that for any n-dimensional plane $\Pi \subset \mathbb{R}^{nm}$ parallel to one of the coordinate planes, $\mathcal{M}_n(A \cap \Pi) = 0$ or 1. Then $\mathcal{M}_{nm}(A) = 0$ or 1.*

† To be exact, they originally only considered PMS of two σ-finite measure spaces, but their principle can be easily extended to PMS of finitely many σ-finite ones using induction methods.

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Proof. For the sake of completeness, we provide a proof by making induction on $m \in \mathbb{N}$. Since there is nothing to do in the $m = 1$ case, we can assume $m \geqslant 2$ and Lemma 2.5 is true for all $\mathbb{I}^{nk}$ with $k < m$. For any $x \in \mathbb{I}^n$ and $y \in \mathbb{I}^{n(m-1)}$, define the sections of A through x and y respectively by

$$A_x = \{y \in \mathbb{I}^{n(m-1)} : (x, y) \in A\}, \qquad A^y = \{x \in \mathbb{I}^n : (x, y) \in A\}.$$

In view of the induction hypothesis, $\mathcal{M}_{n(m-1)}(A_x) = 0$ or 1. By the assumption of this lemma, $\mathcal{M}_n(A^y) = 0$ or 1. We divide $\mathbb{I}^n$ into two parts X_1, X_2, that is, $x \in X_1$ or X_2 according as $\mathcal{M}_{n(m-1)}(A_x) = 0$ or 1. Similarly, we divide $\mathbb{I}^{n(m-1)}$ into two parts Y_1, Y_2, that is, $y \in Y_1$ or Y_2 according as $\mathcal{M}_n(A^y) = 0$ or 1. By Fubini's theorem, $\mathcal{M}_{nm}(A) = \mathcal{M}_n(X_2) = \mathcal{M}_{n(m-1)}(Y_2)$. Since $A = \bigcup_{i,j}(A \cap (X_i \times Y_j))$,

$$\begin{aligned}\mathcal{M}_{nm}(A) &\leqslant \sum_{i,j} \mathcal{M}_{nm}(A \cap (X_i \times Y_j)) = \mathcal{M}_{nm}(A \cap (X_2 \times Y_2)) \\ &\leqslant \mathcal{M}_n(X_2) \cdot \mathcal{M}_{n(m-1)}(Y_2) = \mathcal{M}_{nm}(A)^2,\end{aligned}$$

which easily implies $\mathcal{M}_{nm}(A) = 0$ or 1 as $\mathcal{M}_{nm}(A) \in [0, 1]$. This finishes the proof. □

2.3. *A note on $\mathcal{A}$ and $\mathcal{B}$.* A relook at the definitions of $\mathcal{A}_{n,m}(\Psi_1, \ldots, \Psi_m)$, $\mathcal{B}_{n,m}(\Psi_1, \ldots, \Psi_m)$, $\mathcal{A}^{\times}_{n,m}(\Psi)$, $\mathcal{B}^{\times}_{n,m}(\Psi)$ will be useful to our later study. By letting $\|\alpha\| \triangleq \min\{|\alpha + z| : z \in \mathbb{Z}\}$ for any $\alpha \in \mathbb{R}$, and $\Theta(\mathbf{q}, \mathbf{X}) \triangleq \min\{|\mathbf{q}\mathbf{X} + z| : z \in \mathbb{Z}$ with $\gcd(z, \mathbf{q}) = 1\}$ for any $\mathbf{q} \in \mathbb{Z}^n \backslash \{\mathbf{0}\}$ and any $\mathbf{X} \in \mathbb{I}^n$, we see that

$$\mathcal{A}_{n,m}(\Psi_1, \ldots, \Psi_m), \mathcal{B}_{n,m}(\Psi_1, \ldots, \Psi_m), \mathcal{A}^{\times}_{n,m}(\Psi), \mathcal{B}^{\times}_{n,m}(\Psi)$$

are respectively the sets of $(\mathbf{X}^{(1)}, \ldots, \mathbf{X}^{(m)}) \in \mathbb{I}^{nm}$ for which

$$\|\mathbf{q}\mathbf{X}^{(j)}\| < \Psi_j(\mathbf{q}) \quad (j = 1, \ldots, m), \tag{2.4}$$

$$\Theta(\mathbf{q}, \mathbf{X}^{(j)}) < \Psi_j(\mathbf{q}) \quad (j = 1, \ldots, m), \tag{2.5}$$

$$\prod_{j=1}^{m} \|\mathbf{q}\mathbf{X}^{(j)}\| < \Psi(\mathbf{q}), \tag{2.6}$$

$$\prod_{j=1}^{m} \Theta(\mathbf{q}, \mathbf{X}^{(j)}) < \Psi(\mathbf{q}) \tag{2.7}$$

for infinitely many $\mathbf{q} \in \mathbb{Z}^n \backslash \{\mathbf{0}\}$.

§3. *Zero-one laws in simultaneous Diophantine approximation.*

THEOREM 3.1. *For any non-negative functions $\{\Psi_j\}_{j=1}^m$ on $\mathbb{Z}^n \backslash \{\mathbf{0}\}$, $\mathcal{M}_{nm}(\mathcal{A}_{n,m}(k_1\Psi_1, \ldots, k_m\Psi_m)) \in \{0, 1\}$ is independent of $\{k_j > 0\}_{j=1}^m$.*

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Proof. Obviously, by symmetry and (1.2), to prove Theorem 3.1 it suffices to show that $\mathcal{M}_{nm}(\mathcal{A}_{n,m}(k\Psi_1, \Psi_2, \dots, \Psi_m))$ is independent of $k > 0$. To this aim we first decompose $\mathcal{A}_{n,m}(k\Psi_1, \Psi_2, \dots, \Psi_m)$ as disjoint unions

$$\bigcup_{(\mathbf{X}^{(2)},\dots,\mathbf{X}^{(m)})\in\mathbb{I}^{n(m-1)}} \mathcal{A}_{(\mathbf{X}^{(2)},\dots,\mathbf{X}^{(m)})}(k\Psi_1, \Psi_2, \dots, \Psi_m) \times \{\mathbf{X}^{(2)}\} \times \cdots \times \{\mathbf{X}^{(m)}\},$$

where $\mathcal{A}_{(\mathbf{X}^{(2)},\dots,\mathbf{X}^{(m)})}(k\Psi_1, \Psi_2, \dots, \Psi_m)$ denotes the set of $\mathbf{X}^{(1)} \in \mathbb{I}^n$ for which

$$\|\mathbf{q}\mathbf{X}^{(1)}\| < k\Psi_1(\mathbf{q}) \quad \text{and} \quad \|\mathbf{q}\mathbf{X}^{(j)}\| < \Psi_j(\mathbf{q}) \quad (j = 2, \dots, m)$$

(see §2.3 for the meaning of notation $\|\cdot\|$) for infinitely many $\mathbf{q} \in \mathbb{Z}^n\backslash\{\mathbf{0}\}$. Equivalently, $\mathcal{A}_{(\mathbf{X}^{(2)},\dots,\mathbf{X}^{(m)})}(k\Psi_1, \Psi_2, \dots, \Psi_m)$ is the set of $\mathbf{X}^{(1)} \in \mathbb{I}^n$ for which $\|\mathbf{q}\mathbf{X}^{(1)}\| < k\Psi_1(\mathbf{q})\chi_{\mathbf{W}}(\mathbf{q})$ for infinitely many $\mathbf{q} \in \mathbb{Z}^n\backslash\{\mathbf{0}\}$, where $\mathbf{W}$ is the set of solutions $\mathbf{q} \in \mathbb{Z}^n\backslash\{\mathbf{0}\}$ to $\|\mathbf{q}\mathbf{X}^{(j)}\| < \Psi_j(\mathbf{q})$ $(j = 2, \dots, m)$ with $\chi_{\mathbf{W}}$ denoting its characteristic function. By letting $\omega(\mathbf{q}) = \{1\}$, $M_{\mathbf{q}}(1) = \Psi_1(\mathbf{q}) \cdot \chi_{\mathbf{W}}(\mathbf{q})$ for each $\mathbf{q} \in \mathbb{Z}^n\backslash\{\mathbf{0}\}$, we see that

$$\mathcal{A}_{(\mathbf{X}^{(2)},\dots,\mathbf{X}^{(m)})}(k\Psi_1, \Psi_2, \dots, \Psi_m) = \mathcal{H}_n(\omega, kM).$$

By Lemma 2.1, $\mathcal{M}_n(\mathcal{A}_{(\mathbf{X}^{(2)},\dots,\mathbf{X}^{(m)})}(k\Psi_1, \Psi_2, \dots, \Psi_m))$ is independent of $k > 0$. By Fubini's theorem, $\mathcal{M}_{nm}(\mathcal{A}_{n,m}(k\Psi_1, \Psi_2, \dots, \Psi_m))$ is independent of $k > 0$. This finishes the proof. □

THEOREM 3.2. *For any non-negative functions* $\{\Psi_j\}_{j=1}^m$ *on* $\mathbb{Z}^n\backslash\{\mathbf{0}\}$, $\mathcal{M}_{nm}(\mathcal{B}_{n,m}(k_1\Psi_1, \dots, k_m\Psi_m)) \in \{0, 1\}$ *is independent of* $\{k_j > 0\}_{j=1}^m$.

Proof. The proof of Theorem 3.2 is pretty much the same as that of Theorem 3.1 only with $\mathcal{A}$, $\|\mathbf{q}\mathbf{X}^{(1)}\|$, $\|\mathbf{q}\mathbf{X}^{(j)}\|$, $\omega(\mathbf{q}) = \{1\}$ and $M_{\mathbf{q}}(1) = \Psi_1(\mathbf{q}) \cdot \chi_{\mathbf{W}}(\mathbf{q})$ replaced respectively by $\mathcal{B}$, $\Theta(\mathbf{q}, \mathbf{X}^{(1)})$, $\Theta(\mathbf{q}, \mathbf{X}^{(j)})$, $\omega(\mathbf{q}) = \{\gcd(\mathbf{q})\}$ and $M_{\mathbf{q}}(\gcd(\mathbf{q})) = \Psi_1(\mathbf{q}) \cdot \chi_{\mathbf{W}}(\mathbf{q})$ at suitable places. We leave the details to the interested reader. □

THEOREM 3.3. *For any non-negative functions* $\{\Psi_j\}_{j=1}^m$ *on* $\mathbb{Z}^n\backslash\{\mathbf{0}\}$, $\mathcal{M}_{nm}(\mathcal{C}_{n,m}(k_1\Psi_1, \dots, k_m\Psi_m)) \in \{0, 1\}$ *is independent of* $\{k_j > 0\}_{j=1}^m$.

Proof. Obviously, by symmetry and (1.2), to prove Theorem 3.3 it suffices to show that $\mathcal{M}_{nm}(\mathcal{C}_{n,m}(k\Psi_1, \Psi_2, \dots, \Psi_m))$ is independent of $k > 0$. To this aim we first decompose $\mathcal{C}_{n,m}(k\Psi_1, \Psi_2, \dots, \Psi_m)$ as disjoint unions

$$\bigcup_{(\mathbf{X}^{(2)},\dots,\mathbf{X}^{(m)})\in\mathbb{I}^{n(m-1)}} \mathcal{C}_{(\mathbf{X}^{(2)},\dots,\mathbf{X}^{(m)})}(k\Psi_1, \Psi_2, \dots, \Psi_m) \times \{\mathbf{X}^{(2)}\} \times \cdots \times \{\mathbf{X}^{(m)}\},$$

where $\mathcal{C}_{(\mathbf{X}^{(2)},\dots,\mathbf{X}^{(m)})}(k\Psi_1, \Psi_2, \dots, \Psi_m)$ denotes the set of $\mathbf{X}^{(1)} \in \mathbb{I}^n$ for which

$$|\mathbf{q}\mathbf{X}^{(1)} + p_1| < k\Psi_1(\mathbf{q}) \quad \text{and} \quad |\mathbf{q}\mathbf{X}^{(j)} + p_j| < \Psi_j(\mathbf{q}) \quad (j = 2, \dots, m)$$

for infinitely many pairs $\mathbf{q} \in \mathbb{Z}^n\backslash\{\mathbf{0}\}$, $\mathbf{p} = (p_1, \dots, p_m) \in \mathbb{Z}^m$ with $\gcd(\mathbf{p}, \mathbf{q}) = 1$. For each non-empty solution set $\mathbf{S}_{\mathbf{X}^{(2)},\dots,\mathbf{X}^{(m)};\Psi_2,\dots,\Psi_m}(\mathbf{q})$ of $(p_2, \dots, p_m) \in \mathbb{Z}^{m-1}$ to $|\mathbf{q}\mathbf{X}^{(j)} + p_j| < \Psi_j(\mathbf{q})$ $(j = 2, \dots, m)$, we define a subset of positive

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divisors of $\gcd(\mathbf{q})$ by

$$\omega(\mathbf{q}) = \{\gcd(\mathbf{q}, \alpha) : \alpha \in \mathbf{S}_{\mathbf{X}^{(2)},\dots,\mathbf{X}^{(m)};\Psi_2,\dots,\Psi_m}(\mathbf{q})\}.$$

By letting $\mathbf{W}$ be the support of $\mathbf{S}_{\mathbf{X}^{(2)},\dots,\mathbf{X}^{(m)};\Psi_2,\dots,\Psi_m}$, $\widetilde{\omega}$ be any extension of ω on $\mathbb{Z}^n\backslash\{\mathbf{0}\}$, and $M_{\mathbf{q}}(\cdot)$ be the function taking constant value $\Psi_1(\mathbf{q}) \cdot \chi_{\mathbf{W}}(\mathbf{q})$ on $\mathbb{Z}$ for each $\mathbf{q} \in \mathbb{Z}^n\backslash\{\mathbf{0}\}$, we see that

$$\mathcal{C}_{(\mathbf{X}^{(2)},\dots,\mathbf{X}^{(m)})}(k\Psi_1, \Psi_2, \dots, \Psi_m) = \mathcal{H}_n(\widetilde{\omega}, kM).$$

By Lemma 2.1, $\mathcal{M}_n(\mathcal{C}_{(\mathbf{X}^{(2)},\dots,\mathbf{X}^{(m)})}(k\Psi_1, \Psi_2, \dots, \Psi_m))$ is independent of $k > 0$. By Fubini's theorem, $\mathcal{M}_{nm}(\mathcal{C}_{n,m}(k\Psi_1, \Psi_2, \dots, \Psi_m))$ is independent of $k > 0$. This finishes the proof. □

§4. *Zero-one laws in multiplicative Diophantine approximation.*

THEOREM 4.1. *For any non-negative function* Ψ *on* $\mathbb{Z}^n\backslash\{\mathbf{0}\}$, $\mathcal{M}_{nm}(\mathcal{C}^{\times}_{n,m}(k\Psi)) \in \{0, 1\}$ *is independent of* $k > 0$.

Proof. We first prove the independence property. Note

$$\mathcal{C}^{\times}_{n,m}(k\Psi) = \bigcup_{(\mathbf{X}^{(2)},\dots,\mathbf{X}^{(m)})\in\mathbb{I}^{n(m-1)}} \mathcal{C}^{\times}_{(\mathbf{X}^{(2)},\dots,\mathbf{X}^{(m)})}(k\Psi) \times \{\mathbf{X}^{(2)}\} \times \cdots \times \{\mathbf{X}^{(m)}\},$$

where $\mathcal{C}^{\times}_{(\mathbf{X}^{(2)},\dots,\mathbf{X}^{(m)})}(k\Psi)$ is the set of $\mathbf{X}^{(1)} \in \mathbb{I}^n$ for which $\prod_{j=1}^m |\mathbf{q}\mathbf{X}^{(j)} + p_j| < k\Psi(\mathbf{q})$ for infinitely many pairs $\mathbf{q} \in \mathbb{Z}^n\backslash\{\mathbf{0}\}$, $\mathbf{p} = (p_1, \dots, p_m) \in \mathbb{Z}^m$ with $\gcd(\mathbf{p}, \mathbf{q}) = 1$. For each $\mathbf{q} \in \mathbb{Z}^n\backslash\{\mathbf{0}\}$, let $\omega(\mathbf{q})$ be the set of all positive divisors of $\gcd(\mathbf{q})$, and define

$$J_{\mathbf{q}}(\gamma) = \min\left\{\prod_{j=2}^m |\mathbf{q}\mathbf{X}^{(j)} + p_j| : \gcd(\mathbf{q}, p_2, \dots, p_m) = \gamma\right\} \quad (\forall \gamma \in \omega(\mathbf{q})).$$

It is easy to see that $\mathcal{C}^{\times}_{(\mathbf{X}^{(2)},\dots,\mathbf{X}^{(m)})}(k\Psi)$ is the set of $\mathbf{X}^{(1)} \in \mathbb{I}^n$ for which

$$|\mathbf{q}\mathbf{X}^{(1)} + p_1| \cdot J_{\mathbf{q}}(\gamma) < k\Psi(\mathbf{q}) \tag{4.1}$$

for infinitely many $(\mathbf{q}, p_1, \gamma) \in (\mathbb{Z}^n\backslash\{\mathbf{0}\}) \times \mathbb{Z} \times \mathbb{Z}$ with $\gamma \in \omega(\mathbf{q})$ and $(p_1, \gamma) = 1$. If there exists a pair $(\mathbf{q}, \gamma)$ with $J_{\mathbf{q}}(\gamma) = 0$ and $\Psi(\mathbf{q}) > 0$, then $\mathcal{C}^{\times}_{(\mathbf{X}^{(2)},\dots,\mathbf{X}^{(m)})}(k\Psi) = \mathbb{I}^n$ (♣) as with this $(\mathbf{q}, \gamma)$ we can set arbitrary $\mathbf{X}^{(1)} \in \mathbb{I}^n$ and infinitely many $p_1 \in \mathbb{Z}$ with $(p_1, \gamma) = 1$ to satisfy (4.1). Thus to prove Theorem 4.1, without loss of generality we can exclude all such kind of existences. By assuming $0/0 = 0$ and defining $M_{\mathbf{q}}(\gamma) = \Psi(\mathbf{q})/J_{\mathbf{q}}(\gamma)$ for all $\gamma \in \omega(\mathbf{q})$, we see that

$$\mathcal{C}^{\times}_{(\mathbf{X}^{(2)},\dots,\mathbf{X}^{(m)})}(k\Psi) = \mathcal{H}_n(\omega, kM).$$

By Lemma 2.1, $\mathcal{M}_n(\mathcal{C}^{\times}_{(\mathbf{X}^{(2)},\dots,\mathbf{X}^{(m)})}(k\Psi)) \in \{0, 1\}$ (♠) is independent of $k > 0$. By Fubini's theorem, $\mathcal{M}_{nm}(\mathcal{C}^{\times}_{n,m}(k\Psi))$ is independent of $k > 0$.

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Next we prove the zero-one property. To this purpose, by Lemma 2.5 it suffices to prove that, for example, for any fixed $\mathbf{X}^{(2)}, \dots, \mathbf{X}^{(m)} \in \mathbb{I}^n$, the set of $\mathbf{X}^{(1)} \in \mathbb{I}^n$ for which $\prod_{j=1}^m |\mathbf{q}\mathbf{X}^{(j)} + p_j| < k\Psi(\mathbf{q})$ holds for infinitely many pairs $\mathbf{q} \in \mathbb{Z}^n \backslash \{\mathbf{0}\}$, $\mathbf{p} = (p_1, \dots, p_m) \in \mathbb{Z}^m$ with $\gcd(\mathbf{p}, \mathbf{q}) = 1$, has n-dimensional Lebesgue measure either 0 or 1. Equivalently, we need to show that $\mathcal{C}^{\times}_{(\mathbf{X}^{(2)},\dots,\mathbf{X}^{(m)})}(k\Psi)$ introduced in the previous paragraph has n-dimensional Lebesgue measure either 0 or 1. But since this fact has been proved at the places ♣ and ♠, we are done. □

Remark 4.2. Similar to the proof of the zero-one part of Theorem 4.1, one can replace (1.2) with Lemma 2.5 as a substitute tool in the proofs of the corresponding parts of Theorems 3.1–3.3.

THEOREM 4.3. *For $\mathcal{G} \in \{\mathcal{A}, \mathcal{B}\}$ and any non-negative function Ψ on $\mathbb{Z}^n \backslash \{\mathbf{0}\}$, $\mathcal{M}_{nm}(\mathcal{G}^{\times}_{n,m}(k\Psi)) \in \{0, 1\}$ is independent of $k > 0$.*

Proof. The zero-one property part of Theorem 4.3 is just [**4**, Theorem 4], which was introduced in the introduction. The proof of the independence part of the theorem is pretty much the same as those corresponding parts of Theorems 3.1, 3.2 and 4.1 with obvious modification, and we leave the details to the interested reader. □

As consequences of all the theorems established so far, we have (1.3) as well as

$$\mathcal{M}_{nm}\left(\bigcup_{k\in\mathbb{N}} \mathcal{F}^{\times}_{n,m}(k\Psi)\right) = \mathcal{M}_{nm}\left(\bigcap_{k\in\mathbb{N}} \mathcal{F}^{\times}_{n,m}\left(\frac{\Psi}{k}\right)\right). \tag{4.2}$$

Finally, (1.5) follows immediately from (4.2).

§5. *Further results and questions.*

5.1. *Simultaneous approximation.* For the case $\Psi_1 = \cdots = \Psi_m$ in simultaneous Diophantine approximation, we refer to [**3**] for a survey of a series of conjectures (note in particular the Duffin–Schaeffer conjecture and the Catlin conjecture) and [**7, 13, 15, 16, 18, 21**] for several remarkable progresses. We also highlight the following clear-cut theorems without monotonicity assumptions, due to, respectively, Gallagher–Schmidt–Beresnevich–Velani (see e.g. [**7**]) and Pollington–Vaughan [**16**].

THEOREM 5.1. *Let $mn > 1$ and $\Psi(\mathbf{q}) = \psi(|\mathbf{q}|) : \mathbb{Z}^n \backslash \{\mathbf{0}\} \to [0, \infty)$. Then*

$$\mathcal{M}_{nm}(\mathcal{A}_{n,m}(\Psi, \dots, \Psi)) = 1 \Leftrightarrow \mathcal{M}_{nm}(\mathcal{C}_{n,m}(\Psi, \dots, \Psi)) = 1 \Leftrightarrow \sum_{q=1}^{\infty} q^{n-1}\psi(q)^m = \infty.$$

THEOREM 5.2. *Let $m > 1$ and $\psi : \mathbb{Z} \backslash \{0\} \to [0, \infty)$. Then*

$$\mathcal{M}_m(\mathcal{B}_{1,m}(\psi, \dots, \psi)) = 1 \Leftrightarrow \sum_{q\in\mathbb{Z}\backslash\{0\}} \left(\frac{\psi(q)\varphi(|q|)}{|q|}\right)^m = \infty,$$

where φ is Euler's totient function.

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5.2. *Multiplicative approximation.* For the case $\mathcal{G} \in \{\mathcal{A}, \mathcal{B}\}$ and $n = 1$ in multiplicative Diophantine approximation, we refer to [4] for appropriate conjectures and [4, 12] for Duffin–Schaeffer and Khintchine–Groshev type theorems. For the case $\mathcal{G} = \mathcal{C}$ and $n = 1$ and $m \geqslant 2$, we would like to propose the following conjecture.

Conjecture 5.3. Let $m \geqslant 2$ and $\psi : \mathbb{Z}\backslash\{0\} \to [0, \frac{1}{2}]$. Then

$$\mathcal{M}_m(\mathcal{C}^{\times}_{1,m}(\psi)) = 1 \Leftrightarrow \sum_{q \in \mathbb{Z}\backslash\{0\}} \psi(q) \log^{m-1} \frac{1}{\psi(q)} = \infty.$$

5.3. *Inhomogeneous approximation.* Given

$$\{\Psi_j : \mathbb{Z}^n\backslash\{\mathbf{0}\} \to [0, \infty)\}_{j=1}^m, \qquad \mathbf{b} = (b_1, \ldots, b_m) \in \mathbb{R}^m,$$

we may consider the inhomogeneous Diophantine approximation

$$|\mathbf{q}\mathbf{X}^{(j)} + p_j + b_j| < \Psi_j(\mathbf{q}) \quad (j = 1, \ldots, m), \tag{5.1}$$

and define

$$\begin{aligned}
\mathcal{A}^{\mathbf{b}}_{n,m}(\Psi_1, \ldots, \Psi_m) &= \{\mathbf{X} \in \mathbb{I}^{nm} : (5.1) \text{ holds for i.m. } (\mathbf{p}, \mathbf{q})\},\\
\mathcal{B}^{\mathbf{b}}_{n,m}(\Psi_1, \ldots, \Psi_m) &= \{\mathbf{X} \in \mathbb{I}^{nm} : (5.1) \text{ holds for i.m. } (\mathbf{p}, \mathbf{q})\\
&\qquad \text{with } \gcd(p_j, \mathbf{q}) = 1 \ (\forall j)\},\\
\mathcal{C}^{\mathbf{b}}_{n,m}(\Psi_1, \ldots, \Psi_m) &= \{\mathbf{X} \in \mathbb{I}^{nm} : (5.1) \text{ holds for i.m. } (\mathbf{p}, \mathbf{q})\\
&\qquad \text{with } \gcd(\mathbf{p}, \mathbf{q}) = 1\},
\end{aligned}$$

where as usual

$$\mathbf{X} = (\mathbf{X}^{(1)}, \ldots, \mathbf{X}^{(m)}) \in \mathbb{I}^{nm}, \qquad \mathbf{q} \in \mathbb{Z}^n\backslash\{\mathbf{0}\}, \qquad \mathbf{p} = (p_1, \ldots, p_m) \in \mathbb{Z}^m.$$

In view of quite a few examples in [1–3, 5, 8], it is reasonable to raise the following question.

Question 5.4. Is it true that $\mathcal{M}_{nm}(\mathcal{F}^{\mathbf{b}}_{n,m}(\Psi_1, \ldots, \Psi_m))$ is independent of $\mathbf{b} \in \mathbb{R}^m$?

In much the same way, one can first introduce $\mathcal{A}^{\mathbf{b},\times}_{n,m}(\Psi)$, $\mathcal{B}^{\mathbf{b},\times}_{n,m}(\Psi)$ and $\mathcal{C}^{\mathbf{b},\times}_{n,m}(\Psi)$, then propose a similar question.

Question 5.5. Is it true that $\mathcal{M}_{nm}(\mathcal{F}^{\mathbf{b},\times}_{n,m}(\Psi))$ is independent of $\mathbf{b} \in \mathbb{R}^m$?

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Liangpan Li,
Department of Mathematical Sciences,
Loughborough University,
Loughborough LE11 3TU,
U.K.
E-mail: liliangpan@gmail.com

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