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Mixing with staircase multiplicity functions

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Abstract. Every subgroup of the symmetric group defines a natural factor of the Cartesian power of a transformation. We calculate the set of values of the spectral multiplicity function of such factors (under certain conditions on the transformation) in terms of the number of orbits of diagonal actions of these subgroups. An analogous statement also holds on the unitary level for operators that preserve **1**. In particular, we prove that for every positive integer n, there exists a transformation which is mixing of all orders and has a staircase multiplicity function of length n; that is, the essential values of the spectral multiplicity function are $\{1, 2, \ldots, n\}$.

1. *Introduction*

By a dynamical system T we mean an invertible measure-preserving map (transformation) acting on a non-atomic Lebesgue space (X, μ). We are interested in spectral invariants of T, i.e. the spectrum of a unitary (Koopman) operator induced on $L_2 : \widehat{T} f(x) = f(Tx)$, which have become classical in the theory of dynamical systems. In this paper we study (in a special interesting subclass) the following general problem: determine precise conditions on the spectrum of a unitary operator under which it can be realized by a dynamical system (see [**6**, p. 36]).

We will concentrate our attention on the spectral multiplicity function $M(\widehat{T})$ and its values, denoted by the same symbol. One says that a transformation has a *homogeneous* spectrum of multiplicity n if $M(\widehat{T}) \equiv n$, where the Koopman operator $\widehat{T}$ acts on the zero-mean invariant subspace $L_2^0 = \{f \in L_2 : \int f d\mu = 0\}$. We refer the reader to surveys [**9, 13**], which give more detailed information on the history of spectral invariants in ergodic theory; here we mention only the recent solution of the homogeneous spectrum problem (see [**2, 18**] for the case $n = 2$, and [**4**] for the case of complete generality). However, possible values of the spectral multiplicity function have been little-studied for

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mixing transformations, because many methods applicable to typical transformations do not give stronger results in the subset of mixing transformations.

We propose to study $M(\widehat{T})$ for mixing T. Staircase-like constructions of rank-one transformations are used as examples to construct series of mixing transformations with various multiplicity functions. In particular, we prove that there exist dynamical systems with staircase multiplicity functions. More precisely, we will show the following.

THEOREM 1. *For any n, there exists a mixing transformation T such that*

$$M(\widehat{T}) = \{2, 3, \ldots, n\} \quad \text{on } L_2^0$$

and hence

$$M(\widehat{T}) = \{1, 2, 3, \ldots, n\} \quad \text{on } L_2.$$

Remark 1. For $n = 2$, this result was proved by Ryzhikov (see [**19**]).

Remark 2. Theorem 1 also provides an answer to a question posed by Robinson (see [**16**] and [**9**, §5.3]) about possible values of $m(T) = \max M(\widehat{T})$; he showed in [**16**] that for any n there exists a mixing T with $n < m(T) < \infty$.

Remark 3. Every mixing transformation constructed in this paper is mixing of all orders; in other words, it is k-fold mixing for every $k > 1$ (see Theorem 3).

The paper is organized as follows. In §2 we introduce and study briefly a combinatorial characteristic $D_G(n)$ of a fixed subgroup G of the symmetric group $\mathfrak{S}_n$. In §§3 and 4 we will show that $D_G(n)$ has a direct connection to the spectral multiplicity functions of natural factors of the Cartesian powers; see Theorems 2 and 3. Our main result, Theorem 3 (of which Theorem 1 is a particular case), is proved in §4 as an application of a technical result from §3 that can be viewed as an elaborated version of the classical analysis of multiplicity for Cartesian products (see [**3, 8, 12**]). Roughly speaking, to have a new series of sets (without 1!) as the set of essential values of the spectral multiplicity function on L_2^0 in the mixing case, we just need to calculate $D_G(n)$. This construction can be considered as a source of (mixing) transformations with new (and highly non-homogeneous) spectral multiplicity functions.

2. *Elementary properties of $D_G(n)$*

Let $\mathfrak{S}_n$ be the *symmetric* group acting on $\{1, \ldots, n\}$ by permutations, and let G be some subgroup of $\mathfrak{S}_n$. For $1 \le k \le n$, denote by the same symbol G the *diagonal* action of G on $I_k = \{1, \ldots, n\}^k$. Consider the restriction of the G-action to the G invariant subset

$$I_k' = \{i_k = (i_k(1), \ldots, i_k(k)) \in I_k : i_k(l) = i_k(m) \text{ iff } l = m,\ 1 \le l, m \le k\}.$$

The orbital equivalence relation (on I_k' or I_k) induced by the action of G is given by

$$i_k \sim j_k \Leftrightarrow \text{there exists } g \in G \text{ such that } g(i_k) = j_k.$$

Let us denote by D_k the number of orbits of G on I_k' $(1 \le k \le n)$. Let $D_G(n) = (D_1, \ldots, D_n)$.

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PROPOSITION 1. *Let G be a subgroup of $\mathfrak{S}_n$. Then $D_G(n)$ has the following properties:*

(1) $D_G(n) = (1, \ldots, 1)$ *if and only if* $G = \mathfrak{S}_n$;
(2) $D_G(n) = (n, n(n-1), \ldots, n!, n!)$ *if* $G = \{e\}$;
(3) $D_G(n) = (2, 3, \ldots, n, n)$ *if* $G = \{g \in \mathfrak{S}_n : g(n) = n\}$;
(4) $D_k \le D_m$ *if* $k < m$;
(5) $D_{n-1} = D_n$;
(6) $D_1 = D_2 = \cdots = D_n$ *if and only if* $n = 2$ *or* $G = \mathfrak{S}_n$;
(7) $D_n = n!/\sharp G$;
(8) $D_G(n) \le D_{G_1}(n)$ *if* G_1 *is a subgroup of* G.

We leave the proof of these simple statements to the reader. Note that part (6) follows from the obvious inequality $D_2 \ge D_1(D_1 - 1)$.

Remark 4. $D_G(n)$ is closely related to the average number of fixed points. Indeed, the Cauchy–Frobenius lemma (see [**14**]) implies that if G acts by permutations on a finite set X, and D is the number of its orbits, then

$$D = \frac{1}{\sharp G} \sum_{g \in G} \sharp F(g), \tag{1}$$

where $F(g)$ denotes the set of all fixed points of g. We associate with G the polynomial $P(z) = \sum b_i z^i$, where $b_i = \sharp\{g \in G : \sharp F(g) = i\}$. Obviously, $P(1) = \sharp G$. Using (1), it is easy to see that

$$D_G(n) = (P'(1), \ldots, P^{(n)}(1))/P(1).$$

This expression for $D_G(n)$ gives no extra information concerning possible values of $D_G(n)$, but it is more useful for calculations. For example, if we consider a unique non-trivial normal subgroup G of $\mathfrak{S}_n$, i.e. take $n = 4$ and G to be the set of all permutations with two-cycles of length 2 together with e, then we have $P(z) = 3 + z^4$ and $D_G(n) = (1, 3, 6, 6)$.

Remark 5. $D_G(n)$ can be also expressed in the language of probability as follows.

$$D_G(n) = (E(\xi), E(\xi(\xi - 1)), \ldots, E(\xi(\xi - 1) \cdots (\xi - n + 1))),$$

where G is equipped with the uniform distribution, $E(\eta)$ is the expectation of the random variable η, and $\xi(g)$ is the number of fixed points of g.

3. *Multiplicity functions of parts of tensor products*

Given a subgroup G of $\mathfrak{S}_n$ and $(D_1, \ldots, D_n) = D_G(n)$, let

$$M_G(n) = \{D_1, \ldots, D_n\}.$$

Each element g of G naturally defines a permutation of the coordinates viewed as a transformation on (X^n, μ^n), which we shall also denote by g, given by $g(x_1, \ldots, x_n) = (x_{g(1)}, \ldots, x_{g(n)})$; consequently, it also defines a unitary operator on $L_2(X^n, \mu^n)$. Thus we have a unitary representation of G by operators on $L_2(X^n, \mu^n)$; let us call it V. Define

$$H_{\mathrm{inv}}(G) = \{f \in L_2 : \forall g \in G,\ V_g f = f\}.$$

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Then the subspace of all square-integrable functions which are invariant with respect to any permutation of the coordinates $H_{\mathrm{sym}}(n)$ coincides with $H_{\mathrm{inv}}(\mathfrak{S}_n)$. Obviously, for any unitary operator U acting on $L_2(X, \mu)$ and any $g \in \mathfrak{S}_n$, we have $U^{(n)}V_g = V_g U^{(n)}$ and $U^{(n)}H_{\mathrm{inv}}(G) = H_{\mathrm{inv}}(G)$. Here, $U^{(n)} = U \otimes \cdots \otimes U$ is the n-fold tensor product of U. Put $U_G^{(n)} = U^{(n)}|_{H_{\mathrm{inv}}(G)}$. Let us remark that $U_G^{(n)}$ can be viewed as a natural generalization of the operator adjoint to (measure-theoretical) factors of dynamical systems (see §4).

Given a Hilbert subspace H of $L_2(Y, \nu)$, we will write $H^0 = \{f \in H : \int f d\nu = 0\}$. Obviously, if $\mathbf{1} \in H$, then $H = H^0 \oplus \{c\mathbf{1} : c \in \mathbb{C}\}$.

THEOREM 2. *Suppose that a unitary operator U defined on L_2 preserves $\mathbf{1}$, i.e. $U(\mathbf{1}) = \mathbf{1}$†, and $U^{(n)}$ has simple continuous spectrum on $H^0_{\mathrm{sym}}(n)$; then*

$$M(U_G^{(n)}|_{L_2^0}) = M_G(n). \tag{2}$$

Remark 6. It is not difficult to construct a unitary operator U with $U(\mathbf{1}) = \mathbf{1}$ such that $U^{(n)}$ has simple spectrum on $H_{\mathrm{sym}}(n)$ (and $H^0_{\mathrm{sym}}(n)$) but (2) does not hold for some G. However, in the case of a unitary operator U that comes from a dynamical system, the simplicity of the spectrum of $U^{(n)}$ on $H_{\mathrm{sym}}(n)$ implies that U on L_2^0 has continuous spectrum, and consequently the same is true for the restriction of $U^{(n)}$ to $H^0_{\mathrm{sym}}(n)$.

Denote by σ_U the measure of the maximal spectral type for U on the invariant subspace L_2^0, and denote by σ^n the measure $\sigma_U \times \cdots \times \sigma_U$ (n times) on $\mathbb{T}^n$. The image $\sigma^{(n)}$ of the measure σ^n under the map

$$\pi : (\lambda_1, \ldots, \lambda_n) \to \lambda_1 \cdots \lambda_n,$$

given by $\sigma^{(n)} = \pi(\sigma^n)$, is called the nth *convolution* of measures σ_U.

Let us first prove the following lemma.

LEMMA 1. *Assume that U satisfies the conditions of Theorem 2; then for any $k \le n$ and $i_k \in I_k'$, there exists a (non-zero) Hilbert subspace H_{i_k} in $L_2^0(X^n, \mu^n)$ such that the following properties hold:*

(1) $L_2^0(X^n, \mu^n) = \bigoplus_{k \le n} \bigoplus_{i_k \in I_k'} H_{i_k}$;

(2) *H_{i_k} is orthogonal to H_{j_m} if and only if $i_k \ne j_m$;*

(3) *for any $i_k \in I_k'$, $U^{(n)}H_{i_k} = H_{i_k}$ and, moreover, $U^{(n)}|_{H_{i_k}}$ has simple spectrum with $\sigma_{U^{(n)}|_{H_{i_k}}} \sim \sigma^{(k)}$;*

(4) *for any $g \in \mathfrak{S}_n$ and $i_k \in I_k'$, $V_g H_{i_k} = H_{g(i_k)}$ and if $g(i_k) = i_k$, then $V_g|_{H_{i_k}} = \widehat{E}$ (henceforward E is the identity map).*

Proof. Consider the map $s : L_2^0(X, \mu) \to H^0_{\mathrm{sym}}(n)$ defined by

$$s(\phi) = \frac{1}{\sqrt{n}} \sum_i \phi(x_i).$$

The map s is a unitary isomorphism between the Hilbert spaces $L_2^0(X, \mu)$ and $s(L_2^0(X, \mu))$, and we have $sU = U^{(n)}s$. Therefore the unitary operator U has simple continuous

† Following the same scheme, a certain analog of Theorem 2 can be deduced, *mutatis mutandis*, for unitary operators on a separable Hilbert space that leave a one-dimensional subspace invariant.

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spectrum on $L_2^0(X, \mu)$. This implies that U has simple spectrum on $L_2(X, \mu)$, and σ_U is a continuous measure.

By the spectral theorem, there exists a unitary isomorphism Φ between the Hilbert spaces $L_2(X, \mu)$ and $L_2(\mathbb{T}, \sigma_1)$, where $\sigma_1 = (\sigma_U + \delta)/2$, such that the image of U is just multiplication by λ. Here, δ is the Dirac measure at 1, i.e. $\delta(\{1\}) = 1$. Denote by φ the pre-image of $\mathbf{1}$ under Φ. Since $\sigma_U(\{1\}) = 0$, we have $\Phi(L_2^0(X, \mu)) = \mathbf{1}_{\mathbb{T}\setminus\{1\}} L_2(\mathbb{T}, \sigma_1)$ and $\Phi(\int \varphi \, d\mu) = \mathbf{1}_{\{1\}}$, where, as usual, $\mathbf{1}_B$ denotes the indicator of B.

We will then repeatedly use the unitary isomorphism $\Phi_n = \Phi \otimes \cdots \otimes \Phi$ (n times) between Hilbert spaces $L_2(X^n, \mu^n)$ and $L_2(\mathbb{T}^n, \sigma_1^n)$. Obviously, $\Phi_n \varphi_n = \mathbf{1}$ ($= \mathbf{1}_{\mathbb{T}^n}$), where $\varphi_n(x_1, \ldots, x_n) = \varphi(x_1) \cdots \varphi(x_n)$.

For each $i_k \in I_k'$ ($1 \le k \le n$), let B^{i_k} be the set of all $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{T}^n$ such that $\lambda_j = 1$ if $j \notin \{i_k(1), \ldots, i_k(k)\}$ and $\lambda_j \ne 1$ otherwise, where $i_k = (i_k(1), \ldots, i_k(k))$ and $j = 1, \ldots, n$. We will use the order on $\mathbb{T}\setminus\{1\}$ coming from the usual correspondence between $\mathbb{T}\setminus\{1\}$ and $(0, 1)$. Put

$$B_{i_k} = \{(\lambda_1, \ldots, \lambda_n) \in B^{i_k} : \lambda_{i_k(1)} < \lambda_{i_k(2)} < \cdots < \lambda_{i_k(k)}\}.$$

It is easy to see that the sets B_{i_k} ($i_k \in I_k'$, $1 \le k \le n$) are mutually disjoint. The complement of the union of B_{i_k} ($i_k \in I_k'$, $1 \le k \le n$) and $\{(1, \ldots, 1)\}$ is contained in $\bigcup_{1 \le j_1 < j_2 \le n} R_{j_1, j_2}$, where

$$R_{j_1, j_2} = \{(\lambda_1, \ldots, \lambda_n) \in \mathbb{T}^n : \lambda_{j_1} = \lambda_{j_2} \ne 1\} \quad (j_1 < j_2 \le n).$$

Since σ_U is a continuous measure, for all $j_1 < j_2$ we have

$$\sigma_1^n(R_{j_1, j_2}) = \sigma^2\{(\lambda_1, \lambda_2) \in \mathbb{T}^2 : \lambda_1 = \lambda_2\} = 0.$$

Collecting together all the above remarks, we conclude that $L_2(\mathbb{T}^n, \sigma_1^n)$ can be represented as

$$L_2(\mathbb{T}^n, \sigma_1^n) = \mathbf{1}_{\{(1, \ldots, 1)\}} L_2(\mathbb{T}^n, \sigma_1^n) \oplus \bigoplus_{k \le n} \bigoplus_{i_k \in I_k'} \mathbf{1}_{B_{i_k}} L_2(\mathbb{T}^n, \sigma_1^n).$$

For any $i_k \in I_k'$, now put $H_{i_k} = \Phi_n^{-1}(\mathbf{1}_{B_{i_k}} L_2(\mathbb{T}^n, \sigma_1^n))$ by definition. Part (1) of the lemma follows immediately because $\Phi_n(c\mathbf{1}) = \mathbf{1}_{\{(1, \ldots, 1)\}}$, where $c = (\int \varphi \, d\mu)^n$.

Parts (2) and (4) are obvious, because for any $g \in \mathfrak{S}_n$, an image of V_g via Φ_n is just the unitary operator defined by the permutation g of coordinates on $\mathbb{T}^n$.

In order to prove part (3), first note that for every $i_k \in I_k'$, we have $U^{(n)} H_{i_k} = H_{i_k}$ because the operator $U^{(n)}$ on $L_2(X^n, \mu^n)$ is unitarily equivalent to the operator $\tilde{U}^{(n)} = \lambda_1 \cdots \lambda_n \cdot \widehat{E}$ on $L_2(\mathbb{T}^n, \sigma_1^n)$ via Φ_n. Now fix k. By part (4), the spectral invariants of $U^{(n)}$ on H_{i_k} do not depend on our choice of H_{i_k}. Let $i_k = (1, \ldots, k)$. Obviously, the Hilbert space $\mathbf{1}_{B_{i_k}} L_2(\mathbb{T}^n, \sigma_1^n)$ is naturally isomorphic to $\mathbf{1}_C L_2(\mathbb{T}^k, \sigma^k)$, where $C = \{(\lambda_1, \ldots, \lambda_k) \in \mathbb{T}^k : \lambda_1 < \cdots < \lambda_k\}$, and the image of $\tilde{U}^{(n)}$ under this isomorphism is $\tilde{U}^{(k)}$. Since σ_U is a continuous measure, we have

$$\mathbf{1}_{\mathbb{T}^k} = \sum_{g \in \mathfrak{S}_k} \mathbf{1}_{gC} \quad \sigma^k\text{-a.e.}$$

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As σ^k is a symmetric measure, for all integers p we have

$$\langle \tilde{U}^{(k)p}\mathbf{1}_C, \mathbf{1}_C\rangle = \frac{1}{\sharp \mathfrak{S}_k}\langle \tilde{U}^{(k)p}\mathbf{1}, \mathbf{1}\rangle = \frac{1}{k!}\left(\int \lambda^p \, d\sigma_U\right)^k.$$

Using the fact that the Fourier transform of a convolution of measures is the product of the Fourier transforms of the individual measures, we see that the spectral measure of $\mathbf{1}_C \in L_2(\mathbb{T}^k, \sigma^k)$ is $\sigma^{(k)}/k!$. It is clear that $\sigma^{(k)}$ is the measure of the maximal spectral type of $\tilde{U}^{(k)}$ on $L_2(\mathbb{T}^k, \sigma^k)$; therefore $\sigma^{(k)}$ is the measure of the maximal spectral type of $U^{(n)}$ on H_{i_k}.

Finally, fix $\phi \in H_{i_k}$ and $i'_k \in I'_k$. Put $\phi_{i'_k} = V_g\phi$ if $V_g\phi \in H_{i'_k}$ for some $g \in \mathfrak{S}_n$; note that $\phi_{i'_k}$ is well-defined for every $i'_k \in I'_k$ because of part (4). To show the simplicity of the spectrum of $U^{(n)}$ on H_{i_k}, consider the symmetrization map $\psi : H_{i_k} \to H^0_{\mathrm{sym}}(n)$ defined by $\psi(\phi) = \sum_{i'_k \in I'_k} \phi_{i'_k}$. It is easy to check both that $\psi U^{(n)} = U^{(n)}\psi$ and that $c\psi$ is a unitary isomorphism between H_{i_k} and $\psi(H_{i_k})$ for some positive real c. This implies that $U^{(n)}$ has simple spectrum on H_{i_k}. Thus Lemma 1 is proved. □

Proof of Theorem 2. Given $k \le n$, $i_k, i'_k \in I'_k$, and $f \in H_{i_k}$, let $O_{i_k}(G) = \{g(i_k) : g \in G\}$. Fix $g \in G$ such that $i'_k = g(i_k)$, and denote by $f_{i'_k}$ the function $V_g f$. By Lemma 1, $f_{i'_k} \in H_{i'_k}$, and for any $i'_k \in O_{i_k}(G)$, $f_{i'_k}$ is independent of our choice of g. Consider the symmetrization map $SG_{i_k} : H_{i_k} \to \bigoplus_{i'_k \in O_{i_k}(G)} H_{i'_k}$ given by

$$SG_{i_k}(f) = \frac{1}{\sqrt{\sharp O_{i_k}(G)}} \sum_{i'_k \in O_{i_k}(G)} f_{i'_k}.$$

Put

$$H_{\mathrm{sym}}(i_k, G) = H_{\mathrm{inv}}(G) \cap \bigoplus_{i'_k \in O_{i_k}(G)} H_{i'_k}.$$

It is easy to see that $SG_{i_k}(H_{i_k}) = H_{\mathrm{sym}}(i_k, G)$. Clearly, SG_{i_k} is a unitary isomorphism. Moreover, SG_{i_k} intertwines $U^{(n)}$ and $U^{(n)}_G$ (and equals $U^{(n)}$ on $H_{\mathrm{inv}}(G)$) because, for any g, $V_g U^{(n)} = U^{(n)} V_g$. Thus Lemma 1 part (3) implies that $U^{(n)}_G|_{H_{\mathrm{sym}}(i_k,G)}$ has simple spectrum of maximal spectral type $\sigma^{(k)}$. Consequently, $U^{(n)}_G|_{\bigoplus_{i_k \in I'_k} H_{i_k}}$ has D_k simple (mutually orthogonal) components of maximal spectral type $\sigma^{(k)}$, because $H^0_{\mathrm{inv}}(G)$ is the sum of $U^{(n)}_G$ invariant mutually orthogonal subspaces $H_{\mathrm{sym}}(i_k, G)$, $i_k \in I'_k$, $k = 1, \ldots, n$.

To prove Theorem 2, it remains to show that $\sigma^{(k_1)} \perp \sigma^{(k_2)}$ if $k_1 < k_2 \le n$. Notice first that the $U^{(n)}$ invariant subspaces $H_{\mathrm{sym}}(i_{k_j}, \mathfrak{S}_n)$ $(j = 1, 2)$ of $H_{\mathrm{sym}}(n)$ are mutually orthogonal. The operator $U^{(n)}$ has simple spectrum on $H_{\mathrm{sym}}(n)$. Therefore measures of the maximal spectral type $(\sigma^{(k_j)})$ of $U^{(n)}$ on these subspaces $(j = 1, 2)$ are mutually disjoint. This completes the proof of Theorem 2. □

4. *Applications to mixing transformations*

Every subgroup G of $\mathfrak{S}_n$ naturally acts by permutations of coordinates on X^n. The orbital equivalence relation (on X^n) induced by the action of G is given by

$$x \sim y \;\Leftrightarrow\; \text{there exists } g \in G \text{ such that } g(x) = y.$$

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Since, for any $g \in \mathfrak{S}_n$, the transformations g and $T^{(n)}$ commute, $T^{(n)}$ defines a transformation, which we shall denote $T_G^{(n)}$, on a non-atomic Lebesgue space $(X^n/\sim, \varphi(\mu^n))$, where φ is a canonical map from X^n to $X^n/\sim$. The dynamical system $T_G^{(n)}$ is called a *factor* of $T^{(n)}$. It is easy to see that the Koopman operator $\widehat{T_G^{(n)}}$ is unitarily equivalent to $\widehat{T}_G^{(n)}$.

Theorem 1 is a natural corollary of Proposition 1 and the following theorem.

THEOREM 3. *For any subgroup G of $\mathfrak{S}_n$, there exists a transformation T which is mixing of all orders and which satisfies*

$$M(\widehat{T}|_{L_2^0}) = M_G(n).$$

Proof. It is an immediate consequence of the definitions that for every positive integer k, both Cartesian products and factors of k-fold mixing transformations are also k-fold mixing. Let us also recall that mixing implies mixing of all orders for rank-one transformations (see [**11, 17**]). Therefore, by Theorem 2, to prove Theorem 3 we only need to find a rank-one transformation T which is mixing, such that $\widehat{T}^{(n)}|_{H_{\mathrm{sym}}(n)}$ has simple spectrum for every n. However, such transformations exist even in the class of staircase constructions; this was announced by Ryzhikov (see [**19**, Theorem 5.2]) and proved for $n = 2$. For completeness, we give another collection of such examples (see §4.3), by employing spectral arguments from [**2**] and [**3**]. The proof is also based on explicit bounded non-mixing transformations with simple spectrum of their symmetric products (see §4.2). An appropriate approximation by these bounded transformations gives transformations as required†. □

4.1. *Staircase constructions.* A transformation T has *rank one* if for any m, there exist an integer h_m and a tower (column)

$$A_m, TA_m, \ldots, T^{h_m-1}A_m$$

such that all levels T^jA_m, $0 \le j < h_m$, and the remaining set form a measurable partition ξ_m of X, and $\xi_m \to \varepsilon$ in the sense that for any measurable set B there are ξ_m-measurable sets B_m such that $\mu(B \Delta B_m) \to 0$ as $m \to \infty$.

We will give the standard inductive procedure of *cutting* and *stacking* below, to define staircase transformations as rank-one transformations. We will then consider realizations of the rank-one transformations by piecewise shifts on disjoint unions of subintervals in $\mathbb{R}$, so that every level of every column is a subinterval and, also, for every column the restriction of the transformation to each level except the top one is the shift.

The rank-one transformation T is called a *staircase construction* if there exists a sequence $(r_m)_{m=1}^{\infty}$ of positive integers $r_m > 1$ such that each column C_{m+1} is obtained by cutting C_m into r_m subcolumns, say $C_m(i)$, of equal width, placing $i-1$ spacers only

† I would like to thank the referee for bringing to my attention a very recent paper [**20**], the main goal of which is the exposition of a method to construct (implicitly) mixing transformations with simple spectrum of their symmetric products. The method involves different modifications of unbounded staircase non-mixing transformations for approximations. These are more 'random'; therefore they form a 'larger' class but are less explicit.

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on the subcolumn $C_m(i)$ for $1 \le i < r_m$, and then stacking the subcolumn $C_m(i+1)$ on top of $C_m(i)$ for $1 \le i < r_m$. For the column C_m, let h_m be its height and let d_m be the measure of its one level, i.e. its width, where $m \ge 1$. Note that this is a modification of the usual staircase construction (see [**1**]), because we do not add spacers on the top of the subcolumn $C_m(r_m)$. The pair $(h_1, (r_m)_{m=1}^{\infty})$ defines a unique staircase transformation up to a measure-theoretical isomorphism. Since our further considerations are independent of the initial choice of h_1, we will denote such a transformation by $T = T_{(r_m)}$.

Let $O_m(0) = C_m \setminus C_m(1)$ and define $O_m(j) = C_m(1) \cap \cdots \cap C_{m+j-1}(1) \setminus C_{m+j}(1)$ for $j = 1, 2, \ldots$. From the geometry of the construction we immediately obtain:

(1) each C_m consists of pairwise disjoint subcolumns $O_m(j)$ $(j = 0, 1, 2, \ldots)$ of height h_m and width $(r_{m+j} - 1)d_{m+j+1}$;

(2) each $O_m(j)$ consists of pairwise disjoint subcolumns $O_m(j) \cap C_{m+j}(i)$ $(i = 2, \ldots, r_{m+j})$ of height h_m and equal width;

(3) $T^{-h_m}\Delta_k(i, j) \subset T^{i-2}\Delta_k$, where $\Delta_k(i, j)$ is kth level of $O_m(j) \cap C_{m+j}(i)$, Δ_k is kth level of C_m, $i = 2, \ldots, r_{m+j}$, $k = 1, \ldots, h_m$, and $j = 0, 1, \ldots$.

A priori every staircase transformation constructed above acts on its collection of mutually disjoint intervals (levels) of $\mathbb{R}$. Note that all the properties we are interested in are independent of the mutual location of these intervals in $\mathbb{R}$. We will restrict ourselves to the staircase transformations acting on a space X of finite measure, called *finite* staircase transformations. If the sequence (r_m) does not grow too fast, then the staircase transformation $T = T_{(r_m)}$ is finite. In particular, this is true if (r_m) is bounded, in which case we say that $T_{(r_m)}$ is a *bounded* staircase transformation. By appropriately choosing the width of the first column level (d_1), we can suppose that all finite staircase transformations (up to a measure-theoretical isomorphism) are defined on the unit interval in $\mathbb{R}$ equipped with the usual Lebesgue measure.

4.2. *Subsets of limit polynomials.* Next we will make use of the compactness of the set $\{U \in \mathcal{L}(L_2) : \|U\| \le 1\}$ with respect to the weak operator topology, where $\mathcal{L}(L_2)$ is the space of bounded operators on L_2.

Define

$$P_k(z) = \frac{1}{k-1}\sum_{j=0}^{k-2} z^j \quad (k > 1),$$

and

$$B_{p,\alpha,\beta}(z) = \frac{1}{1+\alpha+\beta}(P_p(z) + \alpha P_{p+1}(z) + \beta P_{p+2}(z)) \quad (\alpha, \beta \in \mathbb{R}).$$

PROPOSITION 2. *For any $p > 1$, there exist a sequence of positive integers r_m, where $|r_m - p - 1| \le 1$, and a Cartesian product of countable subsets I_i $(i = 1, 2)$ of $\mathbb{R}$, such that for every pair $(\alpha, \beta) \in I_1 \times I_2$,*

$$\widehat{T}^{-h_{m_k}} \to B_{p,\alpha,\beta}(\widehat{T}) \quad \text{as } k \to +\infty, \tag{3}$$

where $T = T_{(r_m)}$ and $m_k = m_k(\alpha, \beta) \to +\infty$.

Proof. Fix a sequence (r_m) with $|r_m - p - 1| \le 1$ for every m. It is enough to check that $\langle \widehat{T}_{(r_m)}^{-h_{m_k}} f, g\rangle \to \langle B_{p,\alpha,\beta}(\widehat{T}_{(r_m)}) f, g\rangle$ for every pair (f, g), where f and g are elements of

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some dense set in $L_2(X, \mu)$. Therefore we can assume that f and g are constants, say $f_m(j)$ and $g_m(j)$, on the jth level of C_m, for every j and for sufficiently large m.

Obviously, if $\mu(X_i) \to 0$, then $\mathbf{1}_{X_i}\widehat{T}^{k_i} \to 0$ for any sequence of integers k_i. Thus

$$\widehat{T}^{-h_m} - \sum_j \mathbf{1}_{O_m(j)}\widehat{T}^{-h_m} \to 0 \quad \text{as } m \to +\infty.$$

It is easy to see that on every set $O_m(j) \cap C_{m+j}(i)$ the function $\widehat{T}^{-h_m} f$ is 'almost' $\widehat{T}^{i-2} f$, and hence $\langle \mathbf{1}_{O_m(j)}\widehat{T}^{-h_m} f, g\rangle$ is 'almost' $\mu(O_m(j))\langle P_{r_{m+j}}(\widehat{T}) f, g\rangle$. Moreover,

$$\sum_j \langle \mathbf{1}_{O_m(j)}\widehat{T}^{-h_m} f, g\rangle - \sum_j \mu(O_m(j))\langle P_{r_{m+j}}(\widehat{T}) f, g\rangle \to 0 \quad \text{as } m \to +\infty.$$

Therefore

$$\widehat{T}^{-h_m} - \sum_j \mu(O_m(j)) P_{r_{m+j}}(\widehat{T}) \to 0 \quad \text{as } m \to +\infty.$$

Note that for any j,

$$\mu(O_m(j)) = \frac{r_{m+j} - 1}{r_m \cdots r_{m+j}}\mu(C_m),$$

and $\mu(C_m) \to 1$ as $m \to +\infty$.

Combining the above observations, we conclude that a sequence (r_m) which would suffice is, for example, a *weakly normal* sequence of symbols $\{p, p+1, p+2\}$, i.e. a sequence satisfying the following property: for each finite collection $(\alpha_1, \ldots, \alpha_k) \in \{p, p+1, p+2\}^k$, there exists a countable set of positive integers i_j such that, for every j, $r_{i_j+q} = \alpha_q$ for $q = 0, \ldots, k-1$. Indeed, fix positive integers q and n. Choose a sequence (m_k) such that for every k, $r_{m_k} = r_{m_k+q+n+1} = p$, $r_{m_k+j} = r_{m_k+q+n+1+j} = p+1$ for $j = 1, \ldots, q$, and $r_{m_k+q+i} = r_{m_k+2q+n+1+t} = p+2$ for $i = 1, \ldots, n$ and $t = 1, \ldots, k$. Using the above remarks, it is easy to see that the sequence $(\widehat{T}^{-h_{m_k}})$ tends to some $B_{p,\alpha,\beta}(\widehat{T})$, where α does not depend on n. Varying q and n, we extract an appropriate square $I_1 \times I_2$, and so Proposition 2 is proved. □

Every rank-one transformation has simple spectrum (see [**5**]). Therefore, with the notation described in §3, for any finite staircase transformation T, the isomorphism Φ_n between Hilbert spaces $L_2(X^n, \mu^n)$ and $L_2(\mathbb{T}^n, \sigma_1^n)$ is well-defined. Obviously, $\Phi_n\widehat{T}^{(n)} = \tilde{T}^{(n)}\Phi_n$, where $\tilde{T}^{(n)} = \lambda_1 \cdots \lambda_n \cdot \widehat{E}$ acts on $L_2(\mathbb{T}^n, \sigma_1^n)$.

PROPOSITION 3. *Take $n \in \mathbb{N}$, $p > 1$, and $T = T_{(r_m)}$ for some sequence (r_m) coming from Proposition 2; then*

$$\prod_{i=1}^n B_{p,\alpha,\beta}(\lambda_i) \in \overline{\text{span}}\{\tilde{T}^{(n)j}\mathbf{1} : j \in \mathbb{Z}\} \tag{4}$$

for all pairs (α, β) of real numbers.

Proof. It is clear that if $\widehat{T}^{n_i} \to Q$ for some linear operator Q, then, for any $f \in L_2$, $Qf \in \overline{\text{span}}\{\widehat{T}^j f : j \in \mathbb{Z}\}$. Using (3), we then have that

$$\tilde{T}^{(n)-h_{m_k}} \to \prod_{i=1}^n B_{p,\alpha,\beta}(\lambda_i) \quad \text{as } k \to +\infty$$

and, consequently, (4) is valid for every pair $(\alpha, \beta) \in I_1 \times I_2$.

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Consider a system of canonical conditional measures, say $\nu_n(\cdot|c)$, for σ_1^n naturally defined by a partition of $\mathbb{T}^n$ on the *fibres*

$$F(c) = \{(\lambda_1, \dots, \lambda_n) \in \mathbb{T}^n : \lambda_1 \cdots \lambda_n = c\}.$$

We say that f is *a.e. constant on fibres* if $f = \widetilde{f}$ σ_1^n-a.e., where for all c, $\widetilde{f}|_{F(c)}$ is identically equal to a constant. It is easy to see that $f \in L_2(\mathbb{T}^n, \sigma_1^n)$ is an element of $\overline{\operatorname{span}}\{\tilde{T}^{(n)j}\mathbf{1} : j \in \mathbb{Z}\}$ if and only if f is a.e. constant on fibres. Therefore, $\prod_{i=1}^n B_{p,\alpha,\beta}(\lambda_i)$ is a.e. constant on fibres for every pair $(\alpha, \beta) \in I_1 \times I_2$.

Given $\alpha \in I_1$ and a 'typical' fibre $F(c)$, consider $\prod_{i=1}^n B_{p,\alpha,\beta}(\lambda_i)$ as a polynomial of degree at most n in the variable β on the fibre $F(c)$. Since the number of $\beta \in I_2$ is countable, this polynomial is uniquely defined, i.e. all of its coefficients are equal to constants for almost every $(\lambda_1, \dots, \lambda_n)$ with respect to the measure $\nu_n(\cdot|c)$ on the fibre $F(c)$. Therefore, for every $\beta \in \mathbb{R}$, it is equal to a constant for $\nu_n(\cdot|c)$-a.e. $(\lambda_1, \dots, \lambda_n)$, where $\lambda_1 \cdots \lambda_n = c$. This implies that $\prod_{i=1}^n B_{p,\alpha,\beta}(\lambda_i)$ is a.e. constant on fibres for every $\beta \in \mathbb{R}$. Therefore (4) is valid for all real β.

Given an arbitrary real β, by repeating the proof we deduce that (4) is valid for all real α. Proposition 3 is proved. □

Recall that f is a cyclic vector of an invertible operator U acting on a Hilbert space H if $\overline{\operatorname{span}}\{U^j f : j \in \mathbb{Z}\} = H$.

PROPOSITION 4. *For every sequence (r_m) coming from Proposition 2 and for any positive integer n, $T^{(n)}_{(r_m)}$ has simple spectrum on $H_{\mathrm{sym}}(n)$. Moreover, if φ is a cyclic vector for $\widehat{T}_{(r_m)}|_{L_2}$, then φ_n is a cyclic vector for $\widehat{T}^{(n)}_{(r_m)}|_{H_{\mathrm{sym}}(n)}$, where $\varphi_n(x_1, \dots, x_n) = \varphi(x_1) \cdots \varphi(x_n)$.*

Proof. Given $\beta \in \mathbb{R}$ and $p > 1$, choose an α satisfying

$$\frac{\alpha}{p} + \frac{\beta}{p+1} = -\frac{1}{p-1}.$$

Then we have

$$B_{p,\alpha,\beta}(z) = \left(\frac{1}{1-p} + \frac{1}{1+p}\beta z\right) z^{p-1}.$$

Using Proposition 3, we then conclude that for any positive integer n and $\beta \in \mathbb{R}$,

$$\prod_{i=1}^n \lambda_i^{p-1}\left(\frac{1+p}{1-p} + \beta\lambda_i\right) \in \overline{\operatorname{span}}\{\tilde{T}^{(n)j}\mathbf{1} : j \in \mathbb{Z}\}$$

or, equivalently,

$$P_n(\lambda_1, \dots, \lambda_n, \Theta) = \prod_{i=1}^n (\Theta + \lambda_i) \in \overline{\operatorname{span}}\{\tilde{T}^{(n)j}\mathbf{1} : j \in \mathbb{Z}\} \quad \text{for all } \Theta \in \mathbb{R}.$$

Arguing as in the proof of Proposition 3, we see that $P_n(\lambda_1, \dots, \lambda_n, \cdot)$ has only one collection of zeros for $\nu_n(\cdot|c)$-a.e. $(\lambda_1, \dots, \lambda_n)$, where $\lambda_1 \cdots \lambda_n = c$. By the definition, $(-\lambda_1, \dots, -\lambda_n)$ are zeros of $P_n(\lambda_1, \dots, \lambda_n, \Theta)$. This implies that supp $\nu_n(\cdot|c)$ is a single point modulo permutations of the coordinates. Furthermore, since σ_1^n is a symmetric measure, $\nu_n(\cdot|c)$ is also a symmetric measure for $\sigma_1^{(n)}$-a.e. c.

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Collecting together all the above remarks, we conclude that every function in $\Phi_n H_{\mathrm{sym}}(n)$ is a.e. constant on fibres and, consequently,

$$\Phi_n H_{\mathrm{sym}}(n) = \overline{\mathrm{span}}\{\tilde{T}^{(n)j}\mathbf{1} : j \in \mathbb{Z}\}.$$

It follows that $\widehat{T}^{(n)}_{(r_m)}$ has simple spectrum on $H_{\mathrm{sym}}(n)$.

Given a cyclic vector φ for $\widehat{T}_{(r_m)}|_{L_2}$, consider a unitary isomorphism $\tilde{\Phi} = 1/\Phi\varphi\Phi$ between the Hilbert spaces $L_2(X, \mu)$ and $L_2(\mathbb{T}, \tilde{\sigma}_1)$, where $\tilde{\sigma}_1 = |\Phi\varphi|\sigma_1$. Since $\tilde{\sigma}_1$ is also the measure of the maximal spectral type for $\widehat{T}_{(r_m)}|_{L_2}$, and $\tilde{\Phi}_n \widehat{T}^{(n)}_{(r_m)} = \tilde{T}^{(n)}\tilde{\Phi}_n$ where $\tilde{\Phi}_n = \tilde{\Phi} \otimes \cdots \otimes \tilde{\Phi}$ (n times), we can conclude that

$$\tilde{\Phi}_n H_{\mathrm{sym}}(n) = \overline{\mathrm{span}}\{\tilde{T}^{(n)j}\mathbf{1} : j \in \mathbb{Z}\} \quad \text{in } L_2(\mathbb{T}^n, \tilde{\sigma}_1^n).$$

Obviously, $\tilde{\Phi}_n\varphi_n = \mathbf{1}$, and Proposition 4 is proved. □

Remark 7. There are many classical examples of explicit normal numbers to every base, going back to Champernowne's work [**7**]. Therefore, by means of these examples and Proposition 4, we constructed, in particular, a new series of explicit dynamical systems that satisfy the hypotheses of Theorem 2.

4.3. *Approximations by finite staircase transformations.* There are many ways to clarify the mixing properties of the staircase transformations constructed in Proposition 2. By using, for example, Remark 6 and Proposition 4, we can conclude that the transformations are all weakly mixing; besides, they are not mixing. This can be easily and directly proved by arguing as in the proof of Proposition 2. Indeed, we have $\widehat{T}_{r_m}^{-h_{m_k}} \to P(\widehat{T}_{r_m})$ for some sequence (m_k) and some polynomial $P(\cdot)$. It only remains to mention that a transformation T is mixing if and only if $\widehat{T}^k$ tends to the operator of the orthogonal projection onto the space of constants.

By repeating, with obvious changes, Adams' proof in [**1**] for standard generalizations of classical staircase transformations, it is not difficult to conclude that if (r_m) tends to $+\infty$, (r_m) is 'divergent' in the sense of [**1**] and its growth is not too fast (i.e. $\lim_{n\to\infty} r_m^2/h_m = 0$), then $T_{(r_m)}$ is also mixing.

Let $p_k \to +\infty$. For any $p = p_k$, choose a sequence, say $(r_m(k))$, as given in Proposition 2. We wish to consider limit constructions of some inductively defined sequence of finite staircase transformations $T_{(r_m(k))}$. It is convenient to construct every $T_{(r_m(k+1))}$ starting with an appropriate column of $T_{(r_m(k))}$, where without loss of generality we shall assume that the first column of $T_{(r_m(1))}$ has height 1 and width 1. Therefore $h_1(k+1) = h_{m(k)}(k)$, where the sequence $(m(k))$ will be chosen later. Let $C_{m,k}$ be the mth column of $T_{(r_m(k))}$. We will assume next that $T_{(r_m(k+1))}$ is equal to $T_{(r_m(k))}$ on every level of the column $C_{m(k),k}$ except the top one. It is convenient to choose the location of a union of levels of the column $C_{m,k}$ as an interval $[0, a(m, k))$ in $\mathbb{R}$. Note that, in general, the $T_{(r_m(k))}$ act on different spaces (X_k, μ), where $X_k = [0, a_k)$ with $a_k = \lim_{m\to\infty} a(m, k)$. However, if $m(k)$ tends to $+\infty$ sufficiently fast, then the difference is small, and $\mu(X_k) \to c$ for some $0 < c < \infty$.

Given a sequence $(m(k))$ of positive integers, define an unbounded staircase transformation $T_{(r_m)}$ so that for all k, the transformation $T_{(r_m)}$ is equal to $T_{(r_m(k))}$ on all

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but top levels of the column $C_{m(k),k}$. It is easy to see that $T_{(r_m)}$ is well-defined, acts on $X = [0, a)$ with $a = \lim_{k\to\infty} a(m(k), k)$, $h_1 = h_1(1)$, and

$$(r_1, r_2, \ldots, r_m, \ldots) = (r_1(1), \ldots, r_{m(1)-1}(1), r_1(2), \ldots, r_{m(2)-1}(2), \ldots). \quad (5)$$

PROPOSITION 5. *If the sequence $(m(k))$ tends to $+\infty$ sufficiently fast, then for any n the (mixing) transformation $T^{(n)}_{(r_m)}$ has simple spectrum on $H_{\mathrm{sym}}(n)$, where $T_{(r_m)}$ is the staircase transformation constructed above.*

Proof. Obviously, $r_m \to +\infty$, and for a sequence $(m(k))$ having sufficiently fast growth with respect to p_k, $\mu(X) = a < \infty$ and $r_m^2/h_m \to 0$. Therefore $T_{(r_m)}$ is mixing, and $T^{(n)}_{(r_m)}$ is mixing on every invariant subspace of $L_2(X^n, \mu^n)$.

Given a positive integer n and $L_2([0, 1)^n, \mu^n)$, fix a countable dense subset $\{f_{n,j}\}$ $(j \in \mathbb{N})$ of $H_{\mathrm{sym}}(n) \subseteq L_2([0, 1)^n, \mu^n)$. Let $b > 0$. The map $\psi_{1,b} : [0, 1) \to [0, b)$, $\psi_{1,b}(x) = bx$ defines a certain (non-unitary) isomorphism, say $\psi_{n,b}$, between the Hilbert spaces $L_2([0, 1)^n, \mu^n)$ and $L_2([0, b)^n, \mu^n)$, and between their corresponding symmetric spaces. Let $g_{n,j,b} = \psi_{n,b} f_{n,j}$ $(n, j \in \mathbb{N})$. It is convenient to consider every $g_{n,j,b}$ as an element of $L_2(\mathbb{R}^n, \mu^n)$, where $\operatorname{supp} g_{n,j,b} \subseteq [0, b)^n$.

Fix a sequence (ε_k) of positive numbers such that $\varepsilon_{k+1} < \varepsilon_k/2$ for every k. Given k, $m(k-1)$, transformations $T_{(r_m(s))}$ and a_s $(s \le k)$ are all uniquely defined. Assume we are given numbers $0 < \delta_s < \delta_{s-1}/2$ $(\delta_0 = 1,\ s = 1, \ldots, k-1)$ such that

$$|b - a_s| < \delta_s \ \Rightarrow \text{ for all } n \le s \text{ and all } j \le s,\ \|g_{n,j,b} - g_{n,j,a_s}\|_{L_2} < \varepsilon_s. \quad (6)$$

Define $0 < \delta_k < \delta_{k-1}/2$ to satisfy (6) for $s = k$.

Let us use the following technical proposition.

PROPOSITION 6. *Let T be a weakly mixing rank-one transformation of the Lebesgue space (X, μ), and let (ξ_m) be the corresponding sequence of partitions (see the definition in §4.1). Suppose (ξ_m) is monotonic and φ is the indicator function of an arbitrary non-empty union of levels of some column; then φ is a cyclic vector of $\widehat{T}$ on $L_2(X, \mu)$.*

Proposition 6 can be extracted from considerations in [**15**, p. 129]. However, it is possible to construct a proof (known to the author earlier) by applying only a technique that appeared in [**5**].

Suppose that χ is the indicator function of the first level of the column $C_{1,1}$, with $\chi_n(x_1, \ldots, x_n) = \chi(x_1) \cdots \chi(x_n)$. Assume we are given finite collections of complex numbers $c_q(n, j, s)$ $(s = 1, \ldots, k-1)$ which satisfy, for every pair (n, j),

$$\left\| g_{n,j,a_s} - \sum_q c_q(n, j, s) \widehat{T}^{(n)q}_{(r_m(s))} \chi_n \right\|_{L_2} < \varepsilon_s, \quad (7)$$

and for $n \le s$, $j \le s$, $s \le t \le k-1$,

$$\left\| \sum_q c_q(n, j, s) \widehat{T}^{(n)q}_{(r_m(t))} \chi_n - \sum_q c_q(n, j, s) \widehat{T}^{(n)q}_{(r_m(t+1))} \chi_n \right\|_{L_2} < \varepsilon_{t+1}. \quad (8)$$

Since the first level of $C_{1,1}$ is a union of levels of every column of every $T_{(r_m(s))}$, by virtue of Propositions 4 and 6 we can choose a finite collection of numbers $c_q(n, j, k)$ satisfying (7) for $s = k$.

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Finally, it is easy to check that we can choose $m(k)$ sufficiently large so that, additionally,

$$0 < a_k - a(m(k), k) < \frac{\delta_k}{6}, \tag{9}$$

$$|a_{k+1} - a_k| < \frac{\delta_k}{6}, \tag{10}$$

and for $s \le k$, $n \le s$, $j \le s$,

$$\left\| \sum_q c_q(n, j, s) \widehat{T}^{(n)q}_{(r_m(k))} \chi_n - \sum_q c_q(n, j, s) \widehat{T}^{(n)q}_{(r_m(k+1))} \chi_n \right\|_{L_2} < \varepsilon_{k+1}. \tag{11}$$

Let us consider the sequence $(m(k))$ (and $T_{(r_m)}$) constructed inductively above. Note that $|a - a_k| < \delta_k$ for every k. Indeed, by (9) and (10) we have

$$\begin{aligned} |a - a_k| &= \left| a(m(k), k) - a_k + \sum_{i \ge k} \big(a(m(i+1), i+1) - a(m(i), i) \big) \right| \\ &< \frac{\delta_k}{6} + \sum_{i \ge k} |a_{i+1} - a_i| + \frac{\delta_i}{6} + \frac{\delta_{i+1}}{6} < \delta_k. \end{aligned}$$

Therefore, by using (6), (7), (8), and (11), we conclude that

$$\text{for all } k \text{ and for all } n \le k,\ j \le k,$$
$$\left\| g_{n,j,a} - \sum_q c_q(n, j, k) \widehat{T}^{(n)q}_{(r_m)} \chi_n \right\|_{L_2} < \varepsilon_k + \sum_{j \ge 0} \varepsilon_{k+j} < 3\varepsilon_k.$$

This implies that

$$\text{for all } n \text{ and } j, \quad g_{n,j,a} \in \overline{\text{span}}\{ \widehat{T}^{(n)q}_{(r_m)} \chi_n : q \in \mathbb{Z} \}$$

or, equivalently,

$$\text{for all } n, \quad H_{\text{sym}}(n) = \overline{\text{span}}\{ \widehat{T}^{(n)q}_{(r_m)} \chi_n : q \in \mathbb{Z} \}.$$

Proposition 5 is proved. □

Remark 8. Since every staircase transformation is defined by the pair $(h_1, (r_m))$ uniquely up to a measure-theoretical isomorphism, Proposition 5 is valid for any staircase $T_{(r_m)}$ having the same sequence (r_m) as above (we can choose every h_1). Moreover, it is easy to check that Proposition 5 also holds for a slightly larger subclass of the staircase $T_{(r_m)}$, where (r_m) is constructed in the same manner as in (5) but we place between blocks $(r_1(k), \dots, r_{m(k)-1}(k))$ finite blocks B_k, with certain conditions on B_k.

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